Optimal control of Hamilton-Jacobi-Bellman equations

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Objective

We consider an optimal control problem:

Problem

(1) For a given cost functional K and finite measure dm_0, find

$$\inf \left\{ \int_0^T \int K(t,x,f(t,x))dxdt - \int u(0,x)dm_0(x) \right\}$$

over functions u, f such that $f \in C([0,T] \times \mathbb{T}^N)$ and u solves

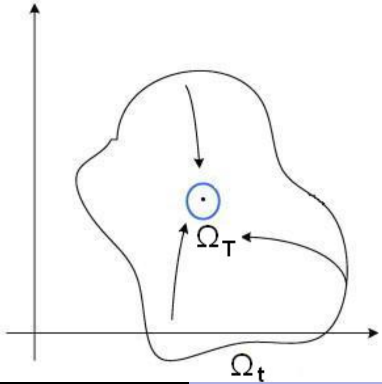
$$-\partial_t u(t,x) + H(x, Du(t,x)) = f(t,x), \quad u(T,x) = u_T(x).$$

(2) Find a PDE characterization of (weak) minimizers.

Motivation

Purpose: to steer an evolving set.
Model: given a set $\mathcal{A}$ of admissible controls and a given region $\Omega_T \subset \mathbb{R}^N$, let Ω_t be the *backward reachable set* at time t defined by

$$\Omega_t := \{y(t) : \dot{y} = c(y,\alpha),\ \alpha \in \mathcal{A}, y(T) \in \Omega_T\}.$$



Model

Strategy: use obstacles.
Obstacle construction: Let $f \geq 0$ be continuous. Define $K_\tau, \tau \in [0, T]$ by

$$K_\tau := \{x : f(\tau, x) > 0\}.$$

Consider the modified "blocked" evolving set

$$\tilde{\Omega}_t = \{y(t) : \dot{y} = c(y, \alpha),\ \alpha \in \mathcal{A}, y(T) \in \Omega_T, y(\tau) \notin K_\tau\ \forall\ \tau \in [t, T]\}.$$

Following [Bokanowski, Forcadel, Zidani 2010], we have a characterization: let $u_T \geq 0$ (continuous) be such that $u_T = 0$ precisely on Ω_T.

$$\tilde{\Omega}_t = \{y(t) : u_T(y(T)) + \int_t^T f(s, y(s)) \leq 0,\ \dot{y} = c(y, \alpha),\ \alpha \in \mathcal{A}\}.$$

Model

$\tilde{\Omega}_t = \{u(t,\cdot) \leq 0\}$ characterized by an optimal control problem:

$$u(t,x) := \inf \left\{ u_T(y(T)) + \int_t^T f(s,y(s))ds : \dot{y} = c(y,\alpha),\ \alpha \in \mathcal{A}, y(t) = x \right\}.$$

u solves a Hamilton-Jacobi-Bellman equation:

$$-\partial_t u(t,x) + H(x, Du(t,x)) = f(t,x), \quad u(T,x) = u_T(x).$$

Conclusion

Steering the front by obstacles is thus related to optimizing solutions to HJB equations.

Related model

A related open problem:

Optimal control via velocity

Given a cost functional $\mathcal{C}$, find

$$\inf \left\{ \mathcal{C}(c) - \int u(T,x) dm_0(x) \right\},$$

where the infimum is taken over u, c satisfying

$$u_t + c(t,x)|Du| = 0, \ \ u(0,x) = u_0(x).$$

Find a characterization of the minimizers.

Very little appears to be known.

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Mass transportation

In 2000, Benamou and Brenier considered a fluid mechanics formulation of the Monge-Kantorovich problem (MKP).
For $\rho_0(x) \geq 0$ and $\rho_T(x) \geq 0$ probability densities on $\mathbb{R}^d$, we say $M : \mathbb{R}^d \to \mathbb{R}^d$ transfers ρ_0 to ρ_T if $\forall A \subset \mathbb{R}^d$ bounded,

$$\int_{x \in A} \rho_T(x)dx = \int_{M(x) \in A} \rho_0(x)dx.$$

The L^p MKP is to find

$$d_p(\rho_0, \rho_T)^p = \inf_M \int |M(x) - x|^p \rho_0(x)dx.$$

d_p is called the L^p Kantorovich (or Wasserstein) distance.

