Microlocal analysis of the HUM operator for a system of wave equations

Jérôme Le Rousseau
Université d'Orléans – Fédération Denis-Poisson

A joint work with
Belhassen Dehman (Faculté des Sciences de Tunis)
and
Matthieu Léautaud (Université Paris 7)

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Let Ω be a Riemannian compact manifold Ω (without boundary)

Δ is the Laplace-Beltrami operator on Ω

We consider the following system

$$\begin{cases} \partial_t^2 u_1 - \Delta u_1 + b u_2 = 0 & \text{in } (0,T)\times\Omega, \\ \partial_t^2 u_2 - \Delta u_2 = \chi g & \text{in } (0,T)\times\Omega, \\ (u_1, \partial_t u_1, u_2, \partial_t u_2)|_{t=0} = (u_1^0, u_1^1, u_2^0, u_2^1) & \text{in } \Omega. \end{cases}$$

$\chi \in \mathscr{C}^\infty(\overline{\Omega})$ and $\omega = \{\chi \neq 0\}$.

We wish to bring the solutions at rest at times $t \geq T$:

$$(u_1, \partial_t u_1, u_2, \partial_t u_2)|_{t=T} = (0,0,0,0).$$

- The system has a cascade structure.
- The control g only acts on one of the equations.
- The second equation can only be controlled through the coupling term bu_2

Let Ω be a Riemannian compact manifold Ω (without boundary)

Δ is the Laplace-Beltrami operator on Ω

Equivalently we consider the following system

$$\begin{cases} \partial_t^2 u_1 - \Delta u_1 + b u_2 = 0 & \text{in } (0,T)\times\Omega, \\ \partial_t^2 u_2 - \Delta u_2 = \chi g & \text{in } (0,T)\times\Omega, \\ (u_1, \partial_t u_1, u_2, \partial_t u_2)|_{t=0} = (0,0,0,0) & \text{in } \Omega. \end{cases}$$

$\chi \in \mathscr{C}^\infty(\overline{\Omega})$ and $\omega = \{\chi \neq 0\}$.

We wish to bring the solutions at time $t \geq T$ to a prescribed state:

$$(u_1, \partial_t u_1, u_2, \partial_t u_2)|_{t=T} = (u_1^0, u_1^1, u_2^0, u_2^1).$$

We consider the following system

$$\begin{cases} \partial_t^2 u_1 - \Delta u_1 + b u_2 = 0 & \text{in } (0,T)\times\Omega, \\ \partial_t^2 u_2 - \Delta u_2 = \chi g & \text{in } (0,T)\times\Omega, \\ (u_1, \partial_t u_1, u_2, \partial_t u_2)|_{t=0} = (0,0,0,0) & \text{in } \Omega. \end{cases}$$

We wish to bring the solutions at time $t \geq T$ to :

$$(u_1, \partial_t u_1, u_2, \partial_t u_2)|_{t=T} = (u_1^0, u_1^1, u_2^0, u_2^1).$$

Regularity Issue

$g \in L^2((0,T)\times\Omega)$,

$\Longrightarrow (u_2, \partial_t u_2) \in \mathscr{C}^0\big([0,T], H_0^1 \times L^2\big)$

$\Longrightarrow (u_1, \partial_t u_1) \in \mathscr{C}^0\Big([0,T], (H^2 \cap H_0^1) \times H_0^1\Big)$

we therefore need

$$(u_1, \partial_t u_1, u_2, \partial_t u_2)|_{t=T} \in (H^2 \cap H_0^1) \times H_0^1 \times H_0^1 \times L^2$$

This is the natural space for the analysis of the control problem.

