Optimal control of Hamilton-Jacobi-Bellman equations

P. Jameson Graber

Commands (ENSTA ParisTech, INRIA Saclay)

17 January, 2014
Table of Contents

1 Introduction
   • Motivation
   • Background

2 Main results
   • Assumptions
   • Existence of minimizers
   • Characterization of minimizers

3 Proofs
   • Duality
   • Equivalence of original and relaxed problems
   • A priori bounds
   • Passing to the limit
   • Existence of weak solutions
   • Uniqueness of weak solutions

4 Further results
   • Well-posedness of weak solutions
   • Long time average behavior
Objective

We consider an optimal control problem:

Problem

(1) For a given cost functional $K$ and finite measure $dm_0$, find

$$\inf \left\{ \int_0^T \int K(t, x, f(t, x)) \, dx \, dt - \int u(0, x) \, dm_0(x) \right\}$$

over functions $u, f$ such that $f \in C([0, T] \times \mathbb{T}^N)$ and $u$ solves

$$-\partial_t u(t, x) + H(x, Du(t, x)) = f(t, x), \quad u(T, x) = u_T(x).$$

(2) Find a PDE characterization of (weak) minimizers.
**Purpose:** to steer an evolving set.

**Model:** given a set $\mathcal{A}$ of admissible controls and a given region $\Omega_T \subset \mathbb{R}^N$, let $\Omega_t$ be the *backward reachable set* at time $t$ defined by

$$\Omega_t := \{y(t) : \dot{y} = c(y, \alpha), \ \alpha \in \mathcal{A}, y(T) \in \Omega_T\}.$$
**Model**

*Strategy*: use obstacles.

Obstacle construction: Let $f \geq 0$ be continuous. Define $K_\tau, \tau \in [0, T]$ by

$$K_\tau := \{x : f(\tau, x) > 0\}.$$ 

Consider the modified “blocked” evolving set

$$\tilde{\Omega}_t = \{y(t) : \dot{y} = c(y, \alpha), \ \alpha \in A, y(T) \in \Omega_T, y(\tau) \notin K_\tau \ \forall \ \tau \in [t, T]\}.$$ 

Following [Bokanowski, Forcadel, Zidani 2010], we have a characterization: let $u_T \geq 0$ (continuous) be such that $u_T = 0$ precisely on $\Omega_T$.

$$\tilde{\Omega}_t = \{y(t) : u_T(y(T)) + \int_t^T f(s, y(s)) \leq 0, \ \dot{y} = c(y, \alpha), \ \alpha \in A\}.$$
Model

\[ \tilde{\Omega}_t = \{ u(t, \cdot) \leq 0 \} \] characterized by an optimal control problem:

\[
u(t, x) := \inf \left\{ u_T(y(T)) + \int_t^T f(s, y(s)) ds : \dot{y} = c(y, \alpha), \; \alpha \in A, y(t) = x \right\}.
\]

\( u \) solves a Hamilton-Jacobi-Bellman equation:

\[-\partial_t u(t, x) + H(x, Du(t, x)) = f(t, x), \quad u(T, x) = u_T(x).\]

Conclusion

Steering the front by obstacles is thus related to optimizing solutions to HJB equations.
A related open problem:

**Optimal control via velocity**

Given a cost functional $C$, find

$$\inf \left\{ C(c) - \int u(T, x) \, dm_0(x) \right\},$$

where the infimum is taken over $u, c$ satisfying

$$u_t + c(t, x)|Du| = 0, \quad u(0, x) = u_0(x).$$

Find a characterization of the minimizers.

Very little appears to be known.
# Table of Contents

1. **Introduction**
   - Motivation
   - Background

2. **Main results**
   - Assumptions
   - Existence of minimizers
   - Characterization of minimizers

3. **Proofs**
   - Duality
   - Equivalence of original and relaxed problems
   - A priori bounds
   - Passing to the limit
   - Existence of weak solutions
   - Uniqueness of weak solutions

4. **Further results**
   - Well-posedness of weak solutions
   - Long time average behavior
Mass transportation

In 2000, Benamou and Brenier considered a fluid mechanics formulation of the Monge-Kantorovich problem (MKP).