PDE characterization of MKP

Theorem (Benamou-Brenier, 2000)

The square of the L^2 Kantorovich distance is equal to the infimum of

$$T \int_{\mathbb{R}^d} \int_0^T \rho(t,x) |v(t,x)|^2 dxdt,$$

among all (ρ, v) satisfying

$$\partial_t \rho + \nabla \cdot (\rho v) = 0$$

for $0 < t < T$ and $x \in \mathbb{R}^d$, and the initial-final condition

$$\rho(0,\cdot) = \rho_0, \quad \rho(T,\cdot) = \rho_T.$$

The optimality conditions are given by $v(t,x) = \nabla \phi(t,x)$, where

$$\partial_t \phi + \frac{1}{2} |\nabla \phi|^2 = 0.$$

Dolbeault-Nazaret-Savaré distances

In 2009, Dolbeault, Nazaret, and Savaré, building on Benamou-Brenier, introduced a new class of transport distances between measures, given by minimizing

$$\int_0^1 \int_{\mathbb{R}^d} |v(t,x)|^2 m(\rho(t,x)) dxdt$$

subject to

$$\partial_t + \nabla \cdot (m(\rho)v) = 0, \ \rho(0,\cdot) = \rho_0, \ \rho(1,\cdot) = \rho_1.$$

In 2013, Cardaliaguet, Carlier, and Nazaret studied geodesics for this class of distances when $m(\rho) = \rho^\alpha$, getting optimality conditions

$$\begin{cases} \partial_t \rho + \nabla \cdot (\frac{1}{2}\rho^\alpha \nabla \phi) = 0, \\ \rho > 0 \Rightarrow \partial_t \phi + \frac{\alpha}{4}\rho^{\alpha-1}|\nabla \phi|^2 = 0, \\ \rho \geq 0, \partial_t \phi \leq 0, \rho(0,\cdot) = \rho_0, \rho(1,\cdot) = \rho_1. \end{cases}$$

Mean Field Games

In 2007, Lasry and Lions give an overview of the following system of equations:

$$\frac{\partial u}{\partial t} - \nu \Delta u + H(x, \nabla u) = V[m] \text{ in } Q \times (0, T)$$
$$\frac{\partial m}{\partial t} + \nu \Delta m + \nabla \cdot \left(\frac{\partial H}{\partial p}(x, \nabla u) m \right) = 0 \text{ in } Q \times (0, T)$$
$$u|_{t=0} = V_0[m(0)] \text{ on } Q, \; m|_{t=T} = m^0 \text{ on } Q.$$

There are two characterizations of the system:

1. asymptotic behavior of differential games with large number of players,
2. optimal control of Fokker-Planck equation, or of Hamilton-Jacobi equation (by duality).

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Assumptions

- *Hamiltonian structure.* $H = H(x, p)$ is Lipschitz in the first variable:

$$|H(x, p) - H(y, p)| \leq L|x - y||p|.$$

  $H(x, \cdot)$ is subadditive and positively homogeneous, hence convex, with linear bounds

$$c_0|p| \leq H(x, p) \leq c_1|p|.$$

  $H(x, p) := \sup\{-c(x, a) \cdot p : a \in A\}$

- *Cost function.* $k = k(t, x, m)$ continuous and strictly increasing with

$$\frac{1}{C}|m|^{q-1} - C \leq k(t, x, m) \leq C|m|^{q-1} + C.$$

  $K^*(t, x, \cdot)$ a primitive of k, $K(t, x, \cdot)$ its Fenchel conjugate. We will assume that $K(t, x, 0) = 0$ and that $K(t, x, f)$ is increasing in f for $f \geq 0$. Note that

$$\frac{1}{C}|f|^p - C \leq K(t, x, f) \leq C|f|^p + C.$$

- *Final-initial conditions.* $u_T : \mathbb{T}^N \to \mathbb{R}$ is Lipschitz, $m_0 \in L^\infty(\mathbb{T}^N)$

Comparison with the literature

1. This is the first study with linearly bounded Hamiltonian.
2. In [Cardaliaguet, 2013] the Hamiltonian is supposed to satisfy

$$\frac{1}{rC}|p|^r - C_2 \leq H(x,p) \leq \frac{C}{r}|p|^r + C$$

for some $r > N(q-1) \vee 1$ and, for some $\theta \in [0, \frac{r}{N+1})$,

$$|H(x,p) - H(y,p)| \leq C|x-y|(|p| \vee 1)^{\theta}.$$

The Hamiltonian $H(x,p) = c(x)|p|^r$ is forbidden by this restriction. By contrast, $H(x,p) = c(x)|p|$ is permitted under our assumptions.

