Minimal control time for some parabolic systems

Farid Ammar Khodja
with: Assia Benabdallah, M. González-Burgos & L. de Teresa

Paris VI - 17 Janvier 2014

The control system

Let $T > 0$, $\omega = (a, b) \subset (0, \pi)$ and consider the following control problem:

$$\begin{cases} y_t - Dy_{xx} + Ay = Bu1_\omega & \text{in } Q_T := (0,\pi) \times (0,T), \\ y(0,\cdot) = Cv, \quad y(\pi,\cdot) = 0 & \text{on } (0,T), \\ y(\cdot,0) = y^0 & \text{in } (0,\pi). \end{cases}$$

- $D = \operatorname{diag}(d_1, d_2)$, $d_i > 0$,
- $A = (a_{ij})_{1 \le i,j \le 2} \in L^\infty(Q_T, \mathcal{L}(\mathbb{R}^2))$,
- $B = \begin{pmatrix} 0 \\ b \end{pmatrix}$ and $C = \begin{pmatrix} 0 \\ c \end{pmatrix}$ are constant vectors of $\mathbb{R}^2$,
- $u \in L^2(Q_T)$ and $v \in L^2(0,T)$: control functions.

Two control issues

Null-controllability (NC) issue.

$T > 0,\ y^0 \in H\left(= L^2(0,\pi) \text{ or } H^{-1}(0,\pi)\right)$ given:

$$\exists?\,(u, v)\ :\ y\left(T, \cdot; y^0, u, v\right) = 0 \text{ on } (0,\pi).$$

□

Approximate controllability (AC) issue:

$T > 0,\ \left(y^0, y^1\right) \in H \times H$ given:

$$\forall \varepsilon > 0,\ \exists?\,(u, v):\ \left\|y\left(T, \cdot; y^0, u, v\right) - y^1\right\| < \varepsilon$$

□

Known results: the distributed case

It is the case $C = 0$:

$$\begin{cases} \left(\partial_t - d_1\partial_x^2 + a_{11}\right) y_1 + a_{12} y_2 = 0 \\ \\ \left(\partial_t - d_2\partial_x^2 + a_{22}\right) y_2 + a_{21} y_1 = u 1_\omega \end{cases}$$

- $a_{12} \in L^\infty(Q_T)$ and $\operatorname{supp}(a_{12}) \cap \omega \times (0, T) \neq \varnothing$ contains an open subset: NC and AC properties are both satisfied (Benabdallah, Dupaix & FAK (2006), Gonzalez-Burgos & de Teresa (2010)) if $a_{12} \geq \sigma > 0$.

Known results: the distributed case

It is the case $C = 0$:

$$\begin{cases} \left(\partial_t - d_1\partial_x^2 + a_{11}\right) y_1 + a_{12} y_2 = 0 \\ \\ \left(\partial_t - d_2\partial_x^2 + a_{22}\right) y_2 + a_{21} y_1 = u 1_\omega \end{cases}$$

- $a_{12} \in L^\infty(Q_T)$ and $\text{supp}(a_{12}) \cap \omega \times (0, T) \neq \varnothing$ contains an open subset: NC and AC properties are both satisfied (Benabdallah, Dupaix & FAK (2006), Gonzalez-Burgos & de Teresa (2010)) if $a_{12} \geq \sigma > 0$.
- a_{12} is a P.D .O. of order ≤ 2 and $\text{supp}(a_{12}) \cap \omega \times (0, T) \neq \varnothing$ contains an open subset: NC and AC properties are both satisfied: (Guerrero (2007), Benabdallah, Cristofol, Gaitan & deTeresa(2013), Mauffrey(2012)) if a_{12} is "invertible" .

These results are proved in any space dimension.

Known results: the distributed case

Problem: What happens if

$$\operatorname{supp}(a_{12}) \cap \omega \times (0, T) = \varnothing \text{ ?}$$

Very few results.

- The NC property is proved to be satisfied in the following settings:

Known results: the distributed case

Problem: What happens if

$$\operatorname{supp}(a_{12}) \cap \omega \times (0, T) = \varnothing \text{ ?}$$

Very few results.