First results known for weak constant coupling (symmetric system) in **[Alabau-Boussouira: 03]**

Alternative formulation

$$\begin{cases} \partial_t^2 u_1 - \Delta u_1 + b(1-\Delta)^{\frac{1}{2}} u_2 = 0 & \text{in } (0,T)\times\Omega, \\ \partial_t^2 u_2 - \Delta u_2 = \chi g & \text{in } (0,T)\times\Omega, \\ (u_1, \partial_t u_1, u_2, \partial_t u_2)|_{t=0} = (0,0,0,0) & \text{in } \Omega. \end{cases}$$

Regularity

$g \in L^2((0,T)\times\Omega)$,

$\Longrightarrow (u_1, \partial_t u_1) \in \mathscr{C}^0\big([0,T], H_0^1 \times L^2\big)$

$\Longrightarrow (u_2, \partial_t u_2) \in \mathscr{C}^0\big([0,T], H_0^1 \times L^2\big)$

We therefore need

$$(u_1, \partial_t u_1, u_2, \partial_t u_2)|_{t=T} \in H_0^1 \times L^2 \times H_0^1 \times L^2$$

$$\begin{cases} \partial_t^2 u_1 - \Delta u_1 + b(1-\Delta)^{\frac{1}{2}} u_2 = 0 & \text{in } (0,T)\times\Omega, \\ \partial_t^2 u_2 - \Delta u_2 = \chi g & \text{in } (0,T)\times\Omega, \\ (u_1, \partial_t u_1, u_2, \partial_t u_2)|_{t=0} = (0,0,0,0) & \text{in } \Omega. \end{cases}$$

Control space $g \in L^2((0,T)\times\Omega) \to$ state space

$$(u_1, \partial_t u_1, u_2, \partial_t u_2) \in H_0^1 \times L^2 \times H_0^1 \times L^2.$$

We also need conditions on

- both the supports of the control function and the coupling term;
- the time T.

Consider the control problem

$$\begin{cases} \partial_t^2 u - \Delta u = \chi g & \text{in} \quad (0,T)\times\Omega \\ u = 0 & \text{on} \quad (0,T)\times\times\partial\Omega \\ (u(0),\partial_t u(0)) = (0,0) \end{cases}$$

Wave symbol: $p = -\tau^2 + R(x,\xi)$
(flat case: $R(x,\xi) = |\xi|^2$, here $R(x,\xi) = |\xi|_x^2$)

- Hamiltonian vector field:

$$H_p = (\partial_\tau p)\partial_t - (\partial_t p)\partial_\tau + (\partial_\xi p)\partial_x - (\partial_x p)\partial_\xi$$

- Integral curves for H_p

$$\frac{dt}{ds} = \partial_\tau p(t,x,\tau,\xi) \qquad \frac{dx}{ds} = \partial_\xi p(t,x,\tau,\xi)$$
$$\frac{d\tau}{ds} = -\partial_t p(t,x,\tau,\xi) = 0 \qquad \frac{d\xi}{ds} = -\partial_x p(t,x,\tau,\xi)$$

- Rays or bicharacteristics: integral curves for H_p within the characteristic set $p=0$.
- Singularities travel along such bicharacteristics (Hörmander)

Geometric control condition **[Rauch-Taylor '74, Bardos-Lebeau-Rauch '92]**

- Case of manifold without boundary.
  (ω, T) is said to satisfy GCC at time T if all bicharacteristics starting from (x, ξ) at time $t = 0$ enter $\omega = \{\chi \neq 0\}$ before $t = T$.
- Case of manifold with boundary.
  (ω, T) is said to satisfy GCC at time T if all generalized bicharacteristics starting from (x, ξ) at time $t = 0$ enter $\omega = \{\chi \neq 0\}$ before $t = T$.

Generalized bicharacteristics are described in **[Melrose-Sjöstrand '78, '82]**

Theorem (Bardos-Lebeau-Rauch '92, Burq-Gérard' 97)

GCC is equivalent to the exact controllability of the wave equation at time T

The system is

$$\begin{cases} \partial_t^2 u_1 - \Delta u_1 + b(1-\Delta)^{\frac{1}{2}} u_2 = 0 & \text{in } (0,T)\times\Omega, \\ \partial_t^2 u_2 - \Delta u_2 = \chi g & \text{in } (0,T)\times\Omega, \\ (u_1, \partial_t u_1, u_2, \partial_t u_2)|_{t=0} = (0,0,0,0) & \text{in } \Omega. \end{cases}$$

with $b \geq 0$.

