Observation of solutions of the Neutron Transport Equation in the diffusion limit.

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Collaboration with Kim Dang Phung

Two approaches.

1 Combine the estimates (now classical involving Green function or Carleman estimates) for observation control and stabilization of solutions of the diffusion equation with the diffusion approximation to obtain related results for the solutions of the neutron transport equation.

2 Use the diffusion approximation to contribute to the understanding of the classical estimates for the diffusion equation.

Transport Equation.

$$\partial_t f(x,v,t) + v \cdot \nabla_x f = \int_{\mathbf{R}^N} [k(x,v',v)f(v') - k(x,v,v')f(v)]dv'$$

$$\text{Detailed balance } \int_{\mathbf{R}^N} k(x,v,v')dv' = \int_{\mathbf{R}^N} k(x,v',v)dv'$$

$(x,v,t) \mapsto f(x,v,t)$ is a function defined on the phase space $\Omega \times \mathbf{R}^N$ and describes the density of particles which at the point x and time t do have the velocity v.
It was as introduced by Lorentz (1905) for a gas of electrons.
Many further applications, electrons in semi conductors , plasma , radiative transfer and interaction of neutrons with uranium kernels.

An elementary version which contains most of the mathematical subtleties $x \in \Omega \quad \omega \in \mathbf{S}^{N-1}$

$$\partial_t u_\epsilon + \frac{\omega \cdot \nabla_x u_\epsilon}{\epsilon} + \frac{1}{\epsilon^2}\sigma_\epsilon(x)(u_\epsilon - \overline{u}_\epsilon) = 0\,, \quad \overline{u} = \frac{1}{|\mathbf{S}^{N-1}|}\int u(x,\omega)d\omega$$

- All the particles have the same kinetic energy $v = \omega \in \mathbf{S}^{N-1}$
- $\Omega \subset \mathbf{R}^N_x$ is either the torus $\mathbf{T}^N = \mathbf{R}^N/\mathbf{Z}^N$ or a domain with smooth boundary and exterior normal $\vec{n}_x$ In this second case a boundary condition has to be prescribed:

Specular :

$$x \in \partial\Omega \Rightarrow u(x,\omega) = u(x,\mathcal{R}(\omega))\,, \mathcal{R}(\omega) = \omega - 2\vec{n_x}(\omega \cdot \vec{n_x})$$

Absorbing :

$$u(x,\omega,t)\Big|(\Gamma_-) = 0\,, \Gamma_- = \{(x,\omega) \in \partial\Omega \times \mathbf{S}^{N-1}\,,\, \omega \cdot \vec{n}_x < 0\,\}$$

Or other types....

Scaling

ϵ is a scaling parameter which will play an important role later and $\sigma_\epsilon(x) \geq 0$ represent the opacity of the media.
The behavior of the solution results from the competition between the advection term $\omega \cdot \nabla_x$ and the relaxation term $\sigma(x)(u - \overline{u})$ and the role of ϵ is to measure the relative strenght of these two effects.
More over such ϵ is motivated by physic.
Examples:

- scattering cross-section of neutrons collisions of non fission type in uranium oxide $\simeq 10\mathrm{cm}^{-1}$
- scattering cross-section of neutrons collisions in water $0.1\mathrm{cm}^{-1}$
- 100 times more in uranium than in water.

The best way to compare effect of the advection and the effect of the relaxation to equilibrium is to consider the the diffusion approximation .

Also useful for the issue of time asymptotic behavior.

This may emphasize the way macroscopic estimates work for control and observation.

Formal estimates for $t > 0$ Valid for all type of boundary condition.

- Maximum principle:

$$\inf_{(x,\omega)\in\Omega\times\mathbf{S}^{N-1}} u_\epsilon(x,\omega,0) \le u_\epsilon(x,\omega,t) \le \sup_{(x,\omega)\in\Omega\times\mathbf{S}^{N-1}} u_\epsilon(x,\omega,0)\,;$$

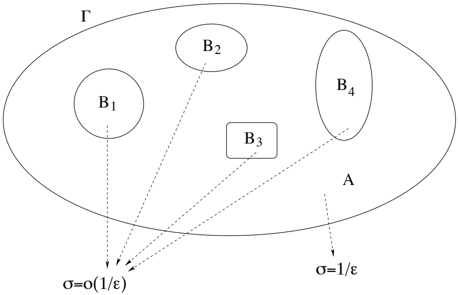
- Energy estimate:

$$\frac{1}{2}\int_{\Omega\times\mathbf{S}^{N-1}} |u_\epsilon(x,\omega,t)|^2 dxd\omega + \int_{\Omega\times\mathbf{S}^{N-1}} \frac{1}{\epsilon^2}\sigma_\epsilon(x)(u_\epsilon - \overline{u}_\epsilon)^2 dxd\omega dx$$
$$= \frac{1}{2}\int_{\Omega\times\mathbf{S}^{N-1}} \| u_\epsilon(x,\omega,0)|^2 dxd\omega\,.$$

Hence the solution is described (in any $L^p\,, 1 < p \le \infty$) by a contraction semi group $e^{t\mathcal{T}_\epsilon}$.

