Null controllability of one-dimensional parabolic equations by the flatness approach

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Let $\Omega \subset \mathbb{R}^N$ be a bounded smooth open set, $\Gamma_0 \subset \partial \Omega$ be a (nonempty) open set, and $T > 0$.

We are concerned with the **null controllability problem**: given $\theta_0$, find a function $u$ s.t. the solution of

\[
\begin{align*}
\theta_t - \Delta \theta &= 0 \quad (t, x) \in (0, T) \times \Omega, \\
\frac{\partial \theta}{\partial \nu} &= 1_{\Gamma_0} u(t, x) \quad (t, x) \in (0, T) \times \Omega, \\
\theta(0, x) &= \theta_0(x), \quad x \in \Omega.
\end{align*}
\]

satisfies

\[
\theta(T, x) = 0 \quad x \in \Omega
\]

Huge literature...
Duality methods (observability estimate for the adjoint eq.)
- Fattorini-Russell ’71, Luxembourg-Korevarr ’71, Dolecki ’73 (1D, using biorthogonal families and complex analysis)
- Lebeau-Robbiano ’95, Imanuvilov-Fursikov 96’ (ND, $\forall(\Omega, \Gamma_0, T)$, using Carleman estimates)

Direct methods
- Jones ’77, Littman ’78 (construction of a fundamental solution with compact support in time, $\Gamma_0 = \partial\Omega$)
- Littman-Taylor ’07 (solution of ill-posed problems)
- Laroche-Martin-Rouchon ’00 (approximate controllability using a flatness approach)

Here, we shall revisit the flatness approach, deriving the null controllability, and show its relevance to numerics.
Introduced in 1995 by M. Fliess, J. Lévine, Ph. Martin, P. Rouchon for (linear or nonlinear) ODE; very useful for motion planning of mechanical systems.

Method applied then by Laroche-Martin-Rouchon to derive the approximate controllability of (i) the 1D heat eq; (ii) the beam equation; (iii) the linearized KdV equation.

The heat control problem reads:

\[
\begin{align*}
\theta_t - \theta_{xx} &= 0, \quad x \in (0, 1) \\
\theta_x(t, 0) &= 0, \quad \theta_x(t, 1) = u(t), \\
\theta(0, x) &= \theta_0.
\end{align*}
\]

They proved in 2000 that for initial data decomposed as

\[
\theta_0(x) = \sum_{i \geq 0} y_i \frac{x^{2i}}{(2i)!}
\]

with

\[
|y_i| \leq C \frac{i!^s}{R^i}, \quad i \geq 0
\]

with \(s \in (1, 2), \ C, \ R > 0\), the system can be driven to 0 with a control \(u(t)\) that is Gevrey of order \(s\).
Take $y = \theta(t, 0)$ as output. It is \textbf{flat}, in the sense that the map $\theta \to y$ is a \textbf{bijection} between appropriate spaces of functions.

Seek a formal solution (analytic in $x$) in the form

$$\theta(t, x) = \sum_{i \geq 0} a_i(t) \frac{x^i}{i!}$$

Plugging this sum in the heat eq. gives $\sum_{i \geq 0} [a_{i+2} - a_i'] \frac{x^i}{i!} = 0$ ($' = d/dt$), and hence

$$a_{i+2} = a_i', \quad i \geq 0.$$  

Since $a_0(t) = \theta(t, 0) = y(t)$ and $a_1(t) = 0$, we arrive to

$$a_{2i+1} = 0, \quad a_{2i} = y^{(i)}, i \geq 0,$$

$$\theta(t, x) = \sum_{i \geq 0} y^{(i)}(t) \frac{x^{2i}}{(2i)!}, \quad u(t) = \theta_x(t, 1) = \sum_{i \geq 1} \frac{y^{(i)}(t)}{(2i - 1)!}.$$
Since $\theta(t, x) = \sum_{i \geq 0} y^{(i)}(t) \frac{x^{2i}}{(2i)!}$, it remains to find $y \in C^\infty([0, T])$ s.t. the series converges and

$$y^{(i)}(0) = y_i, \quad y^{(i)}(T) = 0, \quad i \geq 0.$$ 

Impossible to do with an analytic function, but possible with a function Gevrey of order $s > 1$

