Null controllability of degenerate parabolic equations of hypoelliptic type

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Groupe de travail Contrôle, P6, mai 2015

$A(x) \geqslant 0$ in $\overline{\Omega}$, not > 0 everywhere

$$\begin{cases} \partial_t f - \text{div}[A(x)\nabla f] + b(t,x).\nabla f = 1_\omega(x) u(t,x) & \text{in } (0,T)\times\Omega \\ + \quad B.C. & \text{on } (0,T)\times\Gamma \end{cases}$$

$L = \sum_{j=1}^r X_j^2 + X_0 + c$ where $X_0, ..., X_r$ are C^∞ first-order differential operators and there exists n operators, among $[X_{j_1}, [X_{j_2}, [..., X_{j_k}]...]]$ which are linearly independent at any given point $x \in \Omega$.

Ex of application : Boundary layers in fluids (Prandtl/Crocco eqs).

Question : Null controllability ? (usual sense)
" $\forall f_0 \in L^2(\Omega)$, *there exists* $u \in L^2((0,T)\times\Omega)$ *such that* $f(T) = 0$ "

Comparison parabolic : $\forall \omega \quad \forall T$
hyperbolic : GCC, $T_{min} > 0$

Goal : Understand mechanisms on simple equations/geometries

By Hilbert Uniqueness Method, ...

control system

$$\begin{cases} \Big(\partial_t + L\Big) f(t) = u(t) 1_\omega \\ f(0) = f_0 \end{cases}$$

adjoint system

$$\begin{cases} \Big(-\partial_t + L^*\Big) g(t) = 0 \\ g(T) = g_T \end{cases}$$

- **approximate controllability in time $T > 0$ is equivalent to unique continuation** :

$$g \equiv 0 \text{ on } (0,T) \times \omega \quad \Rightarrow \quad g \equiv 0 \text{ in } Q_T$$

- **null controllability in time $T > 0$ is equivalent to observability** :

$$\exists C_T > 0\,, \quad \|g_0\|^2_{L^2(\Omega)} \leqslant C_T^2 \int_0^T \int_\omega |g(t,x)|^2 dx dt\,, \quad \forall g_T \in L^2(\Omega)\,.$$

$\longrightarrow$ Connection with unique continuation for hypoelliptic operators [Zuily, Alinhac, Bahoury,... 80']

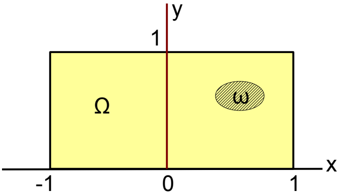
Difficulty : No proof of appropriate Carleman estimates for observability

Plan

1. Grushin-type equations
2. Heisenberg heat equation
3. Kolmogorov-type equations

PART 1 : Grushin-type equations

$$\begin{cases} \left(\partial_t - \partial_x^2 - |x|^{2\gamma}\partial_y^2\right) f(t,x,y) = 1_\omega(x,y) u(t,x,y) & \text{in } \Omega \quad \gamma > 0 \\ f(t,.,.) = 0 & \text{on } \partial\Omega \end{cases}$$

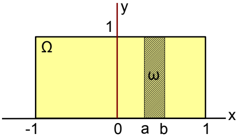


For $\gamma \in \mathbb{N}$, $L = X_1^2 + X_2^2$ with $X_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $X_2 = \begin{pmatrix} 0 \\ x^\gamma \end{pmatrix}$

$$[X_1, X_2] = \begin{pmatrix} 0 \\ \gamma x^{\gamma-1} \end{pmatrix}, \qquad [X_1,[X_1,X_2]] = \begin{pmatrix} 0 \\ \gamma(\gamma-1)x^{\gamma-2} \end{pmatrix}, \dots$$

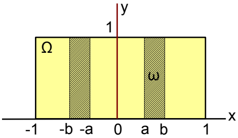
Rk : Unique continuation holds $\forall \gamma > 0$, $\forall \omega$, $\forall T > 0$.

(G) : $\begin{cases} \left(\partial_t - \partial_x^2 - |x|^{2\gamma}\partial_y^2\right) f = 1_\omega u(t,x,y) \\ f(t,.,.) = 0 \text{ on } \partial\Omega \end{cases}$



Thm : [KB-Cannarsa-Guglielmi 2012]

- If $\gamma < 1$ then **(G)** is controllable $\forall T > 0, \forall \omega$.
- If $\gamma = 1$ and $\omega = (a,b) \times (0,1)$ then **(G)** is controllable only in large time : $T_{min} > 0$
- If $\gamma > 1$ then **(G)** is not controllable.



