Time optimal control for Schrödinger and waves type systems

Jérôme Lohéac

BCAM

Journée jeunes contrôleurs

The general Problem

Let X and U be two Banach spaces. Consider the Cauchy problem:

$$\dot{z} = Az + Bu\,, \qquad z(0) = \mathrm{z}_0\,. \qquad (\star)$$

- Given $\mathrm{z}_1 \in L^2(\omega)$, $\mathrm{z}_1 \neq \mathrm{z}_0$, show that there exist a minimal time $\tau = \tau(\mathrm{z}_0, \mathrm{z}_1) > 0$ such that there exists a control $u : [0, \tau] \to L^2(\mathcal{O})$, with

$$\|u\| \leqslant 1\,,$$

  for which the solution z of $(\star)$ satisfies:

$$z(T) = \mathrm{z}_1\,;$$

- Derive optimality conditions (Pontryagin's maximum principle);
- Prove the saturation property, i.e. $\|u\| = 1$.

State of art I

- Finite dimensional systems:
  - R. Bellman, I. Glicksberg and O. Gross,
    On the bang-bang control problem, 1956
  - E. B. Lee and L. Markus,
    Foundations of optimal control theory, 1967
  - A. D. Ioffe et V. M. Tihomirov,
    Theory of extremal problems, 1979
  - ...

State of art II

- Infinite dimensional systems:
  - H. O. Fattorini,
    Time-optimal control of solutions of operational differential equations, 1664
    Infinite dimensional linear control systems, 2005
  - G. Wang,
    L^∞-null controllability for the heat equation and its consequences for the time optimal control problem, 2008
  - K. Kunisch and D. Wachsmuth,
    Time optimal control of the wave equation, its regularisation and numerical realisation, 2011
  - S. Micu, I. Roventa and M. Tucsnak,
    Time optimal boundary controls for the heat equation, 2012
  - G. Wang and E.Zuazua,
    On the equivalence of minimal time and minimal norm controls for internally controlled heat equations, 2012

1. Time optimal control for Schrödinger
   - The problem
   - Optimality conditions
   - Conclusion and open questions

2. Time optimal control for the wave system

The problem

Let consider the Cauchy problem:

$$\dot{z} = i\Delta z + \mathbf{1}_{\mathcal{O}} u \qquad (t > 0\,,\ x \in \Omega)\,, \qquad \text{(S1)}$$

$$z(t,x) = 0 \qquad (t > 0\,,\ x \in \partial\Omega)\,, \qquad \text{(S2)}$$

$$z(0,x) = \mathrm{z}_0(x) \qquad (x \in \Omega)\,, \qquad \text{(S3)}$$

with Ω a bounded domain of $\mathbb{R}^n$ with C^3 boundary, $\mathcal{O} \subset \Omega$ satisfying the geometric optic condition and $\mathrm{z}_0 \in L^2(\Omega)$ given.

Problem

Given $\mathrm{z}_1 \in L^2(\Omega) \setminus \{\mathrm{z}_0\}$, does it exist a minimal time τ such that there exists $u \in L^\infty([0,\tau], L^2(\mathcal{O}))$ with:

$$\|u(t)\|_{L^2(\mathcal{O})} \leqslant 1 \qquad (t \in [0,\, T] \ a.e.)$$

$$\text{and} \quad z(\tau) = \mathrm{z}_1\,,$$

where $z \in C\big([0,\tau], L^2(\Omega)\big)$ is the solution of (S)?

Existence

Lemma

There exists $T > 0$ *and* $u \in L^\infty([0,T], L^2(\mathcal{O}))$ *with* $\|u(t)\|_{L^2(\mathcal{O})} \leqslant 1$ *for almost every* $t \in [0,T]$ *steering* z_0 *at time* $t = 0$ *to* z_1 *at time* $t = T$.

Existence

Lemma

There exists $T > 0$ and $u \in L^\infty([0,T], L^2(\mathcal{O}))$ with $\|u(t)\|_{L^2(\mathcal{O})} \leqslant 1$ for almost every $t \in [0,T]$ steering z_0 at time $t = 0$ to z_1 at time $t = T$.

