Carleman estimates for second order operators of real principal type under weak pseudo convexity

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Seminar: Control theory in Lab. J.-L. Lions,

UPMC, Paris, 7 Nov 2014.

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Plan of the talk:

1. Carleman estimates and pseudo convexity
2. Weak pseudo convexity for the operator of second order and real principal type
3. Limiting Carleman weight in Calderón problem
4. Main result
5. Idea and sketch of proof

(Note: The slides are rearanged after the talk.)

1 Carleman estimates and pseudo convexity

In $x \in \mathbf{R}^n$, let $P = P(x, D) = \sum_{|\alpha| \leq m} a_\alpha(x) D^\alpha$ be a differential operator of order m whose principal symbol is C^∞ function $p = p_m(x, \xi) = \sum_{|\alpha| = m} a_\alpha(x) \xi^\alpha$.

We use the notation $D_{x_j} = -i\partial_{x_j}$ for $1 \leq j \leq n$. For a large parameter λ and a real valued function $\varphi = \varphi(x)$, we want to have the estimate

$$\sum_{|\beta|\le m-1} \lambda^{2(m-|\beta|)-1} \int e^{2\lambda\varphi}|D^\beta u|^2 dx$$
$$\le C \int e^{2\lambda\varphi}|Pu|^2 dx \qquad (1)$$

for $u \in C_0^\infty(\Omega)$, where Ω is a neighborhood of a point x_0. For two functions $f = f(x,\xi)$ and $g = g(x,\xi)$, Poisson bracket between f and g is defined by

$$\{f,g\} = \sum_{j=1}^{n}\Big(\frac{\partial f}{\partial \xi_j}\frac{\partial g}{\partial x_j} - \frac{\partial f}{\partial x_j}\frac{\partial g}{\partial \xi_j}\Big).$$

The condition to get Carleman estimates above is called pseudo convexity for the function φ that appears in the estimate (1).
The hypersurface $S = \{x \in \mathbf{R}^n \mid \psi(x) = \psi(x_0)\}$ which is defined by the function ψ is called strongly pseudo convex (at x_0) if

$$\mathrm{Re}\{\overline{p}, \{p, \psi\}\}(x, \xi) > 0 \tag{2}$$

on the subset on the cotangent bundle $T^*\mathbf{R}^n$

$$\begin{aligned} N_R =& \{(x, \xi) \in T^*\mathbf{R}^n \backslash 0 \mid \\ & p(x, \xi) = \{p, \psi\}(x, \xi) = 0\}, \end{aligned} \tag{3}$$

and

$$\frac{1}{2i\lambda}\{\overline{p}(x,\xi - i\lambda d\psi), p(x,\xi + i\lambda d\psi)\} > 0$$
$$\text{for} \quad \xi \in \mathbf{R}^n,\ \lambda > 0 \tag{4}$$

on the set

$$N_C = \{(x,\xi) \in T^*\mathbf{R}^n\backslash 0 \mid \ p(x,\xi + i\lambda d\psi) = \{p(x,\xi + i\lambda d\psi), \psi\}(x,\xi) = 0\}. \tag{5}$$

The differential operator $P = P(x,D) = \sum_{|\alpha|\leq m} a_\alpha(x) D^\alpha$ is called principally normal if $|\{\overline{p}, p\}| \leq C|\xi|^{m-1}|p|$ for $x \in \Omega$, $\xi \in \mathbf{R}^n$ Elliptic operators and operators with real principal symbols are principally normal.

For the function ψ which define the hypersurface S, we set

$$\psi_\varepsilon(x) = \sum_{|\alpha|\le 2} \frac{1}{\alpha!}\partial^\alpha \psi(x_0)(x-x_0)^\alpha - \varepsilon|x-x_0|^2, \tag{6}$$

$$\varphi(x) = \exp t\psi_\varepsilon(x) \tag{7}$$

for sufficiently small positive ε and sufficiently large t. Under the strongly pseudo convexity of S defined by ψ we have the uniqueness of the solutions by proving the Carleman estimates with the weight φ.

