Spectral inequality and optimal cost of controllability for the Stokes system

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Outline

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$\Omega \subset \mathbb{R}^N$ $(N \geq 2)$ is a bounded connected open set, with $\partial\Omega$ regular. Let $T > 0$ and ω be a nonempty subset of Ω. We consider the controlled Stokes system

$$\left| \begin{array}{ll} y_t - \Delta y + \nabla p = f 1_\omega & \text{in } Q =]0, T[\times\Omega, \\ div\ y = 0 & \text{in } Q, \\ y = 0 & \text{on } \Sigma =]0, T[\times\partial\Omega, \\ y(0) = y_0 & y_0 \in L^2(\Omega)^N,\ div\ y_0 = 0,\ y_0 \cdot \nu|_{\partial\Omega} = 0. \end{array} \right. \tag{1.1}$$

Theorem

For every y_0, there exists $f \in L^2(\omega \times (0, T))$ such that the associated solution of the Stokes system (1.1) satisfies

$$y(T) = 0. \tag{1.2}$$

Moreover, one has

$$\|f\|_{L^2(\omega\times(0,T))} \leq C_{stokes}(T)\|y_0\|_{L^2}, \quad C_{stokes}(T) \leq C_1 e^{C_2/T}. \tag{1.3}$$

Previous results on the null controllability for the Stokes system have been obtained by means of global in time Carleman estimates:

E. Fernandez-Cara, S. Guerrero, O. Yu. Imanuvilov, J.-P. Puel (JMPA 2004) and O. Yu. Imanuvilov, J.-P. Puel, M. Yamamoto (Chin. Ann. Math. Ser. B 2009).

However, with this strategy of proof of null controllability, one obtains:

$$C_{stokes}(T) \leq C_1 e^{C_2/T^4}$$

Therefore our improvement is the behavior of the cost function for $T > 0$ small:

$$C_{stokes}(T) \leq C_1 e^{C_2/T}$$

Outline

Let **V** and **H** be the usual spaces in the context of fluid mechanics

$$\mathbf{V} = \{u \in H_0^1(\Omega)^N;\ div\ u = 0\},$$

$$\mathbf{H} = \{u \in L^2(\Omega)^N;\ div\ u = 0, u \cdot \nu = 0 \text{ on } \partial\Omega\}$$

The Stokes operator A is the unbounded operator on **H** defined by

$$D(A) = \{u \in \mathbf{V}, A(u) \in \mathbf{H}\}$$

$$A(u) = v \quad \text{iff} \quad \int_\Omega \nabla u \nabla g\ dx = \int_\Omega vg\ dx \quad \forall g \in \mathbf{V}$$

Therefore one has $A(u) = v$ iff there exists a pressure p such that

$$-\triangle u + \nabla p = v$$

Observe that the pressure p is an harmonic function in Ω: $\triangle p = 0$.

The Stokes operator A is a selfadjoint positive operator with compact resolvant. Let $\{e_j\}_{j=1}^{\infty}$ be an orthonormal basis of $\mathbf{H}$, given by the eigenvectors of the Stokes equation

$$\left|\begin{array}{ll} -\Delta e_j + \nabla p_j = \mu_j e_j & \text{in } \Omega, \\ div\ e_j = 0 & \text{in } \Omega, \\ e_j = 0 & \text{on } \partial\Omega, \end{array}\right. \tag{2.1}$$

with the sequence of eigenvalues $\{\mu_j\}_{j=1}^{\infty}$ satisfying

$$0 < \mu_1 \leq \mu_2 \leq \dots \qquad \lim_{j\to\infty} \mu_j = \infty$$

Then $\{e_j\}_{j=1}^{\infty}$ is also an orthogonal basis of $\mathbf{V}$, and one has

$$\|curl(e_j)\|_{L^2}^2 = \mu_j$$

Spectral Inequality

Theorem

Let $\omega \subset \Omega$ be a nonempty open set. There exists constants $M > 0$, $K > 0$ such that, for every sequence of complex numbers z_j and every real $\Lambda \geq 1$, we have

$$\sum_{\mu_j \leq \Lambda} |z_j|^2 = \int_\Omega \left| \sum_{\mu_j \leq \Lambda} z_j e_j(x) \right|^2 dx \leq M e^{K\sqrt{\Lambda}} \int_\omega \left| \sum_{\mu_j \leq \Lambda} z_j e_j(x) \right|^2 dx. \quad (2.2)$$

By an argument due to T.I. Seidman (J. Math. Anal. Appl. 2008) and revisited by L. Miller (Discrete Contin.Dyn. Syst. Ser. B, 2010), one deduce from this spectral inequality the null controllability result and the estimate on the cost.

$$C_{stokes}(T) \leq C_1 e^{C_2/T}$$

Spectral inequalities

The spectral inequality has been proven for the eigenfunctions of the scalar Laplace operator with Dirichlet boundary conditions in:

D. Jerison and G. Lebeau. *Nodal sets of sums of eigenfunctions*
Acts of the conference "Harmonic analysis and partial differential equations", 1996. Chicago Lectures in Mathematics.