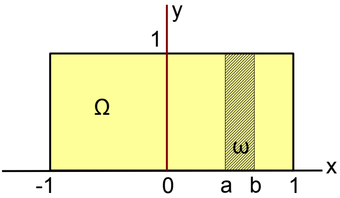
Thm : [KB-Miller-Morancey 2014] $\gamma \geqslant 1$

- If $\gamma = 1$ then $T_{min} = a^2/2$.
- Otherwise, the null controllable initial conditions are functions analytic in y, with L^2-values in x such that

$$\sum_{n\in\mathbb{N}^*} e^{2d\pi n} \|f_n^0\|_{L^2}^2 < \infty \text{ where } d := \begin{cases} \frac{a^{\gamma+1}}{\gamma+1} & \text{if } \gamma > 1 \\ \frac{a^2}{2} - T & \text{if } \gamma = 1, T < \frac{a^2}{2}. \end{cases}$$

- If $\omega = (0,b) \times (0,1)$ then **(G)** is null controllable $\forall T > 0, \forall \gamma > 0$.

Proof when ω is a strip : Fourier decomposition



$$g(t,x,y)=\sum_{n=1}^{\infty} g_n(t,x)\sin(n\pi y)$$

$$\begin{cases} \left(\partial_t^2-\partial_x^2-|x|^{2\gamma}\partial_y^2\right)g=0 \\ g(t,.,.)=0 \quad \text{on } \partial\Omega \end{cases} \qquad (G_n^*)\begin{cases} \partial_t g_n-\partial_x^2 g_n+(n\pi)^2|x|^{2\gamma}g_n=0 \\ g_n(t,\pm 1)=0 \end{cases}$$

Goal : $\int_\Omega g(T,x,y)^2dxdy \leqslant C\int_0^T\int_\omega g(t,x,y)^2dxdydt$

$\Leftrightarrow \quad \sum_{n=1}^{\infty}\int_{-1}^{1} g_n(T,x)^2dx \leqslant C\sum_{n=1}^{\infty}\int_0^T\int_a^b g_n(t,x)^2dxdt$

$\Leftrightarrow$ Uniform observability of (G_n^*) wrt n :

$$\int_{-1}^{1} g_n(T,x)^2dx \leqslant C\int_0^T\int_a^b g_n(t,x)^2dxdt$$

Uniform observability when $\gamma \in (0,1)$ or $[\gamma = 1, T$ large$]$

$$\begin{cases} \partial_t g_n - \partial_x^2 g_n + (n\pi)^2 |x|^{2\gamma} g_n = 0 \\ g_n(t, \pm 1) = 0 \end{cases}$$

1. **Explicit decay rate** for the Fourier components (rescaling)

$$\int_{-1}^{1} g_n(T,x)^2 dx \leqslant e^{-cn^{\frac{2}{1+\gamma}}(T-t)} \int_{-1}^{1} g_n(t,x)^2 dx, \quad \forall t \in [0, T]$$

2. Global **Carleman estimates** with weight $e^{\frac{nT^2\psi(x)}{t(T-t)}}$

$$\int_{T/3}^{2T/3} \int_{-1}^{1} g_n(t,x)^2 dx dt \leqslant e^{Cn} \int_0^T \int_a^b g_n(t,x)^2 dx dt$$

$$\Rightarrow \quad \frac{T}{3} \int_{-1}^{1} g_n(T,x)^2 dx \leqslant e^{Cn - cn^{\frac{2}{1+\gamma}} \frac{T}{3}} \int_0^T \int_a^b g_n(t,x)^2 dx dt$$

$Cn - cn^{\frac{2}{1+\gamma}} \frac{T}{3} \to -\infty$ when $[0 < \gamma < 1]$ or $[\gamma = 1, T > 3/c]$

Rk : The 2D-Grushin-equation is not conjugated by a weight fn !

Proof when ω is arbitrary

Lebeau-Robbiano's strategy : $0 = T_0 < T_1 < \dots < T_j \to T$:

- on $[T_j, T_{j+1/2}]$, one applies a control that steers the 2^j-first components to zero,
- on $[T_{j+1/2}, T_{j+1}]$, no control $\to$ dissipation.