For $\rho_0(x) \geq 0$ and $\rho_T(x) \geq 0$ probability densities on $\mathbb{R}^d$, we say $M : \mathbb{R}^d \rightarrow \mathbb{R}^d$ transfers $\rho_0$ to $\rho_T$ if $\forall A \subset \mathbb{R}^d$ bounded,

$$\int_{x \in A} \rho_T(x) \, dx = \int_{M(x) \in A} \rho_0(x) \, dx.$$ 

The $L^p$ MKP is to find

$$d_p(\rho_0, \rho_T)^p = \inf_M \int |M(x) - x|^p \rho_0(x) \, dx.$$ 

d$_p$ is called the $L^p$ Kantorovich (or Wasserstein) distance.
Theorem (Benamou-Brenier, 2000)

The square of the $L^2$ Kantorovich distance is equal to the infimum of

$$T \int_{\mathbb{R}^d} \int_0^T \rho(t, x) |v(t, x)|^2 \, dx \, dt,$$

among all $(\rho, v)$ satisfying

$$\partial_t \rho + \nabla \cdot (\rho v) = 0$$

for $0 < t < T$ and $x \in \mathbb{R}^d$, and the initial-final condition

$$\rho(0, \cdot) = \rho_0, \quad \rho(T, \cdot) = \rho_T.$$

The optimality conditions are given by $v(t, x) = \nabla \phi(t, x)$, where

$$\partial_t \phi + \frac{1}{2} |\nabla \phi|^2 = 0.$$
Dolbeault-Nazaret-Savaré distances

In 2009, Dolbeault, Nazaret, and Savaré, building on Benamou-Brenier, introduced a new class of transport distances between measures, given by minimizing

$$\int_0^1 \int_{\mathbb{R}^d} |v(t, x)|^2 m(\rho(t, x)) \, dx \, dt$$

subject to

$$\partial_t \rho + \nabla \cdot (m(\rho) v) = 0, \quad \rho(0, \cdot) = \rho_0, \quad \rho(1, \cdot) = \rho_1.$$

In 2013, Cardaliaguet, Carlier, and Nazaret studied geodesics for this class of distances when $m(\rho) = \rho^\alpha$, getting optimality conditions

$$\begin{cases} 
\partial_t \rho + \nabla \cdot (\frac{1}{2} \rho^\alpha \nabla \phi) = 0, \\
\rho > 0 \implies \partial_t \phi + \frac{\alpha}{4} \rho^{\alpha-1} |\nabla \phi|^2 = 0, \\
\rho \geq 0, \partial_t \phi \leq 0, \rho(0, \cdot) = \rho_0, \rho(1, \cdot) = \rho_1. 
\end{cases}$$
Mean Field Games

In 2007, Lasry and Lions give an overview of the following system of equations:

\[
\frac{\partial u}{\partial t} - \nu \Delta u + H(x, \nabla u) = V[m] \quad \text{in} \quad Q \times (0, T)
\]

\[
\frac{\partial m}{\partial t} + \nu \Delta m + \nabla \cdot \left( \frac{\partial H}{\partial p} (x, \nabla u) m \right) = 0 \quad \text{in} \quad Q \times (0, T)
\]

\[
u \frac{\partial V}{\partial t} - \nu \Delta V = 0 \quad \text{in} \quad Q \times (0, T)
\]

\[
V \bigg|_{t=0} = V_0[m(0)] \quad \text{on} \quad Q, \quad m \bigg|_{t=T} = m^0 \quad \text{on} \quad Q.
\]

There are two characterizations of the system:

- asymptotic behavior of differential games with large number of players,
- optimal control of Fokker-Planck equation, or of Hamilton-Jacobi equation (by duality).
Table of Contents

1 Introduction
   - Motivation
   - Background

2 Main results
   - Assumptions
   - Existence of minimizers
   - Characterization of minimizers

3 Proofs
   - Duality
   - Equivalence of original and relaxed problems
   - A priori bounds
   - Passing to the limit
   - Existence of weak solutions
   - Uniqueness of weak solutions

4 Further results
   - Well-posedness of weak solutions
   - Long time average behavior

P. Jameson Graber
Optimal control of Hamilton-Jacobi-Bellman equations
Assumptions

- **Hamiltonian structure.** $H = H(x, p)$ is Lipschitz in the first variable:

\[ |H(x, p) - H(y, p)| \leq L|x - y||p|. \]

$H(x, \cdot)$ is subadditive and positively homogeneous, hence convex, with linear bounds

\[ c_0|p| \leq H(x, p) \leq c_1|p|. \]

$H(x, p) := \sup\{-c(x, a) \cdot p : a \in A\}$

- **Cost function.** $k = k(t, x, m)$ continuous and strictly increasing with

\[ \frac{1}{C}|m|^{q-1} - C \leq k(t, x, m) \leq C|m|^{q-1} + C. \]