$\omega = \{\chi \neq 0\}, \quad \mathcal{O} = \{b > 0\}$

Necessary condition to controllability

- (ω, T_ω) satisfies GCC
- $(\mathcal{O}, T_\mathcal{O})$ satisfies GCC

Definition

Given two sets ω and $\mathcal{O}$ both satisfying GCC, we set $T_{\omega\to\mathcal{O}\to\omega}$ the infimum of times $T > 0$ s.t.

every bicharacteristics traveling at speed one in Ω meets ω in a time $t_0 < T$, meets $\mathcal{O}$ in a time $t_1 \in (t_0, T)$ and meets ω again in a time $t_2 \in (t_1, T)$.

Observe that $\max\left(T_\mathcal{O}, T_\omega\right) \leq T_{\omega\to\mathcal{O}\to\omega} \leq 2T_\omega + T_\mathcal{O}$.

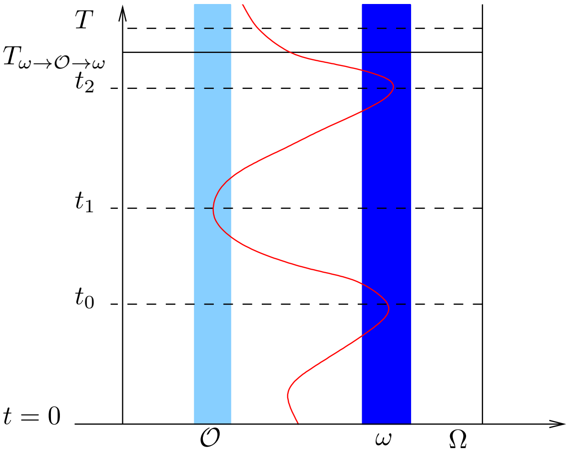
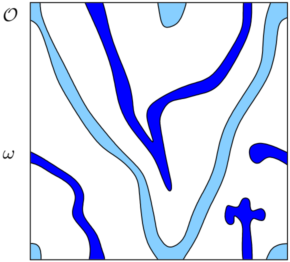


Figure: Geometric condition and time $T_{\omega\to\mathcal{O}\to\omega}$.



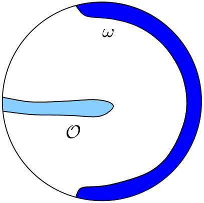


Figure: Examples of open sets $(\Omega, \omega, \mathcal{O})$ s.t. ω and $\mathcal{O}$ both satisfy GCC in Ω: case (a), Ω is the flat torus (or the square), case (b), Ω is the disk.

$$\begin{cases} \partial_t^2 u_1 - \Delta u_1 + b(1-\Delta)^{\frac{1}{2}} u_2 = 0 & \text{in } (0,T)\times\Omega, \\ \partial_t^2 u_2 - \Delta u_2 = \chi g & \text{in } (0,T)\times\Omega, \\ (u_1, \partial_t u_1, u_2, \partial_t u_2)|_{t=0} = (0,0,0,0) & \text{in } \Omega. \end{cases}$$

with $b \geq 0$.

Theorem (Dehman, LR, Léautaud)

Let Ω be a compact manifold without boundary. If

- $\omega = \{\chi \neq 0\}$ *satisfies GCC*
- $\mathcal{O} = \{b > 0\}$ *satisfies GCC*
- $T > T_{\omega\to\mathcal{O}\to\omega}$

then the system is exactly controllable.

If either

- ω *does not satisfy GCC*
- *or* $\mathcal{O}$ *does not satisfy GCC*
- *or* $T < T_{\omega\to\mathcal{O}\to\omega}$

the system is NOT controllable.

Existing results

- [**F. Alabau-Boussouira - M. Leautaud '11**] symmetric systems, weak coupling, long control time.
- [**L. Rosier - L. de Teresa '11**] 1-D, geometric but not sharp control time.
- [**F. Alabau-Boussouira**] Cascade of N equations, long control time.

Adjoint system:

$$\begin{cases} (\partial_t^2 - \Delta) w_1 = 0 & \text{in } (0,T) \times \Omega \\ (\partial_t^2 - \Delta) w_2 = -b(x)(1-\Delta)^{\frac{1}{2}} w_1 & \text{in } (0,T) \times \Omega. \end{cases} \tag{Adj}$$

Controllability is equivalent to the observability inequality:

$$e_0(w_1(0)) + e_0(w_2(0)) \leq C \int_0^T \int_\Omega |\chi w_2|^2 dx\, dt, \tag{Obs}$$

for all (w_1, w_2) solutions of (Adj).