[Benabdallah-Dermenjian-Le Rousseau 07]

Lebeau-Robbiano's inequality : $\exists C > 0$ such that $\forall N$, $\forall b \in l^2$

$$\sum_{k=1}^{N} |b_k|^2 \leqslant e^{CN} \int_c^d \Big| \sum_{k=1}^{N} b_k \sin(ky) \Big|^2 dy$$

Key point :
dissipation $N^{\frac{2}{1+\gamma}}$ $>>>$ cost N **and** Lebeau-Robbiano's cst N
$\longrightarrow$ only when $\gamma < 1$

No uniform observability when $[\gamma > 1]$ or $[\gamma = 1, T \leqslant a^2/2]$

Consider $g_n(t,x) := w_n(x)e^{-\lambda_n t}$ built on the first eigenfunction

$$\begin{cases} -w_n''(x) + (n\pi)^2|x|^{2\gamma} w_n = \lambda_n w_n(x)\,, \quad x \in (-1,1)\,,\ n \in \mathbb{N}^*\,, \\ w_n(\pm 1) = 0\,, \quad w_n \geq 0\,. \end{cases}$$

Then

$$\frac{\int_0^T \int_a^b g_n(t,x)^2 dxdt}{\int_{-1}^1 g_n(T,x)^2 dx} = \frac{e^{2\lambda_n T} - 1}{2\lambda_n} \int_a^b w_n(x)^2 dx \leqslant C e^{c|n|^{\frac{2}{1+\gamma}} T} e^{-2\pi \frac{a^{\gamma+1}}{\gamma+1}|n|}$$

converges to zero.

Key point : Agmon estimates with $h = \frac{1}{n\pi}$ and $V(x) = |x|^{2\gamma}$

$$-h^2 \Delta w + V(x) w = \lambda w \qquad \Rightarrow \qquad w(x) \leqslant C e^{-\frac{d(x)-\epsilon}{h}}$$

$$\begin{cases} \partial_t g_n - \partial_x^2 g_n + (n\pi)^2 |x|^{2\gamma} g_n = 0 \\ g_n(t, \pm 1) = 0 \end{cases}$$

Via 1D transmutation and lateral propagation of energy method on the resulting 1D wave equation, we obtain the **optimal cost estimate**

$$\|g_n(T)\|_{L^2(-1,1)} \leqslant C e^{D^+ |n|} \left(\int_0^T \int_{(-b,-a)\cup(a,b)} |g_n(t,x)|^2 dx \right)^{1/2}$$

$$\text{where} \quad D := \begin{cases} \pi \frac{a^{\gamma+1}}{\gamma+1} & \text{if } \gamma > 1\,, \\ \pi \left(\frac{a^2}{2} - T \right) & \text{if } \gamma = 1\,. \end{cases}$$

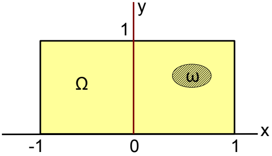
This proves :

- uniform observability wrt n iff $T > T_{min} := a^2/2$ when $\gamma = 1$,
- characterization of null controllable initial conditions :

$$\|u\|^2_{L^2((0,T)\times\Omega)} = \sum_{n\in\mathbb{Z}} \|u_n\|^2_{L^2((0,T)\times(-1,1))} \leqslant \sum_{n\in\mathbb{Z}} C^2 e^{2D^+|n|} \|f_n^0\|^2_{L^2(-1,1)}$$

Further results and open problems about Grushin equation

$$\left(\partial_t - \partial_x^2 - |x|^{2\gamma}\partial_y^2\right) f(t,x,y) = 1_\omega(x,y) u(t,x,y)$$



- Generalizations to $x \in \Omega_1$, $y \in \Omega_2$ with coeff $\|x\|^{2\gamma} b(x)$
  + Inverse-source problem [KB-Cannarsa-Yamamoto 2013]
- **Open :** $\gamma = 1$ and ω arbitrary
- **Open :** general geometry (Ω, ω)
- $\gamma \geqslant 1$ **:** For given control support ω and time T, the regularity of initial conditions that are null controllable (with L^2-controls) depends on ω and T $\longrightarrow$ GCC : **precise dependence ?**

$$\left[\partial_t - \left(\partial_{x_1} - \frac{x_2}{2}\partial_{x_3}\right)^2 - \left(\partial_{x_2} + \frac{x_1}{2}\partial_{x_3}\right)^2\right] g(t,x_1,x_2,x_3) = 1_\omega u(t,x_1,x_2,x_3)$$

$$L = X_1^2 + X_2^2 \text{ with } X_1 = \begin{pmatrix} 1 \\ 0 \\ -\frac{x_2}{2} \end{pmatrix} \quad X_2 = \begin{pmatrix} 0 \\ 1 \\ \frac{x_1}{2} \end{pmatrix} \quad [X_1, X_2] = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