$K^*(t, x, \cdot)$ a primitive of $k$, $K(t, x, \cdot)$ its Fenchel conjugate. We will assume that $K(t, x, 0) = 0$ and that $K(t, x, f)$ is increasing in $f$ for $f \geq 0$. Note that

\[ \frac{1}{C}|f|^p - C \leq K(t, x, f) \leq C|f|^p + C. \]

- **Final-initial conditions.** $u_T : \mathbb{T}^N \rightarrow \mathbb{R}$ is Lipschitz, $m_0 \in L^\infty(\mathbb{T}^N)$.
Comparison with the literature

1. This is the first study with linearly bounded Hamiltonian.
2. In [Cardaliaguet, 2013] the Hamiltonian is supposed to satisfy

\[
\frac{1}{rC} |p|^r - C_2 \leq H(x, p) \leq \frac{C}{r} |p|^r + C
\]

for some \( r > N(q - 1) \lor 1 \) and, for some \( \theta \in [0, \frac{r}{N+1}) \),

\[
|H(x, p) - H(y, p)| \leq C|x - y|(|p| \lor 1)^\theta.
\]

The Hamiltonian \( H(x, p) = c(x)|p|^r \) is forbidden by this restriction. By contrast, \( H(x, p) = c(x)|p| \) is permitted under our assumptions.
Existence theorem

**Relaxed set**

The set $\tilde{K}$ will be defined as the set of all pairs $(u, f) \in BV \times L^p$ such that $u \in L^\infty$, $u(T, \cdot) = u_T$ in the sense of traces, and $-\partial_t u + H(x, Du) \leq f$ in the sense of measures, by which we mean that $f - H(x, Du) + \partial_t u$ is a non-negative Radon measure.

**Relaxed problem:** Find $\inf_{(u,f) \in \tilde{K}} A(u, f)$.

**Lemma**

The relaxed problem has the same infimum as the original.

**Theorem (PJG)**

There exists a minimizer to the relaxed problem.
Characterization of minimizers

Heuristically, use Lagrange multipliers to get system (MFG)

\[-\partial_t u + H(x, Du) = k(t, x, m(t, x)) \quad \text{(state equation)}\]

\[u(T, x) = u_T(x)\]

\[\partial_t m - \text{div} \left( m\partial_p H(x, Du) \right) = 0 \quad \text{(adjoint equation)}\]

\[m(0, x) = m_0(x)\]

which characterizes minimizers \((u, f)\) of the relaxed problem (the optimal choice is \(f = k(t, x, m)\)).

Problems:

- \(H\) not differentiable,
- weak solutions otherwise otherwise hard to define.
Definition of weak solutions

A pair \((u, m) \in BV((0, T) \times \mathbb{T}^N) \times L^q((0, T) \times \mathbb{T}^N)\) is called a weak solution to the MFG system if it satisfies the following conditions.

1. \(u\) satisfies

\[
\int_{\mathbb{T}^N} u(0) m_0 \, dx = \int_{\mathbb{T}^N} u(t) m(t) \, dx + \int_0^t \int_{\mathbb{T}^N} k(s, x, m) \, m \, dx \, ds,
\]

\[
\int_{\mathbb{T}^N} u(t) m(t) \, dx = \int_{\mathbb{T}^N} u_T m(T) \, dx + \int_T^t \int_{\mathbb{T}^N} k(s, x, m) \, m \, dx \, ds,
\]

for almost all \(t \in [0, T]\). We also have

\[-\partial_t u + H(x, Du) \leq k(t, x, m)\]

in the sense of measures.

Moreover, \(u(T, \cdot) = u_T\) in the sense of traces

2. \(m\) satisfies the continuity equation

\[
\partial_t m + \operatorname{div} (m v) = 0 \quad \text{in} \quad (0, T) \times \mathbb{T}^N, \quad m(0) = m_0
\]

in the sense of distributions, where \(v \in L^\infty((0, T) \times \mathbb{T}^N; \mathbb{R}^N)\) is a vector field such that \(v(t, x) \in c(x, A)\) a.e.
Theorem (PJG)

(i) There exists a weak solution \((u, m)\) of \((MFG)\). Moreover, if \(u \in W^{1,p}\), then the Hamilton-Jacobi equation

\[-\partial_t u - \mathbf{v} \cdot D u = f\]

holds pointwise almost everywhere in \(\{m > 0\}\), where \(\mathbf{v}\) is the bounded vector field appearing in the definition.

