Characterization of quadratic growth for strong minima in the optimal control of semi-linear equations

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Outline

Strong and weak minima in calculus of variation

- Let $\ell : [a,b] \times \mathbb{R}^n \times \mathbb{R}^n$ of class C^1, and consider the problem of minimizing the functional (say with fixed end-points):

$$\min_{y(\cdot)\in C^1} J(y) := \int_a^b \ell(t, y(t), \dot{y}(t))dt$$

Definition

We say that y_0 is a local weak minimum *iff $J(y) \geq J(y_0)$ whenever $\|y - y_0\|_{C^1([a,b])} \leq \varepsilon$ and that y_0 is a local* strong minimum *iff $J(y) \geq J(y_0)$ whenever $\|y - y_0\|_{C^0([a,b])} \leq \varepsilon$*

- Example: $J(y) := \int_0^1 [\dot{y}^2 - \dot{y}^4]dt$: $\bar{y} = 0$ is a weak minimum and not a strong minimum.
- Necessary condition: If $y_0 \in C^2([a,b])$ is a weak minimum, then it satisfies Euler-Lagrange equation and Legendre condition.
- Known characterization of strong optimality with quadratic growth (QG): "Weierstrass+QG" + second order optimality condition + QG

State equation

Let $\Omega \subseteq \mathbb{R}^n$ bounded with $C^{1,1}$ boundary and $\varphi : \Omega \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be C^1

(H1) The function φ satisfies:

(i) $D_{(y,u)}\varphi(x,0,0)$ is bounded,

(ii) $D_{(y,u)}\varphi(x,\cdot,\cdot)$ is locally Lipschitz uniformly. on $x \in \Omega$.

(iii) We have $\varphi_y(x,y,u) \geq 0$.

Proposition

Under $\mathbf{(H1)}$, *for every* $u \in L^\infty(\Omega)$ *and* $s \in (\frac{n}{2}, \infty)$, *the equation*

$$\begin{cases} -\Delta y(x) + \varphi(x, y(x), u(x)) & = 0 \qquad \text{a.e. x in } \Omega, \\ y(x) & = 0 \qquad \text{a.e. x in } \partial\Omega, \end{cases}$$

has a unique solution $y_u \in W_0^{1,s}(\Omega) \cap C(\Omega)$. *Moreover, if* $\mathcal{K} \subset L^\infty$ *is a bounded set,* $\exists C_s > 0$ *such that*

$$||y_u||_\infty + ||y_u||_{1,s} \leq C_s, \quad \textit{for all } u \in L^\infty(\Omega) \cap \mathcal{K}.$$

Optimal control problem

Let $\ell : \Omega \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ and suppose that:

(H2) ℓ satisfies **(H1)** except for (iii).

Define the *cost function* $J : L^\infty(\Omega) \to \mathbb{R}$ by

$$J(u) := \int_\Omega \ell(x, y_u(x), u(x))\mathrm{d}x.$$

For $a, b \in C(\Omega)$ with $a \leq b$ define

$$\mathcal{K} = \{u \in L^\infty(\Omega) \mid a(x) \leq u(x) \leq b(x), \quad \text{a.e. x in } \Omega\}$$

Consider the optimal control problem

$$\min J(u) \quad \text{subject to } u \in \mathcal{K}. \tag{$\mathcal{CP}$}$$

Optimal heat source with distributed control

- Let $\Omega \subset \mathbb{R}^3$ heated by electromagnetic induction or by microwaves.
- Assume that the boundary temperature vanishes.
- The optimal control problem becomes (with $N > 0$):

$$\min J(u) := \frac{1}{2} \int_\Omega |y(x) - y_d(x)|^2 + \frac{N}{2} \int_\Omega |u(x)|^2 dx$$

subject to

$$\begin{cases} -\Delta y(x) &= \beta(x) u(x) \quad \text{a.e. x in } \Omega, \\ y(x) &= 0 \qquad\qquad\quad \text{a.e. x in } \partial\Omega, \end{cases}$$

and $u \in \mathcal{K}$, i.e.:

$$a(x) \leq u(x) \leq b(x), \quad \text{a.e. x in } \Omega.$$

Weak and strong minimum

Definition

(i) We say that $\bar{u} \in \mathcal{K}$ is a strong minimum if $\exists \varepsilon > 0$

$$J(u) \geq J(\bar{u}) \quad \text{for all } u \in \mathcal{K} \text{ with } \|y_u - y_{\bar{u}}\|_\infty \leq \varepsilon$$