Here $e_0(w) = \|w\|^2_{L^2(\Omega)} + \|\partial_t w\|^2_{H^{-1}(\Omega)}$.

Adjoint equation for the wave equation:

$$(\partial_t^2 - \Delta)v = 0 \qquad \text{in } (0,T)\times\Omega$$

We set $V = (v, \partial_t v)$

Controllability is equivalent to the observability inequality:

$$e_0(V(0)) \leq C \int_0^T \int_\Omega |\chi v|^2 dx\ dt,$$

for all v solution of the adjoint equation.
Here

$$e_0(V(0)) = \|v(0)\|^2_{L^2(\Omega)} + \|\partial_t v(0)\|^2_{H^{-1}(\Omega)}$$

Controlled equation: $(\partial_t^2 - \Delta)u = \chi g \qquad \text{in } (0,T)\times\Omega$
$U(t) = (u(t), \partial_t u(t)) \in H^1(\Omega)\times L^2(\Omega)$.

Adjoint equation: $(\partial_t^2 - \Delta)v = 0 \qquad \text{in } (0,T)\times\Omega$.

With an integration by parts we have:

$$\langle \chi g, v\rangle_{L^2((0,T)\times\Omega)} = \langle \partial_t u(T), v(T)\rangle_{L^2} - \overbrace{\langle \partial_t u(0), v(0)\rangle_{L^2}}^{=0} - \Big(\langle u(T), \partial_t v(T)\rangle_{H^1,H^{-1}} - \underbrace{\langle u(0), \partial_t v(0)\rangle_{H^1,H^{-1}}}_{=0}\Big)$$

We introduce the maps

$$L : L^2(\Omega)\times H^{-1}(\Omega) \to L^2((0,T)\times\Omega)$$
$$V_T = V(T) = (v(T), \partial_t v(T)) \mapsto \chi v$$

and

$$M : L^2((0,T)\times\Omega) \to H^1(\Omega)\times L^2(\Omega)$$
$$g \mapsto (-u(T), \partial_t u(T))$$

In the HUM approach we seek the control as a solution of the backward wave equation: we set $g = L(V_T) = \chi v$.

Then $(u(T), \partial_t u(T)) = M \circ L(V_T)$

We then find

$$\begin{aligned}\|L(V_T)\|^2_{L^2((0,T)\times\Omega)} &= \langle \partial_t u(T), v(T)\rangle_{L^2} - \langle u(T), \partial_t v(T)\rangle_{H^1,H^{-1}} \\ &= \langle U(T), V_T\rangle_* = \langle M \circ L(V_T), V_T\rangle_*\end{aligned}$$

where

$$\langle U, V\rangle_* = \langle U_2, V_1\rangle_{L^2} + \langle U_1, V_2\rangle_{H^1,H^{-1}}, \qquad U \in H^1 \times L^2,\ V \in L^2 \times H^{-1}$$

$\mathscr{G} = M \circ L$ is the Gramian operator.

We have

$$\|L(V_T)\|^2_{L^2((0,T)\times\Omega)} = \langle \mathscr{G}(V_T), V_T \rangle_*$$

Note that $\mathscr{G} : L^2(\Omega) \times H^{-1}(\Omega) \to H^1(\Omega) \times L^2(\Omega)$

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Note that $\mathscr{G} : L^2(\Omega) \times H^{-1}(\Omega) \to H^1(\Omega) \times L^2(\Omega)$

Observability:

$$\|V_T\|_{L^2\times H^{-1}} \leq C\|L(V_T)\|_{L^2((0,T)\times\Omega)}$$

We have

$$\text{controllability} \quad \Leftrightarrow \quad \text{observability} \quad \Leftrightarrow \quad \text{invertibility of } \mathscr{G}$$

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$$\text{controllability} \quad \Leftrightarrow \quad \text{observability} \quad \Leftrightarrow \quad \text{invertibility of } \mathscr{G}$$

In such case we can solve $\mathscr{G}(V_T) = U_T$ with $U_T = (-u_0, u_1) \in H^1 \times L^2$
The HUM operator is precisely $\mathscr{G}^{-1}$

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The controlled equation is then

$$(\partial_t^2 - \Delta)u = \chi L(V_T) \qquad \text{in } (0,T)\times\Omega \qquad (u(0), \partial_t u(0)) = (0,0)$$

and we obtain $(u(T), \partial_t u(T)) = (u_0, u_1)$.