(ii) If \((u, m)\) and \((u', m')\) are both weak solutions \((MFG)\), then \(m = m'\) almost everywhere while \(u = u'\) almost everywhere in the set \(\{m > 0\}\).
Table of Contents

1 Introduction
   • Motivation
   • Background

2 Main results
   • Assumptions
   • Existence of minimizers
   • Characterization of minimizers

3 Proofs
   • Duality
   • Equivalence of original and relaxed problems
   • A priori bounds
   • Passing to the limit
   • Existence of weak solutions
   • Uniqueness of weak solutions

4 Further results
   • Well-posedness of weak solutions
   • Long time average behavior
Denote by $K_0$ the set of maps $u \in C^1([0, T] \times \mathbb{T}^N)$ such that $u(T, x) = u_T(x)$. The “smooth problem” is to compute the infimum over $K_0$ of the functional

$$A(u) = \int_0^T \int_{\mathbb{T}^N} K(t, x, -\partial_t u + H(x, Du)) \, dx \, dt - \int_{\mathbb{T}^N} u(0, x) \, dm_0(x).$$

**Lemma**

The “smooth problem” is equivalent to the original problem, i.e.

$$\inf_{u \in K_0} A(u) = \inf_{f \in C([0, T] \times \mathbb{T}^N)} J(f).$$

**Proof:** uses the superposition principle.
Define $K_1$ to be the set of all pairs $(m, w) \in L^1((0, T) \times \mathbb{T}^N) \times L^1((0, T) \times \mathbb{T}^N; \mathbb{R}^N)$ such that $m \geq 0$ almost everywhere, $w(t, x) \in m(t, x)c(x, A)$ almost everywhere, and

$$\partial_t m + \text{div} (w) = 0$$

$$m(0, \cdot) = m_0(\cdot)$$

in the sense of distributions. The “dual problem” is to compute the infimum over $K_1$ of

$$B(m, w) = \int_{\mathbb{T}^N} u_T(x)m(T, x)dx + \int_0^T \int_{\mathbb{T}^N} K^*(t, x, m(t, x))dxdt.$$
Proof of duality

Lemma

The smooth and dual problems really are in duality, i.e.

\[ \inf_{u \in K_0} A(u) = - \min_{(m, w) \in K_1} B(m, w). \]

Moreover, the minimum on the right hand side is achieved by a pair \((m, w) \in K_1\), of which \(m\) is unique, and which must satisfy the following:

\((t, x) \mapsto K^*(t, x, m(t, x)) \in L^1((0, T) \times \mathbb{T}^N)\) and \(m \in L^q((0, T) \times \mathbb{T}^N)\).

**Proof** is an application of the Fenchel-Rockafellar duality Theorem. Let

- \(X := C^1([0, T] \times \mathbb{T}^N; \mathbb{R})\),
- \(Y := C([0, T] \times \mathbb{T}^N; \mathbb{R}) \times C([0, T] \times \mathbb{T}^N; \mathbb{R}^N)\).

Let \(\Lambda : X \to Y\) be given by \(\Lambda(u) = (\partial_t u, Du)\).
Proof of duality (cont)

**Lemma**

\[ \inf_{u \in K_0} A(u) = - \min_{(m, w) \in K_1} B(m, w). \]

Then define the functionals \( F : X \to \mathbb{R} \) and \( G : Y \to \mathbb{R} \) by

\[
F(u) = \begin{cases} 
- \int_{\mathbb{R}^N} u(0, x) dm_0(x) & \text{if } u(T, \cdot) = u_T(\cdot) \\
\infty & \text{otherwise}
\end{cases}
\]

\[
G(a, b) = \int_0^T \int_{\mathbb{R}^N} K(t, x, -a + H(x, b)) dx dt
\]

One can show the lemma is equivalent to

\[
\max_{(m, w) \in Y'} - F^*(\Lambda^*(m, w)) - G^*(-(m, w))
\]

\[
= - \min_{(m, w) \in Y'} F^*(\Lambda^*(m, w)) + G^*(-(m, w)).
\]

Apply Fenchel-Rockafellar to finish.
Table of Contents

1 Introduction
   - Motivation
   - Background

2 Main results
   - Assumptions
   - Existence of minimizers
   - Characterization of minimizers

3 Proofs
   - Duality
   - Equivalence of original and relaxed problems
   - A priori bounds
   - Passing to the limit
   - Existence of weak solutions
   - Uniqueness of weak solutions

4 Further results
   - Well-posedness of weak solutions
   - Long time average behavior
Below we prove the following:

**Proposition**

\[
\inf_{(u, f) \in \tilde{K}} \mathcal{A}(u, f) = - \min_{(m, w) \in K_1} \mathcal{B}(m, w).
\]

