Contrôle des équations des water waves

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- We study the **dynamics of water waves**, i.e. of disturbances of the free surface of a liquid. These can be generated by the immersion of a solid body or by impulsive pressures (blowing) applied on the free surface.

- **Question :** which waves can be generated by **impulsive pressures** applied on a **localized** portion of the free surface ?

  Main result : given any time $T > 0$, any two-dimensional, periodic in space, small amplitude gravity-capillary water waves.

- **Local exact controllability** of the Euler equations with free surface.

- The Water Waves system involves several physical parameters and numerous asymptotic (simplified) models can be derived, such as **KdV, NLS, Benjamin-Ono, etc.** (see the book of D. Lannes). There exist many controllability results for such asymptotic equations (see the book of J.-M. Coron).

- We consider here the **full model**, which is **fully nonlinear** instead of semi-linear. In addition, it is not a PDE but instead a pseudo-differential equation, involving the Dirichlet-Neumann operator which is **nonlocal**.

At time $t \geq 0$, the fluid occupies the domain $\Omega(t)$ with a free surface parametrized by a graph :

$$\Omega(t) = \{\, (x,y) \in \mathbb{R} \times \mathbb{R} \,:\, -b < y < \eta(t,x) \,\},$$

where $b \in \mathbb{R}^+ \cup \{+\infty\}$ and $x \mapsto \eta(t,x)$ is part of the unknowns.
The fluid is **incompressible, irrotational**, and the velocity field $v\colon \Omega(t) \to \mathbb{R}^2$ solves

$$\partial_t v + v \cdot \nabla v + \nabla(P + gy) = 0, \quad \operatorname{div} v = 0, \quad \operatorname{curl} v = 0 \quad \text{in } \Omega(t).$$

Boundary conditions :

$$\partial_t \eta = \sqrt{1 + |\partial_x \eta|^2}\, v \cdot n \qquad \text{on } y = \eta(t,x),$$

$$P = \underbrace{\partial_x \left(\frac{\partial_x \eta}{\sqrt{1 + (\partial_x \eta)^2}} \right)}_{=: H(\eta)} + P_{ext}(t,x) \quad \text{on } y = \eta(t,x),$$

$$v_y = 0 \qquad \text{on } y = -b.$$

The equations 2/2

Since curl $v = 0$, we can write $v = \nabla\phi$ and we have

$$\Delta_{x,y}\phi = 0, \quad \partial_t\phi + \frac{1}{2}\left|\nabla_{x,y}\phi\right|^2 + P + gy = 0, \quad \partial_y\phi|_{y=-b} = 0,$$

Finally we set $\psi(t,x) = \phi(t,x,\eta(t,x))$, introduce the Dirichlet-Neumann operator $G(\eta)$ that relates ψ to the normal derivative $\partial_n\phi$ of the potential by

$$(G(\eta)\psi)(t,x) = \sqrt{1+(\partial_x\eta)^2}\,\partial_n\phi|_{y=\eta(t,x)}.$$

Then (η,ψ) solves the Zakharov-Craig-Sulem-Sulem formulation of Water Waves ($t \in \mathbb{R}^+, x \in \mathbb{T}$)

$$\begin{cases} \partial_t\eta = G(\eta)\psi, \\ \partial_t\psi + g\eta + \frac{1}{2}(\partial_x\psi)^2 - \frac{1}{2}\frac{\left(G(\eta)\psi + (\partial_x\eta)(\partial_x\psi)\right)^2}{1+(\partial_x\eta)^2} = H(\eta) + P_{\text{ext}}. \end{cases}$$

Denote $H_0^s := \{u \in H^s, \int_{\mathbb{T}} u\,dx = 0\}$.

Theorem 1 (*Main result*)

Let $T > 0$ and consider a non-empty open subset $\omega \subset \mathbb{T}$. There exist $\sigma > 0, M_0 > 0$ such that, for any (η_{in}, ψ_{in}), $(\eta_{final}, \psi_{final})$ in $H_0^{\sigma+\frac{1}{2}}(\mathbb{T}) \times H^\sigma(\mathbb{T})$ satisfying

$$\|\eta_{in}\|_{H^{\sigma+\frac{1}{2}}} + \|\psi_{in}\|_{H^\sigma} < M_0, \quad \|\eta_{final}\|_{H^{\sigma+\frac{1}{2}}} + \|\psi_{final}\|_{H^\sigma} < M_0,$$

there exists P_{ext} in $C^0([0,T]; H^\sigma(\mathbb{T}))$ satisfying

$$\operatorname{supp} P_{ext}(t,\cdot) \subset \omega, \quad \forall t \in [0,T],$$

such that the Cauchy problem for the Water Waves equations with initial conditions (η_{in}, ψ_{in}) has a unique solution

$$(\eta, \psi) \in C^0([0,T]; H_0^{\sigma+\frac{1}{2}}(\mathbb{T}) \times H^\sigma(\mathbb{T})),$$

and the solution (η, ψ) satisfies $(\eta|_{t=T}, \psi|_{t=T}) = (\eta_{final}, \psi_{final})$.

