Minimization principles in human motions: the inverse optimal control approach

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Outline

1. Inverse optimal control

2. Arm pointing motions
   - Modelling
   - Necessary and sufficient conditions for inactivation
   - Validation/Simulations

3. Goal oriented human locomotion
   - Modelling
   - Analysis of the direct problem
   - Locomotion depends only on $\dot{\theta}$
Inverse optimal control

- Analysis/modelling of human motor control
  → looking for optimality principles

- Subjects under study:
  - Arm pointing motions
  - Goal oriented human locomotion
  - Saccadic motion of the eyes

- Mathematical formulation: inverse optimal control

Given $\dot{X} = \phi(X, u)$ and a set $\Gamma$ of trajectories, find a cost $C(X_u)$ such that every $\gamma \in \Gamma$ is solution of

$$\inf \{C(X_u) : X_u \text{ traj. s.t. } X_u(0) = \gamma(0), X_u(T) = \gamma(T)\}.$$
Difficulties:

- \( \Gamma = \) experimental data (noise, feedbacks, etc)
- Dynamical model not always known (← hierarchical optimal control)
- Limited precision of both dynamical models and costs
  \( \Rightarrow \) necessity of stability (genericity) of the criterion
- Non well-posed inverse problem
- No general method

Validation method: a program in three steps

1. Modelling step: propose a class of optimal control problems
2. Analysis step: enhance qualitative properties of the optimal synthesis
   \( \rightarrow \) reduce the class of problems
   (using geometric control theory)
3. Comparison step: numerical methods
   \( \rightarrow \) choice of the best fitting \( L \) (identification)
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Arm pointing motions

with B. Berret, C. Papaxanthis, T. Pozzo (INSERM Dijon), J.-P. Gauthier (Univ. Toulon) and C. Darlot (CNRS - Telecom ParisTech)

- Pointing motions in a vertical plane (1, 2, or 3 degrees of freedom)
- Fast motions in fixed time
Typical experimental data for 1 dof
Typical experimental data for 2 dof

Arm pointing motions

F. Jean (ENSTA ParisTech)
Inverse optimal control
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Main characteristics

Some strong qualitative characteristics:

- simultaneous inactivations of opposing muscles;
- asymmetric velocity profile
  (acceleration phases shorter than the deceleration ones);

... and more quantitative ones:

- (for 2 et 3 dof) curvature of the finger trajectory;
- (for 3 dof) final configuration of the arm.
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... and more quantitative ones:

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Modelling

Arm = controlled mechanical system described by:
- generalized coordinates $x \in \mathbb{R}^n$ for $n$ dof
- an inertia matrix $M(x)$ (positive definite);

and submitted to two kind of forces:
- $\psi(x, \dot{x}) = \text{gravity} + \text{frictions} + \text{Coriolis}$;
- $\tau(x, u) = \text{action of the muscles (torques)}$;

Here, direct control of each dof, i.e. $\tau(x, u) = u \in \mathbb{R}^n$

$$\rightarrow \quad M(x)\ddot{x} = \psi(x, \dot{x}) + u,$$

$$\Leftrightarrow \quad \dot{X} = \phi(X, u), \quad X = (x, \dot{x}) \in \mathbb{R}^{2n}, \quad u \in \mathbb{R}^n.$$
Muscles dynamics

Several possible modelling for the action of the muscles $u$:

1. Bounds on the torque $u$

$$u \in [u_1^-, u_1^+] \times \ldots \times [u_n^-, u_n^+], \quad u_i^- < 0 < u_i^+$$

2. Gradient constraints: $\dot{u} = v$

$$v \in [v_1^-, v_1^+] \times \ldots \times [v_n^-, v_n^+], \quad v_i^- < 0 < v_i^+$$

3. Agonistic-antagonistic pairs: $u = u_1^1 - u_2^2$

$$0 \leq u_i^1 \leq u_i^+ , \quad 0 \leq u_i^2 \leq -u_i^-$$

4. Pair of muscles with 1st-order dynamic: $u = u_1^1 - u_2^2$

$$\begin{align*}
\dot{u}_i^1 &= -\frac{1}{\sigma_{i1}}u_i^1 + v_i^1 \\
\dot{u}_i^2 &= -\frac{1}{\sigma_{i2}}u_i^2 + v_i^2
\end{align*}$$

$$\text{[control} = (v_i^1, v_i^2 \geq 0)\text{]}$$
Muscles dynamics

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1. **Bounds on the torque $u$**

   $u \in [u_1^-, u_1^+] \times \ldots \times [u_n^-, u_n^+], \quad u_i^- < 0 < u_i^+$