Equivalently,

\[
\inf_{(u, f) \in \tilde{K}} \mathcal{A}(u, f) = \inf_{u \in K_0} \mathcal{A}(u).
\]

**First part of the proof:**

Since for \( u \in K_0 \) we have that \((u, -u_t + H(x, Du)) \in \tilde{K}\), it follows that
\[
\inf_{(u, f) \in \tilde{K}} \mathcal{A}(u, f) \leq \inf_{u \in K_0} \mathcal{A}(u).
\]
It remains to show other inequality.
A technical lemma

Lemma

For $u \in BV$ and $f$ an integrable function, the following are equivalent:

- $-\partial_t u + H(x, Du) \leq f$ in the sense of measures, and
- for every continuous vector field $\tilde{v} = \tilde{v}(t, x) \in c(x, A)$ we have

$$-\int_0^T \int_{\mathbb{T}^N} \phi \partial_t u + \phi \tilde{v} \cdot Du \leq \int_0^T \int_{\mathbb{T}^N} f \phi dt dx.$$

for every continuous function $\phi : [0, T] \times \mathbb{T}^N \to [0, \infty)$.

Remark: the crucial thing is that we only need to consider $\tilde{v}$ continuous.
Proof of technical lemma

Lemma

\[-\partial_t u + H(x, Du) \leq f \iff -\iint \phi (\partial_t u + \tilde{v} \cdot Du) \leq \iint f \phi \quad \forall \tilde{v} \text{ cont.}\]

One direction is obvious. For the other direction, let $\epsilon > 0$. By Lusin’s Theorem, there is a compact set $K$ with $|Du|(\mathbb{R}^n)$ $\leq \epsilon$ and $Du/|Du|$ continuous on $K$. By continuity and convexity of $c(\cdot, A)$ we construct $\tilde{v}(t, x) \in c(x, A)$ by partition of unity so that

\[-\tilde{v} \cdot Du/|Du| \geq H(x, Du/|Du|) - \epsilon.\]

Use this to obtain the estimate

\[\int_{\mathbb{T}^N} \phi H(x, Du) \leq C\epsilon + \int_{\mathbb{T}^N} \phi \tilde{v} \cdot Du.\]

Then let $\epsilon \to 0$. 

Suppose $u \in BV \cap L^\infty$ satisfies $-\partial_t u + H(x, Du) \leq f$ in the sense of measures. Then for any $m \in L^q$ satisfying the continuity equation $\partial_t m + \text{div}(mv) = 0$ for some vector field $v$ with $v(t, x) \in c(x, A)$ a.e., we have

$$\int_{T^N} u(0)m(0)dx \leq \int_{T^N} u(t)m(t)dx + \int_0^t \int_{T^N} fmdxds,$$

$$\int_{T^N} u(t)m(t)dx \leq \int_{T^N} u_T m(T)dx + \int_t^T \int_{T^N} fmdxds,$$

for almost every $t \in (0, T)$.

**Proof:** obtain smooth approximations of $m$ and $mv$ through time scaling and convolution, apply previous lemma and integration by parts.
Second half of the proof: Let \((m, w)\) be a minimizer of the dual problem. It suffices to show that for \((u, f) \in \tilde{K}\), we have \(A(u, f) \geq -B(m, w)\). This essentially follows from the last lemma. Indeed, we have

\[
A(u, f) = \int_0^T \int_{\mathbb{T}^N} K(t, x, f(t, x)) dx dt - \int_{\mathbb{T}^N} u(0, x) m_0(x) dx
\]

\[
\geq \int_0^T \int_{\mathbb{T}^N} f m - K^*(t, x, m) dx dt - \int_{\mathbb{T}^N} u(0, x) m_0(x) dx
\]

\[
\geq - \int_0^T \int_{\mathbb{T}^N} K^*(t, x, m) dx dt - \int_{\mathbb{T}^N} u_T(x) m(T, x) dx = -B(m, w).
\]
Table of Contents

1 Introduction
   • Motivation
   • Background

2 Main results
   • Assumptions
   • Existence of minimizers
   • Characterization of minimizers

3 Proofs
   • Duality
   • Equivalence of original and relaxed problems
   • A priori bounds
   • Passing to the limit
   • Existence of weak solutions
   • Uniqueness of weak solutions

4 Further results
   • Well-posedness of weak solutions
   • Long time average behavior
Lemma

Suppose $u$ is a continuous viscosity solution to the Hamilton-Jacobi equation

$$-\partial_t u + H(x, Du) = f \geq 0$$

in a region $[0, T] \times U$ for some open set $U$ in Euclidean space. Then for any $0 \leq t < s \leq T$ and for any $0 < \beta < 1$, we have the following estimate:

$$u(t, x) - u(s, y) \leq C(1 - \beta^2)^{-N/2p} \|f\|_p |t - s|^\alpha, \quad \forall |x - y| \leq \beta c_0(s - t),$$

where $\alpha = 1 - (N + 1)/p$ and the constant $C$ depends on $p, N,$ and $c_0^{-1}$.

