Wave operators with non-Lipschitz coefficients: energy and observability estimates

Francesco Fanelli
Institut de Mathématiques de Jussieu-Paris Rive Gauche
UNIVERSITÉ PARIS DIDEROT – PARIS 7

JOURNÉE "JEUNES CONTRÔLEURS" 2014
Laboratoire Jacques-Louis Lions
Université Pierre et Marie Curie – Paris 6

Paris – February 13, 2014

Contents of the talk

Wave operators with non-regular coefficients

- The Cauchy problem: energy estimates
  - (i) Operators with non-Lipschitz coefficients
  - (ii) Zygmund condition: a well-posedness result
  - (iii) Sketch of the proof
- The control problem: observability estimates
  - (i) “Classical” observability estimates
  - (ii) Estimates with loss
  - (iii) Remarks and ideas of the proof

THE CAUCHY PROBLEM:

ENERGY ESTIMATES

General setting

$$L\,u \;:=\; \partial_t^2 u \,-\, \sum_{j,k=1}^{N} \partial_j \Big(a_{jk}(t,x)\, \partial_k u \Big)$$

on a strip $[0,T] \times \mathbb{R}^N$, with

$$0 \;<\; \lambda_0\, |\xi|^2 \;\leq\; \sum_{j,k=1}^{N} a_{jk}(t,x)\, \xi_j\, \xi_k \;\leq\; \Lambda_0\, |\xi|^2 \qquad \forall\, \xi \in \mathbb{R}^N \setminus \{0\}$$

▷ **Aim:** studying the Cauchy problem

$$(CP) \qquad \begin{cases} Lu \;=\; f \\ u_{|t=0} \;=\; u_0\,, \quad \partial_t u_{|t=0} \;=\; u_1 \end{cases}$$

in the Sobolev spaces framework

Classical result

Hurd & Sattinger (1968)

$$a_{jk}(t,x) \quad \begin{cases} \text{Lipschitz continuous in } t \\ \text{only bounded with respect to } x \end{cases}$$

$\Longrightarrow$ well-posedness of (CP) in $H^1 \times L^2$

- ▷ More regularity in $x \implies H^s \times H^{s-1}$
- ▷ Key: energy estimate with **no loss of derivatives**

$$\sup_{0\leq t\leq T}\left(\|u(t,\cdot)\|_{H^s} + \|\partial_t u(t,\cdot)\|_{H^{s-1}} \right) \leq$$
$$\leq C_s \left(\|u(0,\cdot)\|_{H^s} + \|\partial_t u(0,\cdot)\|_{H^{s-1}} + \int_0^T \|Lu(t,\cdot)\|_{H^{s-1}}\, dt \right)$$

- De Simon & Torelli (1974): $a_{jk} \in BV_t$
- Counterexamples:
  - ▷ Hurd & Sattinger (1968): discontinuous coefficients
  - ▷ Colombini, De Giorgi & Spagnolo (1979): Hölder coefficients

General idea:

▷ lower regularity assumptions with respect to t

▷ suitable hypothesis on x to compensate it

$\Longrightarrow$ H^∞ well-posedness, but eventually with a

loss of derivatives in the energy estimates

Coefficients depending only on time

- *Integral log-Lipschitz condition*

  Colombini, De Giorgi & Spagnolo (1979)

$$\int_0^{T-\tau} \left|a_{jk}(t+\tau) \,-\, a_{jk}(t)\right| dt \;\leq\; C_0\,\tau\,\log\left(1+\frac{1}{\tau}\right)$$

- *Integral log-Zygmund condition* Tarama (2007)

$$\int_\tau^{T-\tau} \left|a_{jk}(t+\tau)+a_{jk}(t-\tau)-2a_{jk}(t)\right| dt \;\leq\; C_0\,\tau\,\log\left(1+\frac{1}{\tau}\right)$$

Theorem

$$\sup_{0\leq t\leq T}\left(\|u(t,\cdot)\|_{H^{s-\delta}} + \|\partial_t u(t,\cdot)\|_{H^{s-1-\delta}}\right) \leq$$

$$\leq\; C_s\left(\|u(0,\cdot)\|_{H^s} + \|\partial_t u(0,\cdot)\|_{H^{s-1}} + \int_0^T \|Lu(t,\cdot)\|_{H^{s-1-\delta}}\,dt\right)$$