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Observability:

$$\|V_T\|_{L^2\times H^{-1}} \leq C\|L(V_T)\|_{L^2((0,T)\times\Omega)}$$

We have

$$\text{controllability} \quad \Leftrightarrow \quad \text{observability} \quad \Leftrightarrow \quad \text{invertibility of } \mathscr{G}$$

In such case we can solve $\mathscr{G}(V_T) = U_T$ with $U_T = (-u_0, u_1) \in H^1 \times L^2$
The HUM operator is precisely $\mathscr{G}^{-1}$
The controlled equation is then

$$(\partial_t^2 - \Delta)u = \chi L(V_T) \qquad \text{in } (0,T)\times\Omega \qquad (u(0), \partial_t u(0)) = (0,0)$$

and we obtain $(u(T), \partial_t u(T)) = (u_0, u_1)$.

Question: can we prove directly the invertibility of the Gramian operator $\mathscr{G}$?

In part we follow the approach of Dehman-Lebeau

Here, to expose their method, we consider a simplified model to avoid technicalities

We consider the half-wave first-order equation

$$\partial_t u - i\lambda u = \chi g \in L^2(\Omega), \qquad \lambda = \sqrt{-\Delta}, \qquad u(0) = 0$$

λ is pseudo-differential of order 1 with principal symbol $|\xi|$

We have $u \in \mathscr{C}^0([0,T]; L^2(\Omega)) \cap \mathscr{C}^1([0,T]; H^{-1}(\Omega))$

Note that $(\partial_t - i\lambda)(\partial_t + i\lambda) = \partial_t^2 - \Delta$. We have factorized the wave equation
The adjoint equation is

$$\partial_t v - i\lambda v = 0 \in L^2(\Omega), \qquad v(T) = v_T$$

$$\partial_t u - i\lambda u = \chi g \in L^2(\Omega), \qquad u(0) = 0$$

The adjoint equation is

$$\partial_t v - i\lambda v = 0 \in L^2(\Omega), \qquad v(T) = v_T$$

Proceeding as above we find

$$\langle \chi g, v \rangle_{L^2((0,T)\times\Omega)} = \langle u(T), v(T) \rangle_{L^2}.$$

We introduce the maps

$$\begin{aligned} L : L^2(\Omega) &\to L^2((0,T)\times\Omega) \\ v_T = v(T) &\mapsto \chi v \end{aligned}$$

and

$$\begin{aligned} M : L^2((0,T)\times\Omega) &\to L^2(\Omega) \\ g &\mapsto u(T) \end{aligned}$$

and set $\mathscr{G} = M \circ L$.

$$\partial_t u - i\lambda u = \chi g \in L^2(\Omega), \qquad u(0) = 0$$

The adjoint equation is

$$\partial_t v - i\lambda v = 0 \in L^2(\Omega), \qquad v(T) = v_T$$

Setting $g = L(v_T)$ we have

$$\|L(v_T)\|^2_{L^2((0,T)\times\Omega)} = \langle \mathscr{G} v_T, v_T \rangle_{L^2}.$$

controllability $\Leftrightarrow$ observability $\Leftrightarrow$ invertibility of $\mathscr{G}$

$$\partial_t u - i\lambda u = \chi g \in L^2(\Omega), \qquad u(0) = 0$$

Changing $t \to T - t$ in the adjoint equation

$$\partial_t v + i\lambda v = 0 \in L^2(\Omega), \qquad v(0) = v_0$$

We have

$$\|L(v_0)\|^2_{L^2((0,T)\times\Omega)} = \langle \mathscr{G} v_0, v_0 \rangle_{L^2}.$$

controllability $\Leftrightarrow$ observability $\Leftrightarrow$ invertibility of $\mathscr{G}$

Bicharacterisitic flow: we introduce φ solution to

$$\frac{d}{ds}\varphi_s^{\pm}(x,\xi) = H_{\mp|\xi|_x}\big(\varphi_s(x,\xi)\big), \quad \varphi_0^{\pm}(x,\xi) = (x,\xi) \in T^*\Omega \setminus 0.$$

N.B. $H_{\mp|\xi|_x}$ is the vector field: $\mp(\partial_\xi|\xi|_x)\partial_x \pm (\partial_x|\xi|_x)\partial_\xi$.