- $y \in C^\infty([0, T])$ is **Gevrey of order** $s \geq 0$ if there exist $R, C > 0$ such that

$$|y^{(p)}(t)| \leq C \frac{p!^s}{R^p}, \quad \forall p \in \mathbb{N}, \quad \forall t \in [0, T]$$

The larger $s$, the less regular $y$ is ($s = 1 \iff y \in C^\omega$)

- $\theta \in C^\infty([t_1, t_2] \times [0, 1])$ is **Gevrey of order** $s_1$ in $x$ and $s_2$ in $t$ if

$$|\partial_x^{p_1} \partial_t^{p_2} \theta(t, x)| \leq C \frac{(p_1!)^{s_1} (p_2!)^{s_2}}{R_1^{p_1} R_2^{p_2}} \quad \forall p_1, p_2 \in \mathbb{N}, \quad \forall (t, x) \in [t_1, t_2] \times [0, 1]$$
Theorem

Let $\theta_0 \in L^2(0, 1)$ and $T > 0$. Pick $\tau \in (0, T)$ and $s \in (1, 2)$. There exists $y \in C^\infty([\tau, T])$ Gevrey of order $s$ on $[\tau, T]$ such that, setting

$$u(t) = \begin{cases} 
0 & \text{if } 0 \leq t \leq \tau \\
\sum_{i \geq 0} \frac{y^{(i)}(t)}{(2i-1)!} & \text{if } \tau < t \leq T, 
\end{cases}$$

the solution $\theta$ of

$$\begin{align*}
\theta_t - \theta_{xx} &= 0, \quad x \in (0, 1) \\
\theta_x(t, 0) &= 0, \quad \theta_x(t, 1) = u(t), \\
\theta(0, x) &= \theta_0(x)
\end{align*}$$

satisfies $\theta(T, \cdot) = 0$. Furthermore, $u$ is Gevrey of order $s$ in $t$ on $[0, T]$, and $\theta \in C([0, T], L^2(0, 1)) \cap C^\infty((0, T) \times [0, 1])$ is Gevrey of order $s$ in $t$ and $s/2$ in $x$ on $[\varepsilon, T] \times [0, 1]$ for all $\varepsilon \in (0, T)$. \[8/32\]
We apply a null control to smooth the state and reach the class of states for which Laroche-Martin-Rouchon result is valid.

Decomposing the initial state $\theta_0$ as a Fourier series of cosines $\theta_0(x) = \sum_{n \geq 0} c_n \sqrt{2} \cos(n\pi x)$, we obtain

$$\theta(\tau, x) = \sum_{n \geq 0} c_n e^{-n^2 \pi^2 \tau} \sqrt{2} \cos(n\pi x) = \sum_{i \geq 0} y_i \frac{x^{2i}}{(2i)!}$$

where $y_i = \sqrt{2} \sum_{n \geq 0} c_n e^{-n^2 \pi^2 \tau} (-1)^i (n\pi)^{2i}$

**Lemma**

$$|y_i| \leq C ||\theta_0||_{L^1(0,1)} (1 + \tau^{-\frac{1}{2}}) \frac{i!}{\tau^i} \quad \forall i \geq 0$$

for some constant $C > 0$, so that $x \rightarrow \theta(\tau, x)$ is Gevrey of order $1/2$. 
Proposition

(Flatness property) Let \( s \in (1, 2) \) and \( y \in C^\infty([t_1, t_2]) \) \((-\infty < t_1 < t_2 < \infty)\) be Gevrey of order \( s \) on \([t_1, t_2]\). Let

\[
\theta(t, x) := \sum_{i \geq 0} \frac{x^{(2i)}}{(2i)!} y^{(i)}(t).
\]