Remarks

- Proof:
  - ▷ Approximation of the coefficients
  - ▷ Fourier transform
  - ▷ Linking dual variable and approximation parameter
- Hölder coefficients $\Longrightarrow$ solutions in Gevrey classes
  (Colombini, De Giorgi & Spagnolo – 1979)
- Counterexample to distributional solutions
  (Colombini, De Giorgi & Spagnolo – 1979)

Coefficients depending on (t, x)

- *Pointwise log-Lipschitz condition* in all the variables

  Colombini & Lerner (1995)

$$\sup_{(t,x)} \left| a_{jk}(t+\tau, x+y) - a_{jk}(t,x) \right| \leq C\,(\tau + |y|)\, \log\left(1 + \frac{1}{\tau + |y|}\right)$$

- *"Log-Zygmund in time & log-Lipschitz in space" condition*

$$(LZ_t)\quad \sup \left| a_{jk}(t+\tau, x) + a_{jk}(t-\tau, x) - 2a_{jk}(t,x) \right| \leq C_0\,\tau\,\log\left(1 + \tfrac{1}{\tau}\right)$$

$$(LL_x)\quad \sup \left| a_{jk}(t, x+y) - a_{jk}(t,x) \right| \leq C_0 |y| \log\left(1 + \tfrac{1}{|y|}\right)$$

▷ scalar case $x \in \mathbb{R}$: Colombini-Del Santo (2009)

▷ general case $x \in \mathbb{R}^N$: Colombini-Del Santo-F.-Métivier (2013)

Theorem

$$\sup_{0\leq t\leq T^*}\left(\|u(t,\cdot)\|_{H^{s-\beta t}}+\|\partial_t u(t,\cdot)\|_{H^{s-1-\beta t}}\right)\leq$$

$$\leq C_s\left(\|u(0,\cdot)\|_{H^s}+\|\partial_t u(0,\cdot)\|_{H^{s-1}}+\int_0^{T^*}\|Lu(t,\cdot)\|_{H^{s-1-\beta t}}\,dt\right)$$

- loss of derivatives *linearly increasing in time*
- $s\in\,]0,1[$
- Local in time estimates: $T^*\leq T$
- β and C_s depend only on L
- In particular

  $a_{jk}\in\mathcal{C}_b^\infty(\mathbb{R}_x^N)\implies H^\infty$ well-posedness, globally in time

Energy estimates with no loss of derivatives

$$Lu(t,x) \;:=\; \partial_t^2 u \,-\, \sum_{j,k=1}^{N} \partial_j\Big(a_{jk}(t,x)\,\partial_k u\Big)$$

Pointwise Zygmund condition in all the variables:

$$\sup_{(t,x)}\Big|a_{jk}(t+\tau,x+y)+a_{jk}(t-\tau,x-y)-2a_{jk}(t,x)\Big| \;\leq\; C\,(\tau+|y|)$$

Theorem (Colombini, Del Santo, F. & Métivier – 2013)

$$\sup_{0\leq t\leq T}\Big(\|u(t)\|_{H^{1/2}}+\|\partial_t u(t)\|_{H^{-1/2}}\Big) \;\leq$$

$$\leq\; C\,e^{\lambda T}\left(\|u(0)\|_{H^{1/2}}+\|\partial_t u(0)\|_{H^{-1/2}}+\int_0^T e^{-\lambda t}\|Lu(t)\|_{H^{-1/2}}dt\right)$$

Remarks

- Original statement for a complete operator

$$Pu \;=\; \partial_t^2 u \;-\; \sum_{j,k=1}^{N} \partial_j\Big(a_{jk}(t,x)\,\partial_k u\Big) \;+\; B(t,x)\cdot\nabla_{(t,x)}u \;+\; c(t,x)u$$

  - ▷ $B \in L^\infty([0,T]\,;\,\mathcal{C}^\theta(\mathbb{R}^N)) \qquad (\,\theta > 1/2\,)$
  - ▷ $c \in L^\infty([0,T]\times\mathbb{R}^N)$

- Global in time estimate
- Well-posedness of (CP) in $H^{1/2}\times H^{-1/2}$
- Well-posedness in H^∞ if $a_{jk} \in Z([0,T]) \cap \mathcal{C}_b^\infty(\mathbb{R}^N)$

Related results

- Tarama (2007): $a_{jk} = a_{jk}(t) \in Z([0,T])$

  $\Longrightarrow$ no loss of derivatives in any $H^s \times H^{s-1}$

- Cicognani & Colombini (2006)