Discontinuous and infinite diffusion

Both for physical reasons and to contribute to the understanding of relations with observation and control theory it is convenient to consider the degenerate diffusion non only with discontinuous diffusion coefficients but also for degenerate diffusion and to use variational formulation:



The spatial domain $\Omega = A \cup B$ with smooth boundary $\Gamma = \partial\Omega$; here $B = B_1 \cup \ldots \cup B_4$

Well posed Variationnal formulation.

$$M(x) = M(x)^T \in \mathbf{R}^{N\times N}\,, \forall x \in \Omega \quad \xi\cdot M(x)\xi \geq \alpha|\xi|^2$$

Assume $M(x) \in L^\infty(\overline{A})$ infinite in B discontinuous on some closed surfaces $\Sigma \subset A$.

$$\mathcal{H} = L^2(\Omega) \cap \{\nabla v\big|B = 0\}\,,$$

$$\mathcal{V} := \{v \in H^1(\Omega : M^{\frac{1}{2}})\backslash \nabla v\big|B = 0\} = \mathcal{H} \cap H^1(\Omega)$$

$$t \mapsto \rho(t) \in C(\mathbf{R}_{t_+};\mathcal{H}) \cap L^2(\mathbf{R}_{t_+};\mathcal{V}) \,\text{with}\, \partial_t\rho \in L^2(\mathbf{R}_{t_+};\mathcal{V}')\,,\ \rho\big|_{t=0} = \rho^{in}$$

$$\frac{d}{dt}\int_\Omega \rho(t,x)w(x)dx + \int_\Omega \nabla w(x)\cdot M(x)\nabla_x\rho(t,x)dx = 0\,, \forall w \in \mathcal{V}\,.$$

$$\frac{1}{2}\int_\Omega |\rho(t,x)|^2dx + \int_0^t\int_\Omega \nabla_x\rho(s,x)\cdot M(x)\nabla_x\rho(s,x)dxds$$

$$= \frac{1}{2}\int_\Omega |\rho(0,x)|^2dx\,.$$

Equivalent formulation of the variational problem and Convergence theorem

Theorem.

(C. B., E. Bernard, F. Golse , R. Sentis)

$$\begin{cases} \partial_t \rho = \frac{1}{N} \operatorname{div}_x(\sigma(x)^{-1} \nabla_x \rho)\,, \quad x \in A \\ \partial_{\vec{n}} \rho \big|_{\partial\Omega} = 0\,, \\ [< nM \cdot \nabla_x \rho >= \dfrac{\partial \rho}{\partial n_M}]\big|_\Sigma = 0 \quad \rho\big|_{\partial B_l} = \rho_l(t)\,, \quad l = 1, \dots, m \\ \dot{\rho}_l(t) = \dfrac{1}{|B_l|} \displaystyle\int_{\partial B_l} \sigma(x)^{-1} \frac{\partial \rho}{\partial n}(t,x) dS(x) \\ \rho\big|_{t=0} = \rho^{in} \end{cases}$$

Then u_ϵ with initial data $u_\epsilon(x,\omega,0) = \rho^{in} \to \rho$ in $L^2(\Omega \times \mathbf{S}^{N-1})$ uniformly in $t \in [0, T]$.

Operator Formalism

With $\mathcal{V}$ being dense in $\mathcal{H}$ and the evolution equation is described in $\mathcal{H}$ by the analytic semi group $e^{-t\mathcal{A}}$ which preserve the domain of $\mathcal{A}^\alpha, \quad 0 \le \alpha\,.$
In particular

$$\mathcal{V} = D(\mathcal{A}^{\frac{1}{2}}) \quad \text{and} \quad \|e^{-t\mathcal{A}}u\|_{\mathcal{V}} \le \frac{C}{\sqrt{t}}\|u\|\|_{\mathcal{H}}$$

Main ideas in the proof

For simplicity, consider the case where $\sigma \equiv 1$ in A (diffusive region) and $\sigma \equiv 0$ in B_l for $l = 1, \ldots, m$