Existence theorem

Relaxed set

The set $\tilde{\mathcal{K}}$ will be defined as the set of all pairs $(u, f) \in BV \times L^p$ such that $u \in L^\infty$, $u(T, \cdot) = u_T$ in the sense of traces, and $-\partial_t u + H(x, Du) \leq f$ in the sense of measures, by which we mean that $f - H(x, Du) + \partial_t u$ is a non-negative Radon measure.

Relaxed problem: Find $\inf_{(u,f)\in\tilde{\mathcal{K}}} \mathcal{A}(u, f)$.

Lemma

The relaxed problem has the same infimum as the original.

Theorem (PJG)

There exists a minimizer to the relaxed problem.

Characterization of minimizers

Heuristically, use Lagrange multipliers to get system (MFG)

$$\begin{aligned} -\partial_t u + H(x, Du) &= k(t, x, m(t, x)) && \text{(state equation)} \\ u(T, x) &= u_T(x) \\ \partial_t m - \operatorname{div}(m \partial_p H(x, Du)) &= 0 && \text{(adjoint equation)} \\ m(0, x) &= m_0(x) \end{aligned}$$

which characterizes minimizers (u, f) of the relaxed problem (the optimal choice is $f = k(t, x, m)$).
Problems:

- H not differentiable,
- weak solutions otherwise hard to define.

Definition of weak solutions

A pair $(u, m) \in BV((0,T) \times \mathbb{T}^N) \times L^q((0,T) \times \mathbb{T}^N)$ is called a weak solution to the MFG system if it satisfies the following conditions.

1. u satisfies

$$\int_{\mathbb{T}^N} u(0) m_0 dx = \int_{\mathbb{T}^N} u(t)m(t)dx + \int_0^t \int_{\mathbb{T}^N} k(s,x,m) m dx ds,$$

$$\int_{\mathbb{T}^N} u(t)m(t)dx = \int_{\mathbb{T}^N} u_T m(T)dx + \int_t^T \int_{\mathbb{T}^N} k(s,x,m) m dx ds,$$

   for almost all $t \in [0,T]$. We also have $-\partial_t u + H(x, Du) \leq k(t,x,m)$ in the sense of measures. Moreover, $u(T, \cdot) = u_T$ in the sense of traces

2. m satisfies the continuity equation

$$\partial_t m + \mathrm{div}\,(m\mathbf{v}) = 0 \quad \text{in} \quad (0,T) \times \mathbb{T}^N, \qquad m(0) = m_0$$

   in the sense of distributions, where $\mathbf{v} \in L^\infty((0,T) \times \mathbb{T}^N; \mathbb{R}^N)$ is a vector field such that $\mathbf{v}(t,x) \in c(x,A)$ a.e.

Well-posedness

Theorem (PJG)

(i) There exists a weak solution (u, m) *of (MFG). Moreover, if* $u \in W^{1,p}$*, then the Hamilton-Jacobi equation*

$$-\partial_t u - \mathbf{v} \cdot Du = f$$

holds pointwise almost everywhere in $\{m > 0\}$*, where* $\mathbf{v}$ *is the bounded vector field appearing in the definition.*
(ii) If (u, m) *and* (u', m') *are both weak solutions (MFG), then* $m = m'$ *almost everywhere while* $u = u'$ *almost everywhere in the set* $\{m > 0\}$*.*

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Smooth problem

Denote by $\mathcal{K}_0$ the set of maps $u \in C^1([0, T] \times \mathbb{T}^N)$ such that $u(T, x) = u_T(x)$. The "smooth problem" is to compute the infimum over $\mathcal{K}_0$ of the functional

$$\mathcal{A}(u) = \int_0^T \int_{\mathbb{T}^N} K(t, x, -\partial_t u + H(x, Du))dxdt - \int_{\mathbb{T}^N} u(0, x)dm_0(x).$$

Lemma

The "smooth problem" is equivalent to the original problem, i.e.

$$\inf_{u \in \mathcal{K}_0} \mathcal{A}(u) = \inf_{f \in C([0,T] \times \mathbb{T}^N)} \mathcal{J}(f).$$

Proof: uses the superposition principle.

