Un point de vue Riemannien sur la convergence d'observateur en dimension finie.

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version réduite

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We consider a dynamical system

$$\frac{dx}{dt} = \dot{x} = f(x) \quad , \qquad y = h(x) \ .$$

with state x in $\mathbb{R}^n$ and measured output y in $\mathbb{R}^p$.

To ease the presentation, we assume it is complete, time independent, with f and h as smooth as needed and everything is "global".

We want to design an observer, i.e. a dynamical system with the measurements y coming from a real world process as input and with an output supposed to be an estimate of variables involved in a model of the process.

More precisely . . .

Problem :

Design a function $F : \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}^n$ such that the solutions $(X(x,t), \hat{X}((x,\hat{x}),t))$ in $\mathbb{R}^n \times \mathbb{R}^n$ of the extended system :

$$\dot{x} = f(x) \qquad , \qquad \dot{\hat{x}} = F(\hat{x}, h(x))$$

given system — observer

satisfy :

$$\lim_{t\to+\infty} \delta\left(\hat{X}((x,\hat{x}),t)\,,\; X(x,t)\right) \;=\; 0$$

where δ is a distance on $\mathbb{R}^n \times \mathbb{R}^n$.

- We restrict our attention to the case where system state x and observer state $\hat{x}$ evolve in the same space. Nonlinear Luenberger observers for instance do not satisfy this restriction.

- Slotine and Lohmiller (Automatica 98) emphasize the interest of looking at contracting flows. (See Jouffroy CDC 05 for a survey). It turns out that this is too demanding. It is sufficient to have a Riemannian distance between system state x and estimated state $\hat{x}$ which is decreasing along solutions.

- Although primarily Euclidean in this talk, the space, where x lives, is "structured " by the system properties like linear observability. This is why we may need a non flat metric to define the distance between x and $\hat{x}$.
  This is different from using the Riemannian metric linked to the system dynamics (e.g. Aghanan-Rouchon, IEEE TAC 2003). But related to Bonnabel, IEEE TAC 2010.

Plan

A. Necessary conditions

B. Sufficient conditions

C. P such that (f, h) is differentially detectable

D. Conclusions

A. Necessary conditions

A1. Differential detectability

A2. Strong geodesic convexity

Let be given : - the system $\dot{x} = f(x)$,

- the observer $\dot{\hat{x}} = F(\hat{x}, h(x))$
- and a complete Riemannian metric P on $\mathbb{R}^n$

Let $d(\hat{x}, x)$ be the Riemannian distance given by P.

Define the upper right-hand Dini derivative of the Riemannian distance along the (system-observer) solutions as :

$$\mathfrak{D}^+ d(\hat{x}, x) \ = \ \limsup_{t \to 0_+} \frac{d(\hat{X}(\hat{x}, t), X(x,t)) - d(\hat{x}, x)}{t}$$

Let $\mathcal{L}_f P$ denote the Lie derivative in the direction of f of the covariant 2-tensor[1] P

$$\mathcal{L}_f P(x) = \lim_{t \to 0} \frac{\left(I + t\frac{\partial f}{\partial x}(x)\right)^T P(x + tf(x)) \left(I + t\frac{\partial f}{\partial x}(x)\right) \ - \ P(x)}{t}$$