(ii) For $s \in (1, \infty)$, $\bar{u} \in \mathcal{K}$ is a L^s-weak minimum if $\exists \varepsilon > 0$

$$J(u) \geq J(\bar{u}) \quad \text{for all } u \in \mathcal{K} \text{ with } \|u - \bar{u}\|_s \leq \varepsilon$$

(iii) We say that $\bar{u} \in \mathcal{K}$ is a weak minimum if $\exists \varepsilon > 0$

$$J(u) \geq J(\bar{u}) \quad \text{for all } u \in \mathcal{K} \text{ with } \|u - \bar{u}\|_\infty \leq \varepsilon$$

Remark:

- We have (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii). For proving (i) $\Rightarrow$ (ii), we use $\|y_u - y_{\bar{u}}\|_\infty = O(\|u - \bar{u}\|_s)$
- Since $\mathcal{K} \subseteq L^\infty(\Omega)$ is bounded, $L^p \cap \mathcal{K}$ and $L^q \cap \mathcal{K}$ have the same open sets. Thus, every L^s-weak minimum is a L^1-weak minimum.

Weak and strong minimum with quadratic growth

Definition

(i) $\bar{u} \in \mathcal{K}$ is a strong minimum with quadratic growth if $\exists\ \alpha, \varepsilon > 0$

$$J(u) \geq J(\bar{u}) + \alpha\|u - \bar{u}\|_2^2 \quad \text{for all } u \in \mathcal{K} \text{ with } \|y_u - y_{\bar{u}}\|_\infty \leq \varepsilon$$

(ii) For $s \in (1, \infty)$, $\bar{u} \in \mathcal{K}$ is a L^s-weak minimum with quadratic growth if $\exists\ \alpha, \varepsilon > 0$

$$J(u) \geq J(\bar{u}) + \alpha\|u - \bar{u}\|_2^2 \quad \text{for all } u \in \mathcal{K} \text{ with } \|u - \bar{u}\|_s \leq \varepsilon$$

(iii) $\bar{u} \in \mathcal{K}$ is a weak minimum with quadratic growth if $\exists\ \alpha, \varepsilon > 0$

$$J(u) \geq J(\bar{u}) + \alpha\|u - \bar{u}\|_2^2 \quad \text{for all } u \in \mathcal{K} \text{ with } \|u - \bar{u}\|_\infty \leq \varepsilon$$

We have an analogous remark to the previous one.

Outline

Adjoint system

The Hamiltonian $H : \Omega \times \mathbb{R}^3 \to \mathbb{R}$ for $(\mathcal{CP})$ is

$$H(x, y, p, u) = \ell(x, y, u) - p\varphi(x, y, u).$$

Set $\bar{y} := y_{\bar{u}}$. The *adjoint state* $\bar{p}$, associated to $\bar{u}$, is the unique solution of the linear equation:

$$\begin{cases} -\Delta \bar{p} &= H_y(x, \bar{y}, \bar{p}, \bar{u}) \quad \text{in } \Omega, \\ \bar{p} &= 0 \qquad\qquad\qquad \text{on } \partial\Omega. \end{cases}$$

For a nominal $\bar{u} \in \mathcal{K}$:

- Set $\ell(x) := \ell(x, \bar{y}(x), \bar{u}(x))$. Similar conventions for $\varphi(x)$, $H(x)$.
- Given $u \in \mathcal{K}$, set $\delta u := u - \bar{u}$, $\delta y := y_u - \bar{y}$, and

$$\delta H(x) := H(x, \bar{y}(x), \bar{p}(x), u(x)) - H(x, \bar{y}(x), \bar{p}(x), \bar{u}(x)).$$

with similar conventions for $\delta\ell$, $\delta\varphi$ and its derivatives.