- The NC property is proved to be satisfied in the following settings:
  - Alabau and Léautaud (2011): $D = Id$ and $A = -\begin{pmatrix} a(x) & q(x) \\ \delta q(x) & a(x) \end{pmatrix}$, $q \geq 0$, $\omega \times (0, T)$ and $\operatorname{supp}(q)$ satisfies the geometrical condition of control provided $\sqrt{\delta} \, \|q\|_{\infty} \leq \delta^{*}$ in any space dimension.

Known results: the distributed case

Problem: What happens if

$$\operatorname{supp}(a_{12}) \cap \omega \times (0, T) = \varnothing \, ?$$

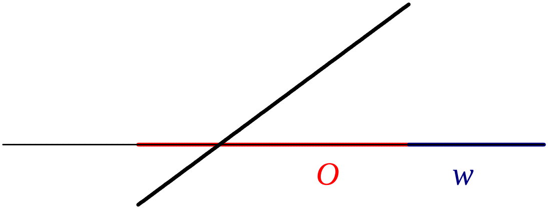
Very few results.

- The NC property is proved to be satisfied in the following settings:
  - Alabau and Léautaud (2011): $D = Id$ and $A = -\begin{pmatrix} a(x) & q(x) \\ \delta q(x) & a(x) \end{pmatrix}$, $q \geq 0$, $\omega \times (0, T)$ and $\operatorname{supp}(q)$ satisfies the geometrical condition of control provided $\sqrt{\delta}\,\|q\|_\infty \leq \delta^*$ in any space dimension.
  - Rosier and de Teresa (2011): $D = Id$ and $A = -\begin{pmatrix} 0 & 1_{\mathcal{O}} \\ 0 & 0 \end{pmatrix}$ with $\omega \cap \mathcal{O} = \varnothing$ in the one-dimensional case.

Known results: the distributed case

The AC property is proved to be satisfied in the following settings:

- Boyer and Olive (2013) : $D = Id$ and $A = -\begin{pmatrix} 0 & q \\ 0 & 0 \end{pmatrix}$ and $\int_0^{\pi} q(x) \sin^2(kx)\, dx \neq 0$ for all $k \geq 1$. The converse is true if $\mathrm{supp}(q) = \overline{\mathcal{O}}$ is contained in a connected component of $\overline{(0,\pi)\backslash\omega}$ that touches the boundary of $(0, \pi)$.



Known results: the boundary case

$$\begin{cases} \left(\partial_t - d_1\partial_x^2 + a_{11}\right) y_1 + a_{12} y_2 = 0 \\ \left(\partial_t - d_2\partial_x^2 + a_{22}\right) y_2 + a_{21} y_1 = 0 \\ y_1 = 0,\ x \in \{0, \pi\}\,;\ y_{2|x=0} = v,\quad y_{2|x=\pi} = 0 \end{cases}$$

- Widely open, even in the one-dimensional case...

Known results: the boundary case

$$\begin{cases} \left(\partial_t - d_1\partial_x^2 + a_{11}\right) y_1 + a_{12} y_2 = 0 \\ \left(\partial_t - d_2\partial_x^2 + a_{22}\right) y_2 + a_{21} y_1 = 0 \\ y_1 = 0,\ x \in \{0,\pi\}\,;\ y_{2|x=0} = v, \quad y_{2|x=\pi} = 0 \end{cases}$$

- Widely open, even in the one-dimensional case...
- Complete solution in the one-dimensional case when the coefficients a_{ij} are constant.

Let $T > 0$, $\omega = (a, b) \subset (0, \pi)$ and:

$$\begin{cases} y_t - y_{xx} = Ay + Bu1_\omega & \text{in } Q_T := (0,\pi) \times (0,T), \\ y(0,\cdot) = y(\pi,\cdot) = 0 & \text{on } (0,T), \\ y(\cdot,0) = y^0 & \text{in } (0,\pi). \end{cases}$$

where $A_0 = \begin{pmatrix} 0 & q(x) \\ 0 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, $q \in L^\infty(0,\pi)$.