Dual problem

Define $\mathcal{K}_1$ to be the set of all pairs $(m, \mathbf{w}) \in L^1((0,T) \times \mathbb{T}^N) \times L^1((0,T) \times \mathbb{T}^N; \mathbb{R}^N)$ such that $m \geq 0$ almost everywhere, $\mathbf{w}(t,x) \in m(t,x)c(x,A)$ almost everywhere, and

$$\partial_t m + \mathrm{div}\,(\mathbf{w}) = 0$$
$$m(0,\cdot) = m_0(\cdot)$$

in the sense of distributions. The "dual problem" is to compute the infimum over $\mathcal{K}_1$ of

$$\mathcal{B}(m,\mathbf{w}) = \int_{\mathbb{T}^N} u_T(x) m(T,x) dx + \int_0^T \int_{\mathbb{T}^N} K^*(t,x,m(t,x)) dx dt.$$

Proof of duality

Lemma

The smooth and dual problems really are in duality, i.e.

$$\inf_{u \in \mathcal{K}_0} \mathcal{A}(u) = - \min_{(m,\mathbf{w}) \in \mathcal{K}_1} \mathcal{B}(m, \mathbf{w}).$$

Moreover, the minimum on the right hand side is achieved by a pair $(m, \mathbf{w}) \in \mathcal{K}_1$, of which m is unique, and which must satisfy the following: $(t,x) \mapsto K^*(t,x,m(t,x)) \in L^1((0,T) \times \mathbb{T}^N)$ and $m \in L^q((0,T) \times \mathbb{T}^N)$.

Proof is an application of the Fenchel-Rockafellar duality Theorem. Let

- $X := C^1([0,T] \times \mathbb{T}^N; \mathbb{R})$,
- $Y := C([0,T] \times \mathbb{T}^N; \mathbb{R}) \times C([0,T] \times \mathbb{T}^N; \mathbb{R}^N)$.

Let $\Lambda : X \to Y$ be given by $\Lambda(u) = (\partial_t u, Du)$.

Proof of duality (cont)

Lemma

$\inf_{u\in\mathcal{K}_0} \mathcal{A}(u) = -\min_{(m,\mathbf{w})\in\mathcal{K}_1} \mathcal{B}(m,\mathbf{w})$.

Then define the functionals $\mathscr{F} : X \to \mathbb{R}$ and $\mathscr{G} : Y \to \mathbb{R}$ by

$$\mathscr{F}(u) = \begin{cases} -\int_{\mathbb{T}^N} u(0,x)dm_0(x) & \text{if} \quad u(T,\cdot) = u_T(\cdot) \\ \infty & \text{otherwise} \end{cases}$$

$$\mathscr{G}(a,\mathbf{b}) = \int_0^T \int_{\mathbb{T}^N} K(t,x,-a+H(x,\mathbf{b}))dxdt$$

One can show the lemma is equivalent to

$$\max_{(m,\mathbf{w})\in Y'} -\mathscr{F}^*(\Lambda^*(m,\mathbf{w})) - \mathscr{G}^*(-(m,\mathbf{w})) = -\min_{(m,\mathbf{w})\in Y'} \mathscr{F}^*(\Lambda^*(m,\mathbf{w})) + \mathscr{G}^*(-(m,\mathbf{w})).$$

Apply Fenchel-Rockafellar to finish.

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Original and relaxed problems equivalent

Below we prove the following:

Proposition

$$\inf_{(u,f)\in\tilde{\mathcal{K}}} \mathcal{A}(u,f) = -\min_{(m,\mathbf{w})\in\mathcal{K}_1} \mathcal{B}(m,\mathbf{w}).$$

Equivalently,

$$\inf_{(u,f)\in\tilde{\mathcal{K}}} \mathcal{A}(u,f) = \inf_{u\in\mathcal{K}_0} \mathcal{A}(u).$$

First part of the proof:
Since for $u \in \mathcal{K}_0$ we have that $(u, -u_t + H(x, Du)) \in \tilde{\mathcal{K}}$, it follows that $\inf_{(u,f)\in\tilde{\mathcal{K}}} \mathcal{A}(u,f) \leq \inf_{u\in\mathcal{K}_0} \mathcal{A}(u)$.
It remains to show other inequality.