First order estimates

The first order *Pontryagin linearization* $z_1[u]$ of $u \to y_u$ in the direction $u - \bar{u}$ is the unique solution of

$$\begin{cases} -\Delta z_1 + \varphi_y(x) z_1 + \delta\varphi(x) & = 0 \ \text{ in } \Omega, \\ z_1 & = 0 \ \text{ on } \partial\Omega. \end{cases}$$

Lemma

Under assumptions (H1)-(H2), for every $s \in (n/2, +\infty)$, we have:

$$\begin{cases} \|\delta y\|_1 = O(\|\delta u\|_1), \ \ \|\delta y\|_2 = O(\|\delta u\|_2), \ \ \|\delta y\|_\infty = O(\|\delta u\|_s), \\ \|z_1\|_1 = O(\|\delta u\|_1), \ \ \|z_1\|_2 = O(\|\delta u\|_2), \ \ \|z_1\|_\infty = O(\|\delta u\|_s), \\ \|z_1 - \delta y\|_1 = O(\|\delta u\|_1 \|\delta u\|_s), \ \ \|z_1 - \delta y\|_2 = O(\|\delta y\|_\infty \|\delta u\|_2). \end{cases}$$

Idea of proof. Use standard regularity results for elliptic equations: if $\alpha \geq 0$ and z satisfies $-\Delta z + \alpha(x) z = f, \text{in } \Omega$ with Dirichlet condition, then $\|z\|_{2,s} \leq c_s \|f\|_s$, $\|z\|_1 \leq c_1 \|f\|_1$.

Pontryagin maximum principle

Lemma

Under **(H1)-(H2)**, *for all $u \in \mathcal{K}$ and $s \in (n/2, \infty)$ we have*

$$\begin{aligned} J(u) - J(\bar{u}) &= \int_\Omega \delta H(x)\mathrm{d}x + O(\|\delta y\|_\infty \|\delta u\|_2), \\ J(u) - J(\bar{u}) &= \int_\Omega \delta H(x)\mathrm{d}x + O(\|\delta u\|_1 \|\delta u\|_s). \end{aligned}$$

Theorem [Raitum '86, BonCas '89]

Let $\overline{u}$ a L^1-weak minimum of J. Then we have:

$$\overline{u}(x) \in \mathrm{argmin}_{v \in [a(x), b(x)]} H(x, \overline{y}(x), \overline{p}(x), v) \quad \text{a.e. } x \in \text{in } \Omega.$$

Idea of proof: Combine lemma and a needle perturbation (Pontryagin-McShane perturbation):

$$u_\varepsilon(x) = \begin{cases} v, & x \in B(x, \varepsilon), \\ \overline{u}(x), & x \in \Omega \backslash B(x, \varepsilon), \end{cases}$$

One has: $\|u_\varepsilon - \overline{u}\|_1 = O(\varepsilon)$.

Outline

Standard second order conditions for semi-linear problems

We suppose

(H3) For $\psi = \varphi, \ell$, we have $\psi(x, \cdot, \cdot)$ is C^2, $D^2_{(y,u)^2}\psi(x, 0, 0)$ is bounded, and $D^2_{(y,u)^2}\psi(x, \cdot, \cdot)$ is locally Lipschitz uniformly on $x \in \Omega$.

For $v \in L^2(\Omega)$, define "weak" linearization $\zeta[v]$ by

$$\begin{aligned} -\Delta\zeta + \varphi_y(x)\zeta + \varphi_u(x)v &= 0, \quad \text{in } \Omega, \\ \zeta &= 0, \quad \text{on } \partial\Omega. \end{aligned}$$

and the quadratic form $Q_2[\bar{u}] : L^2(\Omega) \to \mathbb{R}$ as

$$Q_2[\bar{u}](v) = \int_\Omega \left[H_{yy}(x)(\zeta[v])^2 + 2H_{yu}(x)\zeta[v]v + H_{uu}(x)v^2 \right] \mathrm{d}x.$$

The tangent cone to $\mathcal{K}$ at $\bar{u}$ is

$$\begin{aligned} T_{\mathcal{K}}(\bar{u}) \quad := \big\{ v \in L^2(\Omega) \; ; \; & v(x) \geq 0 \;\text{ if }\; \bar{u}(x) = a(x), \\ & v(x) \leq 0 \;\text{ if }\; \bar{u}(x) = b(x) \big\}. \end{aligned}$$

Second order necessary conditions for semi-linear problems

The critical cone to $\mathcal{K}$ at $\bar{u}$ is

$$C_{\mathcal{K}}(\bar{u}) := \{v \in T_{\mathcal{K}}(\bar{u}) \; ; \; H_u(x)v(x) = 0 \quad \text{a.e. in } \Omega\} \, .$$

We have the following standard necessary conditions (Bonnans '98, Casas-Tröltzsch-Unger '96)