Theorem (Boyer and Olive (2013))

Assume that supp$(q) \subset (0, a)$ *or supp*$(q) \subset (b, \pi)$. *Then*

$$(AC) \Leftrightarrow \left(I_k(q) = \frac{2}{\pi} \int_0^\pi q(x) \sin^2(kx)\, dx \neq 0, \forall k \geq 1 \right).$$

We have:

Theorem

Assume $I_k(q) \neq 0$ for all $k \geq 1$ and let:

$$T_0(q) = T_0 := \limsup_{k \to +\infty} \frac{\log \frac{1}{|I_k(q)|}}{k^2} \in [0, \infty]$$

Then,

1. *If $T > T_0$, the system is null-controllable at time T.*
2. *If $supp(q) \subset (0, a)$ or $supp(q) \subset (b, \pi)$, for any $T < T_0$, the system is not null-controllable at time T.*

Question:

Does there exist q such that $T_0(q) > 0$? □

- Actually:

Question:

Does there exist q such that $T_0(q) > 0$? □

- Actually:

Theorem

For any $\tau \in [0, +\infty]$, *there exists* $q \in L^\infty(0, \pi)$ *such that* $T_0(q) = \tau$.

Question:

Does there exist q such that $T_0(q) > 0$? □

- Actually:

Theorem

For any $\tau \in [0, +\infty]$, there exists $q \in L^\infty(0, \pi)$ such that $T_0(q) = \tau$.

- Note that if $\int_0^\pi q(x)\, dx \neq 0$ then $T_0(q) = 0$. In particular, if $q = 1_{(\alpha,\beta)}$ the previous result recovers the one of Rosier and de Teresa.

Boundary controllability: Results

Consider the following "simple" situation:

$$\begin{cases} y_t = Dy_{xx} + Ay & \text{in } Q_T := (0,\pi) \times (0,T), \\ y(0,\cdot) = C v, \quad y(\pi,\cdot) = 0 & \text{on } (0,T), \\ y(\cdot,0) = y^0 & \text{in } (0,\pi). \end{cases}$$

$$D = \mathrm{diag}\,(1,d)\,, d > 0, \quad A = (a_{ij})_{1 \le i,j \le 2} \in \mathbb{M}_2(\mathbb{R}),$$
$$C = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \in \mathbb{R}^2,\ v \in L^2(0,T)\,.$$

Boundary controllability: Results

Let $A_k^* := -k^2 D + A^*$, $\sigma\left(A_k^*\right) = \left\{-\lambda_{k,1}, -\lambda_{k,2}\right\}$

Theorem (Approximate controllability)

Assume that $d \neq 1$. Then the system is approximately controllable at any $T > 0$ if, and only if:

$$\lambda_{k,i} \neq \lambda_{l,j} \;\; \forall \left(k,i\right) \neq \left(l,j\right), \tag{1}$$

$$\left(\begin{array}{c} \left(-k^2 D + A^*\right) V = \lambda V \\ V \neq 0 \end{array} \Rightarrow C^* D V \neq 0 \right), \;\; \forall k \geq 1 \tag{2}$$

Boundary controllability: Results

Theorem (Null-controllability)

Assume that $d \neq 1$ and that the system is approximately controllable. Then there exists $T_0 = T_0\left(d\right) \geq 0$ such that:

1. *The system is null-controllable if $T > T_0$.*
2. *If $T_0 > 0$, the system is not null-controllable if $T < T_0$.*

The Fattorini-Russell method

- Let φ be a solution to the adjoint problem :

$$\begin{cases} -\dfrac{\partial \varphi}{\partial t} = \left(D \dfrac{\partial^2}{\partial x^2} + A^* \right) \varphi, & \text{in } Q, \\ \varphi(0, \cdot) = \varphi(\pi, \cdot) = 0 & \text{on } (0, T), \\ \varphi(\cdot, T) = \varphi^0 \ \in \mathbb{H}_0^1 (0, \pi) . \end{cases}$$

The Fattorini-Russell method

- Let φ be a solution to the adjoint problem :

$$\begin{cases} -\dfrac{\partial \varphi}{\partial t} = \left(D\dfrac{\partial^2}{\partial x^2} + A^* \right) \varphi, & \text{in } Q, \\ \varphi(0,\cdot) = \varphi(\pi,\cdot) = 0 & \text{on } (0,T), \\ \varphi(\cdot,T) = \varphi^0 \ \in \mathbb{H}_0^1(0,\pi). \end{cases}$$

- If y is a solution of the system associated with $y^0 \in \mathbb{H}^{-1}(0,\pi)$ and $v \in L^2(0,T)$, then:

$$\left\langle y(T), \varphi^0 \right\rangle_{H^{-1},H_0^1} - \left\langle y^0, \varphi(0) \right\rangle_{H^{-1},H_0^1} = \int_0^T v.B^* D\frac{\partial \varphi}{\partial x}(t,0)$$