Strategy of the proof

To simplify, we shall only consider the case $b = +\infty$ (infinite depth).

A. Diagonalization of the equations.
B. Quasi-linear scheme.
C. Controllability/Observability for some linear dispersive equations with non-constant coefficients.
D. Reduction to linear constant coefficient equations (modulo some zero order operators).
E. Ingham type inequalities.

A. Diagonalization of the equations 1/2

Linearized equations around 0 :

$$\begin{cases} \partial_t \eta = |D_x| \psi \\ \partial_t \psi + g\eta - \partial_x^2 \eta = P_{ext} \end{cases}$$

Introduce

$$L = \left((g - \partial_x^2)|D_x|\right)^{\frac{1}{2}} =: \langle D_x \rangle |D_x|^{\frac{1}{2}}.$$

Then $u = \psi - iL|D_x|^{-1}\eta$ satisfies the dispersive equation

$$\partial_t u + iLu = P_{ext}.$$

Similar **diagonalization** of the nonlinear equations, based on

- pseudodifferential calculus (Lannes) ;
- paradifferential calculus (Alazard, Métivier) ;
- symmetrization (Alazard, Burq, Zuily) ;
- normal forms and analysis of quadratic terms (Alazard, Delort) ;
- conjugation (Alazard, Baldi).

A. Diagonalization of the equations 2/2

We follow the analysis of Alazard-Burq-Zuily.

Let $s > 5/2$ and consider $(\eta, \psi) \in C^0\big([0,T]; H_0^{s+\frac{1}{2}}(\mathbb{T}) \times H^s(\mathbb{T})\big)$. There is a symbol p of order 0 and a symbol q of order 1/2 (depending on (η, ψ)) such that

$$u := T_p \psi + i T_q \eta \qquad (\in \mathbb{C})$$

satisfies

$$\boxed{\partial_t u + T_V \partial_x u + i L^{\frac{1}{2}}\big(T_c L^{\frac{1}{2}} u\big) + R(u)u = T_p P_{ext}}$$

with $c := (1 + (\partial_x \eta)^2)^{-\frac{3}{4}}$, $V = (\partial_x \phi)|_{y=\eta}$,

$$\|R(\eta,\psi)u\|_{H^s} \le C\big(\|\eta\|_{H^{s+\frac{1}{2}}}\big) \|\eta\|^{\theta}_{H^{s+\frac{1}{2}}} \Big\{ \|\psi\|_{H^s} + \|\eta\|_{H^{s+\frac{1}{2}}} \Big\}.$$

Moreover, (p, q) can be so chosen that $(\eta, \psi) \mapsto u$ is invertible from $H_0^{s+\frac{1}{2}}(\mathbb{T}) \times H^s(\mathbb{T})$ to $\tilde{H}^s(\mathbb{T}) = \big\{u \in H^s \; : \; \int_{\mathbb{T}} \operatorname{Im} u(x)\, dx = 0\big\}$.

B. Quasi-linear scheme

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- The (u_{n+1}, f_{n+1}) are defined by induction : f_{n+1} is determined by asking that the unique solution u_{n+1} to the Cauchy problem

$$(\partial_t + T_{V(u_n)}\partial_x + iL^{\frac{1}{2}} T_{c(u_n)} L^{\frac{1}{2}})u_{n+1} + R(u_n)u_{n+1} = T_{p(u_n)}\chi_\omega \operatorname{Re} f_{n+1}$$

$$u_{n+1}|_{t=0} = u_{in},$$

satisfies $u_{n+1}|_{t=T} = 0$.

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Then define $P_{ext} := \lim_{n\to+\infty} \operatorname{Re} \chi_\omega f_{n+1}$ and solve the WW system for (η, ψ) with data (η_{in}, ψ_{in}) with this pressure seen as a source term. Then u solves the previous equation so $u|_{t=T} = 0$ which in turn implies that $(\eta, \psi)|_{t=T} = 0$.