Proposition

Under **(H1)-(H3)**, *If $\bar{u}$ is a weak minimum, then:*

(i) $H_u(x)v(x) \geq 0$ *a.e. in* Ω, *for all* $v \in T_{\mathcal{K}}(\bar{u})$.
(ii) $Q_2[\bar{u}](v) \geq 0$ *for all* $v \in C_{\mathcal{K}}(\bar{u})$.

and (Bonnans '98)

Proposition

Under **(H1)-(H3)**, *If $\bar{u}$ is a weak minimum with quadratic growth, then:*

(i) $H_u(x)v(x) \geq 0$ *a.e. in* Ω, *for all* $v \in T_{\mathcal{K}}(\bar{u})$.
(ii) $Q_2[\bar{u}](v) \geq \alpha \|v\|_2$ *for all* $v \in C_{\mathcal{K}}(\bar{u})$.

Sufficient conditions for semi-linear problems

Definition

Given a Hilbert space H, a quadratic form $Q : H \to \mathbb{R}$ is a Legendre form if it is sequentially w.l.s.c. and if h_k converges weakly to h in H and $Q(h_k) \to Q(h)$ then h_k conv. strongly to h.

Example: $\tilde{Q}$ weakly continuous quadratic form $\Longrightarrow x \longmapsto \|x\|^2 + m\tilde{Q}(x)$ Legendre form (Bonnans&Shapiro)

We have the following standard sufficient condition for weak quadratic growth (Bonnans '98)

Theorem

Suppose **(H1)-(H3)** *and that :*

(i) $H_u(x)v(x) \geq 0$ *a.e. in* Ω*, for all* $v \in T_{\mathcal{K}}(\bar{u})$.

(ii) $Q_2[\bar{u}](v) \geq \alpha\|v\|_2$ *for all* $v \in C_{\mathcal{K}}(\bar{u})$.

(iii) *The quadratic for* $Q_2[\bar{u}]$ *is a Legendre form.*

Then $\bar{u}$ *is a weak minimum with quadratic growth.*

For $\tau > 0$ define the *strongly active set*

$$A^\tau(\bar{u}) := \{x \in \Omega \; ; \; |H_u(x)| > \tau\},$$

and the *τ-critical cone*

$$C_\mathcal{K}^\tau(\bar{u}) := \{v \in T_\mathcal{K}(\bar{u}) \; ; \; v(x) = 0 \text{ for } x \in A^\tau(\bar{u})\}.$$

Evidently $C_\mathcal{K}(\bar{u}) \subsetneq C_\mathcal{K}^\tau(\bar{u})$. We have (Casas-Tröltzsch-Unger '96)

Theorem

Suppose **(H1)-(H3)** *and that :*

(i) $H_u(x)v(x) \geq 0$ *a.e. in* Ω, *for all* $v \in T_\mathcal{K}(\bar{u})$.

(ii) *There exists* τ *such that* $Q_2[\bar{u}](v) \geq \alpha \|v\|_2$ *for all* $v \in C_\mathcal{K}^\tau(\bar{u})$.

Then $\bar{u}$ is a weak minimum with quadratic growth.

Outline

A decomposition result for the second order expansion of the cost

Let $u_k \in \mathcal{K}$, set $\delta_k u = u_k - \overline{u}$ and suppose that $\|\delta_k u\|_2 \to 0$. Define $y_k := y_{u_k}$ and a sequence of measurable sets $A_k \subset \Omega$ and $B_k \subset \Omega$ such that

$$|A_k \cup B_k| = |\Omega|, \ \ |A_k \cap B_k| = 0 \ \text{ and } |B_k| \to 0.$$

Decompose u_k into u_{A_k} and u_{B_k} defined by :

$$\begin{cases} u_{A_k} = u_k \quad \text{on } A_k, \quad u_{B_k} = \bar{u} \quad \text{on } A_k, \\ u_{A_k} = \bar{u} \quad \text{on } B_k, \quad u_{B_k} = u_k \quad \text{on } B_k. \end{cases}$$

We set

$$\delta_{A_k} u := u_{A_k} - \overline{u}, \qquad \delta_{B_k} u := u_{B_k} - \overline{u} \quad \text{and hence} \quad \delta_k u = \delta_{A_k} u + \delta_{B_k} u.$$

A decomposition result for the second order expansion of the cost

- Let $\delta H^k(x) := H(x, \bar{y}(x), \bar{p}(x), u_k(x)) - H(x, \bar{y}(x), \bar{p}(x), \overline{u}(x))$.
- We have the following decomposition result