(1) A priori energy estimate

$$\frac{1}{2}\int_\Omega \fint_{\mathbf{S}^{N-1}} u_\epsilon(t,x,\omega)^2 d\omega dx + \frac{1}{\epsilon^2}\int_0^\infty \int_A \fint_{\mathbf{S}^{N-1}} (u_\epsilon - \overline{u}_\epsilon)^2(s,x,\omega) d\omega dx ds \leq \frac{1}{2}\int_\Omega \rho^{in}(x)^2 dx$$

By Banach-Alaoglu's theorem:

$$u_\epsilon \rightharpoonup u \text{ in } L^\infty(\mathbf{R}_+; L^2(\Omega \times \mathbf{S}^{N-1})) \quad \text{and}$$
$$q_\epsilon := \frac{1}{\epsilon}(u_\epsilon - \overline{u}_\epsilon) \rightharpoonup q \text{ in } L^2(\mathbf{R}_+ \times A \times \mathbf{S}^{N-1})$$

(2) Limiting structure of u_ϵ on A as $\epsilon \to 0$:

$$u(t,x,\omega) = \rho(t,x)\,, \qquad \text{for a.e. } \omega \in \mathbf{S}^{N-1}\,,\ x \in A\ t > 0$$

(3) Limiting structure of u_ϵ on B_l as $\epsilon \to 0$:

$$\epsilon \partial_t u_\epsilon + \omega \cdot \nabla_x u_\epsilon = 0 \quad x \in B_l\,,\ |\omega| = 1$$

Therefore

$$\omega \cdot \nabla_x u = 0 \quad x \in B_l\,,\ |\omega| = 1$$

In addition

$$\epsilon\partial_t u_\epsilon + \omega\cdot\nabla_x u_\epsilon = -\mathbf{1}_{x\in A} q_\epsilon \text{ bounded in } L^2_{t,x,\omega}$$

implying equality of the internal and external traces of u on ∂B_l:

$$\omega\cdot\nabla_x u = 0 \quad x\in B_l\,,\ |\omega| = 1 \qquad \text{and } u\big|_{\partial B_l\times \mathbf{S}^{N-1}} = \rho\big|_{\partial B_l}$$

Therefore $x\mapsto u(t,x,\omega)$ is constant along a.e. straight line with direction ω (characteristics), and since B_l is convex

$$\rho(t,x) = \rho_l(t) := \fint_{\partial B_l} \rho(t,x)ds(x)\,, \quad x\in\partial B_l$$

and thus

$$u(t,x,\omega) = u_l(t) := \fint_{B_l} \overline{u}(t,x)dx = \rho_l(t)\,, \quad x\in\partial B_l\,,\ |\omega| = 1$$

(4) Write transport equation in the sense of distributions with test functions $\phi \equiv \phi(x)$ in $\mathcal{V} := \{v \in H^1(\Omega)\,, \nabla v_{|B} = 0\}$, where

$$\frac{d}{dt}\int_\Omega \overline{u}_\epsilon(t,x)\phi(x)dx = \frac{1}{\epsilon}\int_\Omega \omega\cdot\nabla_x\phi(x)\overline{u}_\epsilon(t,x)dx$$
$$= \frac{1}{\epsilon}\int_A \overline{\omega u_\epsilon}(t,x)\cdot\nabla_x\phi(x)dx$$
$$= \int_A \overline{\omega q_\epsilon}(t,x)\cdot\nabla_x\phi(x)dx$$

because $\overline{\omega u}_\epsilon = \overline{\omega(u_\epsilon - \overline{u}_\epsilon)} = \overline{\omega q}_\epsilon$

(5) Observe that, on A

$$\overline{\omega q_\epsilon} = -\overline{\omega^{\otimes 2}\cdot\nabla_x u_\epsilon} - \epsilon\overline{\omega\partial_t u_\epsilon} \rightharpoonup -\frac{1}{N}\nabla_x\rho$$

in $L^2(\mathbf{R}_+ \times A \times \mathbf{S}^{N-1})$

(6) Finally $u_\epsilon \rightharpoonup \rho \equiv \rho(t,x)$ in $L^\infty(\mathbf{R}_+; L^2(\Omega \times \mathbf{S}^{N-1}))$ with

$$\rho \in L^2(\mathbf{R}_+;\mathcal{V}) \cap C_b(\mathbf{R}_+;\mathcal{H})$$

and, for each test function $\phi \in \mathcal{V} = H^1 \cap \{\nabla v\big|_A = 0\}$,

$$\frac{d}{dt}\int_\Omega \overline{u}_\epsilon(t,x)\phi(x)dx = \int_A \frac{1}{N}\nabla_x\rho(t,x)\cdot\nabla_x\phi(x)dx\,.$$

This is the Lions-Magenes variational formulation of the diffusion problem with infinite diffusion in B_l for each $l = 1,\ldots,m$.
Transmission condition results from the fact that both $\rho(t,\cdot)$ and ϕ belong to $\mathcal{H}$.