Modulus of continuity		**Loss of derivatives**
Lipschitz	$\rightsquigarrow$	no loss
intermediate	$\rightsquigarrow$	arbitrarily small loss
log-Lipschitz	$\rightsquigarrow$	finite loss $\sim \beta\, t$

Zygmund functions

Definition: $f \in Z(\mathbb{R}^n)$ if $f \in L^\infty(\mathbb{R}^n)$ and

$$\sup_{z \in \mathbb{R}^n} \left| f(z+\zeta) + f(z-\zeta) - 2f(z) \right| \leq K_0 \, |\zeta|$$

Basic properties

- $Lip(\mathbb{R}^n) \hookrightarrow Z(\mathbb{R}^n) \hookrightarrow logLip(\mathbb{R}^n)$
- Condition on second derivatives: if $f \in \mathcal{C}^2(\mathbb{R})$, then

$$\left| f(z+\zeta) + f(z-\zeta) - 2f(z) \right| = |\zeta|^2 \, f''(\phi_{z,\zeta})$$

- $Z(\mathbb{R}^n) \equiv B^1_{\infty,\infty}(\mathbb{R}^n)$, where

$$\|f\|_{B^1_{\infty,\infty}} := \sup_{\nu \in \mathbb{N}} \left(2^\nu \, \|\Delta_\nu f\|_{L^\infty} \right)$$

Regularization in time

$$f \in Z(\mathbb{R}_t)\,, \quad 0 < \lambda_0 \leq f(t) \leq \Lambda_0$$

Approximation by convolution kernel:

$$f_\varepsilon(t) := (\rho_\varepsilon * f)(t) = \frac{1}{\varepsilon}\int_{\mathbb{R}} \rho\left(\frac{\tau}{\varepsilon}\right) f(t-\tau)\,d\tau$$

Then:

$$\begin{aligned} 0 < \lambda_0 \leq & \; f_\varepsilon \leq \Lambda_0 \\ |f_\varepsilon(t) - f(t)| & \leq C\,\varepsilon \\ |\partial_t f_\varepsilon(t)| & \leq C \log\left(1 + \frac{1}{\varepsilon}\right) \\ \left|\partial_t^2 f_\varepsilon(t)\right| & \leq \frac{C}{\varepsilon} \end{aligned}$$

Littlewood-Paley Theory

- *Littlewood-Paley decomposition*

  Dyadic partition of unity in phase-space:

$$\chi_{-1}(\xi) + \sum_{\nu=0}^{+\infty} \psi_\nu(\xi) \equiv 1$$

$$\operatorname{supp}\chi_{-1} \subset \{|\xi| \leq 1\}\,, \quad \operatorname{supp}\psi_\nu \subset \left\{2^{\nu-1} \leq |\xi| \leq 2^{\nu+1}\right\}$$

$\Longrightarrow$ Operators:

$$\Delta_{-1} := \chi_{-1}(D_x)\,, \quad \Delta_\nu := \psi_\nu(D_x)\,, \quad S_\nu := \sum_{j=-1}^{\nu-1} \Delta_j$$

$$\Longrightarrow \quad \forall\, u \in \mathcal{S}'(\mathbb{R}^N)\,, \qquad u = \sum_{\nu=-1}^{+\infty} \Delta_\nu u$$

- Sobolev spaces

$$u \in H^s \qquad \Longleftrightarrow \qquad \left(2^{s\nu}\, \|\Delta_\nu u\|_{L^2}\right)_{\nu\geq -1} \in \ell^2$$

Paradifferential calculus with parameters

Bony's paraproduct operator: $a\,,\,u\;\in\;\mathcal{S}'(\mathbb{R}^N_x)$

$$T_a u\;:=\;\sum_{\nu\geq 1} S_{\nu-1}a\;\Delta_\nu u$$

▷ Regularization in space

$\Longrightarrow$ well defined also if $a(x)\;\rightsquigarrow\;\alpha(t,x,\xi)$, rough in x

▷ Parameter $\gamma\geq 1$ $\leftrightarrow$ starting from high frequencies

$\Longrightarrow$ $\alpha(t,x,\xi)$ positive symbol $\Longrightarrow$ T_α positive operator