Theorem 1.1 Let P be the operator of principally normal defiend by the symbol $p_m(x,\xi) = \sum_{|\alpha|=m} a_\alpha(x)\xi^\alpha$. Let the hypersurface S is strictly convex at x_0, that is, we have (2) and (4) on (3) and (5), respectively. Then there exist positive constants $\lambda_0 > 0$, $C > 0$ and a neighborhood Ω of x_0 such that we have the inequality (1) for $\lambda \geq \lambda_0$.

The details of this result can be seen in Hörmander [3] and Zuily [16].
Note: I found the nice lecture note written by N. Lerner (UPMC) in his home page.

Roughly speaking, the information of the phase function is considered through the operator $e^{\lambda\varphi}P(x,D)e^{-\lambda\varphi} = P(x, D + i\lambda d\varphi)$. The meaning of the conditions about the strict pseudo convexity is that we have the positivity of the symbol

$$
\begin{aligned}
(|\xi|^2 + \lambda^2)^{m-1} &\le C_1|p(x, \xi + i\lambda d\varphi)|^2 \\
+ \quad & \frac{C_2}{2i\lambda}\{\overline{p}(x, \xi - i\lambda d\varphi), p(x, \xi + i\lambda d\varphi)\} + C_3 r(x, \xi, \lambda)
\end{aligned}
$$

where $r = r(x, \xi, \lambda)$ is a small real symbol of in some sense if P is of principally normal. It follows from the positivity of the symbol above and the sharp Gårding inequality in the theory of pseudo differential operators that Carleman estimate (1) will be obtained.

2 Weak pseudo convexity for the operator of second order and real principal type

We restrict our attention for the operators of second order and of real principal type. So $m = 2$ and the coefficients a_α are real valued smooth functions for $|\alpha| = 2$. The operator P is of principal type if

$$\nabla_\xi p(x,\xi) \neq 0 \quad \text{on} \quad p(x,\xi) = 0,\ \xi \neq 0. \tag{8}$$

Since $p(x,\xi)$ is of second order and satisfies $p(x,t\xi) = t^2 p(x,\xi)$ for $t \in \mathbf{R}$, we have

$\sum_{k=1}^{n} \xi_k \frac{\partial p}{\partial \xi_k}(x, \xi) = 2p(x, \xi)$, so we have $\nabla_\xi p(x, \xi) \neq 0$ for all (x, ξ) with $\xi \neq 0$. In our result the positivity of the symbol

$$|\xi|^2 \leq C \sum_{k=1}^{n} \Big| \frac{\partial p}{\partial \xi_k} \Big|^2$$

is useful. Let the hypersurface $S = \{x \in \mathbf{R} \mid \psi(x) = \psi(x_0)\}$ be noncharacteristic for the operator $P = P(x, D)$, we have $p(x, \nabla\psi) \neq 0$. It is not so difficult to check that one of the conditions about strongly pseudo convexity, that is (4), is automatically satisfied if the hypersurface S is noncharacteristic for the operator of

second order with real principal symbol.
We repeat the classical result about Carleman estimates for the operators of second order and real principal type.

Theorem 2.1 Let $P = P(x, D) = \sum_{j,k=1}^{n} a_{jk}(x) D_j D_k$ and $p = p(x, \xi) = \sum_{j,k=1}^{n} a_{jk}(x) \xi_j \xi_k$. Assume that the operator P is of principal type, that is, $\nabla_\xi p(x, \xi) \neq 0$ when $p(x, \xi) = 0$ and $\xi \neq 0$. Assume that the hypersurface $S = \{x \in \mathbf{R}^n \mid \psi(x) = \psi(x_0)\}$ is noncharactersistic, $p(x, \nabla \psi) \neq 0$, and strongly pseudo convex

$$\{p, \{p, \psi\}\}(x, \xi) > 0 \tag{9}$$

on the set

$$N_R = \{(x,\xi) \in T^*\mathbf{R}^n \backslash 0 \mid \ p(x,\xi) = \{p,\psi\}(x,\xi) = 0\}. \tag{10}$$

Set $\varphi = \varphi(x)$ from the function $\psi = \psi(x)$ as above. Then there exist positive constants $\lambda_0 > 0$, $C > 0$ and a neighborhood Ω of x_0 such that we have

$$\lambda^3 \int e^{2\lambda\varphi}|u|^2 dx + \sum_{j=1}^{n} \lambda \int e^{2\lambda\varphi}|D_j u|^2 dx \leq C \int e^{2\lambda\varphi}|Pu|^2 dx \tag{11}$$

for $\lambda \geq \lambda_0$ and $u \in C_0^\infty(\Omega)$.