A technical lemma

Lemma

For $u \in BV$ and f an integrable function, the following are equivalent:

- $-\partial_t u + H(x, Du) \leq f$ in the sense of measures, and
- for every continuous vector field $\tilde{\mathbf{v}} = \tilde{\mathbf{v}}(t,x) \in c(x, A)$ we have

$$-\int_0^T \int_{\mathbb{T}^N} \phi \partial_t u + \phi \tilde{\mathbf{v}} \cdot Du \leq \int_0^T \int_{\mathbb{T}^N} f \phi dtdx.$$

for every continuous function $\phi : [0, T] \times \mathbb{T}^N \to [0, \infty)$.

Remark: the crucial thing is that we only need to consider $\tilde{\mathbf{v}}$ continuous.

Proof of technical lemma

Lemma

$$-\partial_t u + H(x, Du) \leq f \Leftrightarrow - \iint \phi(\partial_t u + \tilde{\mathbf{v}} \cdot Du) \leq \iint f\phi \quad \forall \tilde{\mathbf{v}} \text{ cont.}$$

One direction is obvious. For the other direction, let $\epsilon > 0$. By Lusin's Theorem, there is a compact set K with $|Du|([0,T] \times \mathbb{T}^N \setminus K) \leq \epsilon$ and $Du/|Du|$ continuous on K. By continuity and convexity of $c(\cdot, A)$ we construct $\tilde{\mathbf{v}}(t,x) \in c(x,A)$ by partition of unity so that $-\tilde{\mathbf{v}} \cdot Du/|Du| \geq H(x, Du/|Du|) - \epsilon$. Use this to obtain the estimate

$$\int_0^T \int_{\mathbb{T}^N} \phi H(x, Du) \leq C\epsilon + \int_0^T \int_{\mathbb{T}^N} \phi \tilde{\mathbf{v}} \cdot Du.$$

Then let $\epsilon \to 0$.

Key lemma

Lemma

Suppose $u \in BV \cap L^\infty$ satisfies $-\partial_t u + H(x, Du) \leq f$ in the sense of measures. Then for any $m \in L^q$ satisfying the continuity equation $\partial_t m + \mathrm{div}(m\mathbf{v}) = 0$ for some vector field $\mathbf{v}$ with $\mathbf{v}(t,x) \in c(x,A)$ a.e., we have

$$\int_{\mathbb{T}^N} u(0)m(0)dx \leq \int_{\mathbb{T}^N} u(t)m(t)dx + \int_0^t \int_{\mathbb{T}^N} fmdxds,$$
$$\int_{\mathbb{T}^N} u(t)m(t)dx \leq \int_{\mathbb{T}^N} u_T m(T)dx + \int_t^T \int_{\mathbb{T}^N} fmdxds,$$

for almost every $t \in (0, T)$.

Proof: obtain smooth approximations of m and $m\mathbf{v}$ through time scaling and convolution, apply previous lemma and integration by parts.

Proof of equivalence

Second half of the proof: Let $(m, \mathbf{w})$ be a minimizer of the dual problem. It suffices to show that for $(u, f) \in \tilde{\mathcal{K}}$, we have $\mathcal{A}(u, f) \geq -\mathcal{B}(m, \mathbf{w})$. This essentially follows from the last lemma. Indeed, we have

$$\begin{aligned}\mathcal{A}(u, f) &= \int_0^T \int_{\mathbb{T}^N} K(t, x, f(t, x))dxdt - \int_{\mathbb{T}^N} u(0, x)m_0(x)dx \\ &\geq \int_0^T \int_{\mathbb{T}^N} fm - K^*(t, x, m)dxdt - \int_{\mathbb{T}^N} u(0, x)m_0(x)dx \\ &\geq -\int_0^T \int_{\mathbb{T}^N} K^*(t, x, m)dxdt - \int_{\mathbb{T}^N} u_T(x)m(T, x)dx = -\mathcal{B}(m, \mathbf{w}).\end{aligned}$$

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Upper bound lemma

Lemma

Suppose u is a continuous viscosity solution to the Hamilton-Jacobi equation

$$-\partial_t u + H(x, Du) = f \geq 0$$

in a region $[0, T] \times U$ for some open set U in Euclidean space. Then for any $0 \leq t < s \leq T$ and for any $0 < \beta < 1$, we have the following estimate:

$$u(t,x) - u(s,y) \leq C(1-\beta^2)^{-N/2p} \|f\|_p |t-s|^\alpha, \quad \forall |x-y| \leq \beta c_0(s-t),$$

where $\alpha = 1 - (N+1)/p$ and the constant C depends on p, N, and c_0^{-1}.

