Optimal placement of sensors, actuators and dampers for waves

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What is the best shape and placement of sensors?
- Reduce the cost of instruments.
- Maximize the efficiency of reconstruction and estimations.
The mathematical theory needed to understand these issues combines:

- Hyperbolic PDEs (possibly on Networks)
- Control Theory
- Optimal Design
- Optimization
- Spectral analysis
- Microlocal analysis
- Numerical analysis
- ...

In this talk we aim to present some toy models and problems, together with some key results and research perspectives.
# The control of waves

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An example: noise reduction

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The control of waves
A toy model

Control of 1−d vibrations of a string

The 1-d wave equation, with Dirichlet boundary conditions, describing the vibrations of a flexible string, with control on one end:

\[
\begin{cases}
  y_{tt} - y_{xx} = 0, & 0 < x < 1, \quad 0 < t < T \\
  y(0, t) = 0; y(1, t) = v(t), & 0 < t < T \\
  y(x, 0) = y^0(x), \quad y_t(x, 0) = y^1(x), & 0 < x < 1
\end{cases}
\]

\[y = y(x, t)\] is the state and \[v = v(t)\] is the control.

The goal is to stop the vibrations, i.e. to drive the solution to equilibrium in a given time \(T\): Given initial data \(\{y^0(x), y^1(x)\}\) to find a control \(v = v(t)\) such that

\[y(x, T) = y_t(x, T) = 0, \quad 0 < x < 1.\]
The control of waves

A toy model

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Waves, Control and Design

LJLL/UMPC, February 2014
The control problem above is equivalent to the following one, on the adjoint wave equation:

\[
\begin{align*}
\varphi_{tt} - \varphi_{xx} &= 0, & 0 < x < 1, 0 < t < T \\
\varphi(0, t) &= \varphi(1, t) = 0, & 0 < t < T \\
\varphi(x, 0) &= \varphi^0(x), \varphi_t(x, 0) = \varphi^1(x), & 0 < x < 1.
\end{align*}
\]

The energy of solutions is conserved in time, i.e.

\[
E(t) = \frac{1}{2} \int_0^1 \left[ |\varphi_x(x, t)|^2 + |\varphi_t(x, t)|^2 \right] dx = E(0), \quad \forall 0 \leq t \leq T.
\]

The question is then reduced to analyze whether the following inequality is true. This is the so called observability inequality:

\[
E(0) \leq C(T) \int_0^T |\varphi_x(1, t)|^2 dt.
\]
The answer to this question is easy to guess: The observability inequality holds if and only if $T \geq 2$.

Wave localized at $t = 0$ near the extreme $x = 1$ propagating with velocity one to the left, bounces on the boundary point $x = 0$ and reaches the point of observation $x = 1$ in a time of the order of 2.
Construction of the Control

Following J.L. Lions’ HUM (Hilbert Uniqueness Method)\(^1\), the control is

\[ v(t) = \varphi_x(1, t), \]

where \( \varphi \) is the solution of the adjoint system corresponding to initial data \((\varphi^0, \varphi^1) \in H_0^1(0, 1) \times L^2(0, 1)\) minimizing the functional

\[
J(\varphi^0, \varphi^1) = \frac{1}{2} \int_0^T |\varphi_x(1, t)|^2 dt + \int_0^1 y^0 \varphi^1 dx - < y^1, \varphi^0 >_{H^{-1} \times H_0^1},
\]

in the space \( H_0^1(0, 1) \times L^2(0, 1) \). \( J \) is convex, continuous (because of the fact that \( \varphi_x(1, t) \in L^2(0, T) \) (hidden regularity)). Moreover,

**COERCIVITY OF \( J \) = OBSERVABILITY INEQUALITY.** \(^2\)

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\(^2\) Norbert Wiener (1894–1964) defined Cybernetics as the science of control and communication in animals and machines.
Pointwise observations

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Take $x_0 \in (0, 1)$. How much energy we can recover from measurements done on $x_0$?

$$\varphi(x_0, t) = \sum_{k \in \mathbb{Z}} a_k e^{i k \pi t} \sin(k \pi x_0).$$

Furthermore, if $T > 2$,

$$\int_0^T \left| \sum a_k e^{i k \pi t} \sin(k \pi x_0) \right|^2 dt \sim \sum \sin^2(k \pi x_0) |a_k|^2.$$

Obviously, two cases:

- The case $x_0 \in \mathbb{Q}$, some of the weights $\sin^2(k \pi x_0)$ vanish and the quadratic term is not a norm.