Then \( \theta \) is Gevrey of order \( s \) in \( t \) and \( s/2 \) in \( x \) on \([t_1, t_2] \times [0, 1]\) and it solves the ill-posed problem

\[
\begin{align*}
th(t, x) & = 0, \quad (t, x) \in [t_1, t_2] \times [0, 1], \\
\theta(t, 0) = y(t), \quad \theta_x(t, 0) & = 0.
\end{align*}
\]

Thus \( u(t) = \theta_x(t, 1) = \sum_{i \geq 1} \frac{y^{(i)}(t)}{(2i-1)!} \) is Gevrey of order \( s \) on \([t_1, t_2]\).

It remains to design a function \( y \in C^\infty([\tau, T]) \) Gevrey of order \( s \in (1, 2) \) such that

\[
y^{(i)}(\tau) = y_i, \quad y^{(i)}(T) = 0, \quad \forall i \geq 0.
\]
Step 2: Design of the control (2)

- For any \( s \in (1, 2) \), we introduce the “Gevrey step function”

\[
\phi_s(t) = \begin{cases} 
1 & \text{if } t \leq 0 \\
\frac{e^{-(1-t)-\kappa}}{e^{-(1-t)-\kappa} + e^{-t-\kappa}} & \text{if } 0 < t < 1 \\
0 & \text{if } t \geq 1.
\end{cases}
\]

where \( \kappa = (s - 1)^{-1} \). Then \( \phi_s \) is Gevrey of order \( s \) on \([-T, T]\) for all \( T > 0 \).

- Let

\[
\bar{y}(t) = \sum_{i \geq 0} y_i \frac{(t - \tau)^i}{i!}
\]

Since \(|y_i| \leq Ci! / \tau^i\), \( \bar{y} \) is Gevrey 1 (analytic) on \([\tau, \tau + R]\) if \( R < \tau \).

Actually, we noticed that \( \bar{y} \) can be extended to \((0, +\infty)\) as an analytic function: indeed, since \( y_i = \sqrt{2} \sum_{n \geq 0} c_n e^{-n^2 \pi^2 \tau} (-1)^i (n\pi)^{2i} \), we have

\[
\bar{y}(t) = \sqrt{2} \sum_{n \geq 0} c_n e^{-n^2 \pi^2 t}
\]

- For \( y \), it is sufficient to pick \( s \in (1, 2) \), \( 0 < R \leq T - \tau \) (where \( 0 < \tau < T \)), and to set

\[
y(t) := \phi_s\left(\frac{t - \tau}{R}\right)\bar{y}(t), \quad t \in [\tau, T].
\]
We are interested in describing the states $\theta_T$ that can be reached at time $T$ from 0 (as in Fattorini-Russell ’71):

$$ \theta_t - \theta_{xx} = 0, \quad x \in (-1, 1), \ t \in (0, T) $$

$$ \theta(t, -1) = f(t), \ \theta(t, 1) = g(t), \ t \in (0, T) $$

$$ \theta(0, x) = 0 $$

$$ \theta(T, x) = \theta_T(x) $$

**Theorem**

If $\theta_T(x) = \sum_{n \geq 0} a_n \frac{x^n}{n!}$ with $\sum_{n \geq 0} |a_n| \frac{R^n}{n!} < \infty$ and $R > R_0 = e^{(2e) - 1}$, then $\theta_T$ is reachable from 0 in time $T$ with $f$ and $g$ Gevrey of order 2 on $[0, T]$.