The adjoint equation is

$$\partial_t v + i\lambda v = 0 \in L^2(\Omega), \qquad \lambda = \sqrt{-\Delta}, \qquad v(0) = v_0$$

The gives $v(0) = e^{-it\lambda} v_0$ (Fourier integral operator)

N.B. Here we ignore the eigenvalue 0
Then we compute the observation

$$\int_0^T \|\chi v\|^2_{L^2(\Omega)} \, dt = \int_0^T \langle e^{it\lambda} \chi^2 e^{-it\lambda} v_0, v_0 \rangle_{L^2(\Omega)} \, dt$$

We thus have $\mathscr{G} v_0 = \int_0^T e^{it\lambda} \chi^2 e^{-it\lambda} v_0 \, dt$.

By the Egorov theorem $e^{it\lambda} \chi^2 e^{-it\lambda}$ is a pseudo-differential operator of order 0 with principal symbol $\sigma(t, x, \xi) = \chi^2 \circ \varphi_t^-(x, \xi)$

The Gramian is thus a pseudo-differential operator of order 0 with symbol

$$\int_0^T \chi^2 \circ \varphi_t^-(x, \xi) \, dt$$

The Gramian is thus a pseudo-differential operator of order 0 with symbol

$$\int_0^T \chi^2 \circ \varphi_t^-(x,\xi)\ dt$$

For the Gramian to be invertible one needs it to be elliptic, that is (order 0)

$$\int_0^T \chi^2 \circ \varphi_t^-(x,\xi)\ dt \geq C, \qquad \forall (x,\xi) \in T^*(\Omega)$$

We thus recover the GC condition of Bardos-Lebeau-Rauch.

Theorem (Dehman-Lebeau)

We have

$$\{\chi > 0\} \text{ satisfies GCC} \quad \Leftrightarrow \quad \text{the operator } \mathscr{G} \text{ is elliptic.}$$

Moreover in such case

1. *the operator* $\mathscr{G}$ *is coercive and invertible;*
2. *the HUM operator* $\mathscr{G}^{-1}$ *can be written as* $\mathscr{G}^{-1} = \Lambda + R$ *when* R *is a regularizing operator and* Λ *is a pseudo-differential operator of order* 0 *with principal symbol*

$$\Big(\int_0^T \chi^2 \circ \varphi_t^-(x, \xi) \; dt \Big)^{-1}$$

The proof of coercivity goes along two steps with a compactness-uniqueness argument as in the original proof of **Bardos-Lebeau-Rauch** for the controllability of the wave equation.

We acknowledge very fruitful discussions with **C. Laurent** on some aspects of the proof.

First-step: from the ellipticity of $\mathscr{G}$ we find with the Gårding inequality:

$$\langle \mathscr{G} v_0, v_0 \rangle_{L^2} \geq C \|v_0\|_{L^2}^2 - C' \|v_0\|_{H^{-\frac{1}{2}}}^2 . \tag{1}$$

We thus have coercivity for the high-frequencies.

Second-step: We consider

$$\mathcal{N}(T) = \{v_0 \in L^2(\Omega); L(v) = \chi v(t,x) = 0 \text{ in } (0,T) \times \omega\},$$

By proving that the unit sphere in $\mathcal{N}(T)$ is compact from (1) we have

Lemma

The space $\mathcal{N}(T)$ is finite dimensional.

Moreover if $v_0 \in \mathcal{N}(T)$ then $\chi v = 0$, implying $\chi \partial_t v = 0$. This implies $L v_0 \in \mathcal{N}(T)$.