Idea of the proof

1. For every $t > 0$ there exists $u \in L^\infty([0,t], L^2(\mathcal{O}))$ steering z_0 to z_1 in time t.

Existence

Lemma

There exists $T > 0$ and $u \in L^\infty([0,T], L^2(\mathcal{O}))$ with $\|u(t)\|_{L^2(\mathcal{O})} \leqslant 1$ for almost every $t \in [0,T]$ steering z_0 at time $t = 0$ to z_1 at time $t = T$.

Idea of the proof

1. For every $t > 0$ there exists $u \in L^\infty([0,t], L^2(\mathcal{O}))$ steering z_0 to z_1 in time t.
   B is bounded. Hence the HUM control is continuous.

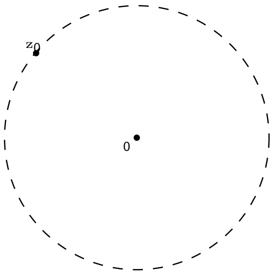
Existence

Lemma

There exists $T > 0$ and $u \in L^\infty([0,T], L^2(\mathcal{O}))$ with $\|u(t)\|_{L^2(\mathcal{O})} \leqslant 1$ for almost every $t \in [0,T]$ steering z_0 at time $t = 0$ to z_1 at time $t = T$.

Idea of the proof

1. For every $t > 0$ there exists $u \in L^\infty([0,t], L^2(\mathcal{O}))$ steering z_0 to z_1 in time t.
2. Let $\mathrm{z}_1 = 0$ then there exists $T = T(\|\mathrm{z}_0\|)$ and a control $u \in L^\infty([0,T], L^2(\mathcal{O}))$, with $\|u\|_{L^\infty([0,T],L^2(\mathcal{O}))} \leqslant 1$ steering z_0 to 0.



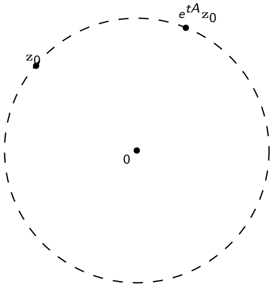
Existence

Lemma

There exists $T > 0$ and $u \in L^\infty([0,T], L^2(\mathcal{O}))$ with $\|u(t)\|_{L^2(\mathcal{O})} \leqslant 1$ for almost every $t \in [0,T]$ steering z_0 at time $t = 0$ to z_1 at time $t = T$.

Idea of the proof

1. For every $t > 0$ there exists $u \in L^\infty([0,t], L^2(\mathcal{O}))$ steering z_0 to z_1 in time t.
2. Let $\mathrm{z}_1 = 0$ then there exists $T = T(\|\mathrm{z}_0\|)$ and a control $u \in L^\infty([0,T], L^2(\mathcal{O}))$, with $\|u\|_{L^\infty([0,T],L^2(\mathcal{O}))} \leqslant 1$ steering z_0 to 0.



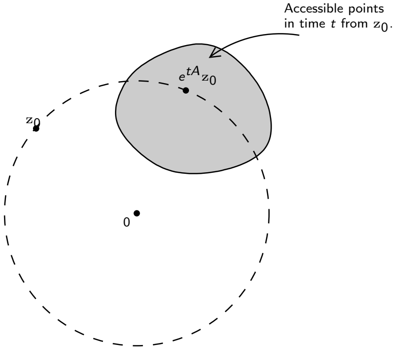
Existence

Lemma

There exists $T > 0$ and $u \in L^\infty([0,T], L^2(\mathcal{O}))$ with $\|u(t)\|_{L^2(\mathcal{O})} \leqslant 1$ for almost every $t \in [0,T]$ steering z_0 at time $t = 0$ to z_1 at time $t = T$.

Idea of the proof

1. For every $t > 0$ there exists $u \in L^\infty([0,t], L^2(\mathcal{O}))$ steering z_0 to z_1 in time t.
2. Let $z_1 = 0$ then there exists $T = T(\|z_0\|)$ and a control $u \in L^\infty([0,T], L^2(\mathcal{O}))$, with $\|u\|_{L^\infty([0,T],L^2(\mathcal{O}))} \leqslant 1$ steering z_0 to 0.