**Example:** $\theta_T(x) = 1/(x^2 + a^2)$.

Reachable if $a > R_0 = e^{(2e) - 1} \sim 1.2$, not reachable if $a < 1$. 
The above result is based on the flatness approach, and it requires a result like Borel-Ritt theorem:

**Proposition**

For any $R > 1$ and any sequence $(a_n)_{n \geq 0}$ of real numbers satisfying

$$|a_n| \leq C \frac{(2n)!}{R^{2n}}$$

one can find a function $f : \mathbb{R} \rightarrow \mathbb{R}$ with compact support in $[-1, 1]$ and Gevrey of order 2, such that

$$f^{(n)}(0) = a_n \quad \forall n \geq 0,$$

$$|f^{(n)}(t)| \leq C' \frac{(2n)!}{(R/R_0)^{2n}} \quad \forall t \in [-1, 1]$$

where $R_0 = e^{(2e)^{-1}} \sim 1.2$. 
Consider now the equation

\[(a(x)\theta_x)_x + b(x)\theta_x + c(x)\theta - \rho(x)\theta_t = 0\]

where \(a, b, c, \rho \in L^1(0, 1)\).

Alessandrini-Escauriaza (2006) proved the null controllability of this equation (with internal or Dirichlet boundary control) when \(a, b, c, \rho \in L^\infty(0, 1)\) with

\[a(x) > K > 0 \quad \text{and} \quad \rho(x) > K > 0 \quad \text{a.e. in} \ (0, 1)\]

**Method of proof:** some extension to quasiregular mappings of classical interpolation results in complex analysis.

We shall see that this result can be extended to parabolic equations with singular or degenerate coefficients by using the flatness approach. (for degenerate eq.: Cannarsa-Martinez-Vancostenoble 2004,...)
Pick $a, b, c, \rho$ with

$$a(x) > 0 \text{ and } \rho(x) > 0 \quad \text{for a.e. } x \in (0, 1)$$

$$\left(\frac{1}{a}, \frac{b}{a}, c, \rho\right) \in [L^1(0, 1)]^4$$

$$\exists K \geq 0, \frac{c(x)}{\rho(x)} \leq K \quad \text{for a.e. } x \in (0, 1)$$

$$\exists p \in (1, \infty], \ a^{1-\frac{1}{p}} \rho \in L^p(0, 1).$$

**Theorem**

Let $(a, b, c, \rho)$ be as above, and $(\alpha_0, \beta_0) \neq (0, 0), \ (\alpha_1, \beta_1) \neq (0, 0)$. Let $\theta_0 \in L^1_{\rho(x)dx}(0, 1)$ and $T > 0$. Pick $\tau \in (0, T)$ and $s \in (1, 2 - p^{-1})$. Then there exists a control $h = h(t)$ Gevrey of order $s$ on $[0, T]$ such that the solution $\theta$ of

$$(a(x) \theta_x)_x + b(x) \theta_x + c(x) \theta - \rho(x) \theta_t = 0, \quad x \in (0, 1)$$

$$\alpha_0 \theta(t, 0) + \beta_0 (a \theta_x)(t, 0) = 0,$$

$$\alpha_1 \theta(t, 1) + \beta_1 (a \theta_x)(t, 1) = h(t),$$

$$\theta(0, x) = \theta_0(x)$$

satisfies $\theta(T, .) = 0$. 
Examples

- $(a(x)\theta_x)_x - \theta_t = 0$, with $a(x) > 0$ and
  \[ a, 1/a \in L^1(0, 1) \]

  Possible: $a \sim (x - x_0)^r$ with $-1 < r < 0$ (singular) or $0 < r < 1$ (degenerate), and not only at a single point $x_0 \in [0, 1]$, but at a sequence of points as well!

  Think about $a(x) = |\sin(x^{-1})|^r$, $-1 < r < 1$.

- $\theta_{xx} + \frac{\mu}{x^2} \theta - \theta_t = 0$, $\mu \leq 1/4$ (no need of Carleman or Hardy inequal.).

- Transmission pb. for the heat eq. (piecewise constant coef.)

  \[
  \begin{align*}
  \rho_0 \theta_t &= a_0 \theta_{xx}, \quad 0 < x < X \\
  \rho_1 \theta_t &= a_1 \theta_{xx}, \quad X < x < 1 \\
  \theta(t, X^-) &= \theta(t, X^+) \\
  a_0 \theta_x(t, X^-) &= a_1 \theta_x(t, X^+) 
  \end{align*}
  \]
So far, the flatness approach was applied to 1D PDE (and for radial solution of 2D problems). The expansion of the solution as a power series in all the spatial coordinates seems not to work well, even in 2D.