Theorem

Under **(H1)-(H3)** *we have that as* $\|\delta_k u\|_2 \to 0$, $\|\delta_{A_k} u\|_\infty \to 0$

$$\begin{aligned} J(u_k) - J(\overline{u}) &= \int_{B_k} \delta H^k(x)\mathrm{d}x + \int_{A_k} H_u(x)\delta_{A_k} u(x)\mathrm{d}x + \tfrac{1}{2} Q_2[\overline{u}](\delta_{A_k} u) + o(\|\delta_k u\|_2^2) \\ &= J(u_{B_k}) - J(\overline{u}) + J(u_{A_k}) - J(\overline{u}) + o(\|\delta_k u\|_2^2). \end{aligned}$$

Proof based on fundamental estimates from the L^s- theory for linear elliptic equations.

Outline

Strict Pontryagin condition

Set $\mathcal{K}(x) := [a(x), b(x)]$, assume that $\min(b - a) > 0$.

Definition

(i) We say that $\bar{u}$ satisfies the strict PMP condition if for all $x \in \Omega$,

$$H(x, \overline{y}(x), \overline{p}(x), \bar{u}(x)) < H(x, \overline{y}(x), \overline{p}(x), v) \quad \text{for all } v \in \mathcal{K}(x),\ v \neq \bar{u}(x).$$

(ii) H satisfies the pointwise global quadratic growth property at $\bar{u}$ if $\exists\ \alpha > 0$ such that for all $x \in \Omega$, $v \in \mathcal{K}(x)$, we have

$$H(x, \overline{y}(x), \overline{p}(x), \bar{u}(x)) + \alpha |v - \bar{u}(x)|^2 \leq H(x, \overline{y}(x), \overline{p}(x), v)$$

Lemma

H satisfies the pointwise global quadratic growth property at $\bar{u}$ iff $\bar{u}$ satisfies the strict PMP condition and $Q_2[\bar{u}]v \geq \alpha \|v\|_2^2$ for all $v \in C_{\mathcal{K}}(\bar{u})$.

Extension of the standard result to the strong case

We have the following extension of Bonnans '98.

Theorem

Suppose **(H1)-(H3)** *and that $\bar{u} \in \mathcal{K}$ satisfies :*

(i) *The strict PMP condition.*

(ii) $Q_2[\bar{u}](v) \geq \alpha \|v\|_2$ *for all* $v \in C_{\mathcal{K}}(\bar{u})$.

(iii) *The quadratic for $Q_2[\bar{u}]$ is a Legendre form.*

Then $\bar{u}$ is a strong minimum with quadratic growth.

Idea of the proof: Suppose that exists $u_k \in \mathcal{K}$ with $\|\delta_k y\|_\infty \to 0$ and $J(u_k) - J(\bar{u}) \leq o(||\delta_k u||_2^2)$, then we easily get that $\|\delta_k u\|_2 \to 0$. Choose

$$A_k := \left\{ x \in \Omega \mid |u_k(x) - \bar{u}(x)| \leq \sqrt{\|\delta_k u\|_1} \right\} \quad \text{and} \quad B_k := \Omega \backslash A_k.$$

Verify that $|B_k| \to 0$, set $\sigma_{A_k} := \|\delta_{A_k} u\|_2$, $\sigma_{B_k} := \|\delta_{B_k} u\|_2$. If $\sigma_{A_k} = o(\sigma_{B_k})$ we get a contradiction with the global growth of H. Otherwise, up to subsequence, $\sigma_{B_k} = O(\sigma_{A_k})$ and we can proceed as in Bonnans '98.

Extension of the standard result to the strong case

End of the proof:

- Recall $\sigma_{A_k} := \|\delta_{A_k} u\|_2$, $\sigma_{B_k} := \|\delta_{B_k} u\|_2$, $\|\delta_k u\|_2^2 = \sigma_{A_k}^2 + \sigma_{B_k}^2$.
- Applying decomposition result yields:

$$\int_{B_k} \delta H^k(x)\mathrm{d}x + \int_{A_k} H_u(x)\delta_{A_k} u(x)\mathrm{d}x + \tfrac{1}{2} Q_2[\overline{u}](\delta_{A_k} u) \leq o(\|\delta_k u\|_2^2).$$

1) If $\sigma_{A_k} = o(\sigma_{B_k})$, then, $Q_2[\overline{u}](\delta_{A_k} u) = O(\sigma_{A_k}^2) = o(\sigma_{B_k}^2)$ and the previous inequality implies

$$\sigma_{B_k}^2 \leq o(\sigma_{B_k}^2),$$

which is a contradiction.
2) If $\sigma_{B_k} = o(\sigma_{A_k})$, we can proceed as previously to obtain a contradiction.