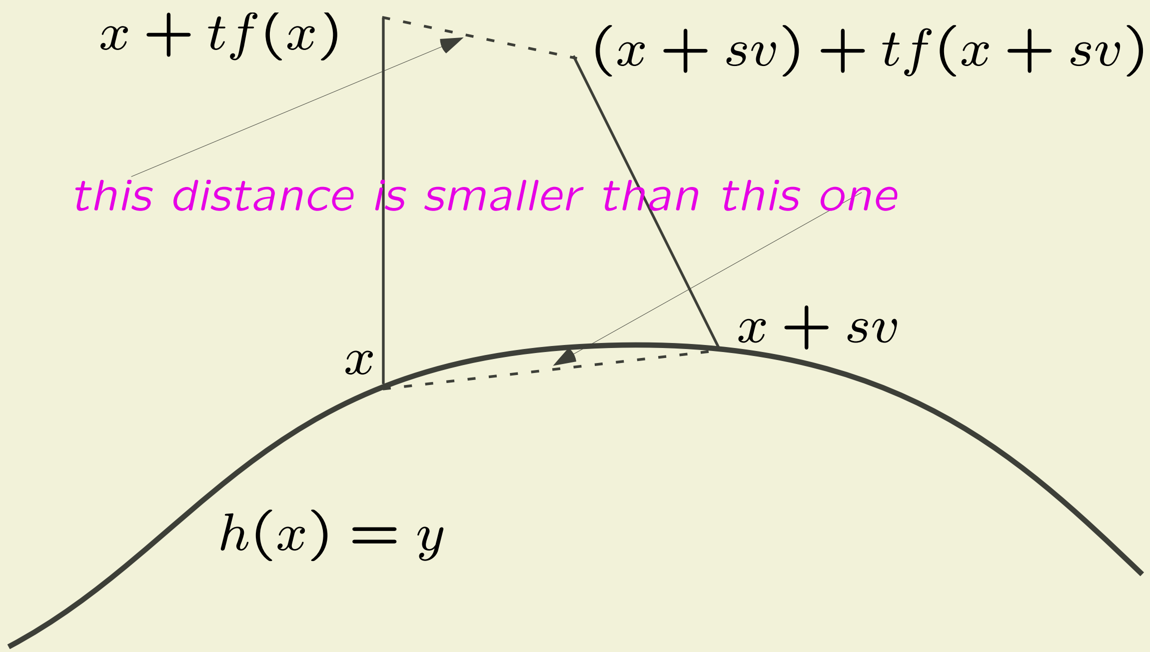
[1]With coordinates, $(\mathcal{L}_f P(x))_{ij} = \sum_k \left[2 P_{ik}(x) \frac{\partial f_k}{\partial x_j}(x) + \frac{\partial P_{ij}}{\partial x_k}(x) f_k(x)\right]$

Proposition: If $\mathfrak{D}^+ d(\hat{x}, x) \leq 0 \qquad \forall (\hat{x}, x) \in \mathbb{R}^n \times \mathbb{R}^n$,

then the vector field f is geodesically weakly monotonic tangentially to the level sets of h[1]
i.e.

$$v^\top \mathcal{L}_f P(x)\, v \leq 0 \qquad \forall (x, v) \in \mathbb{R}^n \times \mathbb{R}^n \quad \text{such that} \quad dh(x)\, v = 0 .$$

[1] This is a coordinate free condition.



f is geodesically monotonic tangentially to the level sets of h

Contraction only in the direction tangent to the level sets of h

In the observer jargon . . .

Definition: The system pair (f, h) is said weakly differentially detectable if there exists a metric P such that the vector field f is geodesically weakly monotonic tangentially to the level sets of the function h.

Proposition (continued): Furthermore, if there exists a function $\sigma : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ such that the function $(\hat{x}, x) \mapsto d(\hat{x}, x)\sigma(\hat{x}, x)$ is a C^2 function on a neighborhood $\mathcal{N}$ of $\{(\hat{x}, x) : \hat{x} = x\}$ with the property that, for some $\varepsilon > 0$,

$$\frac{\partial^2(d\,\sigma)}{\partial \hat{x}^2}(x, x) \geq \varepsilon P(x) \qquad \forall x \in \mathbb{R}^n$$

and

$$\mathfrak{D}^+ d(\hat{x}, x) \leq -\sigma(\hat{x}, x) \qquad \forall (\hat{x}, x) \in \mathcal{N}\ ,$$

then there exists a continuous function $\rho : \mathbb{R}^n \to \mathbb{R}$ and a strictly positive real number q satisfying

$$\mathcal{L}_f P(x) \ \leq \ \rho(x)\, dh(x) \otimes dh(x) - q\, P(x) \qquad \forall x \in \mathbb{R}^n.$$

i.e. the pair (f, h) is strongly differentially detectable.