- The case $x_0 \notin \mathbb{Q}$: $\sin^2(k \pi x_0) \neq 0$ for all $k$ and the quantity under consideration is a norm, i.e. it provides information on all the Fourier components of the solutions.

But, even if, $\sin^2(k \pi x_0) \neq 0$ for all $k$, the norm under consideration is not the $L^2$-one we expect!!!!
Can we explain this in terms of rays, and the propagation of waves (and antiwaves)?

If $x_0$ is rational we can build a finite number of rays and anti-rays that always intersect in $x_0$ for the time interval $(0, 2)$ of periodicity of solutions.
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If $x_0$ is rational we can build a finite number of rays and anti-rays that always intersect in $x_0$ for the time interval $(0, 2)$ of periodicity of solutions.
The case \( x_0 \) irrational.

Note that the following is impossible!!!

\[
\left| \sin(k\pi x_0) \right| \geq \alpha > 0, \quad \forall k?
\]

Indeed, this would mean that

\[
\left| k\pi x_0 - m\pi \right| \geq \beta
\]

for all \( k, m \in \mathbb{Z} \). And this is obviously false.

For suitable irrational numbers \( x_0 \) we can get the optimal lower bound

\[
\left| k\pi x_0 - m\pi \right| \geq \beta / k.
\]

In this case we get an observation inequality but with a loss of one derivative.

But for some other irrational numbers (Liouville ones, for instance) the degeneracy may be arbitrary fast. \(^3\)

1 – $d$ conclusion

D’Alembert $\sim$ Fourier

Ray propagation $\sim$ Spectrum
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Let $\omega$ be an open subset of $\Omega$. Consider:

$$\begin{cases}
y_{tt} - \Delta y = -y_t 1_\omega & \text{in } Q = \Omega \times (0, \infty) \\
y = 0 & \text{on } \Sigma = \Gamma \times (0, \infty) \\
y(x, 0) = y^0(x), y_t(x, 0) = y^1(x) & \text{in } \Omega,
\end{cases}$$

where $1_\omega$ stands for the characteristic function of the subset $\omega$.

The energy dissipation law is then

$$\frac{dE(t)}{dt} = -\int_\omega |y_t|^2 dx.$$

**Question:** Do they exist $C > 0$ and $\gamma > 0$ such that

$$E(t) \leq Ce^{-\gamma t} E(0), \quad \forall t \geq 0,$$

for all solution of the dissipative system?
This is equivalent to an **observability property**⁴: There exists $C > 0$ and $T > 0$ such that

$$E(0) \leq C \int_0^T \int_\omega |y_t|^2 dxdt.$$  

This estimate, together with the energy dissipation law, shows that

$$E(T) \leq \sigma E(0)$$

with $0 < \sigma < 1$. Accordingly the semigroup map $S(T)$ is a strict contraction. By the semigroup property one deduces immediately the exponential decay rate.

The observability inequality and, accordingly, the exponential decay property holds if and only if the support of the dissipative mechanism, $\Gamma_0$ or $\omega$, satisfies the so called the Geometric Control Condition (GCC) (Ralston, Rauch-Taylor, Bardos-Lebeau-Rauch\(^5\), ...)

Rays propagating inside the domain $\Omega$ following straight lines that are reflected on the boundary according to the laws of Geometric Optics. The control region is the red subset of the boundary. The GCC is satisfied in this case. The proof requires tools from Microlocal Analysis.