Once we have the inequality (11), we can add the lower order terms freely as

$$P = P(x, D) = \sum_{j,k=1}^{n} a_{jk}(x) D_j D_k + \sum_{j=1}^{n} b_j(x) D_j + c_0(x)$$

by replacing λ_0 with more large one. Theorem 1.2 is stable under first order perturbations.

When the condition (9) is degererate as

$$\{p, \{p, \varphi\}\}(x, \xi) \geq 0 \tag{12}$$

on N_R, it is very difficult to get the Carleman estimate (11). In fact, we need to impose more conditions on the operators and hypersurface S or to hope weak Carleman estimates.

Example (wave operator with conctant coefficients)

In $\mathbf{R}^3$ near the origin $(0,0,0)$ we study $P(D) = D_{x_1}^2 - D_{x_2}^2 - D_{x_3}^2$ with symbol $p(\xi) = \xi_1^2 - \xi_2^2 - \xi_3^2$. Since we have

$$p(\xi) = \xi_1^2 - \xi_2^2 - \xi_3^2,$$

$$\{p(\xi), \psi(x)\} = 2\xi_1 \frac{\partial \psi}{\partial x_1} - 2\xi_2 \frac{\partial \psi}{\partial x_2} - 2\xi_3 \frac{\partial \psi}{\partial x_3},$$

$$\{p, \{p, \psi\}\} = 4\left(\xi_1^2 \frac{\partial^2 \psi}{\partial x_1^2} + \xi_2^2 \frac{\partial^2 \psi}{\partial x_2^2} + \xi_3^2 \frac{\partial^2 \psi}{\partial x_3^2}\right),$$

$\psi_1(x) = x_1$ and $\psi_2(x) = x_1^2 + x_2^2 + x_3$ are strictly pseudo convex, but $\psi_3(x) = x_3$ is not!

Historically, Lerner and Robbiano [8] studied the problem about unique continuation under weak pseudo convexity. This property is called compact uniqueness or Cauchy compact. After the result [8] by Lerner and Robbiano, Hörmander [3] in his book arranged the proof by Lerner and Robbiano and got the new Carleman estimate under weak pseudo convexity (12) on N_R. We review his Carleman estimate. Let $P = P(x, D) = D_1^2 - R(x, D')$, where $x = (x_1, x') \in \mathbf{R} \times \mathbf{R}^{n-1}$ and $D' = (D_2, \cdots, D_n)$ and $\psi = \psi(x_1) = x_1$. This restriction looks the loss of generality. But we can reduce the general case into this form at least locally. For the function $\varphi(x_1) = x_1 + \frac{1}{2}x_1^2$ and sufficiently small $\varepsilon > 0$ we have for $\lambda \geq \lambda_0$ and

$u \in C_0^\infty(\Omega_0)$

$$\lambda^3 \int e^{2\lambda\varphi}|u|^2 dx + \lambda \int e^{2\lambda\varphi}|D_1 u|^2 dx + \sum_{j=2}^{n} \int e^{2\lambda\varphi}|D_j u|^2 dx \le C \int e^{2\lambda\varphi}|P(\varepsilon x, D)u|^2 dx. \quad (13)$$

We shall compare two inequalities (11) and (13). Smallness of the parameter $\varepsilon > 0$ means that the support of the function u should be very small and the operator $P(x, D)$ looks like as the operator with constant coefficients.