   **[control = $u$]**

2. **Gradient constraints: $\dot{u} = v$**

   $v \in [v_1^-, v_1^+] \times \ldots \times [v_n^-, v_n^+], \quad v_i^- < 0 < v_i^+$

   **[control = $v$]**

3. **Agonistic-antagonistic pairs: $u = u^1 - u^2$**

   $0 \leq u_i^1 \leq u_i^+, \quad 0 \leq u_i^2 \leq -u_i^-$

   **[control = $(u^1, u^2)$]**

4. **Pair of muscles with 1st-order dynamic: $u = u^1 - u^2$**

   $$
   \begin{align*}
   \dot{u}_i^1 &= -\frac{1}{\sigma_i^1} u_i^1 + v_i^1 \\
   \dot{u}_i^2 &= -\frac{1}{\sigma_i^2} u_i^2 + v_i^2
   \end{align*}
   $$

   **[control = $(v_i^1, v_i^2 \geq 0)$]**
Optimal control problem

- **Criterion:** $J(u) = \int_0^T f(X, u) \, dt$.

**Hyp.** $u \mapsto f(X, u)$ strictly convex.

- **Initial data:** $X_s = (x_s, 0)$, target: $X_t = (x_t, 0)$.
- The time $T > 0$ is fixed.

Optimal control problem

$(\mathcal{P})$ minimise the integral cost $J(u)$ among the trajectories of $\dot{X} = \phi(X, u)$ joining $X_s$ to $X_t$ in time $T$.

**Theorem.** The minimum of $(\mathcal{P})$ is reached by some optimal trajectory.
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Necessary condition

***Definition***

\( u \) contains an **inactivation** if one of its components \( u_i \) is \( \equiv 0 \) on a non-empty interval.

\[ \text{Not}^\circ: SC = \text{set of functions } f(X, u) \text{ such that } u \mapsto f(X, u) \text{ is strictly convex and differentiable.} \]

***Theorem***

For a generic cost \( f \in SC \), no minimizing control of \((P)\) contain inactivation.

\[ \Rightarrow \text{the cost } f \text{ is necessarily non differentiable w.r.t. } u \]

(Proof: Pontryagin Maximum Principle + Thom transversality)
Pontryagin Maximum Principle

For $P \in \mathbb{R}^{2n}$ and $\lambda \leq 0$, one defines the *Hamiltonian*:

$$h(\lambda, X, P, u) = \lambda f(X, u) + P^T \phi(X, u),$$

Then, if $(X(t), u^*(t))$ is an optimal trajectory of $(\mathcal{P})$, there exists a curve $P(t) \in \mathbb{R}^{2n}$ (the *adjoint vector*) and $\lambda \leq 0$ such that:

1. $\dot{X}_i = \frac{\partial h}{\partial P_i}$ and $\dot{P}_i = -\frac{\partial h}{\partial X_i}$;

2. $h(\lambda, X(t), P(t), u^*(t))$ is maximal w.r.t $u$, $\forall t$. 
Sufficient condition

Constraints on the cost:

- necessarily non differentiable w.r.t. $u$
- related to energetic consumption

Candidates: functions of the *absolute work* of the controlled forces.

- Work of the controlled forces:
  
  $$w = \int ud\mathbf{x} = \int \sum_{i=1}^{n} u_i dx_i = \int \sum_{i=1}^{n} u_i \dot{x}_i dt.$$  

- Measure of the energetic consumption = absolute work:
  
  $$\dot{Aw} = \int \dot{Aw}(X, u), \quad \dot{Aw}(X, u) = \sum_{i=1}^{n} |u_i \dot{x}_i|, \quad X = (x, \dot{x})$$

  $\rightarrow \dot{Aw}$ non differentiable w.r.t. $u$ when one component $u_i = 0$.  
Form of the costs: $J(u) = \int_0^T f(X, u) dt$ with

$$f(X, u) = \phi(\dot{A}w, X, u), \quad \frac{\partial \phi}{\partial \dot{A}w} \neq 0$$

**Theorem (Inactivation Principle)**

Minimizing such a cost $J(u)$ implies the occurrence of inactivations in every optimal trajectory of $(P)$ when $T$ is small enough.
Proof \((n = 1 \text{ and } f(X, u) = |\dot{x}u| + \text{differentiable fn})\)

Let \(u^*\) be a minimizing control.