A trapped ray scaping the damping region $\omega$ makes it impossible the decay rate to be exponential. Each trajectory tends to zero as $t \to \infty$ but the decay can be arbitrarily slow.
Damped waves
Complexity of the decay rate

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The optimal shape design problem

Optimize $\omega$ (or $\Gamma_0$) and the damping profile $a = a(x)$ to enhance the exponential decay rate:

$$
\begin{align*}
  y_{tt} - \Delta y &= -a(x) y_t 1_\omega \\
  y &= 0 \\
  y(x, 0) &= y^0(x), \\ y_t(x, 0) &= y^1(x)
\end{align*}
$$

in $Q = \Omega \times (0, \infty)$
on $\Sigma = \Gamma \times (0, \infty)$
in $\Omega$.

Within the class of $\omega$ and $a > 0$ such the exponential decay property holds:

$$
E(t) \leq C e^{-\gamma_{a,\omega} t} E(0), \quad \forall t \geq 0
$$

$$
\text{MAXIMIZE} : (\omega, a) \rightarrow \gamma_{a,\omega}.
$$
Overdamping!

Obviously, the decay rate $\gamma_a$ depends on the damping potential $a$. But, against the very first intuition, this map is not monotonic with respect to the size of the damping. A $1 - d$ spectral computation for constant coefficients yields:
Some known results: $1 - d$

- $1 - d$: The exponential decay rate coincides with the spectral abscissa within the class of $BV$ damping potentials. For large eigenvalues $\Re(\lambda) \sim -\int_\omega a(x)dx/2$.\(^6\) Thus:

$$\gamma_a \leq \int_\omega a(x)dx.$$

---

The singular potential

\[ a(x) = \frac{2}{x} \]

produces an arbitrarily fast decay rate. \(^7\)

Connections with:

- Transparent boundary conditions.
- Perfectly matching layers.

---

In the multidimensional case the situation is even more complex. The decay rate is determined as the minimum of two quantities:

- The spectral abscissa;
- The minimum asymptotic average (as \( T \to \infty \)) of the damping potential along rays of Geometric Optics.

The later is in agreement with our intuition of waves traveling along rays of Geometric Optics.

---

Geometric configuration in which the spectral abscissa does not suffice to capture the decay rate. The decay rate vanishes due to a trapped ray, but the spectrum is uniformly shifted in the left complex half space.
Truth = Spectrum + Rays

Truth = Fourier $\cup$ D’Alembert
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Consider the conservative wave equation:

\[
\begin{aligned}
z_{tt} - \nabla^2 z &= 0 & \text{ in } & Q = \Omega \times (0, T) \\
z &= 0 & \text{ for } & x \in \partial\Omega; \quad t \in (0, T) \\
z(x, 0) = z^0(x), z_t(x, 0) = z^1(x) & \text{ in } & (0, \pi).
\end{aligned}
\]

- Optimal placement problems are then of variational nature!
- It corresponds to the analysis of the behavior of the damped system **infinitesimally small damping**.

**Observability:**

\[
\|z^0\|^2_{L^2(\Omega)} + \|z^1\|^2_{H^{-1}(\Omega)} \leq C(\omega, T) \int_0^T \int_{\omega} z^2 \, dx \, dt.
\]

Inspired in previous works by, among others: P. Hébrard & A. Henrot and A. Münch, P. Pedregal & F. Periago.
Fourier series shows that, in general, Fourier modes are mixed in quite an complicated manner thus making the understanding of these issues complex:

\[ z(t) = \sum \hat{z}_k e^{i \sqrt{\lambda_k} t} \phi_k(x). \]

Thus,

\[
\int_0^T \int_\omega |z|^2 dx dt = \sum \sum \hat{z}_k \hat{z}_j \int_\omega \phi_k(x) \phi_j(x) dx \int_0^T e^{i \sqrt{\lambda_k} - i \sqrt{\lambda_j} t}.
\]
Earlier numerical experiments

Optimal design of the billiard: Hébrard-Humbert, 2003\textsuperscript{9}

\textsuperscript{9}P. Hébrard and E. Humbert. The geometrical quantity in damped wave equations on a square. ESAIM:COCV, 12(4), 2006.
Optimal location of the sensor for the first eigenfunction by a level set approach: A. Münch, 2005.\textsuperscript{10}