Here, we shall see that we can deal with the null controllability of the heat equation on a cylinder

\[ \Omega = \omega \times (0,1) \subset \mathbb{R}^N \]

where \( \omega \subset \mathbb{R}^{N-1} \) is a smooth, bounded open set, and \( N \geq 2 \). We thus consider the control problem (\( x = (x', x_N) \))

\[
\begin{align*}
\theta_t - \Delta \theta &= 0, \quad (t, x) \in (0, T) \times \Omega \\
\frac{\partial \theta}{\partial \nu}(t, x', 1) &= u(t, x'), \quad (t, x') \in (0, T) \times \omega \\
\frac{\partial \theta}{\partial \nu}(t, x) &= 0 \quad (t, x) \in (0, T) \times \partial \Omega \setminus \omega \times \{0\} \\
\theta(0, x) &= \theta(x), \quad x \in \Omega
\end{align*}
\]

For \( N = 3 \), this is nothing but the control of the temperature of a metallic rod by the heat flux on one lateral section.
The good way to solve the problem is to consider “hybrid” expansions of $\theta$ mixing Fourier series in $x'$ (no control on $\partial \omega$) and power series in $x_N$ (control at $x_N = 1$).

Introduce an orthonormal basis in $L^2(\omega)$, $(e_j)_{j \geq 0}$, constituted of eigenvectors for the Neumann Laplacian in $\omega \subset \mathbb{R}^{N-1}$, i.e.

\[-\Delta' e_j = \lambda_j e_j \quad \text{in } \omega\]
\[\frac{\partial e_j}{\partial \nu'} = 0 \quad \text{on } \partial \omega\]

where $\Delta' = \partial^2_{x_1} + \cdots \partial^2_{x_{N-1}}$, $\nu' = $ outward unit normal to $\omega$, $0 = \lambda_0 < \lambda_1 \leq \lambda_j \leq \lambda_{j+1} \leq \cdots$.

Decompose $\theta(t, x', 0)$ as

$$\theta(t, x', 0) = \sum_{j \geq 0} z_j(t)e_j(x').$$

We claim that the system is flat, with $(z_j(t))_{j \geq 0}$ as “flat output”. Indeed, given a sequence $(z_j(t))_{j \geq 0}$ of smooth functions, we seek a formal solution of the heat equation in the form

$$\theta(t, x', x_N) = \sum_{i \geq 0} \frac{x_N^i}{i!} a_i(t, x')$$

where the $a_i$’s are still to be defined.
Expansion of the solution (2)

Plugging the formal solution \( \theta = \sum_{i \geq 0} \frac{x_N^i}{i!} a_i \) in the heat equation gives

\[
\sum_{i \geq 0} \frac{x_N^i}{i!} \left[ a_{i+2}(t, x') - (\partial_t - \Delta') a_i(t, x') \right] = 0
\]

so that \( a_{i+2} = (\partial_t - \Delta') a_i \) for all \( i \geq 0 \). Moreover

\[
a_0(t, x') = \theta(t, x', 0) = \sum_{j \geq 0} z_j(t) e_j(x'), \quad a_1(t, x') = 0.
\]

Therefore, for all \( i \geq 0 \)

\[
a_{2i+1} = 0,
\]

\[
a_{2i} = (\partial_t - \Delta')^i a_0 = \sum_{j \geq 0} (\partial_t - \Delta')^i [z_j(t) e_j(x')] = \sum_{j \geq 0} e_j(x') (\partial_t + \lambda_j)^i z_j(t)
\]

\[
= \sum_{j \geq 0} e_j(x') e^{-\lambda_j t} y_j^{(i)}(t)
\]

where we have set \( y_j(t) := e^{\lambda_j t} z_j(t) \). We arrive to

\[
\theta(t, x', x_N) = \sum_{j \geq 0} e^{-\lambda_j t} e_j(x') \sum_{i \geq 0} y_j^{(i)}(t) \frac{x_N^{(2i)}}{(2i)!}
\]
Proposition