3 Limiting Carleman weight in Calderón problem

In the theory of inverse problems, Calderón problem is well known. This is the inverse problem that we determine the information inside as potentials or coefficients in the equations from the maps like the measurements on the boundary. There the special solutions which are called the complex geometrical optics solutions have important roles in many approaches.

For the linear partial differential operator $P(x, D)$ of order 2, we want to construct the solutions $u = u(x; h)$ with complex phase function $\Phi(x) = \varphi(x) + i\psi(x) \in C^\infty(\Omega, \mathbf{C})$

of the form

$$u(x;h) = e^{\frac{1}{h}(\varphi(x)+i\psi(x))}(a_0(x) + hr(x,h)),$$

where h is a small positive parameter.
We can give a simple and important example for this approach. Set $\mathbf{a}, \mathbf{b} \in \mathbf{R}^n$ with $|\mathbf{a}| = |\mathbf{b}|$ and $\langle \mathbf{a}, \mathbf{b} \rangle = 0$

$$\sum_{j=1}^{n} \frac{\partial^2}{\partial x_j^2} e^{\langle \mathbf{a}+i\mathbf{b}, x \rangle} = 0. \tag{14}$$

First we shall construct the phase function $\Phi(x) = \varphi(x) + i\psi(x)$.

The complex eikonal equation is obtained as

$$\begin{aligned} 0 =& p(x, d(\varphi + i\psi)) \\ =& \sum_{j,k=1}^{n} g^{jk}(x)\Big(\frac{\partial\varphi}{\partial x_j} + i\frac{\partial\psi}{\partial x_j}\Big)\Big(\frac{\partial\varphi}{\partial x_k} + i\frac{\partial\psi}{\partial x_k}\Big). \end{aligned}$$

Two real valued functions φ and ψ should satisfy the system of the nonlinear equations of first order

$$\sum_{j,k=1}^{n} g^{jk}\big(\partial_{x_j}\varphi(x)\partial_{x_k}\varphi(x) - \partial_{x_j}\psi(x)\partial_{x_k}\psi(x)\big) = 0,$$

$$\sum_{j,k=1}^{n} g^{jk}(x)\partial_{x_j}\varphi(x)\partial_{x_k}\psi(x) = 0.$$

Once the function φ is fixed, the symbols $a(x,\xi)$ and $b(x,\xi)$ are defined by

$$a(x,\xi) = -\sum_{j,k=1}^{n} g^{jk}\big(\partial_{x_j}\varphi(x)\partial_{x_k}\varphi(x) - \xi_j\xi_k\big)$$

$$b(x,\xi) = \sum_{j,k=1}^{n} g^{jk}(x)\partial_{x_j}\varphi(x)\xi_k$$

For the function φ we want to construct ψ as the solution to

$$a(x,d\psi) = 0, \quad b(x,d\psi) = 0. \tag{15}$$

We review this theory from the point of the pseudo Riemann geometry. Set two symbols $a(x,\xi)$ and $b(x,\xi)$ of real valued from $p(x,\xi+id\varphi(x))=a(x,\xi)+ib(x,\xi)$ where $p(x,\zeta)=\sum_{j,k=1}^{n} g^{jk}(x)\zeta_j\zeta_k$ for $x\in\mathbf{R}^n$ and $\zeta=(\zeta_1,\cdots,\zeta_n)\in\mathbf{C}^n$. We define the notion of limiting Carleman weights.

Definition 3.1 A real valued function φ in $\mathbf{R}^n$ is said to be a limiting Carleman weight if $d\varphi\neq 0$ and it satisfy a kind of Hörmander subelliptic condition

$$\{\overline{p_\varphi},p_\varphi\}=0 \quad \text{when} \quad p_\varphi=0, \tag{16}$$

where $p_\varphi=a(x,\xi)+ib(x,\xi)$.