We obtain the Theorem by **reversibility** of WW.

C. Controllability of a linear equation with non-constant coefficients

Consider the linear op. $P(\underline{u}) = \partial_t + T_{V(\underline{u})}\partial_x + iL^{\frac{1}{2}}\left(T_{c(\underline{u})}L^{\frac{1}{2}}\cdot\right) + R(\underline{u})$
satisfying • $\|V(\underline{u})\| + \|c(\underline{u}) - 1\| + \|R(\underline{u})\|_{\mathcal{L}(L^2)} \lesssim \|\underline{u}\|_{C^0([0,T];H^{s_0})}$.

• If $P(\underline{u})v$ is real valued, then $\frac{d}{dt}\int_{\mathbb{T}} \operatorname{Im} v(t,x)\, dx = 0$.

Proposition 1

Let $T > 0$. There exists s_0 such that, if $\|\underline{u}\|_{C^0([0,T];H^{s_0})}$ is small enough, depending on T, then the following properties hold.

*i) **(Controllability)** For all $\sigma \geq s_0$ and all $u_{in} \in \tilde{H}^\sigma(\mathbb{T})$ there exists f satisfying $\|f\|_{C^0([0,T];H^\sigma)} \leq K(T)\|u_{in}\|_{H^\sigma}$ such that the unique solution u to $P(\underline{u})u = T_p\chi_\omega \operatorname{Re} f \quad ; \; u|_{t=0} = u_{in}$, satisfies $u|_{t=T} = 0$.*

ii) (Stability) Consider another state $\underline{u}'$ with $\|\underline{u}'\|_{C^0([0,T];H^{s_0})}$ small enough and denote by f' the control associated to $\underline{u}'$. Then

$$\left\|f - f'\right\|_{C^0([0,T];H^{\sigma-\frac{3}{2}})} \leq K'(T)\|u_{in}\|_{H^\sigma}\left\|\underline{u} - \underline{u}'\right\|_{C^0([0,T];H^{s_0})}.$$

Write $P = \partial_t + T_V \partial_x + iL^{\frac{1}{2}}\left(T_c L^{\frac{1}{2}} \cdot\right) + R$.
Reduce the analysis by proving that it is enough

- to prove an L^2-result instead of a Sobolev-result ;
- to consider a *classical* equation instead of a *paradifferential* equation.

The idea is to commute the equation with an **elliptic semi-classical operator** $\Lambda_{h,s}$ of order s, satisfying

$$\left\| [\Lambda_{h,s}, P] \Lambda_{h,s}^{-1} \right\|_{\mathcal{L}(L^2)} = O(1),$$

$$\left\| [\Lambda_{h,s}, \chi_\omega] \Lambda_{h,s}^{-1} \right\|_{\mathcal{L}(L^2)} = O(h),$$

which is the reason to introduce the small parameter h.

D. Reduction to a regularized equation 2/3

A possible choice is $\Lambda_{h,s} = I + h^s T_{c^{2s/3}} L^{2s/3}$ which satisfies the following.

i) $\Lambda_{h,s}$ is invertible from H^s to L^2 and

$$\left\| \left[\Lambda_{h,s}, T_V \partial_x \right] \Lambda_{h,s}^{-1} \right\|_{\mathcal{L}(L^2)} = O(1),$$

$$\left\| \left[\Lambda_{h,s}, \chi_\omega \right] \Lambda_{h,s}^{-1} \right\|_{\mathcal{L}(L^2)} \lesssim h \left\| \chi_\omega \right\|_{H^{s+1}} .$$

ii) Setting $\widetilde{P} = \Lambda_{h,s} P \Lambda_{h,s}^{-1}$, we have
$\widetilde{P} = \partial_t + T_V \partial_x + iL^{\frac{1}{2}}(T_c L^{\frac{1}{2}} \cdot) + R_1 = \partial_t + V\partial_x + iL^{\frac{1}{2}}(cL^{\frac{1}{2}} \cdot) + R_2$ with

$$\left\| R_1 \right\|_{\mathcal{L}(L^2)} = O(1),$$

$$R_2 = R_1 + (V - T_V)\partial_x + i(L^{\frac{1}{2}}(cL^{\frac{1}{2}} \cdot) - L^{\frac{1}{2}}(T_c L^{\frac{1}{2}} \cdot)).$$

$$\widetilde{P} = \Lambda_{h,s} P \Lambda_{h,s}^{-1} = \partial_t + V(\underline{u})\partial_x + iL^{\frac{1}{2}}(c(\underline{u})L^{\frac{1}{2}}\cdot) + R_2(\underline{u}).$$