- **Fact N\(^1\):** \(u^*\) is continuous.
- **Fact N\(^2\):** For \(T\) close to \(T_{\text{min}}\), \(u^*\) change of sign when \(\dot{x} > 0\) (change of signs close to the ones of the control at \(T_{\text{min}}\))

Condition 2 of the Maximum Principle: \(u^*(t)\) maximize \(h\)

\[
\Rightarrow 0 \in \partial_u h \quad \text{(or } 0 = \frac{\partial h}{\partial u} \text{ if } h \text{ differentiable w.r.t. } u)\]

where \(h = -f(X, u) + P^T\phi(X, u)\)
Arm pointing motions

Here: \( h = -\dot{x}|u| + g(X, P, u) \) where \( g(X, P, u) \) differentiable w.r.t. \( u \)

\[
\Rightarrow \quad 0 \in \partial_u h = -\dot{x} \partial_u |u^*| + \frac{\partial g}{\partial u}(X, P, u^*)
\]

- \( u^* > 0 \Rightarrow \partial_u |u^*| = 1 \quad \& \quad \frac{\partial g}{\partial u} = \dot{x} \)

- \( u^* < 0 \Rightarrow \partial_u |u^*| = -1 \quad \& \quad \frac{\partial g}{\partial u} = -\dot{x} \)

- \( u^* = 0 \Rightarrow \partial_u |u^*| = [-1, 1] \)

and \( \frac{\partial g}{\partial u} \in [-\dot{x}, \dot{x}] \)

Thus, when the sign of \( u^* \) changes, \( \frac{\partial g}{\partial u} \) passes from \( \dot{x} \) to \( -\dot{x} \) **continuously**

\[\Rightarrow\] inactivation!!
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Choice of the cost

- Integral cost = compromise between $Aw$ and a “comfort term”

$$J(u) = Aw + \int_0^T \tilde{f}(x, y, u) dt.$$  

- For the simulations, $\tilde{f}$ is chosen as the energy of the acceleration:

$$\tilde{f}(X, u) = \sum_{i=1}^{n} \alpha_i (\ddot{x}_i)^2$$

→ simulations do not depend significantly on the parameters $\alpha_i > 0$
Optimal strategies (1 dof)
- for $T$ small, inactivations close to the velocity peak;
- disappearing of the inactivations when $T$ increases.
Asymmetries and inactivations (2 dof)
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Goal-oriented human locomotion

with Y. Chitour, F. Chittaro, and P. Mason (LSS)

Initial point \((x_0, y_0, \theta_0)\)  \rightarrow  Final point \((x_1, y_1, \theta_1)\)

\((x, y \text{ position, } \theta \text{ orientation of the body})\)

QUESTIONS:

Which trajectory is experimentally the most likely?
What criterion is used to choose this trajectory?
Examples of recorded trajectories
(data due to G. Arechavaleta and J-P. Laumond)
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Modelling goal oriented human locomotion

HYPOTHESIS:
the chosen trajectory is solution of a minimization problem

\[
\min \int L(x, y, \theta, \dot{x}, \dot{y}, \dot{\theta}, \ldots) dt
\]
among all “possible” trajectories
joining the initial point to the final one.

→ TWO QUESTIONS:

- What are the possible trajectories? dynamical constraints?
- How to choose the criterion? (inverse optimal control problem)


**Dynamical Constraints**

\[ \text{[Arechavaleta-Laumond-Hicheur-Berthoz, 2006] (=[\text{ALHB}])} \]

\[ \downarrow \]

1st experimental observation: **if target far enough**, the velocity is perpendicular to the body

\[ \dot{x} \sin \theta - \dot{y} \cos \theta = 0 \]  

nonholonomic constraint!

\[ (\text{Dubins}) \rightarrow \begin{cases} 
\dot{x} = v \cos \theta \\
\dot{y} = v \sin \theta 
\end{cases} \quad v = \text{tangential velocity}. \]
Dynamical Model

- 2nd experimental observation in [ALHB]:
  the velocity has a positive lower bound, \( v \geq a > 0 \),
  and is a function (almost constant) of the curvature
  \( \Rightarrow \) The trajectories may be parameterized by arc-length
  and the problem is geometric (the curves do not depend on \( \frac{ds}{dt} \))
  \( \Rightarrow \) \( v \equiv 1 \).