Simulations performed using AMPL + IPOPT
Main results with fixed initial data

- For initial data that are analytic (exponential decay of Fourier coefficients), there is a unique minimizer with a finite number of connected components. \(^{11}\)

- The optimal set always exists but it can be a **Cantor set even for** \(C^\infty\) **smooth data.**

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Reduction to a spectral problem

The problem becomes much simpler in several cases:

- The case $T = \infty$. We then look at the

  $$\lim_{T \to \infty} \frac{1}{T} \int_{-T}^{T} \int_{\omega} |z|^2 dxdt.$$ 

- Randomizing initial data and considering the expected observability constant (Zygmund lemma, recent works by N. Burq et al.\textsuperscript{12})

- In 1-d, $\Omega = (0, \pi)$ in which case solutions are $2\pi$-time periodic.

Cross terms vanish and we are led to the following observability problem:

$$\sum |\hat{z}_k|^2 \leq C(\omega) \sum |\hat{z}_k|^2 \int_{\omega} \phi_k^2(x) dx.$$ 

These issues can be considered, as mentioned above, in two different cases:

- **Fixed initial data**, and therefore fixed weights $|\hat{z}_k|^2$ in $\ell^1$.
- **All possible initial data of finite energy**. Then, the problem becomes that of finding $\omega$ so that the following minimum is maximized:

$$J(\omega) = \inf_k \int_\omega \phi_k^2(x) dx.$$  

$$I = \sup_{|\omega|=L} J(\omega).$$

**Warning!**

This spectral criterium is not sufficient to fully characterize the observability constant since a second microlocal one is also required. We are ignoring the rays!!!

**Spectral criterium = Truth /2**

We are ignoring the ray contribution...
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Optimal placement of sensors and actuators

Main results for the spectral criterium

**Spectral criterium: $1 - d$**

- **Relaxation** occurs (Hébrart-Henrot, 2005): the optimum is achieved by a density function $\rho(x)$ so that $\int_{0}^{\pi} \rho(x) dx = L$ and not by a measurable set with bang-bang densities (except for $L = \pi/2$). The constant density is optimal but is not the unique one.

- **Spillover** occurs $(1 - d)$: The optimal design for the first $N$ Fourier modes is the worst choice for the $N + 1$-th one.

- **No gap!** The infimum over measurable sets and over densities coincides. The functional is not lower semicontinuous!!!

---

Optimal control region with N=1 eigenmodes

The spillover phenomenon
The truncation of the optimal design problem to the first $N$ Fourier modes leads to an efficient approximation algorithm.

\[
J_N(\omega) = \inf_{k=1,\ldots,N} \int_{\omega} \phi_k^2(x) dx.
\]

\[
I_N = \sup_{|\omega|=L} J_N(\omega).
\]

\[
J_N(\omega_N) \to I.
\]

Note this is true despite the lack of weak lower semicontinuity of the functional $J$ and compatible with spillover!
The constant density is optimal and the non-gap result holds under the additional assumption that

\[ \text{the eigenfunctions are uniformly bounded in } L^\infty. \]

Note that this applies to the case where \( \Omega \) is a rectangle, for the classical basis of eigenfunctions in separated variables: \( \sin(kx)\sin(my) \).

Our original proof used an added technical assumption recently removed by Lior Silberman employing the well known convergence of Cesàro means:

\[
\frac{1}{N} \sum_{j=1}^{N} \phi_j^2 \to \frac{1}{|\Omega|}, \text{ as } N \to \infty, \text{ vaguely.}
\]

The \( L^\infty \) bound allows showing that Cesàro means weakly converge in \( L^1 \) which allows testing against any \( L^\infty \) density.
Some references:


Note that the existing results guarantee the converge to the uniform measure for a density-one subsequence of eigenfunctions (Schnirelman Thm).
But in the multi-dimensional case very rarely eigenfunctions are uniformly bounded in $L^\infty$!