Proposition 2

Let $T > 0$. There exists s_0 such that, if $\|\underline{u}\|_{C^0([0,T];H^{s_0})}$ is small enough, then the following properties hold.

i) **(Controllability up to imaginary constants)** *For all $v_{in} \in L^2(\mathbb{T})$ there exists f with $\|f\|_{C^0([0,T];L^2)} \leq K(T)\|v_{in}\|_{L^2}$ such that the unique solution v to $\widetilde{P}(\underline{u})v = \chi_\omega \operatorname{Re} f, \quad v_{|t=0} = v_{in}$ is such that $v(T)$ is an imaginary constant :*

$$\exists b \in \mathbb{R}, \ \forall x \in \mathbb{T}, \quad v(T,x) = ib.$$

ii) (Regularity) Moreover $\|f\|_{C^0([0,T];H^{\frac{3}{2}})} \leq K(T)\|v_{in}\|_{H^{\frac{3}{2}}}$.
iii) (Stability) Consider another state $\underline{u}'$ with $\|\underline{u}'\|_{C^0([0,T];H^{s_0})}$ small enough and denote by f' the control associated to $\underline{u}'$. Then

$$\|f - f'\|_{C^0([0,T];L^2)} \leq K'(T)\|v_{in}\|_{H^{\frac{3}{2}}}\|\underline{u} - \underline{u}'\|_{C^0([0,T];H^{s_0})}.$$

D. Reduction to a regularized equation 3/3

Let $v_{in} \in \tilde{H}^s$ and seek $f \in C^0([0,T]; H^s(\mathbb{T}))$ such that

$$Pv = T_p \chi_\omega \operatorname{Re} f, \quad v(0) = v_{in} \Longrightarrow \quad v(T) = 0.$$

Assume that Prop 2, i) holds.

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Assume that Prop 2, i) holds. For all $y \in L^2(\mathbb{T})$ there is $\widetilde{f} \in C^0([0, T]; L^2(\mathbb{T}))$ s.t. the unique solution u_1 to

$$\widetilde{P} u_1 = \chi_\omega \operatorname{Re} \widetilde{f}, \quad u_1|_{t=0} = y \implies u_1(T, x) = ib, \ b \in \mathbb{R}.$$

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Define $\mathcal{K} y = u_2(0)$ where u_2 is the solution to

$$\widetilde{P} u_2 = \left(\Lambda_{h,s} T_p \chi_\omega \Lambda_{h,s}^{-1} - \chi_\omega\right) \operatorname{Re} \widetilde{f}, \quad u_2|_{t=T} = 0.$$

Then, with $f := \Lambda_{h,s}^{-1} \widetilde{f}$ and $v := \Lambda_{h,s}^{-1}(u_1 + u_2)$,

$$Pv = T_p \chi_\omega \operatorname{Re} f, \quad v(0) = \Lambda_{h,s}^{-1}(y + \mathcal{K} y), \quad v(T,x) = ib, \ b \in \mathbb{R}.$$

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But $I + \mathcal{K}$ is invertible for h small, and $y \in L^2$ can be so chosen that

$$y + \mathcal{K} y = \Lambda_{h,s} v_{in}.$$

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Then, with $f := \Lambda_{h,s}^{-1} \widetilde{f}$ and $v := \Lambda_{h,s}^{-1}(u_1 + u_2)$,

$$Pv = T_p \chi_\omega \operatorname{Re} f, \quad v(0) = \Lambda_{h,s}^{-1}(y + \mathcal{K} y), \quad v(T,x) = ib, \; b \in \mathbb{R}.$$

But $I + \mathcal{K}$ is invertible for h small, and $y \in L^2$ can be so chosen that

$$y + \mathcal{K} y = \Lambda_{h,s} v_{in}.$$

Since $\frac{d}{dt} \int_{\mathbb{T}} \operatorname{Im} v(t,x)\, dx = 0$ and $\int_{\mathbb{T}} \operatorname{Im} v(0,x)\, dx = 0$, we deduce $v(T) = 0$. Which is the control result of the first prop.