It has been used by G. Lebeau and E. Zuazua in the study of null-controllability of a system of linear thermoelasticity (Arch. Rational Mech. Anal., 1998).

It has been recently extended to the bi-laplacian with Dirichlet boundary condition by J. Le Rousseau and L. Robbiano.

Outline

We introduce the sets

$$Z = (0,1) \times \Omega \text{ and } W = (1/3, 2/3) \times \Omega.$$

Let $\Lambda \geq 1$. For a given sequence $(z_j)_j$ of complex numbers, set

$$u(s,x) = \sum_{\mu_j \leq \Lambda} z_j \frac{sinh(s\sqrt{\mu_j})}{\sqrt{\mu_j}} e_j(x),$$

The "classical proof" of the spectral inequality will be to glue together local Carleman estimates for the pair $(u,p) = \sum_{\mu_j \leq \Lambda} a_j(s)\big(e_j(x), p_j(x)\big)$,

$a_j(s) = z_j \frac{sinh(s\sqrt{\mu_j})}{\sqrt{\mu_j}}$, which is a solution of

$$\left| \begin{array}{ll} -\partial_{ss}^2 u - \Delta_x u + \nabla_x p = 0 & \text{in } Z, \\ div_x \ u = 0 & \text{in } Z, \\ u(s,x) = 0 & \text{on } (0,1) \times \partial\Omega, \\ u(0,x) = 0 & \text{in } \Omega. \end{array} \right. \tag{3.1}$$

Unfortunately, system (3.1) does not have local unique continuation property. Indeed, for any function $q = q(s,x)$ such that $\Delta_x q = 0$, the pair

$$(u,p) = (\nabla_x q, \partial^2_{ss} q)$$

is a solution of the two first lines of (3.1). Therefore, we cannot obtain a local Carleman estimate for system (3.1).

Thus we will work with $v = curl\ u$ which is a solution of the system

$$\left| \begin{array}{ll} -\partial^2_{ss} v - \Delta_x v = 0 & \text{in } Z, \\ v(0,x) = 0 & \text{in } \Omega. \end{array} \right. \tag{3.2}$$

Now, the **major problem** is that $v = curl(u)$ is equal to

$$v(s,x) = \sum_{\mu_j \leq \Lambda} z_j \frac{sinh(s\sqrt{\mu_j})}{\sqrt{\mu_j}} curl(e_j)(x).$$

v **satisfies no boundary condition on the lateral boundary** $(0,1) \times \partial\Omega$**.**

To overcome the preceding lack of boundary condition for v on $(0,1) \times \partial\Omega$, we introduce a small parameter h related to Λ by

$$h = \frac{\delta}{\sqrt{\Lambda}} \quad \text{with} \quad \delta > 0 \quad \text{small.}$$

Informally, $x \mapsto v(s,x)$ is concentrated at frequencies $\leq \sqrt{\Lambda}$. Thus, for the semiclassical analysis with semiclassical parameter h, the spectrum of v is concentrated near

$$|\xi| \leq h\sqrt{\Lambda} = \delta$$

Therefore we force localization near $\xi = 0$

The Main Inequality

Theorem

There exists $h_0 > 0$, $\mu > 0$, $A > 0$, $\alpha \in (0,1)$, *and for* $\delta > 0$ *small enough* $C_\delta > 0$, *such that* $v = curl\ u$ *satisfies for all* $h \in]0, h_0]$, *all* $(z_j)_j$ *and all* $\Lambda \in [1, (\delta/h)^2]$

$$\|v\|_{H^1(W)^N} \leq C_\delta \Big(e^{-\mu/h} \|v\|_{H^1(Z)^N} + e^{A/h} \|v\|^{1-\alpha}_{H^1(Z)^N} \|\partial_s v(0,x)\|^{\alpha}_{L^2(\omega)^N} \Big) \tag{3.3}$$

$$(\sum_{\mu_j \leq \Lambda} |z_j|^2)^{1/2} \leq C \|v\|_{H^1(W)^N}, \quad \|v\|_{H^1(Z)^N} \leq C\Lambda^M (\sum_{\mu_j \leq \Lambda} |z_j|^2)^{1/2} e^{\sqrt{\Lambda}}$$

$$\|\partial_s v(0,x)\|^2_{L^2(\omega)^N} \leq C\Lambda^M (\sum_{\mu_j \leq \Lambda} |z_j|^2)^{1/2} \|\partial_s v(0,x)\|_{L^2(\tilde{\omega})^N} \quad \omega \subset\subset \tilde{\omega}$$