Theorem.

On B $\sigma_\epsilon(x) = o(\epsilon^2)$, *on A* $\sigma_\epsilon(x) \geq a > 0$, $u_\epsilon(x,\omega,0) = \rho(x,0) \Rightarrow$ $u_\epsilon \to \rho$ *in* $C(\mathbf{R}^t_t; L^2(\Omega)) \cap L^2(\mathbf{R}_t; \mathcal{V})$ *strong*.

$$\frac{1}{2}\int_\Omega |\rho(t,x)|^2 dx + \int_0^t \int_\Omega \frac{1}{N\sigma(x)} |\nabla_x \rho(s,x)|^2 dxds = \frac{1}{2}\int_\Omega |\rho(0,x)|^2 dx$$

$$\frac{1}{2}\int_{\Omega\times\mathbf{S}^{N-1}} |u_\epsilon(x,\omega,t)|^2 dxd\omega + \int_0^t \int_{\Omega\times\mathbf{S}^{N-1}} \frac{1}{\epsilon^2}\sigma_\epsilon(x)(u_\epsilon - \overline{u}_\epsilon)^2 dxd\omega ds$$

$$= \frac{1}{2}\int_{\Omega\times\mathbf{S}^{N-1}} |\rho(x,0)|^2 dx\,.$$

$$\lim_{\epsilon\to 0} \frac{1}{2}\int_{\Omega\times\mathbf{S}^{N-1}} |u_\epsilon(x,\omega,t)|^2 dxd\omega \geq \frac{1}{2}\int_\Omega |\rho(t,x)|^2 dx$$

$$\lim_{\epsilon\to 0} \int_0^t \int_{\Omega\times\mathbf{S}^{N-1}} \frac{1}{\epsilon^2}\sigma_\epsilon(x)(u_\epsilon - \overline{u}_\epsilon)^2 dxd\omega ds \geq$$

$$\int_0^t \int_{\Omega\times\mathbf{S}^{N-1}} (\text{weak} \lim_{\epsilon\to 0}(\frac{\sqrt{\sigma_\epsilon(x)}(u_\epsilon - \overline{u}_\epsilon)}{\epsilon}))^2 d\omega dxds$$

End of Proof

$$\text{On } \; B \; \sigma_\epsilon(x) = o(\epsilon^2) \Rightarrow \frac{\sqrt{\sigma_\epsilon(x)}(u_\epsilon - \overline{u}_\epsilon)}{\epsilon} \rightharpoonup 0$$

$$\text{On } \; A \; \frac{\epsilon}{\sqrt{\sigma_\epsilon(x)}}\partial_t u_\epsilon + \frac{1}{\sqrt{\sigma_\epsilon(x)}}\omega \cdot \nabla_x u_\epsilon + \frac{\sqrt{\sigma_\epsilon(x)}(u_\epsilon - \overline{u}_\epsilon)}{\epsilon} = 0$$

$$\Rightarrow \frac{\sqrt{\sigma_\epsilon(x)}(u_\epsilon - \overline{u}_\epsilon)}{\epsilon} \rightharpoonup -\frac{1}{\sqrt{\sigma_\epsilon(x)}}\omega \cdot \nabla_x \rho$$

$$\lim_{\epsilon \to 0} \int_0^t \int_{\Omega \times \mathbf{S}^{N-1}} \frac{1}{\epsilon^2}\sigma_\epsilon(x)(u_\epsilon - \overline{u}_\epsilon)^2 dx d\omega ds$$

$$\geq \int_0^t \int_\Omega \frac{1}{\sigma(x)} \int_{\mathbf{S}^{N-1}} (\omega \cdot \nabla_x \rho(x,s))^2 d\omega dx ds$$

$$= \int_0^t \int_\Omega \frac{1}{N\sigma(x)} |\nabla_x \rho(s,x)|^2 dx ds \, .$$

Improving the error estimate with $B = \emptyset$

Theorem.