D. Reduction to constant coefficients 1/2

The next step is to replace $\tilde{P} = \partial_t + V\partial_x + iL^{\frac{1}{2}}(cL^{\frac{1}{2}}\cdot) + R_2$ by

$$Q := \Phi \tilde{P} \Phi^{-1} = \partial_t + W\partial_x + iL + R_3,$$

where $W = W(t,x)$ satisfies $\int_{\mathbb{T}} W(t,x)\,dx = 0$ and

$$\|W\|_{C^0([0,T];H^{s_0-d})} \leq C\,\|V\|_{C^0([0,T];H^{s_0})} + \|c-1\|_{C^0([0,T];H^{s_0})}$$

where $d > 0$ is a universal constant, and R_3 is of order zero.
Φ is obtained as a composition of the 3 change of variables (preserving the L^2-norm in x)

$$(1+\partial_x\kappa(t,x))^{\frac{1}{2}}\,h(t,x+\kappa(t,x)),$$

$$h(a(t),x),$$

$$h(t,x-b(t)).$$

(Cancellation of a term of order 1/2.)

D. Reduction to constant coefficients 1/2

We now study **observability** for the **dual** operator

$$Q^* := -\partial_t - W\partial_x - iL + R_4.$$

We seek an operator $\mathcal{A}$ such that

$$i\big[\mathcal{A}, L\big] + W\partial_x\mathcal{A} \quad \text{is a zero order operator.}$$

As in Alazard-Baldi, we find an (non-local) operator of the form

$$\mathcal{A} = \mathrm{Op}\left(q(t,x,\xi)e^{i\beta(t,x)|\xi|^{\frac{1}{2}}}\right)$$

such that

$$\big(\partial_t + W\partial_x + iL + R_4\big)\mathcal{A} = \mathcal{A}\big(\partial_t + iL + R_5\big)$$

with R_5 of order 0 satisfying

$$\|R_5\|_{C^0([0,T];\mathcal{L}(L^2))} \lesssim \|V\|_{C^0([0,T];H^{s_0})} + \|c-1\|_{C^0([0,T];H^{s_0})} + \|R\|_{C^0([0,T];\mathcal{L}(L^2))}.$$

E. Ingham type inequalities 1/3

We now study Ingham inequalities for sums of oscillatory functions whose phases differ from the one of the linearized equation. For some given real-valued function $\beta \in C^3(\mathbb{R})$, set

$$\mu_n(t) = \mathrm{sign}(n) \left[\langle n \rangle |n|^{\frac{1}{2}} t + \beta(t) |n|^{\frac{1}{2}} \right].$$

Not perturbative since $e^{i\beta(t)|n|^{\frac{1}{2}}} - 1$ is not small.

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Not perturbative since $e^{i\beta(t)|n|^{\frac{1}{2}}} - 1$ is not small.
We follow the usual strategy and first prove a **high frequency result** : there exists $\delta_1 > 0$ such that, if

$$\left\|(\partial_t\beta, \partial_t^2\beta)\right\|_{L^\infty} \leq \delta_1,$$

then, for any $T > 0$, there is $N_0 \in \mathbb{N}$ such that,

$$\boxed{\frac{T}{2}\sum_{\substack{n\in\mathbb{Z}\\ |n|\geq N_0}} |w_n|^2 \leq \int_0^T \Big|\sum_{\substack{n\in\mathbb{Z}\\ |n|\geq N_0}} w_n e^{i\mu_n(t)}\Big|^2 dt.}$$

For $\beta = 0$ this result holds with $N_0 = 0$: Kahane, Ball-Slemrod, Haraux.

We then prove upper bounds :

- for any sequence (w_n) (not only for high frequencies) ;
- with **amplitudes** ζ_n depending on time **whose derivatives in time of order** k **can grow with** n **as** $|n|^{k/2}$.

Proposition 3

There are $\delta_2, A > 0$ *such that, if* $\left\|(\partial_t \beta, \partial_t^2 \beta, \partial_t^3 \beta)\right\|_{L^\infty} \leq \delta_2$, *then,*

$$\int_0^T \left| \sum_{n \in \mathbb{Z}} w_n \zeta_n(t) e^{i\mu_n(t)} \right|^2 dt \leq A N(\zeta)^2 T \sum_{n \in \mathbb{Z}} |w_n|^2$$

with

$$N(\zeta) := \sum_{0 \leq k \leq 2} \sup_{n \in \mathbb{Z}} \sup_t \left| (1 + |n|)^{-\frac{k}{2}} \partial_t^k \zeta_n(t) \right| .$$

Using the upper bound, we finally get :

Proposition 4 (*Sharp Ingham type inequality*)