Thus $y(T) = 0$ iff $\exists v$ s.t.:

$$\int_0^T v.C^* D\frac{\partial \varphi}{\partial x} \,|_{x=0} = -\left\langle y^0, \varphi(0) \right\rangle_{H^{-1},H_0^1}, \ \forall \varphi^0 \in \mathbb{H}_0^1(0,\pi).$$

Let $\{V_{k,1}, V_{k,2}\}$ the normalized eigenvectors of $-k^2 D + A^*$ associated with $\{-\lambda_{k,1}, -\lambda_{k,2}\}$. Then

$$\sigma\left(D\frac{\partial^2}{\partial x^2} + A^*\right) = \cup_{k\geq 1}\{-\lambda_{k,1}, -\lambda_{k,2}\}$$

and $\left\{\Phi_{k,i} = \sqrt{\frac{2}{\pi}} V_{k,i} \sin kx\right\}$ is the associated family of eigenfunctions.

$(\Phi_{k,i})$ is a (Riesz) basis of $\mathbb{L}^2(0,\pi)$.

Choosing $\varphi^0 = \Phi_{k,i}$ in

$$\int_0^T v.C^* D \frac{\partial \varphi}{\partial x} \mid_{x=0} = -\left\langle y^0, \varphi(0) \right\rangle_{H^{-1}, H_0^1}, \ \forall \varphi^0 \in \mathbb{H}_0^1(0, \pi).$$

and using the assumption $C^* DV_{k,i} \neq 0$, this relation is reduced to the problem of moments

$$\int_0^T v(T-t) e^{-\lambda_{k,i} t} = m_{k,i}, \forall (k, \ i).$$

with

$$-\frac{\left\langle y^0, \Phi_{k,i} \right\rangle_{H^{-1}, H_0^1}}{C^* DV_{k,i} \sqrt{\frac{2}{\pi}} k} e^{-\lambda_{k,i} T} = m_{k,i}$$

- If $\left(e^{-\lambda_{k,i}t}\right)$ admits a biorthogonal family $(q_{k,i})$ in $L^2(0,T)$, i.e. if:

$$\int_0^T e^{-\lambda_{k,i}t} q_{l,j}\,dt = \begin{cases} 1, & \text{if } (k,i) = (l,j) \\ 0, & \text{if } (k,i) \neq (l,j) \end{cases}$$

then v is formally computed:

$$v(T-t) = \sum_{k\geq 1} m_{k,i} q_{k,i}(t).$$

- If $\left(e^{-\lambda_{k,i}t}\right)$ admits a biorthogonal family $\left(q_{k,i}\right)$ in $L^2\left(0, T\right)$, i.e. if:

$$\int_0^T e^{-\lambda_{k,i}t} q_{l,j} dt = \begin{cases} 1, & \text{if } (k,i) = (l,j) \\ 0, & \text{if } (k,i) \neq (l,j) \end{cases}$$

then v is formally computed:

$$v(T-t) = \sum_{k \geq 1} m_{k,i} q_{k,i}(t).$$

- The question is then: $\sum_{k \geq 1} m_{k,i} q_{k,i}(t) \in L^2\left(0, T\right)$?

- If $\left(e^{-\lambda_{k,i}t}\right)$ admits a biorthogonal family $\left(q_{k,i}\right)$ in $L^2\left(0,T\right)$, i.e. if:

$$\int_0^T e^{-\lambda_{k,i}t} q_{l,j} dt = \begin{cases} 1, & \text{if } (k,i) = (l,j) \\ 0, & \text{if } (k,i) \neq (l,j) \end{cases}$$

then v is formally computed:

$$v(T-t) = \sum_{k\geq 1} m_{k,i} q_{k,i}(t).$$

- The question is then: $\sum_{k\geq 1} m_{k,i} q_{k,i}(t) \in L^2\left(0,T\right)$?
- We are led to: $\left\|q_{k,i}\right\|_{L^2(0,T)} \underset{k\to\infty}{\sim} ?$