- The whole problem is invariant by rototranslations
  \( \Rightarrow \) \( L \) independent of \( (x, y, \theta) \)
  + the initial point can be chosen as \( (x, y, \theta)(0) = 0 \)
Dynamical Model

- Previous observations: $L = L(\dot{\theta}, \ddot{\theta}, \ldots)$
- Trajectories obtained through a minimization procedure
  
  \[ \Rightarrow \text{trajectories in a complete functional space, } \theta \in W^{k,p}, \text{ and} \]
  \[ L = L(\dot{\theta}, \ldots, \theta^{(k)}) \text{ convex w.r.t. } \theta^{(k)} \]

\[ \rightarrow \text{the trajectories are solutions of the optimal control problem} \]

\[ \min C_L(u) = \int_0^T L(\dot{\theta}, \ldots, \theta^{(k)}) dt \]

among all trajectories of

\[ \begin{aligned}
\dot{x} &= \cos \theta \\
\dot{y} &= \sin \theta \\
\theta^{(k)} &= u
\end{aligned} \quad u \in L^p, \]

s.t. \((x, y, \theta)(0) = 0 \quad \text{and} \quad (x, y, \theta)(T) = (x_1, y_1, \theta_1) := X_1. \]
**The class of admissible costs**

### Definition

\( \mathcal{L}_k \) = set of functions \( L = L(\dot{\theta}, \ldots, \theta^{(k)}) \) such that:

- \( L \) is smooth (at least \( C^2 \))
- 0 is the unique minimum of \( L \) (normalization \( L(0) = 1 \))
- \( L \) is strictly convex w.r.t. \( \theta^{(k)} \)
- \( L(\dot{\theta}, \ldots, \theta^{(k-1)}, u) \geq C|u|^p \) for \( |u| > R \)

Physiologically, \( k \geq 3 \) is not reasonable \( \Rightarrow \) \( k = 1 \) or \( k = 2 \)

\( \rightarrow \) The class of admissible costs is \( \mathcal{L} = \mathcal{L}_1 \cup \mathcal{L}_2 \)
Inverse Optimal Control Problem

Given experimental data, infer a cost function $L \in \mathcal{L}_k$, $k = 1$ or 2, such that the recorded trajectories are optimal solutions of

$$
\begin{align*}
\min C_L(u) &= \int_0^T L(\dot{\theta}, \ldots, \theta^{(k)}) \, dt \\
\text{subject to} \quad &
\begin{cases}
\dot{x} = \cos \theta \\
\dot{y} = \sin \theta \\
\theta^{(k)} = u
\end{cases}
\\
\text{with} \quad & (x, y, \theta)(0) = 0 \quad \text{and} \quad (x, y, \theta)(T) = X_1, \quad T \text{ not fixed.}
\end{align*}
$$

MAIN QUESTIONS

- Stability of the direct problem
- What is the value of $k$?
  (+ asymptotic analysis)
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Analysis of $P_k(L)$ – General Results

Proposition

- For every target $X_1$, there exists an optimal trajectory of $P_k(L)$.
- Every optimal trajectory satisfies the Pontryagin Maximum Principle.

REMARKS:

- The control system is controllable
- The proof of existence uses standard arguments (cf. Lee & Markus)
- The optimal control does not belong \textit{a priori} to $L^\infty([0,T])$. But:

Lemma

For $(x_1, y_1)$ far away from 0, the optimal control is uniformly bounded.

→ not necessary to put an a priori bound on the control
**CASE** \( L = L(\dot{\theta}) \in \mathcal{L}_1 \)

\[
H = p_1 \cos \theta + p_2 \sin \theta + p_3 \dot{\theta} - \nu L(\dot{\theta}) \equiv 0
\]

- No abnormal extremals \( \rightarrow \) optimal traj. are \( C^\infty \) \( (\nu = 1) \)
- **Adjoint Equation**: \((p_1, p_2)\) are constant and
  using \( \frac{\partial H}{\partial u} = 0 \), the adjoint equation writes as an ODE:
  \[
  \ddot{\theta} = G_L(\theta, \dot{\theta}; (p_1, p_2)), \quad \theta(0) = 0
  \]

**CASE** \( L = L(\dot{\theta}, \ddot{\theta}) \in \mathcal{L}_2 \)

\[
H = p_1 \cos \theta + p_2 \sin \theta + p_3 \dot{\theta} + p_4 \ddot{\theta} - \nu L(\dot{\theta}, \ddot{\theta}) \equiv 0
\]

- No abnormal extremals \( \rightarrow \) optimal traj. are \( C^\infty \) \( (\nu = 1) \)
- **Adjoint Equation**: \((p_1, p_2)\) are constant and
  \[
  \theta^{(4)} = F_L(\theta, \dot{\theta}, \ddot{\theta}, \theta^{(3)}; (p_1, p_2)),
  \]
  with initial data: \((\theta, \theta^{(3)})(0) = (0, 0)\) \[transversality condition\]
Stability results

We say that $L_\varepsilon \in \mathcal{L}$ converges to $L_0 \in \mathcal{L}$ if:

- $|L_\varepsilon(\dot{\theta}, \ddot{\theta}) - L_0(\dot{\theta}, \ddot{\theta})| \text{ or } |L_\varepsilon(\dot{\theta}, \ddot{\theta}) - L_0(\dot{\theta})| \leq C_\varepsilon |\ddot{\theta}|^p$ when $L_\varepsilon \in \mathcal{L}_2$,
- $|L_\varepsilon(\dot{\theta}) - L_0(\dot{\theta})| \leq C_\varepsilon |\dot{\theta}|^p$ when $L_0$ and $L_\varepsilon \in \mathcal{L}_1$.