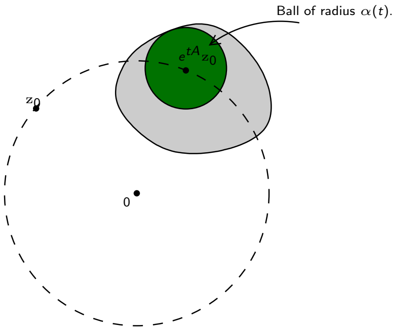
Existence

Lemma

There exists $T > 0$ and $u \in L^\infty([0,T], L^2(\mathcal{O}))$ with $\|u(t)\|_{L^2(\mathcal{O})} \leqslant 1$ for almost every $t \in [0,T]$ steering z_0 at time $t = 0$ to z_1 at time $t = T$.

Idea of the proof

1. For every $t > 0$ there exists $u \in L^\infty([0,t], L^2(\mathcal{O}))$ steering z_0 to z_1 in time t.
2. Let $\mathrm{z}_1 = 0$ then there exists $T = T(\|\mathrm{z}_0\|)$ and a control $u \in L^\infty([0,T], L^2(\mathcal{O}))$, with $\|u\|_{L^\infty([0,T],L^2(\mathcal{O}))} \leqslant 1$ steering z_0 to 0.



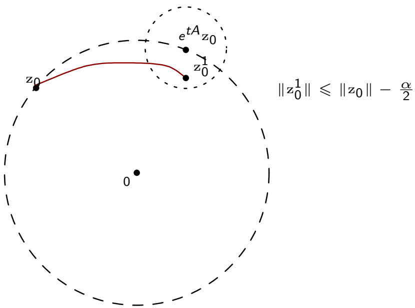
Existence

Lemma

There exists $T > 0$ and $u \in L^\infty([0,T], L^2(\mathcal{O}))$ with $\|u(t)\|_{L^2(\mathcal{O})} \leqslant 1$ for almost every $t \in [0,T]$ steering z_0 at time $t = 0$ to z_1 at time $t = T$.

Idea of the proof

1. For every $t > 0$ there exists $u \in L^\infty([0,t], L^2(\mathcal{O}))$ steering z_0 to z_1 in time t.
2. Let $z_1 = 0$ then there exists $T = T(\|z_0\|)$ and a control $u \in L^\infty([0,T], L^2(\mathcal{O}))$, with $\|u\|_{L^\infty([0,T],L^2(\mathcal{O}))} \leqslant 1$ steering z_0 to 0.



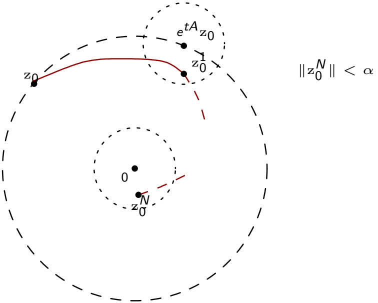
Existence

Lemma

There exists $T > 0$ and $u \in L^\infty([0,T], L^2(\mathcal{O}))$ with $\|u(t)\|_{L^2(\mathcal{O})} \leqslant 1$ for almost every $t \in [0,T]$ steering z_0 at time $t = 0$ to z_1 at time $t = T$.

Idea of the proof

1. For every $t > 0$ there exists $u \in L^\infty([0,t], L^2(\mathcal{O}))$ steering z_0 to z_1 in time t.
2. Let $\mathrm{z}_1 = 0$ then there exists $T = T(\|\mathrm{z}_0\|)$ and a control $u \in L^\infty([0,T], L^2(\mathcal{O}))$, with $\|u\|_{L^\infty([0,T],L^2(\mathcal{O}))} \leqslant 1$ steering z_0 to 0.



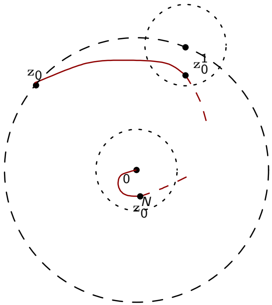
Existence

Lemma

There exists $T > 0$ and $u \in L^\infty([0,T], L^2(\mathcal{O}))$ with $\|u(t)\|_{L^2(\mathcal{O})} \leqslant 1$ for almost every $t \in [0,T]$ steering z_0 at time $t = 0$ to z_1 at time $t = T$.