Second-step (continued):

Hence, $\mathcal{N}(T)$ is stable under λ and is finite dimensional.

If $\mathcal{N}(T) \neq \{0\}$, this implies that there exist λ and $w \neq 0$, such that

$$w \in \mathcal{N}(T), \qquad \lambda w = \lambda w, \qquad \chi w = 0,$$

We thus have $-\Delta w = \lambda^2 w$ and $\chi w = 0$. A classical unique continuation result yields $w = 0$

Lemma

We have $\mathcal{N}(T) = \{0\}$.

Second-step (continued):
We assume that coercivity does not hold:

There exists $(v_0^{(n)}) \subset L^2(\Omega)$ such that

$$\|v_0^{(n)}\| = 1 \qquad \|\mathscr{G} v_0^{(n)}\|_{L^2(\Omega)} \to 0$$

We have $v_0^{(n)} \rightharpoonup v \in L^2(\Omega)$.

Continuity of the half wave equation yields $v \in \mathcal{N}(T)$, that is $v = 0$

In particular we have $v_0^{(n)} \to 0$ in $H^{-1}(\Omega)$.
The high-frequency result:

$$\langle \mathscr{G} v_0, v_0 \rangle_{L^2} \geq C \|v_0\|_{L^2}^2 - C' \|v_0\|_{H^{-\frac{1}{2}}}^2$$

then implies the contradiction

$$0 \geq C - C' \times 0.$$

Again we consider a simplified model here for the sake of exposition: Set $\lambda = \sqrt{-\Delta}$ on L^2_+, projecting onto the orthogonal of the space of constant functions.
Consider:

$$\begin{cases} (\partial_t - i\lambda)u_1 - \frac{1}{2i} b u_2 = 0 & \text{in } (0,T)\times\Omega, \\ (\partial_t - i\lambda)u_2 = \chi f & \text{in } (0,T)\times\Omega. \end{cases}$$

Adjoint system:

$$\begin{cases} (\partial_t + i\lambda)v_1 = 0 & \text{in } (0,T)\times\Omega, \\ (\partial_t + i\lambda)v_2 + \frac{1}{2i} b v_1 = 0 & \text{in } (0,T)\times\Omega, \\ (v_1(0), v_2(0)) = (g,h) \in L^2(\Omega;\mathbb{C}^2) \end{cases}$$

The observability inequality reads

$$\|g\|^2_{L^2(\Omega)} + \|h\|^2_{L^2(\Omega)} \le C \int_0^T \|\chi v_2\|^2_{L^2(\Omega)} dt.$$

The Gramian operator is given by

$$\int_0^T \|\chi v_2\|^2_{L^2(\Omega)} dt = \big(\mathscr{G}(g,h),(g,h)\big)_{L^2(\Omega;\mathbb{C}^2)}.$$

If the wave system is exactly controllable then $\mathscr{G}$ is invertible and the HUM operator is $\mathscr{G}^{-1}$

Theorem

There exists $G \in \Psi^0(\Omega;\mathbb{C}^{2\times 2})$, and R an infinitely smoothing operator on Ω such that

$$\mathscr{G} = G + R,$$

where the principal symbol (in $S^0(T^*\Omega,\mathbb{C}^{2\times 2})$) of G is

$$\begin{pmatrix} \frac{1}{4}\int_0^T \chi^2 \circ \varphi_t^- \Big(\int_0^t b \circ \varphi_\sigma^- d\sigma\Big)^2 dt & \frac{1}{2i}\int_0^T \chi^2 \circ \varphi_t^- \Big(\int_0^t b \circ \varphi_\sigma^- d\sigma\Big) dt \\ -\frac{1}{2i}\int_0^T \chi^2 \circ \varphi_t^- \Big(\int_0^t b \circ \varphi_\sigma^- d\sigma\Big) dt & \int_0^T \chi^2 \circ \varphi_t^- dt \end{pmatrix}$$

Theorem (continued)