▷ Symbolic calculus for Zygmund continuous symbols

Proof of the energy estimate

(i) $a_{jk} \rightsquigarrow a_{jk,\varepsilon} \rightsquigarrow \alpha_\varepsilon(t,x,\xi,\gamma) := \sum_{j,k} a_{jk,\varepsilon}(t,x)\,\xi_j\,\xi_k + \gamma^2$

with $\varepsilon = (\gamma^2 + |\xi|^2)^{-1/2}$

(ii) Approximation of the operator

$$Lu = \partial_t^2 u + \operatorname{Re} T_{\alpha_\varepsilon} u + Ru$$

$R : H^s \longrightarrow H^{s-1}$ for any $s \in]0,1[$

(iii) Energy $E(t) := \|v(t)\|_{L^2}^2 + \|w(t)\|_{L^2}^2$, with

$$v(t,x) := T_{\alpha_\varepsilon^{-1/4}} \partial_t u - T_{\partial_t\left(\alpha_\varepsilon^{-1/4}\right)} u$$

$$w(t,x) := T_{\alpha_\varepsilon^{1/4}} u$$

In particular, $E(t) \sim \|\partial_t u(t)\|^2_{H^{-1/2}} + \|u(t)\|^2_{H^{1/2}}$

(if $\gamma \geq 1$ large enough)

(iv) Differentiation in time $\Longrightarrow$ cancellations

- ▷ Tarama's cancellation (definition of $E(t)$)
- ▷ Paradifferential operator $\mathrm{Re}\, T_{\alpha_\varepsilon}$

(v) Gronwall's inequality to conclude

- Remarks
  - ▷ Cancellations only for $s = 1/2$
  - ▷ $s \neq 1/2$ not clear

THE CONTROL PROBLEM: OBSERVABILITY ESTIMATES

Setting

$N = 1$, coefficient just depending on x

$$\begin{cases} \omega(x)\partial_t^2 u - \partial_x^2 u = 0 & \text{in } [0,1]\times[0,T] \\ u(t,0) = u(t,1) = 0 & \text{in } [0,T] \\ u(0,x) = u_0(x)\,, \quad \partial_t u(0,x) = u_1(x) & \text{in } [0,1] \end{cases}$$

- $0 < \omega_* \leq \omega(x) \leq \omega^*$
- $T > T^*$ (in our case, $T^* \sim \|\sqrt{\omega}\|_{L^1_x(0,1)}$)
- Energy:

$$E(t) := \frac{1}{2}\int_0^1 \left(\omega(x)\,|u_t(t,x)|^2 + |u_x(t,x)|^2\right) dx$$

$\Longrightarrow$ $E(t) \equiv E(0)$ on $[0,T]$

Observability estimates

- *Internal observability*: for any $\Omega :=]l_1, l_2[\subset [0,1]$,

$$E(0) \leq C \int_0^T \int_{l_1}^{l_2} \left(\omega(x)\, |u_t(t,x)|^2 + |u_x(t,x)|^2 \right) dx\, dt$$

- *Boundary observability*: $\Omega = \{0,1\}$ (or a subset),

$$E(0) \leq C \int_0^T \left(|u_x(t,0)|^2 + |u_x(t,1)|^2 \right) dt$$

Remarks:

▷ Observability $\iff$ "Geometric Control Condition" for Ω

(Bardos, Lebeau & Rauch – 1992 ; Burq & Gérard – 1997)

▷ $N \geq 2$:

(i) Microlocal analysis $\Longrightarrow$ $\mathcal{C}^2$ regularity

(ii) Carleman estimates $\Longrightarrow$ $\mathcal{C}^1$ regularity

(Duyckaerts, Zhang, Zuazua – 2008)

Previous results

- ω Lipschiz $\Longrightarrow$ observability estimates:

$$E(0) \leq C \int_0^T |u_x(t,0)|^2 \, dt$$

- Avellaneda, Bardos & Rauch (1992):

$$\omega_\varepsilon(x) := \omega(x/\varepsilon) \quad \Longrightarrow \quad \lim_{\varepsilon \to 0} C_\varepsilon = +\infty$$

- Fernández-Cara & Zuazua (2002):

$$\omega \in BV(0,1) \quad \Longrightarrow \quad \text{observability estimates}$$

- Castro & Zuazua (2003):

$$\omega \in \mathcal{C}^s(0,1) \quad \Longrightarrow \quad \text{NO observability estimates}$$

The Zygmund case

$$\begin{cases} \omega(x)\partial_t^2 u \,-\, \partial_x^2 u \,=\, 0 & \text{in } [0,1]\times[0,T] \\ u(t,0) \,=\, u(t,1) \,=\, 0 & \text{in } [0,T] \\ u(0,x) \,=\, u_0(x)\,, \quad \partial_t u(0,x) \,=\, u_1(x) & \text{in } [0,1] \end{cases}$$