Idea of the proof

1. For every $t > 0$ there exists $u \in L^\infty([0,t], L^2(\mathcal{O}))$ steering z_0 to z_1 in time t.
2. Let $\mathrm{z}_1 = 0$ then there exists $T = T(\|\mathrm{z}_0\|)$ and a control $u \in L^\infty([0,T], L^2(\mathcal{O}))$, with $\|u\|_{L^\infty([0,T],L^2(\mathcal{O}))} \leqslant 1$ steering z_0 to 0.



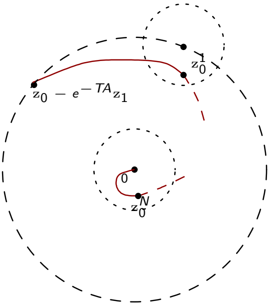
Existence

Lemma

There exists $T > 0$ *and* $u \in L^\infty([0,T], L^2(\mathcal{O}))$ *with* $\|u(t)\|_{L^2(\mathcal{O})} \leqslant 1$ *for almost every* $t \in [0,T]$ *steering* z_0 *at time* $t = 0$ *to* z_1 *at time* $t = T$*.*

Idea of the proof

1. For every $t > 0$ there exists $u \in L^\infty([0,t], L^2(\mathcal{O}))$ steering z_0 to z_1 in time t.
2. Let $\mathrm{z}_1 = 0$ then there exists $T = T(\|\mathrm{z}_0\|)$ and a control $u \in L^\infty([0,T], L^2(\mathcal{O}))$, with $\|u\|_{L^\infty([0,T],L^2(\mathcal{O}))} \leqslant 1$ steering z_0 to 0.
3. Take $\mathrm{z}_0 - e^{-TA}\mathrm{z}_1$, with $T = T(\|\mathrm{z}_0\| + |\mathrm{z}_1\|)$.



Existence

Lemma

There exists $T > 0$ and $u \in L^\infty([0,T], L^2(\mathcal{O}))$ with $\|u(t)\|_{L^2(\mathcal{O})} \leqslant 1$ for almost every $t \in [0,T]$ steering z_0 at time $t = 0$ to z_1 at time $t = T$.

By extracting a minimizing sequence, we obtain:

Proposition

There exists a minimal time $\tau = \tau(\mathrm{z}_0, \mathrm{z}_1)$ for which there exist a time optimal control $u \in L^\infty([0,\tau], L^2(\mathcal{O}))$ with $\|u\|_{L^\infty([0,\tau],L^2(\mathcal{O}))} \leqslant 1$ steering z_0 to z_1 in time τ.

Weak Pontryagin maximum principle I

Lemma

If τ is the minimal time, then every optimal control $u \in L^\infty([0,\tau], L^2(\mathcal{O}))$ satisfies:

$$\|u\|_{L^\infty([0,\tau],L^2(\mathcal{O}))} = 1\,.$$

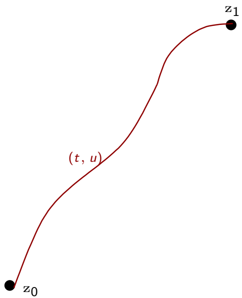
Weak Pontryagin maximum principle I

Lemma

If τ is the minimal time, then every optimal control $u \in L^\infty([0,\tau], L^2(\mathcal{O}))$ satisfies:

$$\|u\|_{L^\infty([0,\tau],L^2(\mathcal{O}))} = 1\,.$$

Idea of the proof



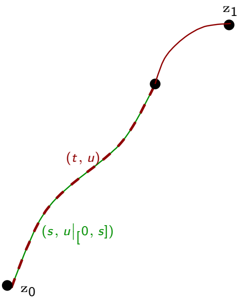
Weak Pontryagin maximum principle I

Lemma

If τ is the minimal time, then every optimal control $u \in L^\infty([0,\tau], L^2(\mathcal{O}))$ satisfies:

$$\|u\|_{L^\infty([0,\tau],L^2(\mathcal{O}))} = 1 \, .$$

Idea of the proof



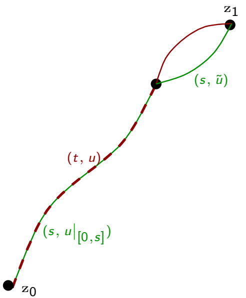
Weak Pontryagin maximum principle I

Lemma

If τ is the minimal time, then every optimal control $u \in L^\infty([0,\tau], L^2(\mathcal{O}))$ satisfies:

$$\|u\|_{L^\infty([0,\tau],L^2(\mathcal{O}))} = 1\,.$$

Idea of the proof



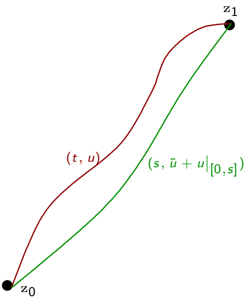
Weak Pontryagin maximum principle I

Lemma

If τ is the minimal time, then every optimal control $u \in L^\infty([0,\tau], L^2(\mathcal{O}))$ satisfies:

$$\|u\|_{L^\infty([0,\tau],L^2(\mathcal{O}))} = 1\,.$$

Idea of the proof



Weak Pontryagin maximum principle II

Theorem

There exists $\eta \in L^2(\Omega)$ *such that if* $u \in L^\infty([0,\tau], L^2(\mathcal{O}))$ *is a time optimal control steering* z_0 *to* z_1, *then:*

$$\int_0^\tau \langle u(t), w(t)|_{\mathcal{O}} \rangle \,\mathrm{d}t \geqslant \int_0^\tau \langle v(t), w(t)|_{\mathcal{O}} \rangle \,\mathrm{d}t \qquad (v \in L^\infty([0,\tau], L^2(\mathcal{O}))),$$

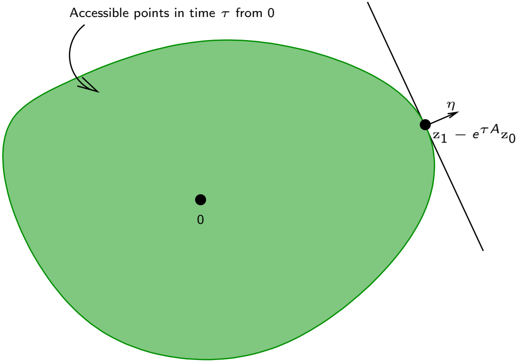
with w *solution of:*

$$\begin{aligned} \dot{w} &= i\Delta w && (t \in (0,\tau),\ x \in \Omega), \\ w(t,x) &= 0 && (t \in (0,\tau),\ x \in \partial\Omega), \\ w(\tau,x) &= \eta(x) && (x \in \Omega), \end{aligned}$$

Weak Pontryagin maximum principle III

Idea of the proof

Use Hahn-Banach and duality.



Saturation property

Lemma

Let $T > 0$, if there exists $e \subset [0, T]$, a set of positive measure, such that the solution w of:

$$\begin{aligned} \dot{w} &= i\Delta w && (t \in (0,T)\,,\ x \in \Omega)\,, \\ w(t,x) &= 0 && (t \in (0,T)\,,\ x \in \partial\Omega)\,, \\ w(0,x) &= w_0(x) && (x \in \Omega)\,, \end{aligned}$$

satisfies:

$$w(t,x) = 0 \qquad (t \in e\,,\ x \in \Omega)\,,$$

then,

$$w_0 = 0\,.$$

The proof is based on an unique continuation theorem for complex analytic functions and the approximate observability of the system.

Strong Pontryagin maximum principle and uniqueness

Consequently, we have:

Theorem

There exists $\eta \in L^2(\Omega)$ and a unique time optimal control $u \in L^\infty([0,\tau], L^2(\mathcal{O}))$ steering z_0 to z_1, then:

$$u(t,x) = \frac{w(t,x)}{\|w(t,\cdot)\|_{L^2(\mathcal{O})}} \qquad (x \in \mathcal{O}\,,\ t \in [0,\tau]\ a.e.)\,,$$

with w solution of:

$$\begin{aligned} \dot{w} &= i\Delta w && (t \in (0,\tau)\,,\ x \in \Omega)\,,\\ w(t,x) &= 0 && (t \in (0,\tau)\,,\ x \in \partial\Omega)\,,\\ w(\tau,x) &= \eta(x) && (x \in \Omega)\,, \end{aligned}$$

Summary

In more generality, we need:

- (A, B) exactly controllable in any time $T > 0$ with L^∞ in time controls;
- There exists $u \in L^\infty([0, T], L^2(\mathcal{O}))$ with $\|u\|_{L^\infty([0,T],L^2(\mathcal{O}))} \leqslant 1$ steering z_0 to z_1;

to obtain the weak maximum principle.
If in addition, if

- (A, B) is approximatively controllable from any set e of positive measure;

then we have the strong maximum principle and uniqueness of the optimal control.