- If $\left(e^{-\lambda_{k,i}t}\right)$ admits a biorthogonal family $\left(q_{k,i}\right)$ in $L^2\left(0,T\right)$, i.e. if:

$$\int_0^T e^{-\lambda_{k,i}t} q_{l,j}dt = \begin{cases} 1, & \text{if } (k,i) = (l,j) \\ 0, & \text{if } (k,i) \neq (l,j) \end{cases}$$

then v is formally computed:

$$v(T-t) = \sum_{k\geq 1} m_{k,i} q_{k,i}(t).$$

- The question is then: $\sum_{k\geq 1} m_{k,i} q_{k,i}(t) \in L^2\left(0,T\right)$?
- We are led to: $\left\|q_{k,i}\right\|_{L^2(0,T)} \underset{k\to\infty}{\sim} ?$
- At this level, a number $c\left(\Lambda\right) \in [0,\infty]$ associated with the sequence $\Lambda = \{\lambda_{k,i}\}$ takes place: the index of condensation of Λ.

- Let's go back to the question:

$$\left\| q_{k,i} \right\|_{L^2(0,T)} \underset{k \to \infty}{\sim} {\color{red}?}$$

- Let's go back to the question:

$$\|q_{k,i}\|_{L^2(0,T)} \underset{k\to\infty}{\sim} ?$$

- Actually, the family $\{q_{k,i}\}$ may be constructed using for instance the Laplace transform and Blaschke products (L. Schwartz, Fattorini-Russell, etc) such that:

$$\|q_{k,i}\|_{L^2(0,T)} \underset{k\to\infty}{\sim} Ce^{\mathrm{Re}(\lambda_{k,i})(\varepsilon+c(\Lambda))}.$$

- $$\left|m_{k,i}\right| \left\|q_{k,i}\right\|_{L^2(0,T)} \underset{k\to\infty}{\sim} \frac{\left|\left\langle y^0, \Phi_{k,i}\right\rangle_{H^{-1},H_0^1}\right|}{\sqrt{\frac{2}{\pi}}k\left|C^* D V_{k,i}\right|} e^{-\mathrm{Re}(\lambda_{k,i})(\mathbf{T}-\varepsilon-\mathbf{c}(\Lambda))}$$

-
$$\left|m_{k,i}\right| \left\|q_{k,i}\right\|_{L^2(0,T)} \underset{k\to\infty}{\sim} \frac{\left|\left\langle y^0, \Phi_{k,i}\right\rangle_{H^{-1},H_0^1}\right|}{\sqrt{\frac{2}{\pi}}k\left|C^* DV_{k,i}\right|} e^{-\mathrm{Re}(\lambda_{k,i})(\mathbf{T}-\varepsilon-\mathbf{c}(\Lambda))}$$

- Now, it can be checked that $k\left|B^* DV_{k,i}\right| \geq e^{-\varepsilon \mathrm{Re}(\lambda_{k,i})}$, and then:

$$T > c(\Lambda) \Rightarrow v(T-t) = \sum_{k\geq 1} m_{k,i} q_{k,i}(t) \in L^2(0,T).$$

-

$$\left|m_{k,i}\right| \left\|q_{k,i}\right\|_{L^2(0,T)} \underset{k\to\infty}{\sim} \frac{\left|\left\langle y^0, \Phi_{k,i}\right\rangle_{H^{-1},H_0^1}\right|}{\sqrt{\frac{2}{\pi}} k \left|C^* DV_{k,i}\right|} e^{-\mathrm{Re}(\lambda_{k,i})(\mathbf{T}-\varepsilon-\mathbf{c}(\Lambda))}$$

- Now, it can be checked that $k\left|B^* DV_{k,i}\right| \geq e^{-\varepsilon\,\mathrm{Re}(\lambda_{k,i})}$, and then:

$$T > c\left(\Lambda\right) \Rightarrow v(T-t) = \sum_{k\geq 1} m_{k,i} q_{k,i}(t) \in L^2\left(0, T\right).$$

- Thus: $T_0\left(d\right) = c\left(\Lambda\right)$.

- The *index of condensation* of a sequence $\Lambda = (\lambda_k) \subset \mathbb{C}$ is a number $c(\Lambda) \in [0, +\infty]$ associated with Λ that "measures" the condensation at infinity.