The symbols $a(x,\xi)$ and $b(x,\xi)$ are defined by the pseudo distance and pseudo inner product on $T^*\mathbf{R}^n$ as

$$a(x,\xi) = |\xi|_g^2 - |d\varphi|_g^2, \quad b(x,\xi) = 2\langle d\varphi, \xi\rangle$$

for $d\varphi = \sum_{j=1}^{n}(\partial_{x_j}\varphi)dx^j$, $\xi = \sum_{j=1}^{n}\xi_j dx^j$. The condition of the limiting Carleman weight means that the manifold $J = \{(x,\xi) \in T^*\mathbf{R}^n |\ a(x,\xi) = b(x,\xi) = 0\}$ is involutive, that is,

$$\{a, b\}(x,\xi) = 0$$

on $J = \{(x,\xi) \in T^*\mathbf{R}^n |\ a(x,\xi) = b(x,\xi) = 0\}$.

We shall check this condition from pseudo convexity. For the function φ with $p(x, d\varphi) = 1$ on Ω, we have

$$a(x, \xi) = p(x, \xi) - 1, \quad b(x, \xi) = \{p, \varphi\}(x, \xi),$$

that is $\{a, b\}(x, \xi) = \{p, \{p, \varphi\}\}(x, \xi)$.
Carleman estimate for the limiting Carleman weights had obtained by Sylvester-Uhlmann [12] for the linear phase. Kenig, Sjöstrand and Uhlmann [6] used the new limiting Carleman weight like $\varphi(x) = \log|x|$ for the flat Laplacian. Laplace Beltrami oparator, that is the second order elliptic operator with real symmetric coefficients, has been studied after their result. We can see them in [1]. They based on elliptic Carleman estimate. In this talk we want to study non-elliptic operators.

4 Main result

4.1 Nonlocal Carleman estimate under weak pseudo convexity

We give the main result for the operator of second order and real principal type.

Theorem 4.1 (Ferreira-Rousseau-Takuwa) Let $x = (x_1, x')$, $(x' = (x_2, \cdots, x_n))$. For the second order operator

$$\begin{aligned} P(x, D) =& D_{x_1}^2 - R(x, D_{x'}) \\ =& D_{x_1}^2 - \sum_{j,k=2}^{n} D_{x_j}(g^{jk}(x) D_{x_k}) \end{aligned}$$

with coefficients g^{jk} of real valued, $g^{jk} = g^{kj}$ and $\det(g^{jk}) \neq 0$. Set the principal symbol $r(x, \xi') = \displaystyle\sum_{j,k=2}^{n} g^{jk}(x)\xi_j\xi_k$ for the operator $R(x, D_{x'})$.

Assume that the operator $P(x, D)$ is of principal type, that is $\nabla_{\xi'} r(x, \xi) \neq 0$ for $\xi' \neq 0$. We also assume the weak pseudo convexity

$$\frac{\partial r}{\partial x_1}(x, \xi') \geq 0 \quad \text{when } r(x, \xi') = 0. \tag{17}$$

Assume $|\partial_{x_l} g^{jk}(x)|$ are small in Ω. (We do not give pricise assumption here.)

Then there exists $\lambda_0 > 0$ such that we have

$$\lambda^3 \int_\Omega e^{2\lambda\varphi(x)}|u|^2 dx + \lambda \int_\Omega e^{2\lambda\varphi(x)}|D_{x_1}u|^2 dx + \int_\Omega e^{2\lambda\varphi(x)}|D_{x'}u|^2 dx \le C \int_\Omega e^{2\lambda\varphi(x)}|\{D_{x_1}^2 - R(x, D_{x'})\}u|^2 dx, \quad (18)$$

where $\varphi = \varphi(x_1) = x_1$ for $\lambda \ge \lambda_0$ and $u \in C_0^\infty(\Omega)$.

Note: Non selfadjoint form can be treated in our proof:

$$R(x, D_{x'}) = \sum_{j,k=2}^{n} g^{jk}(x) D_{x_j} D_{x_k}.$$

L. Hörmander [3] and N. Lerner-L. Robbiano [8] had studied Carleman estimates that were obtained only for a sufficiently small neighborhood Ω. Because of weakness of the convexity of the weight functions. It is very delicate to have Carleman type estimates. Theorem 4.1 shows the new Carleman estimate without the restriction of the size of the domain Ω. In fact, we can consider that Theorem 4.1 is the improvement of the original Carleman estimates proved by L. Hörmander for operators of second order real principal type.