- $0 < \omega_* \leq \omega(x) \leq \omega^*\,, \qquad \omega \in \mathcal{Z}:$

$$\int \left|\omega(x+y) + \omega(x-y) - 2\,\omega(x)\right| dx \leq K\,|y|$$

- $T > T^* := 2\,\|\sqrt{\omega}\|_{L^1_x(0,1)}$

Theorem (F. & Zuazua – 2013)

$$\|u_0\|^2_{H^1_0(\Omega)} + \|u_1\|^2_{L^2(\Omega)} \leq C \int_0^T |\partial_x u(t,0)|^2\, dt$$

Observability estimates with loss

(*LL*) Integral log-Lipschitz condition:

$$\int \left|\omega(x+y) - \omega(x)\right| dx \leq C\,|y|\,\log\left(1+\frac{1}{|y|}\right)$$

(*LZ*) Integral log-Zygmund condition:

$$\int \left|\omega(x+y) + \omega(x-y) - 2\omega(x)\right| dx \leq C\,|y|\,\log\left(1+\frac{1}{|y|}\right)$$

Theorem (F. & Zuazua – 2013)

$$\|u_0\|^2_{H^1_0(\Omega)} + \|u_1\|^2_{L^2(\Omega)} \leq C \int_0^T \left|\partial_t^m \partial_x u(t,0)\right|^2 dt$$

- ▷ Right-hand side finite for smooth enough data

About the first variation

Modulus of continuity – Observability estimates

- ω Lipschitz

  $\Longrightarrow$ no loss

- ω between Lipschitz and log-Lipschitz

  $\Longrightarrow$ arbitrarly small loss $\Longrightarrow$ $m = 1$

- ω log-Lipschitz

  $\Longrightarrow$ finite loss

- ω worse than log-Lipschitz

  $\Longrightarrow$ infinite loss

Proof of observability

Sidewise energy estimates

(i) Sidewise energy:

$$F(x) \;:=\; \frac{1}{2}\int_{(T^*/2)x}^{T-(T^*/2)x} \Big(\omega(x)\,\left|\partial_t u(t,x)\right|^2 \,+\, \left|\partial_x u(t,x)\right|^2\Big)\,dt$$

In particular, $F(0) \;=\; (\omega(0)/2)\,\int_0^T \left|\partial_x u(t,0)\right|^2\,dt$

(ii) Zygmund – log-Zygmund $\rightsquigarrow$ Tarama (2007)

log-Lipschitz $\rightsquigarrow$ Colombini-De Giorgi-Spagnolo (1979)

(thanks to finite propagation speed)

(iii) integration in space

On the counterexamples

Castro & Zuazua (2003):

$$\omega \in \mathcal{C}^s(0,1) \quad \Longrightarrow \quad \text{NO} \quad \text{observability estimates}$$

Proof: counterexample (ideas from Colombini & Spagnolo (1989))

- "Fractal" partition of $[0,1]$
- Construction of the oscillating coefficient:
  - ▷ more and more oscillations for $x \sim 0$
  - ▷ energy decreasing for $x \to 0$
  - ▷ energy exponentially concentrated inside the subintervals

$\Longrightarrow$ energy too "small" at $x = 0$

Construction ok also for $x \sim 1$ and for internal observability

Remarks

- $\mathcal{Z} \equiv B^1_{1,\infty} \quad \Longrightarrow \quad W^{1,1} \hookrightarrow \mathcal{Z}$
  - ▷ Example by Tarama (2007) $\quad \Longrightarrow \quad BV \hookrightarrow \mathcal{Z}$
- Controllability results

$$\begin{cases} \omega(x)\partial_t^2 y - \partial_x^2 y = 0 & \text{in } [0,1]\times[0,T] \\ y(t,0) = f(t)\,, \quad y(t,1) = 0 & \text{in } [0,T] \\ y(0,x) = y_0(x)\,, \quad \partial_t y(0,x) = y_1(x) & \text{in } [0,1] \end{cases}$$

(i) $\omega \in \mathcal{Z} \quad \Longrightarrow \quad f \in L^2(0,T)$

(ii) $\omega \in \mathcal{LZ} - \mathcal{LL} \quad \Longrightarrow \quad f \in H^{-m}(0,T)$

THANK YOU !