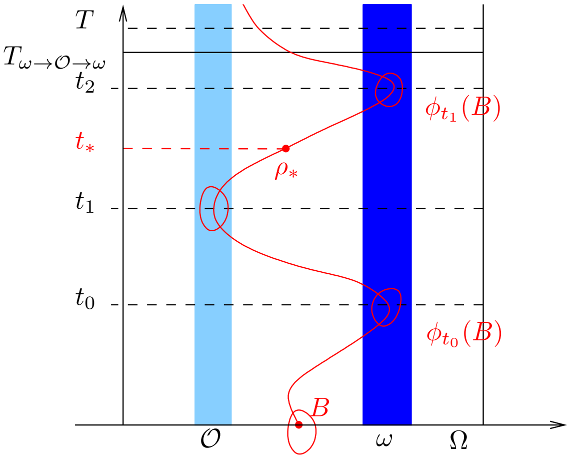
- *In particular, we have*

$$\det(\sigma_0(G)) = \frac{1}{8}\int_0^T\int_0^T (\chi^2\circ\varphi_{t_1}^-)(\chi^2\circ\varphi_{t_2}^-)\Big(\int_{t_1}^{t_2} b\circ\varphi_\sigma^- d\sigma\Big)^2 dt_1 dt_2 \in S^0(T^*\Omega).$$

- *The operator* $\mathscr{G}$ *is coercive on* $L^2(\Omega;\mathbb{C}^2)$.
- *The operator* $\mathscr{G}$ *is invertible in* $L(L^2(\Omega))$. *Its inverse* $(\mathscr{G})^{-1}$, *the HUM operator, can be decomposed as* $(\mathscr{G})^{-1} = \Lambda + R$ *where* R *is smoothing and* $\Lambda \in \Psi^0(T^*\Omega, \mathbb{C}^{2\times 2})$, *with principal symbol*

$$\det(\sigma_0(G))^{-1} \times \begin{pmatrix} \int_0^T \chi^2\circ\varphi_t dt & \frac{1}{2i}\int_0^T \chi^2\circ\varphi_t^-\Big(\int_0^t b\circ\varphi_\sigma^- d\sigma\Big)dt \\ -\frac{1}{2i}\int_0^T \chi^2\circ\varphi_t^-\Big(\int_0^t b\circ\varphi_\sigma^- d\sigma\Big)dt & \frac{1}{4}\int_0^T \chi^2\circ\varphi_t^-\Big(\int_0^t b\circ\varphi_\sigma^- d\sigma\Big)^2 dt \end{pmatrix}$$

$$\det(\sigma_0(G)) = \frac{1}{8}\int_0^T\int_0^T (\chi^2\circ\varphi^-_{t_0})(\chi^2\circ\varphi^-_{t_2})\Big(\int_{t_0}^{t_2} b\circ\varphi^-_\sigma \, d\sigma\Big)^2 dt_0 dt_2 \in S^0(T^*\Omega).$$



Observe that the geometric condition appears clearly in $\det(\sigma_0(G))$.

Sketch of proof. The Duhamel formula gives

$$v_1(t) = e^{-it\lambda} g, \qquad v_2(t) = e^{-it\lambda} h - \frac{1}{2i} \int_0^t e^{-i(t-\sigma)\lambda} b v_1(\sigma) d\sigma.$$

We compute

$$\int_0^T \|\chi v_2\|^2_{L^2(\Omega)} dt = \big((G+R)(g,h),(g,h)\big)_{L^2}$$

with

$$G = \begin{pmatrix} \frac{1}{4}\int_0^T (B_t)^* e^{it\lambda}\chi^2 e^{-it\lambda} B_t dt & +\frac{1}{2i}\int_0^T (B_t)^* e^{it\lambda}\chi^2 e^{-it\lambda} dt \\ -\frac{1}{2i}\int_0^T e^{it\lambda}\chi^2 e^{-it\lambda} B_t dt & \int_0^T e^{it\lambda}\chi^2 e^{-it\lambda} dt \end{pmatrix},$$

where

$$B_t := \int_0^t e^{i\sigma\lambda} b e^{-i\sigma\lambda} d\sigma$$

The conclusion follows from the Egorov Theorem.

Thank you for your attention.

Further details in:

B. Dehman, J. Le Rousseau, and M. Léautaud. *Controllability of two coupled wave equations on a compact manifold*, preprint 2012, 53 pages.
http://hal.archives-ouvertes.fr/hal-00686967