Let $s \in (1, 2)$, $-\infty < t_1 < t_2 < \infty$, and let $y = (y_j)_{j \geq 0}$ in $C^\infty([t_1, t_2])$ satisfy for some constants $M, R > 0$

$$|y_j^{(i)}(t)| \leq M \frac{i!^s}{R^i}, \quad \forall i, j \geq 0, \forall t \in [t_1, t_2].$$

Then the function

$$\theta(t, x', x_N) = \sum_{j \geq 0} e^{-\lambda_j t} e_j(x') \sum_{i \geq 0} y_j^{(i)}(t) \frac{x_N^{(2i)}}{(2i)!}$$

is well defined in $[t_1, t_2] \times \overline{\Omega}$, and it is Gevrey of order $s$ in $t$, $1/2$ in $x_1, \ldots, x_{N-1}$ and $s/2$ in $x_N$. It solves the ill-posed problem

$$\theta_t - \Delta \theta = 0, \quad (t, x) \in [t_1, t_2] \times \overline{\Omega},$$

$$\theta(t, x', 0) = \sum_{j \geq 0} e^{-\lambda_j t} y_j(t) e_j(x'),$$

$$\theta_{x_N}(t, x', 0) = 0.$$
Null controllability of the heat equation on cylinders

Consider the control system

\[
(S) \begin{cases}
    \theta_t - \Delta \theta = 0, & (t, x) \in (0, t) \times \Omega \\
    \frac{\partial \theta}{\partial \nu}(t, x', 1) = u(t, x'), & (t, x') \in (0, T) \times \omega \\
    \frac{\partial \theta}{\partial \nu}(t, x) = 0 & (t, x) \in (0, T) \times \partial \Omega \setminus \omega \times \{0\} \\
    \theta(0, x) = \theta(x), & x \in \Omega
\end{cases}
\]

**Theorem**

Let \( \Omega = \omega \times (0, 1) \subset \mathbb{R}^{N-1} \times \mathbb{R} \) be as above, and let \( \theta_0 \in L^2(\Omega) \) and \( T > 0 \) be given. Pick any \( \tau \in (0, T) \) and any \( s \in (1, 2) \). Then there exists a sequence \((y_j)_{j \geq 0}\) of functions in \( C^\infty([\tau, T]) \) which are Gevrey of order \( s \) on \([\tau, T]\) and such that the control input

\[
u(t, x') = \begin{cases}
0 & \text{if } 0 \leq t \leq \tau, \\
\sum_{i,j \geq 0} e^{-\lambda_j t} \frac{y_j^{(i)}(t)}{(2i-1)!} e_j(x') & \text{if } \tau \leq t \leq T,
\end{cases}
\]

is Gevrey of order \( s \) in \( t \) and \( 1/2 \) in \( x_1, \ldots, x_{N-1} \) on \([0, T] \times \overline{\omega}\), and the solution \( \theta \) of \( (S) \) satisfies \( \theta(T, .) = 0 \).

Furthermore, \( \theta \in C([0, T], L^2(\Omega)) \cap C^\infty((0, T] \times \overline{\Omega}) \), and \( \theta \) is Gevrey of order \( s \) in \( t \), \( 1/2 \) in \( x_1, \ldots, x_{N-1} \) and \( s/2 \) in \( x_N \) on \([\epsilon, T] \times \overline{\Omega} \) for all \( \epsilon \in (0, T) \).
Numerical approximation

Assume given $T > 0$, $\tau \in (0, T)$, $s \in (1, 2)$, and $\theta_0 \in L^2(\Omega)$ decomposed as

$$\theta_0(x', x_N) = \sum_{j, n \geq 0} c_{j, n} e_j(x') \sqrt{2} \cos(n\pi x_N).$$