Main Inequality $\Longrightarrow$ Spectral inequality

In order to estimate $\|v\|_{H^1(W)^N}$, we will estimate $\|v\|_{H^1(B_r(z)\cap Z)^N}$ for any $z \in (0,1) \times \overline{\Omega}$.
There is three cases:

1) $z = (s,x) \in (0,1) \times \omega$, with $s > 0$ small;
2) $z = (s,x) \in (0,1) \times \Omega$;
3) $z = (s,x) \in (0,1) \times \partial\Omega$.

Each one of the cases above are proved using appropriate Carleman inequalities. Actually, since in *cases* 1 and 2 we are away from the lateral boundary, the proof for such cases is performed exactly as in the case of the heat equation (see G. Lebeau, L. Robbiano , CPDE 1995), and we obtain the better **pure interpolation estimate**: there exists $r = r_z > 0$, C_z, and $\alpha_z \in]0,1[$ such that

$$\|v\|_{H^1(B_r(z)\cap Z)^N} \le C_z \|v\|^{1-\alpha_z}_{H^1(Z)^N} \|\partial_s v(0,x)\|^{\alpha_z}_{L^2(\omega)^N} \tag{3.4}$$

Let $r = dist(x, \partial\Omega)$. For $s_0 \in]1/4, 3/4[$ small, set

$$\psi(s, r) = r - (s - s_0)^2$$

and consider $\varphi = e^{D\psi}$. For D large enough, φ satisfies Hörmander's sub-ellipticity condition for the operator A in the variables $(s, r, y), y \in \partial\Omega$ and $\partial_r \varphi > 0$ near $r = 0$. Let $0 < r_0 < s_*^2$, and introduce a cutoff function $\chi(s, r) = \chi_0(s)\chi_1(r)$, with $\chi_0 \in C_0^\infty(]s_0 - 2s_*, s_0 + 2s_*[)$, $\chi_0 \equiv 1$ for $|s - s_*| \leq 3s_*/2$ and $\chi_1 \in C_0^\infty([0, r_0[)$ with $\chi_1 \equiv 1$ on $[0, r_0/2]$.

For the proof of *case* 3, we use the following Carleman estimate for the operator $P = -\triangle_z = -(\partial_s^2 + \triangle_x)$.
There exist C and $h_1 > 0$ such that for every $h \in (0, h_1]$, one has

$$h||e^{\varphi/h}\chi v||_{L^2}^2 + h^3||e^{\varphi/h}\nabla_{s,x}(\chi v)||_{L^2}^2$$

$$\leq C\left(h||e^{\varphi/h}\chi v||_{L^2(r=0)}^2 + h^3||e^{\varphi/h}\nabla_{s,y}(\chi v)||_{L^2(r=0)}^2 + h^4||e^{\varphi/h}P(\chi v)||_{L^2}^2\right)$$

One has $P(v) = 0$, and in *cases* 1 and 2 we have a pure interpolation estimate. Thus the third term on the right

$$h^4 ||e^{\varphi/h} P(\chi v)||^2_{L^2} = h^4 ||e^{\varphi/h} [P, \chi] v||^2_{L^2}$$

will give at the end a contribution which will be easily estimated by the right hand side of the Main Inequality.

The main problem is thus the control of the two first term in the right hand side of the Carleman estimate:

$$h||e^{\varphi/h} \chi v||^2_{L^2(r=0)} + h^3 ||e^{\varphi/h} \nabla_{s,y}(\chi v)||^2_{L^2(r=0)} \tag{3.5}$$

Localization near $\eta = 0$

Set $a_j(s) = a_j sinh(s\sqrt{\mu_j})/\sqrt{\mu_j}$, $v_0 = v|_{\partial\Omega}$ and $\varphi_0(s) = \varphi(s,0)$.

$$G(s,r,y) = \chi_0(s)\chi_1(r)e^{\varphi(s,r)/h}v(s,r,y)$$

$$G_0(s,y) = \chi_0(s)e^{\varphi_0(s)/h}v_0(s,y) = \chi_0(s)e^{\varphi_0(s)/h}\sum_{\mu_j\leq\Lambda} a_j(s)curl(\ e_j)|_{\partial\Omega}(y),$$

