On a conjecture of Coron and Guerrero

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Outline

The transport-diffusion equation controlled on one side
- Introduction
- The transport equation
- Uniform controllability in the vanishing viscosity limit

The heat equation controlled on one side
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- Link between the heat equation and the transport-diffusion equation

Conclusion and perspectives
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Notations

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- diffusion coefficient $\varepsilon > 0$,
- length $L > 0$,
- $y^0 \in L^2(0, L)$ initial condition,
- control $v \in L^2(0, T)$,
- $Q := (0, T) \times (0, L)$.

The equation

$$\begin{cases} y_t - \varepsilon y_{xx} + My_x & = 0 \text{ in } Q, \\ y(.,0) & = v(t) \text{ in } (0,T), \\ y(.,L) & = 0 \text{ in } (0,T). \end{cases} \qquad \text{(T-D)}$$

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Well-posedness: given initial condition $y^0 \in L^2(0,L)$, unique solution $y \in C^0([0,T], L^2(0,L)) \cap C^1((0,T], L^2(0,L))$ and for all $t > 0$, $y(t,.) \in H^2(0,L)$.

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$\Rightarrow$ We are interested only here in null controllability.

A result of control

Theorem (Fattorini-Russell'71, Fursikov-Imanuvilov'95)

For all $y^0 \in L^2(0,L)$, there exists a control $v \in L^2(0,T)$ such that the solution y of (T-D) *verifies $y(T,.) = 0$.*

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Question: what about the uniform controllability when $\varepsilon \to 0$?

The case $\varepsilon = 0$ and $M > 0$ (1)

We assume $M > 0$.

$$\begin{cases} y_t + My_x & = 0 \text{ in } Q, \\ y(.,0) & = v(t) \text{ in } (0,T). \end{cases}$$

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Initial condition: $y^0 \in L^2(0,L)$. We extend y^0, v on $\mathbb{R}^+$ by 0 and the unique weak solution is $y^0(x - Mt) + v(t - x/M)$ (restricted on Q). Area of influence of y^0: $x > Mt$. Area of influence of v: $x < Mt$.

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1. $T \geqslant L/M$: $x - MT < 0$ for $x \in (0, L)$ so $y(T, x) \equiv 0$ on $(0, L)$ for $v = 0$.
   $\Rightarrow$ Null-controllable, the cost of the control is 0.

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2. $T < L/M$: $x - MT \in (0, L)$ and $x \in (0, L)$ iff $x \in (MT, L)$ so $y(T, x)$ is null for $x \in (0, MT)$ but not necessarily on (MT, L) as soon as $y^0 \not\equiv 0$ on $(0, MT - L)$.
   $\Rightarrow$ Never null-controllable, the cost of the control is $+\infty$.

The case $\varepsilon = 0$ and $M < 0$ (1)

We assume $M < 0$.

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2. $T < L/|M|$: $x - MT \in (0, L)$ and $x \in (0, L)$ iff $x \in (0, L - |M|T)$ so $y(T, x)$ is null for $x \in (L - |M|T, L)$ but not necessarily on $(0, L - |M|T)$ as soon as $y^0 \not\equiv 0$ on $(0, L - |M|T)$. $\Rightarrow$ Never null-controllable, the cost of the control is $+\infty$.

What should we expect and what is true?

Definition

Cost of the control for (T-D)*:*

$$C_{TD}(T,L,M,\varepsilon) := \sup_{y^0} \inf_{(y,v) \text{ ver. (T-D) and } y(T)=0} \frac{||v||_{L^2(0,T)}}{||y^0||_{L^2(0,T)}}.$$

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Natural conjecture concerning (T-D):

Conjecture

$C_{TD}(T,L,M,\varepsilon) \to 0$ *when* $\varepsilon \to 0$ *for* $T \geqslant L/|M|$ *and*
$C_{TD}(T,L,M,\varepsilon) \to \infty$ *when* $\varepsilon \to 0$ *for* $T < L/|M|$.

What should we expect and what is true? (2)

Conjecture (partially) false:

Theorem (Coron-Guerrero'05)

1. *If* $M > 0$*, then* $C_{TD}(T, L, M, \varepsilon) \geqslant Ce^{\frac{C}{\varepsilon}}$ *when* $\varepsilon \to 0$ *for* $T < L/|M|$.
2. *If* $M < 0$*, then* $C_{TD}(T, L, M, \varepsilon) \geqslant Ce^{\frac{C}{\varepsilon}}$ *when* $\varepsilon \to 0$ *for* $T < 2L/|M|$.