Cf. [Cardaliaguet 2013]. This lemma gives us upper bounds.
Let \((u_n, f_n)\) be a minimizing sequence. WLOG, \(u_n\) and \(f_n\) are Lipschitz, hence differentiable a.e., and \(f_n \geq 0\). We have:

- \(f_n\) bounded in \(L^p\) by estimates on \(K\), so \(f_n \rightarrow f\) weakly.
- This implies \(u_n\) pointwise bounded.
- The above implies \(H(x, Du_n)\) is bounded in \(L^1\), so \(Du_n\) is as well.
- The above implies \(\partial_t u_n\) is bounded in \(L^1\), so \(u_n\) is bounded in \(BV\).
- \((\partial_t u_n, Du_n) \rightarrow (\partial_t u, Du)\) in the weak measure sense, while \(u_n \rightarrow u\) in \(L^1\).

Conclusion: \((u, f)\) is the minimizer.
Table of Contents

1 Introduction
   - Motivation
   - Background

2 Main results
   - Assumptions
   - Existence of minimizers
   - Characterization of minimizers

3 Proofs
   - Duality
   - Equivalence of original and relaxed problems
   - A priori bounds
   - Passing to the limit
   - Existence of weak solutions
   - Uniqueness of weak solutions

4 Further results
   - Well-posedness of weak solutions
   - Long time average behavior
Minimizers are weak solutions

Suppose \((u, f) \in \tilde{K}\) is a minimizer of relaxed problem and \((m, w) \in K_1\) is a minimizer of dual problem.

- We know that \(-\partial_t u + H(x, Du) \leq f\) in the sense of measures, so
  \[
  \int_0^t \int fm + \int u(t)m(t) - u(0)m_0 \geq 0,
  \]
  \[
  \int_t^T \int fm + \int u_T m(T) - u(t)m(t) \geq 0.
  \]

- The main point is to get \(\int_0^T \int fm + \int u_T m(T) - u(0)m_0 = 0\), so that the above inequalities become equality.

- Everything else comes directly from the definition of \(\tilde{K}\) and \(K_1\) and the properties of minimizers already proved.
Minimizers are weak solutions, part 2

$(u, f) \in \tilde{\mathcal{K}}$ is a minimizer of relaxed problem and $(m, w) \in \mathcal{K}_1$ is a minimizer of dual problem. By the duality theorem,

\[
0 = \int_0^T \int K(t, x, f) + K^*(t, x, m) + \int u_T m(T) - u(0)m_0 \geq \int_0^T \int fm + \int u_T m(T) - u(0)m_0 \geq 0,
\]

which implies the desired result.

It also implies $K(t, x, f(t, x)) + K^*(t, x, m(t, x)) = f(t, x)m(t, x)$ a.e. so $f(t, x) = k(t, x, m)$. 

---

P. Jameson Graber  
Optimal control of Hamilton-Jacobi-Bellman equations
Weak solutions are also minimizers

Suppose \((u, m)\) is a weak solution. We set \(w = mv\) and \(f = k(\cdot, \cdot, m)\). Then \((u, f)\) is a minimizer of the relaxed problem and \((m, w)\) a minimizer of the dual problem. For example:

\[
\mathcal{A}(u', f') = \iint K(t, x, f') - \int u'(0)m_0 \\
\geq \iint K(t, x, f) + \partial_f K(t, x, f)(f' - f) - \int u'(0)m_0 \\
= \iint K(t, x, f) + m(f' - f) - \int u'(0)m_0 \\
= \iint K(t, x, f) + mf' + \int m(T)u_T - m_0u(0) - u'(0)m_0 \\
\geq \iint K(t, x, f) - \int m_0u(0) = \mathcal{A}(u, f).
\]
Suppose \((u_1, m_1)\) and \((u_2, m_2)\) are both weak solutions.

1. \(m_1\) and \(m_2\) are both minimizers of dual problem, so \(m_1 = m_2 =: m\).