Fourier series in x_3 lead to a 2D heat-equation with complex-valued coefficients, on which the previous strategy fails.

Change of variables : $x = x_1 \qquad y = x_2 \qquad z = x_3 + \frac{x_1 + x_2}{2}$

$$\left(\partial_t - \partial_x^2 - (x\partial_z + \partial_y)^2\right) g(t,x,y,z) = 1_{\widetilde{\omega}} \widetilde{u}(t,x,y,z)$$

This formulation is well adapted to Fourier transform in variables (y, z). We will study this eq on

- the rectangle $(x, y, z) \in (-1, 1) \times \mathbb{T} \times \mathbb{T}$
- the unbounded domain $(x, y, z) \in (-1, 1) \times \mathbb{T} \times \mathbb{R}$

Results

$$\textbf{(H)} : \begin{cases} \Big(\partial_t - \partial_x^2 - (x\partial_z + \partial_y)^2\Big) f(t,x,y,z) = 1_\omega u(t,x,y,z) \\ f(t,\pm 1, y, z) = 0\,, \end{cases}$$

Thm : [KB-Cannarsa 2015]

- On the rectangle $\Omega = (-1,1) \times \mathbb{T} \times \mathbb{T}$ with $\omega = (a,b) \times \omega_y \times \mathbb{T}$,
- On $\Omega = (-1,1) \times \mathbb{T} \times \mathbb{R}$ with $\omega = (a,b) \times \omega_y \times \mathbb{R}$,

(H) is null controllable in large time $T_{min} \geqslant \frac{1}{8} \max\{(1+a)^2, (1-b)^2\}$.

Remarks :

- Unique continuation holds $\forall T > 0$ (Holmgren principle).
- T_{min} is always > 0 and related to the distance between $\partial\Omega$ and ω.

Key points of the proof

$$\begin{cases} \Big(\partial_t - \partial_x^2 - (x\partial_z + \partial_y)^2\Big) g(t,x,y,z) = 0\,, & (x,y,z) \in (-1,1) \times \mathbb{T} \times \mathbb{T} \\ g(t,\pm 1, y, z) = 0\,, \end{cases}$$

Fourier series : $y \mapsto n \in \mathbb{Z}$ $z \mapsto p \in \mathbb{Z}$

$$\begin{cases} \Big(\partial_t - \partial_x^2 + (px+n)^2\Big) g_{n,p}(t,x) = 0\,, & x \in (-1,1) \\ g_{n,p}(t,\pm 1) = 0\,, \end{cases}$$

1. **Decay rate :** $\lambda_{n,p} \geqslant \begin{cases} c|p| \\ cn^2 \text{ if } |n| \geqslant 2|p| \end{cases}$
2. **Optimal weight** for Carleman estimates : $e^{(|n|+|p|)\frac{T^2\psi(x)}{t(T-t)}}$

With p fixed :
dissipation N^2 >>> cost N **and** Lebeau-Robbiano's cst N

- Lebeau-Robbiano's strategy wrt n, with p fixed : localization wrt y
- Quantification of the cost $e^{|p|(c_1 - c_2 T)}$: uniform wrt p for large T

Further results, open problems

$$\left(\partial_t - \partial_x^2 - (x\partial_z + \partial_y)^2\right) g(t,x,y,z) = 0$$

- Lipschitz stability estimate for the inverse source-problem [KB-Cannarsa 2015]
- **Open problems :**
  - explicit value of the minimal time
  - ω arbitrary (localization in z)
  - observability inequality with $\Omega = \mathbb{R} \times \mathbb{T} \times \mathbb{T}$
    $\lambda_{n,p} = c|p|$ (invariance under translation in x)
    $\longrightarrow$ no localization in y should be possible.
- **Conjecture :** If the first Lie bracket is sufficient to satisfy Hörmander's condition, then null controllability holds in large time.