Example

- Let f, $y_d \in C(\Omega)$, $g \in C^2(\mathbb{R})$ s.t. $g_y \geq 0$ and g_{yy} local Lipschitz.
- Consider the following data for $(\mathcal{CP})$,

$$\ell(x,y,u) = \frac{1}{2}|u|^2 + \frac{1}{2}|y - y_d(x)|^2, \qquad \varphi(x,y,u) = g(y) + u + f.$$

- We have $H(x,y,p,u) = \frac{1}{2}|u|^2 + \frac{1}{2}|y - y_d(x)|^2 - p(g(y) + u + f)$ and:
  - $Q_2[\bar{u}]$ is a Legendre form: $Q_2[\bar{u}](v) = \|v\|_2^2 + \tilde{Q}(v)$
  - H is strictly convex with respect to u.

- Thus:

(I) For all $v \in T_{\mathcal{K}}(\bar{u})$ we have $H_u(x)v(x) \geq 0$ a.e. in Ω
and
(II) $Q_2[\bar{u}](v) \geq \alpha\|v\|^2$ for all $v \in C_{\mathcal{K}}(\bar{u})$

are a sufficient condition for a strong minimum with quadratic growth.

Extension to the strong case without Legendre condition

Using analogous arguments we have the following extension of Casas-Tröltzsch-Unger '96.

Theorem

Suppose **(H1)-(H3)** *and that* $\bar{u} \in \mathcal{K}$ *satisfies :*

(i) *The strict PMP condition.*

(ii) *There exists* τ *such that* $Q_2[\bar{u}](v) \geq \alpha \|v\|_2$ *for all* $v \in C_{\mathcal{K}}^{\tau}(\bar{u})$

Then $\bar{u}$ *is a strong minimum with quadratic growth.*

Outline

An improved result

Theorem

Suppose **(H1)-(H3)** *and that* $\bar{u} \in \mathcal{K}$
Then $\bar{u}$ *is a strong minimum with quadratic growth iff the two conditions below hold:*

(i) *The Hamiltonian satisfies the global quadratic growth at* $\bar{u}$*.*

(ii) *We have* $Q_2[\bar{u}](v) \geq \alpha \|v\|_2$ *for all* $v \in C_{\mathcal{K}}(\bar{u})$

Idea of the proof: Suppose that exists $u_k \in \mathcal{K}$ with $\|\delta_k y\|_\infty \to 0$ and $J(u_k) - J(\bar{u}) \leq o(||\delta_k u||_2^2)$, then we easily get that $\|\delta_k u\|_2 \to 0$. Let $\varepsilon_k \downarrow 0$, $B_k = B_k^1 \cup B_k^2$ and $A_k := \Omega \setminus B_k$ where

$$B_k^1 := \left\{ x \in \Omega \ : \ |u_k(x) - \bar{u}(x)| \ \geq \sqrt{\|\delta_k u\|_1} \right\},$$
$$B_k^2 := \{ x \in \Omega \ : |H_u(x)| \leq \varepsilon_k \},$$

Set $\sigma_{A_k} := \|\delta_{A_k} u\|_2$, $\sigma_{B_k} := \|\delta_{B_k} u\|_2$. If $\sigma_{A_k} = o(\sigma_{B_k})$ we get a contradiction with the global growth of H. Otherwise,up to subsequence, $\sigma_{B_k} = O(\sigma_{A_k})$. Setting $h_k := \delta_{A_k} u / \sigma_{A_k}$ we get $\alpha \|h_k\|_2^2 \leq Q_2[\bar{u}](h_k) \leq o(1)$ which is impossible.

Outline

Extensions ?

- Finitely many integral constraints (non unique multiplier)

$$G_i(y,u) := \int_\Omega g_i(x, y(x), u(x))\mathrm{d}x \le 0, \quad i = 1, \dots, q.$$

- Mixed state and control constraints

$$g_i(x, y(x), u(x))\mathrm{d}x \le 0, \quad \text{for a.a. } x \in \Omega,\ i = 1, \dots, q.$$

- Parabolic problems
- Sensitivity analysis

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