- The *index of condensation* of a sequence $\Lambda = (\lambda_k) \subset \mathbb{C}$ is a number $c(\Lambda) \in [0, +\infty]$ associated with Λ that "measures" the condensation at infinity.
- This notion has been :

- The *index of condensation* of a sequence $\Lambda = (\lambda_k) \subset \mathbb{C}$ is a number $c(\Lambda) \in [0, +\infty]$ associated with Λ that "measures" the condensation at infinity.
- This notion has been :
  - introduced by Vl. Bernstein in 1933: (Leçons sur les progrès récents de la théorie des séries de Dirichlet) for real sequences and,

- The *index of condensation* of a sequence $\Lambda = (\lambda_k) \subset \mathbb{C}$ is a number $c(\Lambda) \in [0, +\infty]$ associated with Λ that "measures" the condensation at infinity.
- This notion has been :
  - introduced by Vl. Bernstein in 1933: (Leçons sur les progrès récents de la théorie des séries de Dirichlet) for real sequences and,
  - extended by J. R. Shackell in 1967 for complex sequences.

Index of condensation

Let $\Lambda = (\lambda_k) \subset \mathbb{C}$ be a sequence such that

$$k \neq l \Rightarrow \lambda_k \neq \lambda_l$$
$$\exists \delta > 0 : \Re(\lambda_k) \geq \delta |\lambda_k| > 0, \ \forall k \geq 1,$$
$$\sum_{k \geq 1} \frac{1}{|\lambda_k|} < \infty.$$

With this sequence is associated the interpolating function:

$$E(z) = \prod_{k=1}^{\infty} \left(1 - \frac{z^2}{\lambda_k^2}\right), \ E'(\lambda_k) = -\frac{2}{\lambda_k} \prod_{j \neq k}^{\infty} \left(1 - \frac{\lambda_k^2}{\lambda_j^2}\right)$$

Index of condensation

Definition

The index of condensation of $\Lambda = (\lambda_k)$ is the number:

$$c(\Lambda) = \lim_{k\to\infty} \sup \frac{\ln\left|\frac{1}{E'(\lambda_k)}\right|}{\Re(\lambda_k)} \in [0, +\infty].$$

Theorem

$|\lambda_k - \lambda_l| \geq \rho\,|k - l| \Rightarrow c(\Lambda) = 0$

Index of condensation: examples

$$\begin{aligned}\Lambda &= \left\{\lambda_{3k+\nu} = k^2 + \nu e^{-\beta_\nu k^\alpha},\ 0 \leq \nu \leq 2, k \geq 1\right\} \\ \rightarrow &\quad c(\Lambda) = \begin{cases} 0 & \alpha < 2 \\ (\beta_1 + \beta_2) & \alpha = 2 \\ \infty & \alpha > 2 \end{cases}\end{aligned}$$

Note that in this example:

$$\inf_{k \neq l} \frac{|\lambda_k - \lambda_l|}{|k - l|} = 0$$

Index of condensation: examples

$$\Lambda = \left\{\lambda_{k^2+1+\nu} = k^2 + \nu e^{-k^\alpha},\ 0 \le \nu \le 2k-1, k \ge 1\right\}$$

$$\rightarrow \quad c(\Lambda) = \begin{cases} 0 & \text{si } \alpha < 1 \\ 2 & \text{si } \alpha = 1 \\ +\infty & \text{si } \alpha > 1 \end{cases}$$

Note again that:

$$\inf_{k \neq l} \frac{|\lambda_k - \lambda_l|}{|k-l|} = 0$$

The non controllability result

The null-controllability property is equivalent to the observability inequality:

$$\left\| \varphi\left(0\right) \right\|_{H_0^1}^2 \leq C_T \int_0^T \left| C^* D \frac{\partial \varphi}{\partial x}\left(t, 0\right) \right|^2 dt.$$

for any solution of the adjoint problem

$$\begin{cases} -\dfrac{\partial \varphi}{\partial t} = \left(D \dfrac{\partial^2}{\partial x^2} + A^* \right) \varphi, & \text{in } Q, \\ \varphi(0, \cdot) = \varphi(\pi, \cdot) = 0 & \text{on } (0, T), \\ \varphi(\cdot, T) = \varphi^0 \in \mathbb{H}_0^1\left(0, \pi\right). \end{cases}$$

Idea.

Construct a sequence (φ_n) of solutions to the adjoint problem such that:

$$\int_0^T \left| C^* D \frac{\partial \varphi_n}{\partial x}(t,0) \right|^2 dt \to 0, \quad \|\varphi_n(0)\|^2_{H_0^1} \geq \delta > 0.$$

□

Changing the numbering of the eigenvalues and eigenfunctions, we denote them: (λ_k, Φ_k).