For $u_\epsilon(x, v, t)$ solution of the rescaled transport equation (with, in the presence of boundary, reflection or absorbing condition) and initial data $u_\epsilon(x, v, 0) = \rho(x, 0)$ in $\mathcal{V}$, the convenient space ($H^1_0(\Omega)$ for absorbing boundary condition and $H^1(\Omega)$ for reflection boundary condition) one has:

$$\|u_\epsilon(x, v, t) - \rho(x, t)\|_{L^2(\Omega\times \mathbf{S}^{N-1})} \leq \epsilon^{\frac{1}{2}} C \|\rho_0\|^{\frac{1}{2}}_{\mathcal{V}} \|\rho_0\|^{\frac{1}{2}}_{L^2(\Omega)} .$$

Proof

$$\partial_t(u_\epsilon - (\rho - \epsilon\omega\cdot\nabla_x\rho)) + \frac{\omega\cdot\nabla_x(u_\epsilon - (\rho - \epsilon\omega\cdot\nabla_x\rho))}{\epsilon}$$
$$+\frac{1}{\epsilon^2}\sigma_\epsilon(x)((u_\epsilon - (\rho - \epsilon\omega\cdot\nabla_x\rho)) - \overline{((u_\epsilon - (\rho - \epsilon\omega\cdot\nabla_x\rho))}$$
$$= \epsilon\partial_t\omega\cdot\nabla_x\rho + (-\partial_t\rho + (\omega\cdot\nabla_x)^{\otimes^2}\rho)\,.$$

With ρ solution of the limit equation and

$$(u_\epsilon - (\rho - \epsilon\omega\cdot\nabla_x\rho)) = (u_\epsilon - \overline{u_\epsilon}) + (\overline{u_\epsilon} - \rho + \epsilon\omega\cdot\nabla_x\rho)$$

one has:

$$\int_{\Omega\times\mathbf{S}^{N-1}}(-\partial_t\rho + (\omega\cdot\nabla_x)^{\otimes^2}\rho)((u_\epsilon - \overline{u_\epsilon}) + (\overline{u_\epsilon} - \rho + \epsilon\omega\cdot\nabla_x\rho))dxd\omega$$
$$= \int_{\Omega\times\mathbf{S}^{N-1}}(-\partial_t\rho + (\omega\cdot\nabla_x)^{\otimes^2}\rho)(u_\epsilon - \overline{u_\epsilon})dxd\omega\,.$$

And eventually:

$$|\int_0^t\int_{\Omega\times\mathbf{S}^{N-1}}(-\partial_t\rho+(\omega\cdot\nabla_x)^{\otimes^2}\rho)(u_\epsilon-\overline{u_\epsilon})dxd\omega ds|$$
$$\leq \epsilon(\int_0^t\int_{\Omega\times\mathbf{S}^{N-1}}(-\partial_t\rho+(\omega\cdot\nabla_x)^{\otimes^2}\rho)^2dxd\omega ds)^{\frac12}$$
$$(\int_0^t\int_{\Omega\times\mathbf{S}^{N-1}}\frac{1}{\epsilon^2}(u_\epsilon-\overline{u_\epsilon})^2dxd\omega ds)^{\frac12}$$
$$\leq C\epsilon\|\rho(.,0)\|_{H^1(\Omega)}\|\rho(.,0)\|_{L^2(\Omega)}$$

By the same token

$$\int_0^t\int_{\Omega\times\mathbf{S}^{N-1}}\epsilon\partial_t(\omega\cdot\nabla_x\rho)(u_\epsilon-\rho+\epsilon\omega\cdot\nabla_x\rho)dxd\omega ds$$
$$=\epsilon^2\int_0^t\int_{\Omega\times\mathbf{S}^{N-1}}\partial_t(\omega\cdot\nabla_x\rho)\frac{u_\epsilon-\overline{u_\epsilon}}{\epsilon}dxd\omega ds$$
$$\frac{\epsilon^2}{2}\int_{\Omega\times\mathbf{S}^{N-1}}((\omega\cdot\nabla_x\rho(x,t))^2-(\omega\cdot\nabla_x\rho(x,0))^2)dxd\omega|$$
$$\leq C\epsilon^2\|\rho(.,0)\|_{H^1(\Omega)}\|\rho(.,0)\|_{L^2(\Omega)}\,.$$

Observation of Transport via diffusion

- For use of radiative material for non destructive analysis.
- For potential application for control and stabilisation.
- For a better understanding of the relations between different estimates closely related to solutions.
- The only thing available is to combine classical estimates for diffusion approximation with the above error estimate.

