

# On the determination of the principal coefficient from boundary measurements in a KdV equation

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## Presentation of the problem

The KdV equation with non-constant coefficient  $a = a(x)$  is given as

$$\left\{ \begin{array}{ll} y_t + a(x)y_{xxx} + y_x + yy_x = g, & \forall (x, t) \in (0, L) \times (0, T), \\ y(0, t) = g_0(t), \quad y(L, t) = g_1(t), & \forall t \in (0, T), \\ \quad \quad \quad y_x(L, t) = g_2(t), & \forall t \in (0, T), \\ y(0, x) = y_0(x), & \forall x \in (0, L), \end{array} \right.$$

where the initial data  $y_0$ , the source term  $g$ , and the functions  $g_0, g_1, g_2$  are assumed to be known.

The KdV equation is used to describe approximately long waves in water of relatively shallow depth.

In this context, the principal coefficient  $a = a(x)$  is related with the shape of the bottom of the channel where the water wave propagates. See Samer Israwi's PhD thesis under supervision of David Lannes (Bordeaux, 2010).

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In general, in an inverse problem, we want to get some information of a system (conductivity, diffusion, **shape of the bottom of a channel**, etc) from partial knowledge (observations/measurements) of the solution.

### Inverse Problem

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### Inverse Problem (Uniqueness)

Given some boundary measurements  $meas(y)$ , is there a unique  $a = a(x)$  associated?

$$i.e. \quad meas(y) = meas(\tilde{y}) \implies a = \tilde{a}?$$

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Given some measurement  $meas(y)$ , is it possible to reconstruct the coefficient  $a = a(x)$ ?

In this talk, we are concerned with the stability problem (and consequently with the uniqueness problem!).

Remark: This kind of inverse problem is called a single-measurement IP



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## Calderón's Problem, multi-measurement IP

It is the problem of recovering the conductivity  $\gamma$  in a body  $\Omega$  from current-voltage measurements

$$\begin{cases} \operatorname{div}(\gamma \nabla u) &= 0, & \Omega \\ u &= f, & \partial\Omega \end{cases}$$

The Dirichlet-to-Neumann map is defined by:

$$\Lambda_\gamma f = \gamma \frac{\partial u}{\partial \nu}$$

Is the operator

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- 3 **Stability.** If  $\Lambda_{\gamma_1}$  is close to  $\Lambda_{\gamma_2}$ , then  $\gamma_1$  and  $\gamma_2$  are close (in a suitable sense).
- 4 **Partial data.** If  $\Gamma \subset \partial\Omega$  and if  $\Lambda_{\gamma_1}|_{\Gamma} = \Lambda_{\gamma_2}|_{\Gamma}$  for all boundary voltages  $f$ , then  $\gamma_1 = \gamma_2$ .

Starting from the work of Calderón in 1980, the inverse conductivity problem has been studied intensively. In the case where  $n \geq 3$  and all conductivities are in  $C^2(\Omega)$ , the following positive results are some examples of what can be proven.

- Theorem. (Sylvester-Uhlmann 1987). **Uniqueness.**
- Theorem. (Nachman 1988) There is an algorithm for **reconstructing**  $\gamma$  from  $\Lambda_{\gamma}$ .
- Theorem. (Alessandrini 1988). **Logarithmic stability.**
- Theorem. (Kenig-Sjostrand-Uhlmann 2007) **Partial data.** Assume that  $\Omega$  is convex and  $\Gamma$  is any open subset of  $\partial\Omega$ . If  $\Lambda_{\gamma_1}|_{\Gamma} = \Lambda_{\gamma_2}|_{\Gamma}$  for all  $f \in H^{\frac{1}{2}}(\partial\Omega)$ , and if  $\gamma_1|_{\partial\Omega} = \gamma_2|_{\partial\Omega}$ , then  $\gamma_1 = \gamma_2$ .

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## BMK method

We follow ideas of Bukhgeim, Klibanov (1981), and Klibanov, Malinsky (1991).

If we set:

- $u = y - \tilde{y}$  and
- $\sigma = \tilde{a} - a$

then  $u$  solves the following KdV equation:

$$\begin{cases} u_t + a(x)u_{xxx} + (1 + \tilde{y})u_x + \tilde{y}_x u + uu_x = \sigma \tilde{y}_{xxx}, & \forall (x, t) \in (0, L) \times (0, T), \\ u(0, t) = 0, \quad u(L, t) = 0, \quad u_x(L, t) = 0 & \forall t \in (0, T), \\ u(x, 0) = 0, & \forall x \in (0, L). \end{cases}$$

Then  $z = u_t$  satisfies the following equation:

$$\begin{cases} z_t + a(x)z_{xxx} + (1 + y)z_x + y_x z = f_\sigma, & \forall (x, t) \in (0, L) \times (0, T), \\ z(0, t) = 0, \quad z(L, t) = 0, \quad z_x(L, t) = 0 & \forall t \in (0, T), \\ z(x, 0) = \sigma(x)y_{0,xxx}(x), & \forall x \in (0, L), \end{cases}$$

where

$$f_\sigma = \sigma(x)\tilde{y}_{xxx} - \tilde{y}_{xt}u - \tilde{y}_t u_x.$$

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We follow ideas of Bukhgeim, Klibanov (1981), and Klibanov, Malinsky (1991).

If we set:

- $u = y - \tilde{y}$  and
- $\sigma = \tilde{a} - a$

then  $u$  solves the following KdV equation:

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$$\|z(x, 0)\|_X \leq C\|f_\sigma\|_Y + (\text{boundary terms})$$

where  $C$  can be chosen small !!