2. \(f := k(\cdot, \cdot, m)\), then both \((u_1, f)\) and \((u_2, f)\) are minimizers of relaxed problem.

3. \(u := \max\{u_1, u_2\}\); we show \(-\partial_t u + H(x, Du) \leq f\) in the sense of measures.

4. Then \((u, f)\) is a minimizer of relaxed problem, so

\[
\iint K(t, x, f) - \int u_i(0)m_0 = \iint K(t, x, f) - \int u(0)m_0.
\]

5. Combine with previous inequalities and use \(u \geq u_i\) to get \(u = u_i\) in \(\{m > 0\}\).
Table of Contents

1 Introduction
   - Motivation
   - Background

2 Main results
   - Assumptions
   - Existence of minimizers
   - Characterization of minimizers

3 Proofs
   - Duality
   - Equivalence of original and relaxed problems
   - A priori bounds
   - Passing to the limit
   - Existence of weak solutions
   - Uniqueness of weak solutions

4 Further results
   - Well-posedness of weak solutions
   - Long time average behavior
Recent work

The following is joint work with Pierre Cardaliaguet. We study

\[
\begin{align*}
(i) & \quad -\partial_t \phi + H(x, D\phi) = f(x, m) \\
(ii) & \quad \partial_t m - \text{div} (mD_p H(x, D\phi)) = 0 \\
(iii) & \quad \phi(T, x) = \phi_T(x), \ m(0, x) = m_0(x)
\end{align*}
\]

and the corresponding \textit{ergodic problem}

\[
\begin{align*}
(i) & \quad \overline{\lambda} + H(x, D\overline{\phi}) = f(x, \overline{m}(x)) \\
(ii) & \quad -\text{div} (\overline{m}D_p H(x, \overline{D\phi})) = 0 \\
(iii) & \quad \overline{m} \geq 0, \ \int_{\mathbb{T}^d} \overline{m} = 1
\end{align*}
\]
Assumptions

1. (Initial-final conditions) $m_0 \in C(\mathbb{T}^d)$, $m_0 > 0$, $\phi_T : \mathbb{T}^d \to \mathbb{R}$ Lipschitz.

2. (Conditions on the Hamiltonian) $H : \mathbb{T}^d \times \mathbb{R}^d \to \mathbb{R}$ is continuous in both variables, convex and differentiable in the second variable, with $D_pH$ continuous in both variables. Also,

$$\frac{1}{rC}|p|^r - C \leq H(x, p) \leq \frac{C}{r}|p|^r + C.$$  

3. (Conditions on the coupling) Let $f$ be continuous on $\mathbb{T}^d \times (0, \infty)$, strictly increasing in the second variable, satisfying

$$\frac{1}{C}|m|^{q-1} - C \leq f(x, m) \leq C|m|^{q-1} + C \ \forall \ m \geq 1.$$  

4. The relation holds between the growth rates of $H$ and of $F$:

$$r > \max\{d(q - 1), 1\}.$$
Optimal control problems

Optimal control of HJ equation: find the infimum of

$$
A(\phi) = \int_0^T \int_{\mathbb{T}^d} F^*(x, -\partial_t \phi + H(x, D\phi)) dx dt - \int_{\mathbb{T}^d} \phi(0, x) dm_0(x)
$$

over the set of maps $\phi \in C^1([0, T] \times \mathbb{T}^d)$ such that $\phi(T, x) = \phi_T(x)$.

Optimal control of transport equation: find the infimum of

$$
B(m, w) = \int_{\mathbb{T}^d} \phi_T m(T) dx + \int_0^T \int_{\mathbb{T}^d} mH^*(x, -\frac{w}{m}) + F(x, m) dx dt
$$

over $(m, w) \in L^1((0, T) \times \mathbb{T}^d) \times L^1((0, T) \times \mathbb{T}^d; \mathbb{R}^d)$, $m \geq 0$ a.e., $\int_{\mathbb{T}^d} m(t, x) dx = 1$ a.e. $t \in (0, T)$, and

$$
\partial_t m + \text{div} (w) = 0, \ m(0, \cdot) = m_0(\cdot)
$$

in the sense of distributions.