Observation and frequency

$$u'(t) + \mathcal{A}u = 0 \quad \mathcal{A} = \mathcal{A}^*\,, \quad (\mathcal{A}(u), v) = a(u, v) \quad a(u, u) \geq \|u\|^2_{\mathcal{V}}\,,$$

$$\mathrm{f}_k(u) = \mathrm{f}_k(u_0) = \frac{(\mathcal{A}^{k+1}(u_0), u_0)}{(\mathcal{A}^k(u_0), u_0)}$$

$$N_k(t) = \frac{(\mathcal{A}^{k+1}(u)(t), u(t))}{(\mathcal{A}^k(u)(t), u(t))}\,, \quad \frac{dN^k, (t)}{dt} < 0 \Rightarrow N^k(t)(u) \leq f_k(u)\,,$$

$$(\mathcal{A}^k u(0), u(0)) \leq e^{T\mathrm{f}_k(u_0)}(\mathcal{A}^k u(T), u(T))\,.$$

- If a solution $u(x,T)=e^{T\mathcal{A}}u_0\,,\, u_0\in\mathcal{H}$ is equal to 0 for any $x\in\Omega$ it is identically equal to zero on $\Omega\times\mathbf{R}_t^+$.
- If $\partial_x^k u(x,T)=\partial_x^k(e^{T\mathcal{A}}u_0)(x)=0$ for all k at one point $x_0\in A$ $u(x,t)\equiv 0$ on $A\times\{T\}$.

Hence every where (Landis and Oleinik (1968-1974). In particular if $u(x,T)$ is zero on an open set $\tilde\Omega\subset\Omega$

Theorem.

Let U being a domain of unique continuation (for sake of simplicity $U=\tilde\Omega\times\{T\}$ or $U=\tilde\Omega\times(T-\delta,T)$ with $\tilde\Omega\subset A$.
Then for any $s>0$ there exists a bounded positive function $f_s(\|u_0\|_{D(\mathcal{A}^s)})$ such that one has:

$$\|u_0\|_{L^2(\Omega)}\le f_s(U,\|u_0\|_{D(\mathcal{A}^s)})\|e^{t\mathcal{A}}u_0\|_{L^2(U)}$$

Proof By contradiction and compactness.

An example of general but " conditional " observation theorem.

Theorem.

For any $\tilde{\Omega} \subset \Omega$ and any $s > 0$ there exists a bounded function $f(\tilde{\Omega}, s, \|\rho_0\|_{D(\mathcal{A}^s)})$ such that any sequence of solutions $u_\epsilon(x, \omega, t)$ of the transport equation with well prepared initial data $u_\epsilon(x, \omega, 0) = \rho_0(x) \in D(\mathcal{A}^s)$ satisfies the relation:

$$\|\rho_0\|_{L^2(\Omega)} \leq f(\tilde{\Omega}, s, \|\rho_0\|_{D(\mathcal{A}^s)}) \lim_{\epsilon \to 0} \|u_\epsilon\|_{L^2(\tilde{\Omega} \times \mathbf{S}^{N-1})} \tag{1}$$

Proof

If (??) would be wrong there would exists a sequence of $u_{\epsilon(n)}(x,\omega,t)$ of solutions of the transport equation with initial data $u_{\epsilon(n)}(x,\omega,0) = \rho_n(x,0)$ and $\|\rho(x,0)\|_{D(\mathcal{A}^s)}) \leq C < \infty$ such that

$$\|\rho_n(x,0)\|_{L^2(\Omega)} \geq \alpha > 0 \text{ and } \lim_{\epsilon\to 0} \|u_{\epsilon(n)}\|_{L^2(\tilde{\Omega}\times \mathbf{S}^{N-1})} = 0$$

Then with the point wise (in time) convergence of the diffusion approximation and the compact injection of $D(\mathcal{A}^s)$ in $\mathcal{H}$ the sequence $\rho_n(x,0) \to \rho(x,0) \neq 0$ which will be the initial data for a solution $e^{-t\mathcal{A}}(\rho(x,0))$ which vanishes on the set of uniqueness $\tilde{\Omega}\times\{T\}$ Hence the contradiction.

Remark

Assume $\Omega = A \cup B$ with $B \neq \emptyset$ then for the diffusion equation the $u \equiv 0$ propagates from any subset of unique continuation to $A \times \mathbf{R}_t^+ \Rightarrow \rho = 0$ on $\partial B \times \mathbf{R}_t^+$ and therefore on B since $\rho(x,t)$ does not depends on x in B and therefore one has the global unique continuation as used above.
On the other hand $\rho = 0$ in $B \times \mathbf{R}_t^+$ implies that $\rho(x,t) = 0$ for $x \in \partial B$.
However for $\partial_n \rho_A(x,t)$ one has only

$$\frac{1}{|B_l|}\int_{\partial B_l} \sigma(x)^{-1}\frac{\partial \rho_A}{\partial n}(t,x)dS(x) = 0$$

which does not seems enough to obtain the unique continuation in A .