Despite of this an in depth use of the structure of the eigenfunctions in the disk allows shows that the constant density is optimal!
We put in evidence the intimate relations between optimal design problems, the high frequency behavior of the spectrum and the (quantum) ergodicity properties of the dynamical system associated to the billiard of the domain.

Optimal designs in $2 - d$ for the square with volume fraction 1/2 in the spectral criterium. This is an exceptional case where classical optimal domains exist.
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7. **Conclusions and Perspectives**
Similar questions can be formulated for parabolic equations, and in particular for the linear heat equation. Once more a randomized observability criterium leads to a spectral criterium. Due to the intrinsic damping in the heat equation the high frequency components are then penalized:

$$C_{T,\text{rand}}(\chi_\omega) = \inf_{j \in \mathbb{N}^*} \frac{e^{2\lambda_j T} - 1}{2\lambda_j} \int_\omega \phi_j(x)^2 \, dx,$$

(1)

One is then able to show that the optimal design is determined by a finite number of eigenfunctions, the relevant number of them diminishing as $T$ increases. The proof uses recent results on the lack of concentration of eigenfunctions on measurable sets. Namely, that

$$\int_\omega |\phi_j(x)|^2 \, dx \geq c_1 \exp(-c_2 \sqrt{\lambda_j}).$$

uniformly over sets $\omega$ of a given measure.

According to [42, Theorem 2.1], there exists $C > 0$ such that
\[ \lim_{j \to +1} \int_{\Omega_{jk}} \varphi \, d\nu = \int_{\Omega_1} \varphi \, d\nu \] and
\[ \int_{\Omega_{jk}} R_{jk}(\varphi) \, d\nu \leq C j^{-2/3} \int_{\Omega_{jk}} \varphi \, d\nu. \]

The first estimate ensures that, for a given $k \in \mathbb{N}^*$ and for $j$ large enough, $\Phi^j$ is contained in $\Omega_{jk}$.

Using the inequalities $j \leq k \leq (2j)^2$ for all $j \in \mathbb{N}$ and $k \in \mathbb{N}^*$, we easily infer that
\[ \lim_{j \to +1} \int_{\Omega_{jk}} \varphi \, d\nu = 0, \]
for every $k \in \mathbb{N}^*$, which raises a contradiction. The conclusion follows.

### 2.6 Several numerical simulations

We provide hereafter several numerical simulations. The truncated problem of order $N$ is obtained by considering all couples $(j, k)$ such that $j \neq N$ and $k \neq N$. The simulations are made with a primal-dual approach combined with an interior point line search filter method.

On Figure 1 (resp., on Figure 2), we compute the optimal domain $\Phi^N$ for the operator $A_0 = 4$, the Dirichlet-Laplacian (resp., the Neumann-Laplacian on the domain defined with zero average) on the square $\Omega = (0, \pi)^2$. We can observe the expected stationarity property of the sequence of optimal domains $\Phi^N$ from $N = 4$ on (i.e., 16 eigenmodes).

**Figure 1:** On this figure, $\Omega = (0, \pi)^2$, $L = 0$, $T = 0$, and $A_0$ is the Dirichlet Laplacian. Row 1, from left to right: optimal domain $\Phi^N$ (in green) for $N = 1, 2, 3$. Row 2, from left to right: optimal domain $\Phi^N$ (in green) for $N = 4, 5, 6$.

More precisely, we used the optimization routine IPOPT (see [59]) combined with the modeling language AMPL (see [15]) on a standard desktop machine.

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Conclusions

- There is a **rich (and difficult!)** field to be explored at the intersection of the following topics:
  - Networks
  - Damped wave equations
  - Control theory
  - Optimal design and Optimization
  - Spectral theory
  - Ergodicity of billiards
  - Microlocal analysis
  - Fine numerics

- At the continuous level there is a huge **need of better spectral understanding**.

- Also of gaining comprehension of **complex dynamics when spectral analysis does not suffice**: What about the optimal placement of sensors for the original time-evolution problem?

- Lots to be done on the **numerics** of these problems.