Cf. [Cardaliaguet 2013]. This lemma gives us *upper bounds*.

Passing to the limit

Let (u_n, f_n) be a minimizing sequence. WLOG, u_n and f_n are Lipschitz, hence differentiable a.e., and $f_n \geq 0$. We have:

- f_n bounded in L^p by estimates on K, so $f_n \to f$ weakly.
- This implies u_n pointwise bounded.
- The above implies $H(x, Du_n)$ is bounded in L^1, so Du_n is as well.
- The above implies $\partial_t u_n$ is bounded in L^1, so u_n is bounded in BV.
- $(\partial_t u_n, Du_n) \to (\partial_t u, Du)$ in the weak measure sense, while $u_n \to u$ in L^1.

Conclusion: (u, f) is the minimizer.

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Minimizers are weak solutions

Suppose $(u, f) \in \tilde{\mathcal{K}}$ is a minimizer of relaxed problem and $(m, \mathbf{w}) \in \mathcal{K}_1$ is a minimizer of dual problem.

- We know that $-\partial_t u + H(x, Du) \leq f$ in the sense of measures, so

$$\int_0^t \int fm + \int u(t)m(t) - u(0)m_0 \geq 0,$$
$$\int_t^T \int fm + \int u_T m(T) - u(t)m(t) \geq 0.$$

- The main point is to get $\int_0^T \int fm + \int u_T m(T) - u(0)m_0 = 0$, so that the above inequalities become equality.
- Everything else comes directly from the definition of $\tilde{\mathcal{K}}$ and $\mathcal{K}_1$ and the properties of minimizers already proved.

Minimizers are weak solutions, part 2

$(u, f) \in \tilde{\mathcal{K}}$ is a minimizer of relaxed problem and $(m, \mathbf{w}) \in \mathcal{K}_1$ is a minimizer of dual problem.
By the duality theorem,

$$\begin{aligned} 0 &= \int_0^T \int K(t,x,f) + K^*(t,x,m) + \int u_T m(T) - u(0) m_0 \\ &\geq \int_0^T \int fm + \int u_T m(T) - u(0) m_0 \geq 0, \end{aligned}$$

which implies the desired result.
It also implies $K(t,x,f(t,x)) + K^*(t,x,m(t,x)) = f(t,x)m(t,x) a.e.$ so $f(t,x) = k(t,x,m)$.

Weak solutions are also minimizers

Suppose (u,m) is a weak solution. We set $\mathbf{w} = m\mathbf{v}$ and $f = k(\cdot,\cdot,m)$. Then (u,f) is a minimizer of the relaxed problem and $(m,\mathbf{w})$ a minimizer of the dual problem. For example:

$$\begin{aligned}
\mathcal{A}(u',f') &= \iint K(t,x,f') - \int u'(0)m_0 \\
&\geq \iint K(t,x,f) + \partial_f K(t,x,f)(f'-f) - \int u'(0)m_0 \\
&= \iint K(t,x,f) + m(f'-f) - \int u'(0)m_0 \\
&= \iint K(t,x,f) + mf' + \int m(T)u_T - m_0 u(0) - u'(0)m_0 \\
&\geq \iint K(t,x,f) - \int m_0 u(0) = \mathcal{A}(u,f).
\end{aligned}$$

Weak solutions are unique

Suppose (u_1, m_1) and (u_2, m_2) are both weak solutions.