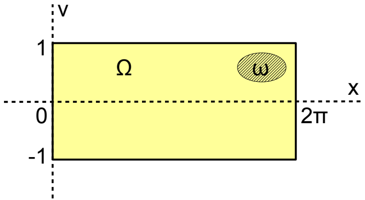
$$\left(\partial_t + v^\gamma \partial_x - \partial_v^2\right) f(t,x,v) = 1_\omega(x,v) u(t,x,v)$$

For $\gamma \in \mathbb{N}^*$, $\quad L = X_1^2 + X_0$ with $X_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ and $X_0 = \begin{pmatrix} v^\gamma \\ 0 \end{pmatrix}$

$\sim$ linearizations of Prandtl and Crocco equations for fluids.

We will consider this equation on

- the whole space : $(x,v) \in \mathbb{R} \times \mathbb{R}$ with $\gamma = 1$
- the rectangle : $(x,v) \in \mathbb{T} \times (-1,1)$
  - with Dirichlet boundary conditions in v, $\forall \gamma \in \mathbb{N}^*$
  - with periodic-type boundary conditions in v with $\gamma = 1$



$$\left(\partial_t + v^\gamma \partial_x - \partial_v^2\right) f(t,x,v) = 1_\omega(x,v) u(t,x,v)$$

Thm : [KB 2014 + KB-Helffer-Henry-Robbiano] $(x,v) \in \mathbb{T} \times (-1,1)$

- Periodic/v + $(\gamma = 1)$: null controllable $\forall T > 0$, $\forall \omega$,
- Dirichlet/v + $\left(\omega = \mathbb{T} \times (a,b)\right)$
  - $\gamma = 1$: null controllable $\forall T > 0$
  - $\gamma = 2$: null controllable only in large time : $T_{min} \geqslant \frac{a^2}{2}$
  - $\gamma \geqslant 3$: not null controllable.

Rk : Unique continuation holds $\forall \gamma$, but not for all zero-order C^∞-perturbation [Alinhac-Zuily 1980]

Thm : [Moyano-Le Rousseau 2015] $\gamma = 1$ $(x,v) \in \mathbb{R}^{2d}$ $\omega = \omega_x \times \omega_v$
If both ω_x and ω_v satisfy

$$\exists \delta, r > 0 / \forall y \in \mathbb{R}^d, \exists y' \in \omega_x / B(y', r) \subset \omega_x \text{ and } \|y - y'\| < \delta .$$

then null controllability holds $\forall T > 0$.

Key points of the proof on the rectangle

$$\Big(\partial_t + v^\gamma \partial_x - \partial_v^2\Big) g(t,x,v) = 0 \quad (x,v) \in \mathbb{T} \times (-1,1)$$

Fourier series : $x \mapsto n \in \mathbb{Z}$

$$\Big(\partial_t + inv^\gamma - \partial_v^2\Big) g_{n,p}(t,x,v) = 0 \quad (x,v) \in \mathbb{T} \times (-1,1)$$

1. **Decay rate :** periodic/v and $\gamma = 1$: $|n|^2$ (explicit)
   Dirichlet/v : $|n|^{\frac{2}{2+\gamma}}$ (subtle spectral analysis)
2. **Optimal weight** for Carleman estimates : $e^{\sqrt{|n|}\frac{T^2 \psi(x)}{t(T-t)}}$

Periodic/v and $\gamma = 1$:
dissipation N^2 $>>>$ cost $\sqrt{N}$ **and** Lebeau-Robbiano's cst N
$\longrightarrow$ observability $\forall \omega$

Dirichlet/v :
dissipation $N^{2/3}$ or $\sqrt{N}T$ $>>>$ cost $\sqrt{N}$ **but not** L-R's cst N
$\longrightarrow$ this strategy does not allow localization in variable x.