The exact solution $\theta$ of the previous control problem such that $\theta(T, .) = 0$ was given as

$$\theta(t, x', x_N) = \sum_{j, n \geq 0} c_{j, n} e^{-(\lambda_j + n^2 \pi^2)t} e_j(x') \sqrt{2} \cos(n\pi x_N), \quad 0 \leq t \leq \tau,$$

$$\theta(t, x', x_N) = \sum_{j \geq 0} e^{-\lambda_j t} e_j(x') \sum_{i \geq 0} y_j^{(i)}(t) \frac{x_N^{2i}}{(2i)!}, \quad \tau \leq t \leq T,$$

where

$$y_j(t) = \phi(t) \sum_{n \geq 0} c_{j, n} e^{-n^2 \pi^2 t}, \quad \tau \leq t \leq T,$$

$$\phi(t) = \phi_s\left(\frac{t - \tau}{T - \tau}\right), \quad \tau \leq t \leq T.$$

In practice, only partial sums can be computed. They prove to give accurate approximations of both the trajectory and the control. Exponentially small errors.
Initial state: $\theta_0 := 1_{(1/2,1)}(x) - 1_{(0,1/2)}(x)$
Parameters: $\tau = 0.3$, $R = 0.2$, $T = \tau + R = 0.5$, $s = 1.6$

Fig.1. $\bar{\theta}(t, x)$

Computations by Philippe Martin
Numerical simulations ($N=1$) Control

Initial state: $\theta_0 := 1_{(1/2,1)}(x) - 1_{(0,1/2)}(x)$

Parameters: $\tau = 0.3, R = 0.2, T = \tau + R = 0.5, s = 1.6$

Fig. 2. $\bar{u}(t)$ (blue) and $\|\bar{u}(t)\|_{L^2(0,t)}$ (green)

Computations by Philippe Martin
Numerical simulations \((N=2)\) Trajectory

Initial state: \(\theta_0 := (1_{(1/2,1)}(x_1) - 1_{(0,1/2)}(x_1))(1_{(0,1/2)}(x_2) - 1_{(1/2,1)}(x_2))\)

Parameters: \(\tau = 0.05, R = 0.25, T = \tau + R = 0.3, s = 1.65\)
Initial state: $\theta_0 := \left(1_{(1/2,1)}(x_1) - 1_{(0,1/2)}(x_1)\right)\left(1_{(0,1/2)}(x_2) - 1_{(1/2,1)}(x_2)\right)$

Parameters: $\tau = 0.05$, $R = 0.25$, $T = 0.35$, $s = 1.65$

Fig. 4. $\bar{u}(t, x_1)$

Computations by Philippe Martin
Numerical simulations ($N=1$, discontinuous coefficients) Trajectory

Initial state: $\theta_0 := \frac{1}{2}1_{(1/2,1)}(x) - \frac{1}{2}1_{(0,1/2)}(x)$

Parameters: $\tau = 0.3$, $T = 0.35$, $s = 1.6$, 
$(a_0, \rho_0, a_1, \rho_1) = (10/19, 15/8, 10, 1/8)$

Fig.1. $\bar{\theta}(t, x)$
With this approach, we have to compute \( N \geq 20 \) derivatives of some functions, e.g.

\[
\varphi(t) = \exp(-t^{-k}(1 - t)^{-k})
\]

where \( k = (s - 1)^{-1} \).

Purely numerical methods (using e.g. finite differences) not appropriate!

Formal methods limited to \( N \leq 20 \).

We compute the derivatives **by induction** as follows: Derivating \( \varphi \) yields

\[
p^{k+1} \frac{d}{dt} \varphi = kp \frac{d}{dt} \varphi
\]

where \( p(t) = t(1 - t) \).