1. m_1 and m_2 are both minimizers of dual problem, so $m_1 = m_2 =: m$.
2. $f := k(\cdot, \cdot, m)$, then both (u_1, f) and (u_2, f) are minimizers of relaxed problem.
3. $u := \max\{u_1, u_2\}$; we show $-\partial_t u + H(x, Du) \leq f$ in the sense of measures.
4. Then (u, f) is a minimizer of relaxed problem, so

$$\iint K(t,x,f) - \int u_i(0) m_0 = \iint K(t,x,f) - \int u(0) m_0.$$

5. Combine with previous inequalities and use $u \geq u_i$ to get $u = u_i$ in $\{m > 0\}$.

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Recent work

The following is joint work with Pierre Cardaliaguet. We study

$$\begin{cases} (i) & -\partial_t \phi + H(x, D\phi) = f(x, m) \\ (ii) & \partial_t m - \mathrm{div}\,(m D_p H(x, D\phi)) = 0 \\ (iii) & \phi(T, x) = \phi_T(x), m(0, x) = m_0(x) \end{cases}$$

and the corresponding *ergodic problem*

$$\begin{cases} (i) & \overline{\lambda} + H(x, D\overline{\phi}) = f(x, \overline{m}(x)) \\ (ii) & -\mathrm{div}(\overline{m} D_p H(x, \overline{D}\phi)) = 0 \\ (iii) & \overline{m} \geq 0, \ \int_{\mathbb{T}^d} \overline{m} = 1 \end{cases}$$

Assumptions

1. (Initial-final conditions) $m_0 \in C(\mathbb{T}^d)$, $m_0 > 0$, $\phi_T : \mathbb{T}^d \to \mathbb{R}$ Lipschitz.
2. (Conditions on the Hamiltonian) $H : \mathbb{T}^d \times \mathbb{R}^d \to \mathbb{R}$ is continuous in both variables, convex and differentiable in the second variable, with $D_p H$ continuous in both variables. Also,
$$\frac{1}{rC}|p|^r - C \leq H(x,p) \leq \frac{C}{r}|p|^r + C.$$
3. (Conditions on the coupling) Let f be continuous on $\mathbb{T}^d \times (0,\infty)$, strictly increasing in the second variable, satisfying
$$\frac{1}{C}|m|^{q-1} - C \leq f(x,m) \leq C|m|^{q-1} + C \quad \forall\, m \geq 1.$$
4. The relation holds between the growth rates of H and of F:
$$r > \max\{d(q-1), 1\}.$$

Optimal control problems

Optimal control of HJ equation: find the infimum of

$$\mathcal{A}(\phi) = \int_0^T \int_{\mathbb{T}^d} F^*(x, -\partial_t \phi + H(x, D\phi))dxdt - \int_{\mathbb{T}^d} \phi(0,x)dm_0(x)$$

over the set of maps $\phi \in C^1([0,T] \times \mathbb{T}^d)$ such that $\phi(T,x) = \phi_T(x)$.
Optimal control of transport equation: find the infimum of

$$\mathcal{B}(m,w) = \int_{\mathbb{T}^d} \phi_T m(T)dx + \int_0^T \int_{\mathbb{T}^d} mH^*\left(x, -\frac{w}{m}\right) + F(x,m)dxdt$$

over $(m,w) \in L^1((0,T) \times \mathbb{T}^d) \times L^1((0,T) \times \mathbb{T}^d; \mathbb{R}^d)$, $m \geq 0$ a.e., $\int_{\mathbb{T}^d} m(t,x)dx = 1$ a.e. $t \in (0,T)$, and

$$\partial_t m + \operatorname{div}(w) = 0, \; m(0,\cdot) = m_0(\cdot)$$

in the sense of distributions.

Theorem (Cardaliaguet, PJG)

The optimal control problems above are in duality, i.e.

$$\inf_{\phi \in \mathcal{K}_0} \mathcal{A}(\phi) = - \min_{(m,w) \in \mathcal{K}_1} \mathcal{B}(m,w).$$

Relaxed problem

$\mathcal{K}$ will be defined as the set of all pairs $(\phi, \alpha) \in BV \times L^1$ such that $D\phi \in L^r((0,T) \times \mathbb{T}^d)$, $\phi(T, \cdot) \leq \phi_T$ in the sense of traces, $\alpha_+ \in L^p((0,T) \times \mathbb{T}^d)$, $\phi \in L^\infty((t,T) \times \mathbb{T}^d)$ for every $t \in (0,T)$, and

$$-\partial_t \phi + H(x, D\phi) \leq \alpha.$$

in the sense of distribution. Find

$$\inf_{(\phi,\alpha)\in\mathcal{K}} \mathcal{A}(\phi,\alpha) = \inf_{(\phi,\alpha)\in\mathcal{K}} \int_0^T \int_{\mathbb{T}^d} F^*(x, \alpha(t,x))dxdt - \int_{\mathbb{T}^d} \phi(0,x) m_0(x) dx.$$