We consider a cut-off function $\theta \in C_0^\infty(]-2,2[)$ satisfying $0 \leq \theta \leq 1$, and $\theta \equiv 1$ in a neighborhood of $[-\sqrt{3},\sqrt{3}]$. Let $\triangle_{\partial\Omega}$ be the Laplace operator on the boundary $\partial\Omega$ acting on vector fields. Let ε_j be an orthonormal basis of eigenfunctions of $\triangle_{\partial\Omega}$ with $-\triangle_{\partial\Omega}\varepsilon_j = \tau_j^2\varepsilon_j$.

$$\Theta(\sum_j f_j\varepsilon_j) = \sum_j \theta\Big(\sqrt{1+\Lambda^{-1}\tau_j^2}\Big) f_j\varepsilon_j\ .$$

$$\sigma(\Theta) = \theta\Big(\sqrt{1+|\eta|_y^2}\Big) Id$$

$$G_0(s,y) = G_1(s,y) + G_2(s,y), \quad G_1 = \Theta(G_0), \quad G_2 = (1-\Theta)(G_0)$$

Localization near $\eta = 0$

Lemma

Let $Q(x, D_x)$ be a differential operator defined in a neighborhood of $\partial\Omega$. For all $M, N \in \mathbb{N}$ there exists constants C_M and $D_{M,N}$ such that for all $\Lambda > 0$ and $u(x) = \sum_{\mu_j \leq \Lambda} a_j e_j(x)$ with $a_j \in \mathbb{C}$, one has, with $v = Q(u)\Big|_{\partial\Omega}$ and $w = (Id - \Theta)v$

$$||\Delta_{\partial\Omega}^M v||_{L^2(\partial\Omega)} \leq C_M \Lambda^{2M+1} \left(\sum_{\mu_j \leq \Lambda} |a_j|^2 \right)^{1/2}$$

$$||\Delta_{\partial\Omega}^M w||_{L^2(\partial\Omega)} \leq D_{M,N} \Lambda^{-N} \left(\sum_{\mu_j \leq \Lambda} |a_j|^2 \right)^{1/2}$$

Since φ is independent of y, the principal symbol p_φ of the conjugate operator

$$P_\varphi g = h^2 e^{\varphi/h} P(e^{-\varphi/h} g)$$

is equal to

$$p_\varphi = \big(\sigma + i\partial_s\varphi\big)^2 + \big(\tau + i\partial_r\varphi\big)^2 + \eta^2.$$

where $\xi = (\tau, \eta)$ is the dual variable of $x = (r, y)$, and σ the dual variable of s. The roots of p_φ with respect to τ are given by

$$\tau_\pm = -i\partial_r\varphi \pm i\sqrt{\eta^2 + \big(\sigma + i\partial_s\varphi\big)^2}.$$

Close to $s = s_0$, $\sigma = 0$ and $\eta = 0$, since $\partial_r\varphi(s_0, 0) > 0$ and $\partial_s\varphi(s_0, 0) = 0$, we get $Im\ \tau_\pm < 0$. From elliptic boundary estimates, the localization near $\sigma = 0$ of (3.5) will be absorbed by the left hand side of the Carleman estimate.

It remains to estimate the contribution of the localization near $\eta = 0$ and $|\sigma| \geq c > 0$ of

$$h||e^{\varphi/h}\chi v||^2_{L^2(r=0)} + h^3||e^{\varphi/h}\nabla_{s,y}(\chi v)||^2_{L^2(r=0)} \tag{3.6}$$

Thus we need an estimate on the Fourier transform:

$$J = \int e^{-it\sigma/h}\tilde{\chi}(t)e^{-\frac{1}{h}(\theta(t)\pm(t+s_0)\sqrt{\nu_j})}dt = \int e^{i\phi(t)/h}\tilde{\chi}(t)dt, \tag{3.7}$$

with $\sqrt{\nu}_j = h\sqrt{\mu}_j \leq \delta$ and the phase

$$\phi = -t\sigma + i(\theta(t) \pm (t+s_0)\sqrt{\nu_j}), \quad \theta(t) = \varphi(t+s_0, 0) - \varphi(s_0, 0)$$

This is an **analytic function of** t and $|\phi'(0) + \sigma| \leq \delta$.
This allows to prove that the contribution of the localization near $\eta = 0$ and $|\sigma| \geq c > 0$ of (3.6) is estimated for some $\mu = \mu_c$ by

$$e^{-2\mu/h} \|v\|^2_{H^1(Z)^N}$$

It is here that we use **analyticity in s of the coefficient of $\partial_s^2 + \triangle_x$**.
This conclude the proof of the Main Inequality.

Observe that we use the boundary condition

$$e_j|_{\partial\Omega} = 0$$

in order to conclude

$curl(e_j)|_{\partial\Omega}$ **is concentrated near** $\eta = 0$.