Notation: $\mathcal{T}(L, X_1) = \text{the set of trajectories } (x, y, \theta, \dot{\theta}) \text{ s.t.}$

- $(x, y, \theta)$ is optimal for $P_1(L)$ with final point $X_1$ if $L \in \mathcal{L}_1$;
- $(x, y, \theta, \dot{\theta})$ is optimal for $P_2(L)$ with final point $X_1$ if $L \in \mathcal{L}_2$.

**Theorem**

If $X_1^\varepsilon \rightarrow X_1$, $L_\varepsilon$ converges to $L_0$, and $(x_\varepsilon, y_\varepsilon, \theta_\varepsilon, \dot{\theta}_\varepsilon) \in \mathcal{T}(L_\varepsilon, X_1^\varepsilon)$, then

$$d_{\text{unif}}((x_\varepsilon, y_\varepsilon, \theta_\varepsilon, \dot{\theta}_\varepsilon), \mathcal{T}(L_0, X_1)) \rightarrow 0$$

(+ uniform convergence of $\ddot{\theta}_\varepsilon$ and $\theta_\varepsilon^{(3)}$ if $L_\varepsilon \in \mathcal{L}_2$ under add. hypothesis).
Stability results

→ Optimal trajectories + adjoint equations are stable under perturbations of the cost.

Consequences
- Stability of the direct problem
- Our modelling is compatible with the physiology
- A solution “up to perturbations” is sufficient

Question
Can we choose, up to perturbations, a cost in $\mathcal{L}_1$? (i.e. a cost that depends only on the curvature)
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Remark on the problem $P_1(L)$

If $L \in \mathcal{L}_1$, optimal solutions parameterized by $(p_1, p_2, \dot{\theta}(0))$ and:

$$H = p_1 \cos \theta + p_2 \sin \theta + \dot{\theta} L'(\dot{\theta}) - L(\dot{\theta}) \equiv 0$$

If $\dot{\theta}(t_0) = 0$, then $p_1 \cos \theta(t_0) + p_2 \sin \theta(t_0) = 1$

$\Rightarrow$ all minimizers deduced from the ones parameterized by $(1, p_2, 0)$

$\rightarrow$ one-parameter family of curves (up to rotation + translation)

- For $L \in \mathcal{L}_k$, $\mathcal{M}(L) = \{\text{minimizers } (x, y, \theta) \text{ of } P_k(L) \text{ s.t. } \dot{\theta} = 0 \text{ once}\}$
- Given $t$, define the transformation $\Phi_t : \mathcal{M}(L) \rightarrow \mathbb{R}^3$ by

$$\Phi_t(x, y, \theta) = (\bar{x}(t), \bar{y}(t), \bar{\theta}(t))$$

(only depends on $(x(t_0), y(t_0), \theta(t_0))$ where $t_0$ is such that $\dot{\theta}(t_0) = 0$)

Proposition

If $L \in \mathcal{L}_1$, for every fixed $t$, the set $\Phi_t(\mathcal{M}(L))$ is a curve in $\mathbb{R}^3$. 
Numerical test

- Numerical test: apply the transformation $\Phi_t$ to the recorded curves. Does it give a curve?

**YES!**

($\Phi_t$ applied at different times to $\sim 200$ recorded trajectories)
Validity of the test

Consider a cost $L \in \mathcal{L}_2$.

**Theorem**

- There exists an open set $\tilde{M}$ of minimizers $(x, y, \theta)$ of $P_2(L)$ s.t. $\dot{\theta} = 0$ at some time ($\tilde{M}$ close to the straight line).
- For every $t > 0$, the map $\Phi_t$ is of rank $\geq 2$ on $\tilde{M}$.

**Idea of the proof:**

- Parameterization of minimizers by initial adjoint vector
- Continuity of the adjoint vector w.r.t. the goal
  ($\leftarrow$ stability results)
Conclusion

Models with $L = L(\dot{\theta})$ should be sufficient to describe human locomotion.
References