Theorem (Cardaliaguet, PJG)

The optimal control problems above are in duality, i.e.

$$
\inf_{\phi \in \mathcal{K}_0} A(\phi) = - \min_{(m, w) \in \mathcal{K}_1} B(m, w).
$$
Relaxed problem

$\mathcal{K}$ will be defined as the set of all pairs $(\phi, \alpha) \in BV \times L^1$ such that $D\phi \in L^r((0, T) \times \mathbb{T}^d)$, $\phi(T, \cdot) \leq \phi_T$ in the sense of traces, $\alpha_+ \in L^p((0, T) \times \mathbb{T}^d)$, $\phi \in L^\infty((t, T) \times \mathbb{T}^d)$ for every $t \in (0, T)$, and

$$-\partial_t \phi + H(x, D\phi) \leq \alpha.$$

in the sense of distribution. Find

$$\inf_{(\phi, \alpha) \in \mathcal{K}} \mathcal{A}(\phi, \alpha) = \inf_{(\phi, \alpha) \in \mathcal{K}} \int_0^T \int_{\mathbb{T}^d} F^*(x, \alpha(t, x)) dx dt - \int_{\mathbb{T}^d} \phi(0, x) m_0(x) dx.$$

**Theorem (Cardaliaguet, PJG)**

*The relaxed problem has a minimizer. Moreover,*

$$\inf_{(\phi, \alpha) \in \mathcal{K}} \mathcal{A}(\phi, \alpha) = -\min_{(m, w) \in \mathcal{K}_1} \mathcal{B}(m, w) = \inf_{\phi \in \mathcal{K}_0} \mathcal{A}(\phi).$$
Definition of weak solution

A pair \((\phi, m) \in BV((0, T) \times \mathbb{T}^d) \times L^q((0, T) \times \mathbb{T}^d)\) is called a weak solution if it satisfies the following conditions.

1. \(D\phi \in L^r\) and the maps \(mf(x, m), mH^*(x, -D_pH(x, D\phi))\) and \(mD_pH(x, D\phi)\) are integrable,

2. \(\phi\) satisfies \(-\partial_t \phi + H(x, D\phi) \leq f(x, m)\) in the sense of distributions, the boundary condition \(\phi(T, \cdot) \leq \phi_T\) in the sense of trace and the following equality

\[
\iint m(H(x, D\phi) - \langle D\phi, D_pH(x, D\phi) \rangle - f(x, m)) \, dx \, dt = \int (\phi_T m(T) - \phi(0)m_0) \, dx
\]

3. \(m\) satisfies the continuity equation

\[
\partial_t m - \text{div} (mD_pH(x, D\phi)) = 0 \quad \text{in} \quad (0, T) \times \mathbb{T}^d, \quad m(0) = m_0
\]

in the sense of distributions.
Theorem (Cardaliaguet, PJG)

(i) If \((m, w) \in \mathcal{K}_1\) is a minimizer of the dual problem and \((\phi, \alpha) \in \tilde{\mathcal{K}}\) is a minimizer of the relaxed problem, then \((\phi, m)\) is a weak solution and 
\[\alpha(t, x) = f(x, m(t, x))\]
almost everywhere.

(ii) Conversely, if \((\phi, m)\) is a weak solution, then there exist functions 
\(w, \alpha\) such that \((\phi, \alpha) \in \mathcal{K}\) is a minimizer of the relaxed problem and 
\((m, w) \in \mathcal{K}_1\) is a minimizer of the dual problem.

(iii) If \((\phi, m)\) and \((\phi', m')\) are both weak solutions, then \(m = m'\) almost 
everywhere while \(\phi = \phi'\) almost everywhere in the set \(\{m > 0\}\).
Main result

Let $\phi_f \in C^1$ on $\mathbb{T}^d$.
Let $(\phi^T, m^T)$ be a “good” solution of

$$
\begin{align*}
-\partial_t \phi + H(x, D\phi) &= f(x, m) \\
\partial_t m - \text{div} (m D_p H(x, D\phi)) &= 0 \\
\phi(T, x) &= \phi_f(x), m(0, x) = m_0(x).
\end{align*}
$$

and $(\bar{\lambda}, \bar{\phi}, \bar{m})$ be a solution of the ergodic MFG system. Define

$$
\psi^T(s, x) = \phi^T(sT, x), \quad \mu^T(s, x) = m^T(sT, x) \quad \forall (s, x) \in (0, 1) \times \mathbb{T}^d.
$$

Theorem (Cardaliaguet, PJG)

As $T \to +\infty$,

- $(\mu^T)$ converges to $\bar{m}$ in $L^\theta((0, 1) \times \mathbb{T}^d)$ for any $\theta \in [1, q)$,
- $\psi^T/T$ converges to the map $s \to \bar{\lambda}(1 - s)$ in $L^\theta((\delta, 1) \times \mathbb{T}^d)$ for any $\theta \geq 1$ and any $\delta \in (0, 1)$.