Open questions

- Whow can we compute numerically the time and optimal control?
- What about unbounded control operators?
  $\rightarrow$ Schrödinger with boundary control operator.
- If the minimal time T_* to ensure controllability is strictly positive?
  $\rightarrow$ Wave equation.

The problem

Let consider the Cauchy problem:

$$\ddot{z} = \Delta z + \mathbf{1}_{\mathcal{O}} u \qquad (t > 0\,,\ x \in \Omega)\,, \qquad \text{(W1)}$$

$$z(t,x) = 0 \qquad (t > 0\,,\ x \in \partial\Omega)\,, \qquad \text{(W2)}$$

$$(z,\dot{z})(0,x) = \mathrm{z}_0(x) \qquad (x \in \Omega)\,, \qquad \text{(W3)}$$

with Ω a bounded domain of $\mathbb{R}^n$ with C^2 boundary, $\mathcal{O} \subset \Omega$ satisfying the geometric optic condition and $\mathrm{z}_0 \in H^1_0(\Omega) \times L^2(\Omega)$ given.

Problem $(\mathcal{P})$

Given $\mathrm{z}_1 \in \big(H^1_0(\Omega) \times L^2(\Omega)\big) \setminus \{\mathrm{z}_0\}$, *does it exist a minimal time* τ *such that there exists* $u \in L^2([0,\tau], L^2(\mathcal{O}))$ *with:*

$$\|u\|_{L^2(L^2([0,\tau],L^2(\mathcal{O}))} \leqslant 1$$

$$\text{and} \quad (z,\dot{z})(\tau,\cdot) = \mathrm{z}_1\,,$$

where $z \in C\big([0,\tau], H^1_0(\Omega) \times L^2(\Omega)\big)$ *is the solution of (W)?*

Optimality conditions

Let T_* be the time given by the optic geometric condition.

Theorem

There exists en optimal time $\tau > 0$ *and a time optimal control* u *steering* z_0 *to* z_1 *in time* τ.
In addition, if $\tau > T_*$, *the time optimal control* u *is unique,* $\|u\|_{L^2([0,\tau],\mathcal{O})} = 1$ *and*

$$u(t,x) = \dot{w}(t,x) \qquad (t \in [0,\tau]\,,\ x \in \mathcal{O})\,,$$

where w *is solution of:*

$$\begin{aligned} \ddot{w} &= \Delta w && (t > 0\,,\ x \in \Omega)\,,\\ w(t,x) &= 0 && (t > 0\,,\ x \in \partial\Omega)\,,\\ (w,\dot{w})(\tau,x) &= \eta(x) && (x \in \Omega)\,, \end{aligned}$$

with $\eta \in \left(H_0^1(\Omega) \times L^2(\Omega)\right) \setminus \{0\}$.

The finite dimensional problem

Let $(\varphi_n)_{n\in\mathbb{N}}$ be the eigenvectors of the Dirichlet-Laplacian operator. For every $N\in\mathbb{N}$, we define $V_N=\mathrm{Span}\{\varphi_0,\dots,\varphi_N\}$ and $P_N\in\mathcal{L}(L^2(\Omega))$ the orthogonal projection on V_N. Let also define the problems set for every $N\in\mathbb{N}$:

Problem $(\mathcal{P}_N)$

Given $\mathrm{z}_0,\mathrm{z}_1\in H_0^1(\Omega)\times L^2(\Omega)$, $\mathrm{z}_0\neq\mathrm{z}_1$ *Find the minimal time* $\tau_N\geqslant 0$ *such that there exists a control* $u_N\in L^2([0,\tau_N],L^2(\mathcal{O}))$ *satisfying* $\|u_N\|_{L^2([0,\tau_N],L^2(\mathcal{O}))}\leqslant 1$ *and* $z_N(\tau_N)=P_N\mathrm{z}_1$, *where* z *is the solution of:*

$$\ddot{z}_N=\Delta z_N+P_N\mathbf{1}_{\mathcal{O}}u_N\,,\qquad (z_N,\dot{z}_N)(0)=P_N\mathrm{z}_0\,,\qquad (\star_N)$$

with Dirichlet boundary condition.