A. Necessary conditions

A1. Differential detectability

A2. Strong geodesic convexity

Definition [infinite gain margin] The observer

$$\dot{\hat{x}} = F(\hat{x}, y)$$

is said to have an infinite gain margin with respect to P if

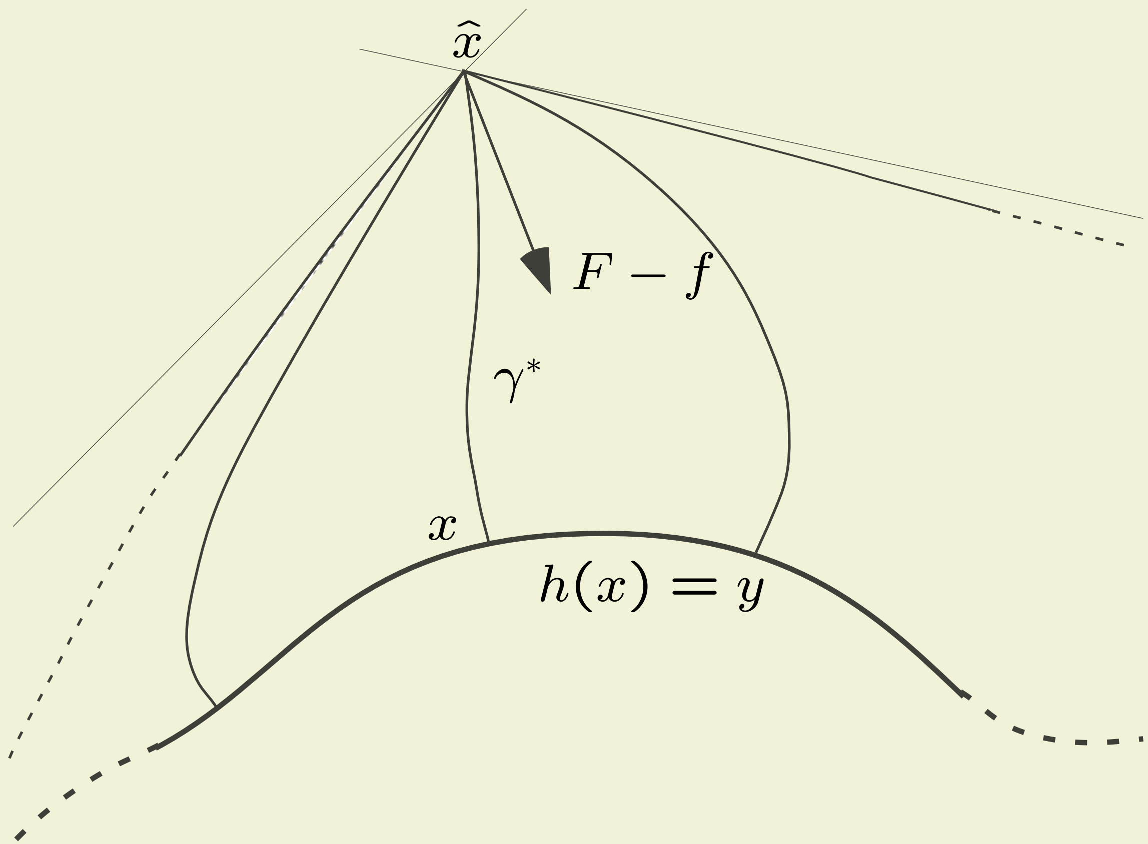
$$F(x, h(x)) = f(x) \ ,$$

and, for any geodesic γ^*, minimal on $[0, \hat{s})$, we have

$$\frac{d\gamma^*}{ds}(s) P(\gamma^*(s)) \left[F\left(\gamma^*(s), h(\gamma^*(0))\right) - f(\gamma^*(s))\right] \ < \ 0 \qquad \forall s \in (0, \hat{s}) \ .$$