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Remark: This kind of inequality is called observability in control theory

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$$\begin{cases} z_t - a(x)z_{xx} = f_\sigma, & \forall (x, t) \in (0, L) \times (0, T), \\ z(0, t) = 0, \quad z(L, t) = 0 & \forall t \in (0, T), \\ z(x, 0) = \sigma(x)y_{0,xx}(x), & \forall x \in (0, L), \end{cases}$$

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- From a control point of view, in some cases it is parabolic and in others hyperbolic.
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Remark: Some symmetry conditions have to be imposed.

## BMK method - KdV equation

$$\begin{cases} z_t + a(x)z_{xxx} + (1+y)z_x + y_x z = f_\sigma, & \forall (x,t) \in (0,L) \times (0,T), \\ z(0,t) = 0, \quad z(L,t) = 0, \quad z_x(L,t) = 0 & \forall t \in (0,T), \\ z(x,0) = \sigma(x)y_{0,xxx}(x), & \forall x \in (0,L), \end{cases}$$

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## BMK method - Extension for negative time

Symmetric extension to  $(0, L) \times (-T, T)$  of  $g$  defined on  $(0, L) \times (0, T)$ :

$$g^s(x, t) = \begin{cases} g(x, t) & \text{if } x \in [0, L], t \in [0, T], \\ g(L - x, -t) & \text{if } x \in [0, L], t \in [-T, 0). \end{cases}$$

Anti-symmetric extension to  $(0, L) \times (-T, T)$  of  $g$  defined on  $(0, L) \times (0, T)$ :

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Defining  $v = z^s$ , we obtain:

$$\begin{cases} v_t + a(x)v_{xxx} + (1 + y^s)v_x + (y_x)^a v = f_\sigma^a, & \forall x \in (0, L), t \in (-T, T), \\ v(0, t) = 0, \quad v(L, t) = 0, & \forall t \in (-T, T), \\ v_x(L, t) = \begin{cases} 0, & \forall t \in (0, T), \\ -z_x(0, -t), & \forall t \in (-T, 0). \end{cases} \\ v(x, 0) = \sigma(x)y_{0,xxx}(x), & \forall x \in (0, L). \end{cases}$$

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## BMK method - Extension for negative time

The solution of

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satisfies a Carleman estimate which allows to prove

$$\|v(x, 0)\|_X \leq C \|f_\sigma\|_Y + (\text{boundary terms})$$

with  $C$  small.

## Carleman estimates - $Lv = v_t + av_{xxx} = f$

Any  $v \in L^2(-T, T; H^3 \cap H_0^1(0, L))$  and a weight function  $\phi(x, t) = \frac{\beta(x)}{(T+t)(T-t)}$ .

$$w = e^{-\lambda\phi}v, \quad \text{and} \quad L_\phi w = e^{-\lambda\phi}L(e^{\lambda\phi}w)$$

where  $\lambda$  is a large parameter to be chosen later.

The obtained Carleman estimate is an inequality like

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## Carleman estimates for KdV.

- Rosier [2004]. Null control of the surface of a water wave by means of a wavemaker at the left end-point.
- Glass-Guerrero [2008]. Cost of the null control of KdV by means of a control at the left end-point.
- Both papers prove Carleman estimates with one parameter  $\lambda > 0$ .
- For us, it is important a second parameter. Look at one dominating term:

$$\lambda^5 \iint \phi_x^4 (-a_x \phi_x - 5a \phi_{xx} + 4a^2 \phi_{xx}) |w|^2$$

This impose bad conditions of kind  $\|a_x/a\|_{L^\infty} \leq M$ .

- Solution is to choose  $\phi$  such that  $\phi_{xx} \approx s^2 \varphi$  with a second parameter  $s > 0$ .

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## Main Result.

$$\left\{ \begin{array}{l} y_t + a(x)y_{xxx} + y_x + yy_x = g, \\ y(0, t) = g_0(t), \quad y(L, t) = g_1(t), \\ \qquad \qquad \qquad y_x(L, t) = g_2(t), \\ y(0, x) = y_0(x), \end{array} \right. \quad \begin{array}{l} \forall (x, t) \in (0, L) \times (0, T), \\ \forall t \in (0, T), \\ \forall t \in (0, T), \\ \forall x \in (0, L). \end{array}$$

Data  $(g, g_k, y_0)$  fixed and regular enough!

### Theorem

Let  $|y_{0,xxx}(x)| \geq \delta > 0$ , also symmetric wrt  $L/2$ . Let

$$\Sigma = \left\{ a \text{ symmetric wrt } L/2 \mid a \geq a_0 > 0, \|a\|_{W^{3,\infty}} \leq M_1, \text{ and } \|y(a)\|_{W^{1,\infty}(Q)} \leq M_2 \right\}$$

There exists a constant  $C = C(L, T, a_0, M_1, M_2, \delta) > 0$  such that for any  $a, \tilde{a} \in \Sigma$ :

$$C\|a - \tilde{a}\|_{L^2(0,L)} \leq \|y_x(0, t) - \tilde{y}_x(0, t)\|_{H^1(0,T)} + \|y_{xx}(0, t) - \tilde{y}_{xx}(0, t)\|_{H^1(0,T)} \\ + \|y_{xx}(L, t) - \tilde{y}_{xx}(L, t)\|_{H^1(0,T)}$$



Thank you for your attention!