Convergence I

Proposition

For every $N \in \mathbb{N}$, $(\mathcal{P}_N)$ admits a unique solution (τ_N, u_N). In addition, the sequence (τ_N) is increasing and convergent to τ and $(u_N)_{N\in\mathbb{N}}$ is weakly convergent (up to an extraction) to an element $u \in L^2([0,\tau], L^2(\mathcal{O}))$ and (τ, u) is solution of the optimal control problem $(\mathcal{P})$.

Convergence I

Proposition

For every $N \in \mathbb{N}$, $(\mathcal{P}_N)$ admits a unique solution (τ_N, u_N). In addition, the sequence (τ_N) is increasing and convergent to τ and $(u_N)_{N\in\mathbb{N}}$ is weakly convergent (up to an extraction) to an element $u \in L^2([0,\tau], L^2(\mathcal{O}))$ and (τ, u) is solution of the optimal control problem $(\mathcal{P})$.

Proposition

Up to an extraction, the convergence of $\big((\tau_n, u_N)\big)_N$ to a solution (τ, u) of $(\mathcal{P})$ is strong. In addition,

$$\|u\|_{L^2([0,\tau],L^2(\mathcal{O}))} = 1 \quad \text{and} \quad \exists \eta \in H_0^1(\Omega) \times L^2(\Omega) \text{ s.t. } u = \dot{w}\big|_{\mathcal{O}},$$

with w solution of:

$$\begin{aligned} \ddot{w} &= \Delta w && (t > 0,\ x \in \Omega), \\ w(t,x) &= 0 && (t > 0,\ x \in \partial\Omega), \\ (w, \dot{w})(\tau, x) &= \eta(x) && (x \in \Omega). \end{aligned}$$

Convergence II

Idea of the proof

For every $N \in \mathbb{N}$, the time optimal control u_N is the one obtained by the HUM (Hilbert Uniqueness Method), that is:

$$u_N = (P_N \dot{w}_N)\big|_{\mathcal{O}}$$

with w_N solution of:

$$\ddot{w}_N = \Delta w_N\,, \qquad (w_N, \dot{w}_N)(\tau) = \eta_N(x)\,.$$

where $\eta_N \in V_N$ minimize:

$$J_N(\zeta) = \frac{1}{2}\int_0^{\tau_N} \|P_N \dot{w}_\zeta(t,\cdot)\|^2_{L^2(\mathcal{O})}\,\mathrm{d}t - \langle P_N(\mathrm{z}_1 - \mathbb{T}_{\tau_n}\mathrm{z}_0), \zeta\rangle$$
$$(\zeta \in H^1(\Omega) \times L^2(\Omega))\,,$$

with w_z given by:

$$\ddot{w}_\zeta = \Delta w_\zeta\,, \qquad (w_\zeta, \dot{w}_\zeta)(\tau_N, x) = \zeta(x)\,.$$

Convergence III

But (J_N) is Γ-convergent to the function J:

$$J(\zeta) = \frac{1}{2}\int_0^\tau \|\dot{w}_\zeta(t,\cdot)\|^2_{L^2(\mathcal{O})}\,\mathrm{d}t - \langle \mathrm{z}_1 - \mathbb{T}_\tau \mathrm{z}_0, \zeta\rangle \qquad (\zeta \in H^1_0(\Omega)\times L^2(\Omega))\,,$$

with w_z given by:

$$\ddot{w}_\zeta = \Delta w_\zeta\,, \qquad (w_\zeta, \dot{w}_\zeta)(\tau, x) = \zeta(x)\,.$$

- Results are true for boundary control.

- Results are true for boundary control.

- Is the time optimal control unique and of maximal norm for $\tau \leqslant T_*$?

- How can we compute numerically the time optimal control?
  - How can we describe the set of accessible points in time $t < T_*$?
  - Can we adapt the filtration method for the controls in time $t < T_*$?