Quantification wrt n of the decay rate of Fourier modes

Goal : $\partial_t g_n + inv^\gamma g_n - \partial_v^2 g_n = 0\,, \quad v \in (-1,1)\,, \quad g_n(t,\pm 1) = 0$

$$\Rightarrow \exists K, c > 0 \quad \|g_n(t)\|_{L^2(-1,1)} \leqslant K \|g_n^0\|_{L^2(-1,1)} e^{-cn^{\frac{2}{2+\gamma}} t}\,, \quad \forall n, g_n(0)$$

After rescaling $g_n(t,v) = h\left(\tau = n^{\frac{2}{2+\gamma}} t, y = n^{\frac{1}{2+\gamma}} v\right)$, $R = n^{\frac{1}{2+\gamma}}$, we need

$$\partial_\tau h + iy^\gamma h - \partial_y^2 h = 0\,, y \in (-R,R) \qquad h(\tau, \pm R) = 0$$

$$\Rightarrow \exists K, c > 0 \quad \|h(\tau)\|_{L^2(-R,R)} \leqslant K \|h^0\|_{L^2(-R,R)} e^{-c\tau}\,, \quad \forall R, h_0$$

Strategy : From resolvent bounds to semigroup bounds
[Helffer-Sjöstrand 2009] a quantitative Gearhardt-Prüss statement

Key points : The Davies operator $\left(-\frac{d^2}{dy^2} + iy^2\right)$ on the whole line $\mathbb{R}$ and the Dirichlet realization of the complex Airy operator $\left(-\frac{d^2}{dy^2} + iy\right)$ on the half line $(0,\infty)$ have discrete spectrum and ad hoc resolvent estimates.
For $\gamma = 1$: transformations $y \mapsto \pm y \pm R$ only affect $\Im$(eigenvalues).

Key ideas of the proof on the whole space

[Moyano-Le Rousseau 2015]

$$\left(\partial_t + v\partial_x - \partial_v^2\right)g(t,x,v) = 0\,, \quad (x,v) \in \mathbb{R}^2$$

$$\text{Fourier transform : } \quad x \mapsto \xi \in \mathbb{R}$$

$$\left(\partial_t + i\xi v - \partial_v^2\right)g_\xi(t,v) = 0\,, \quad (x,v) \in \mathbb{R}^2$$

1. **Decay rate :** $e^{-\xi^2 t^3}$ (explicit solution)
2. **Optimal weight** for Carleman estimates : $e^{\sqrt{|\xi|}\frac{T^2\psi(x)}{t(T-t)}}$

dissipation N^2 >>> cost $\sqrt{N}$ **and** Lebeau-Robbiano's cst N

Spectral inequality : There exists $C>0$ such that, for every $N>0$,

$$\int_{\mathbb{R}^d} |f(x)|^2 dx \leqslant e^{CN} \int_{\omega_x} |f(x)|^2 dx\,, \quad \forall f \in L^2(\mathbb{R}^d)/\text{Supp}(\hat{f}) \subset B_{\mathbb{R}^d}(0,N)$$

Further results, open problems

$$\Big(\partial_t + v^\gamma \partial_x - \partial_v^2\Big) g(t,x,v) = 0 \quad (x,v) \in \mathbb{T} \times (-1,1)$$

- **Conjecture :** Null controllability does not hold with $\omega = (a,b) \times (-1,1)$ and Dirichlet/v $\longrightarrow$ **GCC**
  Proof on a toy model $\partial_x \longleftarrow \sqrt{-\Delta_x}$: [KB-Helffer-Henry-Robbiano]
- Characterization of **null controllable data** in term of Gevrey-2 regularity with $\omega = \mathbb{T} \times (a,b)$ (optimal costs)
- For given control support ω and time T, the regularity of initial conditions that are null controllable (with L^2-controls) depends on ω and T. **Precise dependence ?**
- **General geometry ?** Direct 2D approach ?
- **Linearized Crocco eq :** $+$ degenerate diffusion in v

2D Navier Stokes

$\Downarrow$ boundary layer over a flat plate

Prandtl system

$\Downarrow$ Crocco transformation : $(x, y) \in (0, L) \times (0, 1)$

Crocco eq : $\left(\partial_t + \alpha(t) y \partial_x + \beta(t)(1-y)\partial_y - \nu w^2 \partial_y^2\right) w(t,x,y) = 0$

Control strategy : global approximate controllability

$+$ local exact controllability via perturbation argument

$\Rightarrow$ global exact controllability

Crocco eq has stationary solutions vanishing at $y = 1$ $\Rightarrow$ linearized syst :

$$\left(\partial_t + \alpha(t) y \partial_x + \beta(t)(1-y)\partial_y - (y-1)^2 \partial_y^2\right) w(t,x,y) = 0$$

We will study **Kolmogorov eq :** $\left(\partial_t + y\partial_x - \partial_y^2\right) f(t,x,y) = 0$