For any $T \in (0,1]$ *there are* $C(T)$ *and* $\delta(T)$ *such that, if*

$$\left\|(\partial_t \beta, \partial_t^2 \beta, \partial_t^3 \beta)\right\|_{L^\infty} \leq \delta(T),$$

then

$$C(T) \sum_{n \in \mathbb{Z}} |w_n|^2 \leq \int_0^T \left| \sum_{n \in \mathbb{Z}} w_n e^{i\mu_n(t)} \right|^2 dt.$$

Remainder on the strategy

A. Diagonalization of the equations.

B. Quasi-linear scheme.

C. **Controllability/Observability for some linear dispersive equations with non-constant coefficients.**

D. Reduction to linear constant coefficient equations (modulo some zero order operators).

E. Ingham type inequalities.

Proposition 5 (*Observability, as a consequence of Ingham*)

Consider $\omega \subset \mathbb{T}$ *and* $T > 0$. *There are* $\varepsilon_0 > 0$ *and* $K > 0$ *such that, if*

$$\sup_{t\in[0,T]} \sum_{1\leq k\leq 3} \left\| \partial_t^k W(t) \right\|_{H^1} + \sup_{t\in[0,T]} \| W(t) \|_{H^r} \leq \varepsilon_0,$$

then, for any initial data v_0 *whose mean value* $\langle v_0 \rangle = \int_{\mathbb{T}} v_0(x)\,dx$ *satisfies*

$$|\mathrm{Re}\langle v_0\rangle| \geq \frac{1}{2}\,|\langle v_0\rangle| - \varepsilon\|v_0\|_{L^2}, \qquad (\Delta)$$

the solution v *of*

$$\partial_t v + iLv = 0, \quad v(0) = v_0,$$

satisfies

$$\int_0^T \int_\omega |\mathrm{Re}(\mathcal{A}v)(t,x)|^2 \, dxdt \geq K \int_{\mathbb{T}} |v_0(x)|^2 \, dx.$$

Assume $W = 0$ so that $\mathcal{A} = I$. Then $(*)$ is false if $v(t,x) = C$ with $C \in i\mathbb{R}$.

C. HUM Method 1/2

One deduces an observability result for $Q^* = -\partial_t - W\partial_x - iL + R_4$.
Now let M such that $M - 1$ is small enough and introduce

$$L^2_M := \left\{ \varphi \in L^2(\mathbb{T};\mathbb{C})\,;\, \operatorname{Im} \int_{\mathbb{T}} M(x)\varphi(x)\,dx = 0 \right\}.$$

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(Functions of L^2_M satisfy (Δ).)
For any $w_{in} \in L^2(\mathbb{T})$, there exists a unique $f_1 \in L^2_M$ such that,

$$\forall \phi_1 \in L^2_M, \quad \mathrm{Re} \int_0^T \big(\chi_\omega \,\mathrm{Re}\, f(t), \phi(t)\big)\,dt = -\,\mathrm{Re}(w_{in}, \phi(0)),$$

where f and ϕ are the unique functions in $C^0(\mathbb{R}; L^2(\mathbb{T}))$ satisfying

$$Q^* f = 0, \qquad f_{|t=T} = f_1,$$
$$Q^* \phi = 0, \qquad \phi_{|t=T} = \phi_1.$$

C. HUM Method 2/2

- H.U.M. method : There exists a control f such that the solution of

$$Qw = \partial_t w + W\partial_x w + iLw + R_3 w = \chi_\omega \operatorname{Re} f, \quad w(0) = w_{in},$$

satisfies $w(T,x) = ibM(x)$ for some constant $b \in \mathbb{R}$.

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• Since $Q = \Phi^{-1}\widetilde{P}\,\Phi$, Φ is a change of variable (**local** transformation), one deduces L^2-controllability for $\widetilde{P}$ from L^2-controllability for Q (applied with a smaller control domain $\omega' \subset \omega$).

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Choosing $M(x) = \Phi(1)$ (which is not $M = 1$), the final state is constant. There exists a control f such that the solution of

$$\partial_t v + V\partial_x v + iL^{\frac{1}{2}}\left(cL^{\frac{1}{2}}v\right) + R_2 v = \chi_\omega \operatorname{Re} f, \quad v(0) = v_{in},$$

satisfies

$$v(T,x) = ib,$$

for some constant $b \in \mathbb{R}$. This proves the control result of Prop. 2.
For the stability estimates, some extra work is needed...

Merci pour votre attention !