Motivation: if the observer $\dot{\hat{x}} = F(\hat{x}, y)$ makes $t \mapsto d(\hat{X}((x, \hat{x}), t), X((x, \hat{x}), t)$ nonincreasing and the above condition is satisfied, then, for any real number $\ell \geq 1$, the same holds for the observer

$$\dot{\hat{x}} = f(\hat{x}) \ + \ \ell \ [F(\hat{x}, y) - f(\hat{x})]$$



Since our only knowledge is $\widehat{x}$ and $h(x) = y$, we need the level set of h $\{x \in \mathbb{R}^n : h(x) = y\}$ to be "seen" from $\widehat{x}$ within a cone whose aperture is less than π.

Proposition: If the observer has an infinite gain margin with respect to P and

$$\mathfrak{D}^+ d(\hat{x}, x) \leq 0 \quad \forall (\hat{x}, x) \in \mathbb{R}^n \times \mathbb{R}^n$$

then, for any y in $\mathbb{R}^p$, each level set of h is strongly geodesically convex[1], i.e., for any pair of points $(x_a, x_b) \in \{x \in \mathbb{R}^n : h(x) = y\}^2$, any minimal geodesic γ^* between $x_a = \gamma^*(s_a)$ and $x_b = \gamma^*(s_b)$ satisfying

$$\gamma^*(s) \in \{x \in \mathbb{R}^n : h(x) = y\} \qquad \forall s \in [s_a, s_b].$$

Definition : The set $\{x \in \mathbb{R}^n : h(x) = y\}$ is said totally geodesic if any geodesic γ satisfying :

$$h(\gamma(0)) = y \quad , \qquad \frac{dh \circ \gamma}{ds}(0) = 0$$

satisfies :

$$h(\gamma(s)) = y \qquad \forall s$$

[1] This is a coordinate free property.

Proposition :

1. When the number p of outputs is 1, if the level sets of h are totally geodesic, then they are strongly geodesically convex and h has the following function geodesic monotonicity property[1] : there exists a C^2 function $\delta : \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}_{\geq 0}$ satisfying

$$\delta(h(x), h(x)) = 0 \quad , \qquad \left.\frac{\partial^2 \delta}{\partial y_a^2}(y_a, y_b)\right|_{y_a = y_b = h(x)} > 0 \qquad \forall x \in \mathbb{R}^n ,$$

such that, for any pair (x_a, x_b) in $\mathbb{R}^n \times \mathbb{R}^n$ satisfying

$$h(x_a) \neq h(x_b)$$

and any minimal geodesic γ^* between $x_a = \gamma^*(s_a)$ and $x_b = \gamma^*(s_b)$, with $s_a \leq s_b$, we have :

$$\frac{d}{ds}\delta(h(\gamma^*(s)), h(\gamma^*(s_a))) > 0 \qquad \forall s \in (s_a, s_b] .$$

[1] Question : Is there a name for this property (weaker than affine) ?

2. For any $p = dim(h)$, the function geodesic monotonicity of h implies its level sets are strongly geodesically convex and totally geodesic.

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Proposition : We assume there exists a complete Riemannian metric P on $\mathbb{R}^n$ for which the pair (f,h) is strongly differentially detectable

Semi-global convergence: If the output function h is geodesically monotonic, then, for any positive real number E there exists a continuous function $k_E : \mathbb{R}^n \to \mathbb{R}$ such that, with the observer given by[1]

$$F(\hat{x},y) \;=\; f(\hat{x}) \;-\; k_E(\hat{x})\,\mathrm{grad}_P h(\hat{x})\frac{\partial \delta}{\partial y_1}(h(\hat{x}),y)\ ,$$

the following holds

$$\mathfrak{D}^+ d(\hat{x},x) \;\leq\; -\frac{q}{4}\, d(\hat{x},x) \qquad \forall (x,\hat{x}) \in \{(x,\hat{x})\ :\ d(\hat{x},x) < E\}\ .$$

and it gives an observer with infinite gain margin.