What should we expect and what is true? (3)

On the other hand, upper bounds:

Theorem (Coron-Guerrero'05)

1. *If $M > 0$, then $C_{TD}(T, L, M, \varepsilon) \leqslant Ce^{\frac{-C}{\varepsilon}}$ when $\varepsilon \to 0$ for $T \geqslant 4.3L/|M|$.*
2. *If $M < 0$, then $C_{TD}(T, L, M, \varepsilon) \leqslant Ce^{\frac{-C}{\varepsilon}}$ when $\varepsilon \to 0$ for $T \geqslant 57.2L/|M|$.*

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Method: Carleman estimates and adapted dissipation estimate on the adjoint system.

What should we expect and what is true? (4)

Improvement:

Theorem (Glass'09)

1. *If* $M > 0$*, then* $C_{TD}(T, L, M, \varepsilon) \leqslant Ce^{\frac{-c}{\varepsilon}}$ *when* $\varepsilon \to 0$ *for* $T \geqslant 4.2L/|M|$*.*
2. *If* $M < 0$*,* $C_{TD}(T, L, M, \varepsilon) \leqslant Ce^{\frac{-c}{\varepsilon}}$ *when* $\varepsilon \to 0$ *for* $T \geqslant 6.1L/|M|$*.*

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Method: similar to the moment method but on the adjoint equation.

What is conjectured

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The numbers given in the previous theorems come from technical restrictions. $\Rightarrow$ New "natural" conjecture concerning (T-D):

Conjecture (Coron-Guerrero'05)

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2. *If* $M < 0$*,* $C_{TD}(T, L, M, \varepsilon) \to 0$ *when* $\varepsilon \to 0$ *for* $T \geqslant 2L/|M|$*.*

The heat equation

$$\begin{cases} z_t - z_{xx} & = 0 \text{ in } Q, \\ z(.,0) & = w(t) \text{ in } (0,T), \\ z(.,L) & = 0 \text{ in } (0,T), \end{cases} \tag{Heat}$$

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Well-posedness: unique solution
$z \in C^0([0,T], L^2(0,L)) \cap C^1((0,T], L^2(0,L))$ with initial condition
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As before, only interested on null-controllability.

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Theorem (Fattorini-Russell'71, Fursikov-Imanuvilov'95, Lebeau-Robbiano'95)

For all $z^0 \in L^2(0, L)$, *there exists a control* $w \in L^2(0, T)$ *such that the solution* z *of* (Heat) *verifies* $z(T, .) = 0$.

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Question: what about the cost of the control when $T \to 0$?

The cost of fast controls for the heat equation

We introduce

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Conjecture (Miller'04, Ervedoza-Zuazua'11, ...)

$\alpha^* := L^2/4$, *i.e.* $C_H(T,L) \approx e^{(L^2/4)^+/T}$.

An relevant changing of unknowns

The fondamental remark is the following:

Lemma

y verifies (T-D) with initial condition y^0 and control v iff

$$z(t,x) = e^{\frac{M^2 t}{4\varepsilon^2} - \frac{Mx}{2\varepsilon}} y(\frac{t}{\varepsilon}, x)$$

verifies (Heat) posed on $(0, \varepsilon T) \times (0, L)$ with initial condition $z^0(x) := e^{-\frac{Mx}{2\varepsilon}} y^0(x)$ and control $w(t) := e^{\frac{M^2 t}{4\varepsilon^2}} v(\frac{t}{\varepsilon})$.

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Proof: direct computations (differentiate z with respect to t and x and compare with y.)

$\Rightarrow$ We can estimate $C_{TD}(T, L, M, \varepsilon)$ in the vanishing viscosity limit by estimating $C_H(T, L)$ in small time!