This indicates that the rate function $f_s(U, \|u_0\|_{D(\mathcal{A}^s)})$ *may be very sensitive to the variations of* $\sigma(x)$ *even when* $\sigma(x) \geq \alpha > 0$. *and that in presence of the relaxation term the things may not be so simple depending on the size of the action of this operator.*

Start from estimations of observation for the genuine diffusion equation....Obtained with Carleman estimates or / with comparison with the Green function of the heat equation... For instance I will start from (Phung and Wang) : For some c one has:

$$\|e^{T\Delta}\rho_0\|_{L^2(\Omega)} \leq De^{\frac{c^2}{T}}\|\rho_0\|_{L^2(\Omega)}^{\frac{c}{1+c}}\|e^{T\Delta}\rho_0\|_{L^2(\tilde\Omega)}^{\frac{1}{1+c}}$$

With $T \geq \delta > 0$ and frequency estimate this gives:

$$\|\rho_0\|_{H^1(\Omega)} \leq f_1(\rho_0)\|\rho_0\|_{L^2(\Omega)} \leq Df_1(\rho_0)e^{(c_1 \mathrm{f}_0(\rho_0)T)}\left(\int_{\tilde\Omega}(e^{T\Delta}\rho_0)(x))^2dx\right)^{\frac{1}{2}}$$

Evaluation of f in term of frequency and for $B = \emptyset$ II

Theorem.

For any $\tilde{\Omega} \subset \Omega$ there exists 4 "universal constants" δ, c, C, D such that for $0 < \delta < T$ any solution $u_\epsilon(x, \omega, t)$ of the rescaled transport equation well prepared initial data $u_\epsilon(x, \omega, 0) = \rho_0(x) \in H^1_0(\Omega)$ satisfies the relation:

$$\|\rho_0\|_{H1(\Omega)}(1 - \sqrt{\epsilon} C D \mathrm{f}_1(\rho_0) e^{c \mathrm{f}_0(\rho_0) T}) \leq D \mathrm{f}_1(\rho_0) e^{c \mathrm{f}_0(\rho_0) T} \|u_\epsilon\|_{L^2(\tilde{\Omega} \times \mathbf{S}^{N-1})} .$$

In all what follows C represent a constant related to the diffusion approximation and D to local and frequency estimates for the diffusion equation.

Corollary.

For ϵ small enough with respect to the frequency of the solution

$$\epsilon \leq (CD\mathrm{f}_1(\rho_0))^{-2} e^{-2c\mathrm{f}_0(\rho_0)T} \alpha^2 \quad \textit{with } 0 < \alpha < 1$$

one has the following estimate on the source in term of observation on an arbitrarily small small open set $\tilde{\Omega} \subset \Omega$ at any time $T \geq \delta > 0$

$$\|\rho_0\|_{H^1(\Omega)} \leq \frac{D\mathrm{f}_1(\rho_0)}{1-\alpha} e^{c\mathrm{f}_0(\rho_0)T} \|u_\epsilon\|_{L^2(\tilde{\Omega}\times \mathbf{S}^{N-1})} \,.$$

Proof

Start from

$$\|\rho_0\|_{H^1(\Omega)} \leq D\mathrm{f}_1(\rho_0)e^{c\mathrm{f}_0(\rho_0)T}\|\rho(x,T)\|_{L^2(\tilde{\Omega})}$$

$$\leq D\mathrm{f}_1(\rho_0)e^{c\mathrm{f}_0(\rho_0)T}\|\rho(x,T)-u_\epsilon\|_{L^2(\tilde{\Omega}\times \mathbf{S}^{N-1})} + D\mathrm{f}_1(\rho_0)e^{c\mathrm{f}_0(\rho_0)T}\|u_\epsilon\|_{L^2(\tilde{\Omega}}$$

$$\leq \sqrt{\epsilon}CD\mathrm{f}_1(\rho_0)e^{c\mathrm{f}_0(\rho_0)T}\|\rho_0\|_{H^1(\Omega)} + D\mathrm{f}_1(\rho_0)e^{2c\mathrm{f}_1(\rho_0)T}\|u_\epsilon\|_{L^2(\tilde{\Omega}\times \mathbf{S}^{N-1})}$$

or finally:

$$\|\rho_0\|_{H^1(\Omega)}(1 - CD\mathrm{f}_1(\rho_0)\sqrt{\epsilon}e^{c\mathrm{f}_0(\rho_0)T}) \leq D\mathrm{f}_1(\rho_0)e^{c\mathrm{f}_1(\rho_0)T}\|u_\epsilon\|_{L^2(\tilde{\Omega}\times \mathbf{S}^{N-1})}\,.$$

Comments I

• Without the advection term the information would not propagate from one part of the domain to the rest of this domain therefore the advection operator plays an important role.

• On the other hand no hypothesis of geometric control is present. This is due to the fact that the effect of the relaxation term is enhanced when $\epsilon \to 0$ then all directions are involved.