Lemma

There exists $\Delta = (G_k)_{k\geq 1} = \left((\lambda_n)_{n\in N_k}\right)_{k\geq 1}$ *with* $\mathrm{card} N_k = p_k$ *such that:*

$$\overline{\lim_{k\to\infty}} \max_{m\in N_k} \left\{ \frac{1}{|\lambda_m|} \log \frac{p_k!}{\left|\prod'_{G_k}(\lambda_m - \lambda_n)\right|} \right\} = c(\Lambda).$$

and

$$\lim_{k\to\infty} \int_0^\infty \left| \sum_{m\in N_k} \frac{p_k! e^{-\lambda_m t}}{\prod'_{G_k}(\lambda_m - \lambda_n)} \right|^2 dt = 0.$$

If (Ψ_k) is the biorthogonal basis associated with (Φ_k) and $a_m = p_k! e^{-\lambda_m t} / \prod'_{G_k} (\lambda_m - \lambda_n)$ for $m \in N_k$, choose

$$\langle \varphi_k^0, \Psi_m \rangle = \begin{cases} \dfrac{a_m}{b_m}, & \text{if } m \in N_k. \\ 0 & \texttt{otherwise}. \end{cases}$$

where (b_m) is chosen so that:

$$C^* D \frac{\partial \varphi_k}{\partial x}(t,0) = \sum_{m \in N_k} a_m e^{-\lambda_m t}$$

Then

$$\int_0^T \left| \sum_{m \in N_k} e^{-\lambda_m t} \langle \varphi_k^0, \Psi_m \rangle b_m \right|^2 dt \leq \int_0^\infty \left| \sum_{m \in N_k} a_m e^{-\lambda_m t} \right|^2 dt \to 0.$$

- On the other hand at least for a subsequence

$$\begin{aligned} e^{-\Re\left(\lambda_{m_k}\right)T}\left|\left\langle \varphi_k^0, \Phi_{m_k}\right\rangle\right| &\geq e^{-\Re\left(\lambda_{m_k}\right)(T-\varepsilon)} a_{m_k} \\ &\geq C e^{-\Re\left(\lambda_{m_k}\right)(T-\varepsilon-c(\Lambda))} \underset{k\to\infty}{\longrightarrow} +\infty \end{aligned}$$

- On the other hand at least for a subsequence

$$\begin{aligned} e^{-\Re\left(\lambda_{m_k}\right)T}\left|\left\langle \varphi_k^0, \Phi_{m_k}\right\rangle\right| &\geq e^{-\Re\left(\lambda_{m_k}\right)(T-\varepsilon)} a_{m_k} \\ &\geq C e^{-\Re\left(\lambda_{m_k}\right)(T-\varepsilon-c(\Lambda))} \underset{k\to\infty}{\longrightarrow} +\infty \end{aligned}$$

- This proves that the observability inequality fails for $T < c\left(\Lambda\right)$.

Example

Question:

Is it possible that $c(\Lambda) > 0$? □

- Consider the case $A = 0$. Then:

$$\lambda_{k,1} = k^2,\ \lambda_{k,2} = dk^2,\ V_{k,1} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, V_{k,2} = \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

$$[\lambda_{k,i} \neq \lambda_{l,j}\ \ \forall (k,i) \neq (l,j)] \Leftrightarrow \sqrt{d} \notin \mathbb{Q},$$

$$C^* D V_{k,i} \neq 0 \Leftrightarrow c_1 c_2 \neq 0.$$

Example

Example

Theorem

Let $\Lambda = \left(k^2,\ dk^2\right)$. *For any* $\delta \in [0, \infty]$, *there exists* d *such that* $\sqrt{d} \in \mathbb{R} \backslash \mathbb{Q}$ *tel que* $\delta = c(\Lambda)$.

Example

Theorem

Let $\Lambda = \left(k^2,\ dk^2\right)$. *For any* $\delta \in [0, \infty]$, *there exists* d *such that* $\sqrt{d} \in \mathbb{R} \backslash \mathbb{Q}$ *tel que* $\delta = c\left(\Lambda\right)$.

A consequence: There exists a positive number d such that $c\left(\Lambda\right) = \infty$. Thus for such a number, the system is never null-controllable while it is approximately controllable at any $T > 0$!

- Thank you for your attention!