Estimations on the costs of controls

Link between the two costs:

Lemma

For all $T_0 < T$, one has

$$\begin{cases} C_{TD}(T,L,M,\varepsilon) \leqslant \dfrac{e^{-\frac{M^2 T_0}{4\varepsilon}}}{\sqrt{\varepsilon}} C_D(\varepsilon(T-T_0),L) & \text{if } M>0, \\ C_{TD}(T,L,M,\varepsilon) \leqslant \dfrac{e^{-\frac{M^2 T_0}{4\varepsilon}+\frac{|M|L}{2\varepsilon}}}{\sqrt{\varepsilon}} C_D(\varepsilon(T-T_0),L) & \text{if } M<0. \end{cases}$$

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Proof: use the definitions of the costs, the previous changing of unknowns and let System (T-D) evolve naturally on $(0,T_0)$ (i.e. the control is 0 on $(0,T_0)$).

Improving the constants (1)

We can deduce the following result for $M > 0$:

Theorem (Lissy'12)

Assume that $M > 0$ and $T > \frac{2\sqrt{3}L}{M} \geqslant 3.6L/M$. Then there exists some constant $K > 0$, there exists some constant $C > 0$ such that, for all $\varepsilon > 0$ and all $y^0 \in L^2(0,L)$, there exists a solution $(y_\varepsilon, v_\epsilon)$ of the control problem (T-D) verifying $y_\varepsilon(T,.) = 0$ and

$$||v_\varepsilon||_{L^2(0,T)} \leqslant Ce^{-\frac{K}{\varepsilon}}||y^0||_{L^2(0,L)}.$$

Moreover, if we assume that the conjecture $\alpha^* = L^2/4$ is verified, then one can state the same result as soon as

$$T > \frac{2L}{|M|}.$$

Improving the constants (2)

We have a similar result for $M < 0$:

Theorem (Lissy'12)

Assume that $M < 0$ *and* $T > \frac{(2\sqrt{3}+2)L}{|M|} \geqslant 5.6L/M$. *Then there exists some constant* $K > 0$, *there exists some constant* $C > 0$ *such that, for all* $\varepsilon > 0$ *and all* $y^0 \in L^2(0,L)$, *there exists a solution* $(y_\varepsilon, v_\epsilon)$ *of the control problem* (T-D) *verifying* $y_\varepsilon(T,.) = 0$ *and*

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Moreover, if we assume that the conjecture $\alpha^* = L^2/4$ *is verified, then one can state the same result as soon as*

$$T > \frac{4L}{|M|}.$$

Improving the constants (3)

Proof: use the previous Lemma and Tenenbaum-Tucsnak'07 ($\alpha^* = 3L^2/4$) and optimize T_0 (for $M > 0$, $T_0 = T/2$, for $M < 0$, $T_0 = 1/2 + 1/(2 + 2\sqrt{3})$). See what happens if we assume $\alpha^* = L^2/4$.

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Every better upper bound on α^* will automatically provide a better lower bound for the time needed to ensure exponential decay of $C_{TD}(T, L, M, \varepsilon)$.
Unfortunately, even with the best α^* possible we cannot recover the conjecture of Coron and Guerrero!

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- The following observability estimate is known for (Heat) (Ervedoza-Zuazua'11):

$$\int_0^T e^{-\frac{L^2}{2t}}||z(t,.)||^2_{L^2(0,L)}dt \leqslant C(T)\int_0^T |\partial_x z(t,0)|^2 dt.$$

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$\Rightarrow$ We would obtain easily the Conjecture above if $C(T)$ is less than exponential in $1/T$, but it is an open question...

What could possibly be done for (Heat)

- Lebeau-Robbiano: even if we obtain the best estimate for the spectral inequality, we lose for the moment a factor 2 in the computations of the cost of the control (Miller'10).
- Moment method: Tenenbaum-Tucsnak can be improved?
- The following observability estimate is known for (Heat) (Ervedoza-Zuazua'11):

$$\int_0^T e^{-\frac{L^2}{2t}} ||z(t,.)||^2_{L^2(0,L)} dt \leqslant C(T) \int_0^T |\partial_x z(t,0)|^2 dt.$$

$\Rightarrow$ We would obtain easily the Conjecture above if $C(T)$ is less than exponential in $1/T$, but it is an open question...

- Other method: Use of quantities under the form $\int_0^L |\nabla z|^2 G(t,x)dx / \int_0^L |z|^2 G(t,x)dx$ where G is a backward heat kernel (Wang-Phung'11).

Conclusion

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Thank you for your attention!