5 Idea and sketch of proof

Step 1: Convexification of the phase function $\varphi(x_1) = x_1$

$$\widetilde{\varphi}_\delta(x_1) = f_\delta\big(\varphi(x_1)\big) = x_1 + \frac{1}{2}\delta x_1^2,$$
$$f_\delta(s) = s + \frac{1}{2}\delta s^2, \quad 0 \le \delta \le \frac{1}{2}.$$

By setting $\delta = \dfrac{\mu}{\lambda}$, where $\mu > 0$ is small enough, the commutators

$$[e^{\frac{1}{2}\delta x_1^2}, D_1], \quad [e^{\frac{1}{2}\delta x_1^2}, D_1^2], \quad [e^{\frac{1}{2}\delta x_1^2}, R(x, D_{x'})] = 0$$

are nice terms (by the boundedness in x_1).

Note:

- $\widetilde{\varphi}_\delta(x_1)$ is still weak pseudo convex phase function.
- The new parameter $\delta > 0$ is introduced to control the size of the support of $u \in C_0^\infty(\Omega)$.
- Thanks to the large parameter λ, $U = e^{\frac{1}{2}\delta x_1^2}$ and its inverse U^{-1} are nice bounded operator on the weighted space of L^2 and H^1 with parameter in (18).

Step 2: Scalling argument

For $\varepsilon_1 > 0$, we set the new variables.
$\Psi_{\varepsilon_1} : \mathbf{R} \times \mathbf{R}^{n-1} \to \mathbf{R} \times \mathbf{R}^{n-1}$ as $(x_1, x') \mapsto (y_1, y')$ by $y_1 = \frac{1}{\varepsilon_1} x_1$, $y' = x'$. In the final step of the proof the small parameter $\varepsilon_1 > 0$ will be fixed.

We study

$$\frac{1}{\varepsilon_1^3}\widetilde{I} = I = \int_\Omega e^{2\lambda\widetilde{\varphi}_\delta(x_1)}|\{D_{x_1}^2 - R(x_1, x', D_{x'})\}u|^2 dx$$

where

$$\begin{aligned}\widetilde{I} &= \int_{\widetilde{\Omega}} e^{2\widetilde{\lambda}\widetilde{\varphi}_\delta(y_1)}|\{D_{y_1}^2 - \varepsilon_1^2 R(\varepsilon_1 y_1, y', D_{y'})\}\widetilde{u}|^2 dy \\ &= \int_{\widetilde{\Omega}} |T^2 - \varepsilon_1^2 R(\varepsilon_1 y_1, y', D_{y'})\}\widetilde{v}|^2 dy \\ &= ||P_\lambda \widetilde{v}||^2_{L^2(\widetilde{\Omega})},\end{aligned}$$

and

$$
\begin{aligned}
P_\lambda =& T^2 - \varepsilon_1^2 R(\varepsilon_1 y_1, y', D_{y'}),\\
T =& T(y_1, D_1) = D_{y_1} + i\widetilde{\lambda}(1 + \widetilde{\delta} y_1),\\
\widetilde{\lambda} =& \varepsilon_1 \lambda, \quad \widetilde{\delta} = \varepsilon_1 \delta,\\
\widetilde{\Omega} =& \Psi_{\varepsilon_1}(\Omega) \subset \left\{ y \in \mathbf{R}^n | \ |y_1| \le \frac{M_1}{\varepsilon_1}, |y'| \le M_2 \ \right\}.
\end{aligned}
$$

Step 3: Commutator argument between P_λ and P_λ^*

$$||P_\lambda \widetilde{v}||^2_{L^2(\widetilde{\Omega})} - ||P_\lambda^* \widetilde{v}||^2_{L^2(\widetilde{\Omega})} = \langle [P_\lambda, P_\lambda^*]\widetilde{v}, \widetilde{v} \rangle_{L^2(\widetilde{\Omega})}$$

for $\widetilde{v} \in C_0^\infty(\widetilde{\Omega})$, where

$$\begin{aligned} [P_\lambda, P_\lambda^*] =& 4\widetilde{\lambda}\widetilde{\delta}(TT^* + T^*T) - \varepsilon_1^2 J_1 + \varepsilon_1^4 [R^*, R] \\ J_1 =& [(T^*)^2, R] - [T^2, R^*]. \end{aligned}$$

By following the argument by Hörmander [3], we can show the estimate as below.