Theorem (Cardaliaguet, PJG)

The relaxed problem has a minimizer. Moreover,

$$\inf_{(\phi,\alpha)\in\mathcal{K}} \mathcal{A}(\phi,\alpha) = -\min_{(m,w)\in\mathcal{K}_1} \mathcal{B}(m,w) = \inf_{\phi\in\mathcal{K}_0} \mathcal{A}(\phi).$$

Definition of weak solution

A pair $(\phi, m) \in BV((0,T)\times\mathbb{T}^d)\times L^q((0,T)\times\mathbb{T}^d)$ is called a weak solution if it satisfies the following conditions.

1. $D\phi \in L^r$ and the maps $mf(x,m)$, $mH^*\left(x, -D_pH(x,D\phi)\right)$ and $mD_pH(x,D\phi)$ are integrable,
2. ϕ satisfies $-\partial_t\phi + H(x,D\phi) \leq f(x,m)$ in the sense of distributions, the boundary condition $\phi(T,\cdot) \leq \phi_T$ in the sense of trace and the following equality
$$\iint m\left(H(x,D\phi) - \langle D\phi, D_pH(x,D\phi)\rangle - f(x,m)\right)dxdt = \int(\phi_T m(T) - \phi(0)m_0)dx$$
3. m satisfies the continuity equation
$$\partial_t m - \mathrm{div}\,(mD_pH(x,D\phi)) = 0 \quad \text{in} \quad (0,T)\times\mathbb{T}^d, \quad m(0) = m_0$$
in the sense of distributions.

Well-posedness

Theorem (Cardaliaguet, PJG)

(i) If $(m, w) \in \mathcal{K}_1$ *is a minimizer of the dual problem and* $(\phi, \alpha) \in \tilde{\mathcal{K}}$ *is a minimizer of the relaxed problem, then* (ϕ, m) *is a weak solution and* $\alpha(t,x) = f(x, m(t,x))$ *almost everywhere.*

(ii) Conversely, if (ϕ, m) *is a weak solution, then there exist functions* w, α *such that* $(\phi, \alpha) \in \mathcal{K}$ *is a minimizer of the relaxed problem and* $(m, w) \in \mathcal{K}_1$ *is a minimizer of the dual problem.*

(iii) If (ϕ, m) *and* (ϕ', m') *are both weak solutions, then* $m = m'$ *almost everywhere while* $\phi = \phi'$ *almost everywhere in the set* $\{m > 0\}$.

Main result

Let $\phi_f \in C^1$ on $\mathbb{T}^d$.
Let (ϕ^T, m^T) be a "good" solution of

$$\begin{cases} -\partial_t \phi + H(x, D\phi) = f(x, m) \\ \partial_t m - \mathrm{div}\,(m D_p H(x, D\phi)) = 0 \\ \phi(T, x) = \phi_f(x), m(0, x) = m_0(x). \end{cases}$$

and $(\bar{\lambda}, \bar{\phi}, \bar{m})$ be a solution of the ergodic MFG system. Define

$$\psi^T(s, x) = \phi^T(sT, x), \qquad \mu^T(s, x) = m^T(sT, x) \qquad \forall (s, x) \in (0, 1) \times \mathbb{T}^d.$$

Theorem (Cardaliaguet, PJG)

As $T \to +\infty$,

- *(μ^T) converges to $\bar{m}$ in $L^\theta((0, 1) \times \mathbb{T}^d)$ for any $\theta \in [1, q)$,*
- *ψ^T / T converges to the map $s \to \bar{\lambda}(1 - s)$ in $L^\theta((\delta, 1) \times \mathbb{T}^d)$ for any $\theta \geq 1$ and any $\delta \in (0, 1)$.*

Thank you!

P. Cardaliaguet and P. Graber, "Mean field games systems of first order," submitted to ESAIM: Control, Optimization, and Calculus of Variations. Available on the arXiv.

P. Graber, "Optimal control of first-order Hamilton-Jacobi equations with linearly bounded Hamiltonians," to appear in Applied Mathematics and Optimization. Also available on the arXiv.