Derivating \( i \) times that identity and using Leibniz’ rule results in

\[
p^{k+1} \varphi^{(i+1)} + \sum_{j=1}^{i} \binom{i}{j} (p^{k+1})^{(j)} \varphi^{(i+1-j)} = k(p \varphi^{(i)}) + i \ddot{p} \varphi^{(i-1)}
\]

This equation gives \( \varphi^{(i+1)} \) in terms of \( \varphi^{(0)}, ..., \varphi^{(i)} \), and those of \( p^{k+1} \).

In practice, \( N = 140 \) derivatives can be computed on line.
For the sake of simplicity, we limit ourselves to the 1D case

\[ i\theta_t + \theta_{xx} = 0, \quad 0 < x < 1 \]

\[ \theta(t, 0) = 0, \quad \theta(t, 1) = u(t) \]

\[ \theta(0, x) = \theta_0(x). \]

The null (\(\iff\) exact) controllability can be established by the same flatness approach as for the heat eq. However, the first step (smoothing effect) has to be modified, for the application of a null boundary control does not smooth out the solution as for the heat eq.

Following an idea in Rosier-Zhang (2009), we notice that a strong smoothing effect occurs if we consider Schrödinger equation on the whole line with a compactly supported initial data:

\[ iv_t + v_{xx} = 0, \quad -\infty < x < \infty \]

\[ v(0, x) = v_0(x) := \begin{cases} 
\theta_0(x) & \text{if } x \in (0, 1) \\
-\theta_0(-x) & \text{if } x \in (-1, 0), \\
0 & \text{if } x \in (-\infty, -1) \cup (1, +\infty)
\end{cases} \]

Fact: for any given \(\theta_0 \in L^2(0, 1)\) and \(\tau > 0\), the function \(x \rightarrow v(\tau, x)\) is Gevrey of order \(1/2\) on segments.
Flatness applied to Schrödinger

**Theorem**

Let \( \theta_0 \in L^2(0, 1) \) and \( T > 0 \). Pick \( \tau \in (0, T) \) and \( s \in (1, 2) \). There exists \( y \in C^\infty([\tau, T]) \) Gevrey of order \( s \) on \([\tau, T]\) such that, setting

\[
u(t, 1) - \sum_{k \geq 1} (-i)^k \frac{y^{(k)}(t)}{(2k+1)!}
\]

if \( \tau < t \leq T \),

the solution \( \theta \) of

\[
i\theta_t + \theta_{xx} = 0, \quad x \in (0, 1)
\]

\[
\theta(t, 0) = 0, \quad \theta(t, 1) = u(t),
\]

\[
\theta(0, x) = \theta_0(x)
\]

satisfies \( \theta(T, .) = 0 \). Furthermore, \( u \) is in \( L^4(0, T) \) and is Gevrey of order \( s \) in \( t \) on \([\varepsilon, T] \), and \( \theta \in C([0, T], L^2(0, 1)) \cap C^\infty((0, T] \times [0, 1]) \) is Gevrey of order \( s \) in \( t \) and \( s/2 \) in \( x \) on \([\varepsilon, T] \times [0, 1]\) for all \( \varepsilon \in (0, T) \).
Numerical simulations \((N=1)\)

Initial state: \(\theta_0 := 1_{(0.5,1)}(x) + i 1_{(0.2,0.7)}(x)\)

Parameters: \(\tau = 0.35, \ T = 0.5, \ s = 1.6\)

real part

imaginary part

Computations by Philippe Martin
The flatness approach allows to recover the null controllability of the heat equation in cylinders, with explicit controls and trajectories easy to approximate. Extension to parabolic equation with nonsmooth coefficients.

Similar results have been obtained for the control of Schrödinger equation. Smoothing effect (step 1) obtained in a different way.

Work in progress
- Extension to any pair \((\Omega, \Gamma_0)\).
- Controllability of the Korteweg-de Vries equation

Future direction of research
- Exact controllability results for linear/nonlinear equations
- Numerical investigation of the cost of the control in terms of the parameters \(\tau\) (free evolution), \(R\) (active control), \(s\) (Gevrey regularity)