Lemma 5.1

$$
\begin{aligned}
&4\widetilde{\lambda}\widetilde{\delta}\Big(||T\widetilde{v}||^2 + ||T^*\widetilde{v}||^2\Big) + ||P_\lambda^*\widetilde{v}||^2 \\
&\quad + 4\varepsilon_1^3\widetilde{\lambda}\Big\langle(1+\widetilde{\delta}y_1)\frac{\partial r}{\partial x_1}(\varepsilon_1 y_1, y', D_{y'})\widetilde{v}, \widetilde{v}\Big\rangle \\
&\leq ||P_\lambda\widetilde{v}||^2 + \varepsilon_1^2\langle J_1^{(2)}\widetilde{v}, \widetilde{v}\rangle + \varepsilon_1^4\langle J_2\widetilde{v}, \widetilde{v}\rangle,
\end{aligned}
$$

where

$$
\begin{aligned}
&|\langle J_1^{(1)}\widetilde{v}, \widetilde{v}\rangle| \leq C\varepsilon_1\big(||T\widetilde{v}|| + ||T^*\widetilde{v}||\big)||\widetilde{v}||_{(0,1)} \\
&|\langle J_2\widetilde{v}, \widetilde{v}\rangle| \leq C||\widetilde{v}||^2_{(0,1)},
\end{aligned}
$$

and $||\widetilde{v}||_{(0,1)} = ||\widetilde{v}||_{L^2} + \sum_{j=2}^{n}||D_{y_j}\widetilde{v}||_{L^2}$.

Step 4: Weak pseudo convexity

Special case: $\{p, \{p, x_1\}\} = 0$ (Limiting Carleman weight)
The symbol $r(x_1, x', \xi')$ is independent of x_1. We have

$$\left\langle (1 + \widetilde{\delta} y_1) \frac{\partial r}{\partial x_1} (\varepsilon_1 y_1, y', D_{y'}) \widetilde{v}, \widetilde{v} \right\rangle = 0.$$

Gereral case: $\{p, \{p, x_1\}\} \geq 0$ (weak pseudo convexity)

The term as above becomes small in a sense by the construction of the symbols and the sharp Gårding inequality. Here we have used pseudo differential operators. (We skip this part today.)

Step 5: Estimate for $||\widetilde{v}||$ and $||D_{y_1}\widetilde{v}||$

It is not so difficult to get the estimates

$$\widetilde{\lambda}^2||\widetilde{v}||^2 \leq 2\Big(||T\widetilde{v}||^2 + ||T^*\widetilde{v}||^2\Big)$$

$$\widetilde{\lambda}^2||D_{y_1}\widetilde{v}||^2 \leq ||P_\lambda\widetilde{v}||^2 + ||P_\lambda^*\widetilde{v}||^2$$

$$+ 4\widetilde{\lambda}^2\widetilde{\delta}^2||\widetilde{v}||^2 + \varepsilon_1^6||\widetilde{v}||^2_{(0,1)}.$$

Step 6: Estimate for $||D_{y_j}\widetilde{v}||$ $(2 \leq j \leq n)$

Instead of the pseudo convexity, we control $||D_{y_j}\widetilde{v}||$ in y' direction by the principal type condition. To make our idea clear, we review the idea of Hörmander's calculation in [3].

Original approach If we have a time, we give a explanation.

Our approach If we have a time, we give a explanation.

In the talk we had to skip this part. The original Hörmander's calculation in [3] needed to make $|x'|$ small enough. So we introduce the new commutator argument in our result.

Combining the 6 steps, we have obtained Theorem 4.1.

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