[1] this observer expression is intrinsic

Conclusion :

We are very much interested in finding a metric P such that

1. that the pair (f, g) is differentially detectable;

2. when $p = 1$, the level sets of h are totally geodesic

These are sufficient properties to design a convergent observer and they are “not far” from being necessary.

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C. P such that (f, h) is differentially detectable

Link with linear detectability/observability

C. P such that (f, h) is differentially detectable. Link with linear detectability/observability

To each x, we associate the functions

$$t \mapsto A_x(t) = \frac{\partial f}{\partial x}(X(x,t)) \quad , \qquad t \mapsto C_x(t) = \frac{\partial h}{\partial x}(X(x,t)) \ .$$

Proposition : If P is such that the pair (f, g) is strongly differentially detectable and satisfies

$$0 < \underline{p}\, I \leq P(x) \leq \overline{p}\, I \quad \forall x$$

then, for each x, there exists a continuous function $t \mapsto K_x(t)$ such that the origin of

$$\dot{\xi} = (A_x(t) - K_x(t) C_x(t))\, \xi$$

is uniformly exponentially stable.

In other words, the pair (f, g) is strongly differentially detectable $\Rightarrow$ linear detectability

C. P such that (f, h) is differentially detectable. Link with linear detectability/observability

Conversely ...

Proposition: Assume there exists coordinates such that f and h have bounded differential. and strictly positive real numbers τ and ϵ such that we have[1] :

$$\int_{-\tau}^{0} \Phi_x(s,0)^\top C_x(s)^\top C_x(s) \Phi_x(s,0) ds \;\geq\; \epsilon\, I \qquad \forall x \;,$$

= Uniform linear observability

Then, for any real strictly positive real number number λ, there exist a continuous symmetric covariant 2-tensor P with positive definite values which admits a Lie derivative $\mathcal{L}_f P$ satisfying

$$\mathcal{L}_f P(x) \;=\; dh(x) \otimes dh(x) \;-\; \lambda\, P(x)$$

[1] Φ_x is the transition matrix of $\dot{\xi} = A_x(t)\xi$.

Numerical scheme to evaluate P on a grid

Given x in a grid, integrate

$$\dot{x} = f(x)$$

backwards up to some time T sufficiently large to get $t \in [-T, 0] \to X(x,t)$.

Then integrate forward

$$\dot{\Pi} = -\Pi \frac{\partial f}{\partial x}(X(x,t)) - \frac{\partial f}{\partial x}(X(x,t))^{\top} \Pi + \frac{\partial h}{\partial x}(X(x,t))^{\top} \frac{\partial h}{\partial x}(X(x,t)) - \lambda \Pi$$

from 0 at time $-T$ to get $\Pi((0,-T),0)$ at time 0

The result is that $\Pi((0,-T),0)$ is an approximation of $P(x)$

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The case with a constant Riemannian metric is encountered in most publications.

But it may not exist and it requires specific coordinates.

To design an observer from a Riemannian metric, we need this metric P to satisfy

1. $\mathcal{L}_f P(x) < \rho(x)\, dh(x) \otimes dh(x)$ i.e. P makes the pair (f, h) differentially detectable.

2. be such that h is geodesically monotonic, respectively when $p = dim(h) = 1$, that the level sets of h are totally geodesic.

Point 1 is always possible as soon as the first order approximation along any solution is (uniformly) observable or the system is strongly differentially observable.

Point 2 is necessary only for observers with infinite gain margin.

For the time being we have no design procedure giving a mteric satisfying the two properties