• It is only at the limit that the action of the diffusion is determining. An interpretation of the connection between these two aspect would be to consider the operator

$$\frac{\sigma(x)}{\epsilon^2}(u(x,\omega) - \overline{u(x)})$$

as a "jump process" similar to a stochastic process $\sigma\partial\mathcal{B}_t$ in the derivation of the Brownian motion.

The competition between advection and relaxation may also appear in the following approach.

$$\partial_t u_\epsilon + \frac{1}{\epsilon}\omega \cdot \nabla_x u_\epsilon + \frac{1}{\epsilon^2}(u_\epsilon(x,\omega) - \overline{u_\epsilon}) = 0$$

$$\partial_t \overline{u_\epsilon} + \nabla_x \frac{\overline{\omega u_\epsilon}}{\epsilon} = 0$$

$$\partial_t \overline{u_\epsilon} - \overline{(\omega \cdot \nabla_x)^{\otimes^2} u_\epsilon} = \overline{\partial_t \omega \cdot \nabla u_\epsilon}$$

$$\epsilon \partial_t^2 \overline{u_\epsilon} + \overline{\partial_t \omega \cdot \nabla u_\epsilon} = 0$$

$$\partial_t \overline{u_\epsilon} + \epsilon \partial_t^2 \overline{u_\epsilon} - \frac{1}{N}\Delta \overline{u_\epsilon} = \overline{(\omega \cdot \nabla_x)^{\otimes^2}(u_\epsilon - \overline{u_\epsilon})} = \epsilon r_\epsilon \,.$$

A perturbation of

$$\epsilon \partial_t^2 u_\epsilon - \Delta u_\epsilon + \partial_t u_\epsilon = 0$$

Then Lopez, Zhang and Zuazua J. Math. Pures Appl. 79, 8 (2000)

Comment III

- The proof of the convergence estimates works also for a piecewise σ with constant diffusion with discontinuities on a surface Σ

$$0 < \alpha \le \sigma_\epsilon(x) \le \beta < \infty$$

reason is that with $\quad q_+ = q_- \quad$ and $\quad \frac{1}{\sigma_+}\partial_n q_+ = \frac{1}{\sigma_-}\partial_n q_-$

the interface term

$$\frac{1}{\epsilon}\int_\Sigma \int_{\mathbf{S}^{N-1}} \omega\cdot\vec{n}[(u_\epsilon - \rho_+ + \epsilon\frac{1}{\sigma_+}\omega\cdot\nabla_x\rho_+)^2 \\ -(u_\epsilon - \rho_- + \epsilon\frac{1}{\sigma_-}\omega\cdot\nabla_x\rho_-)^2]d\omega d\Sigma = 0\,.$$

Then Le Rousseau Robbiano Invent math (2011).

- I do not know how to extend the above result to the case of $x \in B \neq \emptyset \Rightarrow \sigma_\epsilon(x) \to 0$. There is the above rate of convergence f but no explicit estimate!

What about Le Rousseau -Robbiano in the presence of transparent region??

Above estimates are made for finite time $0 < t < T$ and $\epsilon \to 0$ to compare with more classical results consider in $\Omega \times \mathbf{S}^{N-1}$ (Ω convex and bounded and reflecting boundary conditions, or $\Omega = (\mathbf{R}/\mathbf{Z})^d$ i.e. periodic boundary conditions) solutions of total mean value zero:

$$\partial_t u_\epsilon + \frac{\omega \cdot \nabla u_\epsilon}{\epsilon} + \frac{\sigma_\epsilon}{\epsilon^2}(u_\epsilon - \overline{u_\epsilon}) = 0$$

$$\int\int u_\epsilon(x,\omega)dxd\omega = 0$$

Then the solution is given by a contraction semi group $e^{-t\mathcal{T}_\epsilon}$

1 If $\forall x \in \Omega \quad \sigma_\epsilon(x) \geq \alpha > 0$ then spectra$(e^{-t\mathcal{T}_\epsilon}) \cap \{e^{-t\alpha} < |z| < 1\}$ is a set of eigenvalues of finite multiplicity with the leading eigenvalue

$$e^{t\lambda_\epsilon}$$

being simple real and and strictly less than 1 Moreover when $\epsilon \to 0$ this eigenvalue converge to the leading eigenvalue of the diffusion equation. (Spectral analysis goes back to Ghidouche-Point-Ukai) convergence to diffusion toSentis and B. -Santos-Sentis.

2 If $\sigma_\epsilon(x) > 0$ on an open set $\tilde{\Omega} \subset \Omega$ of positive measure then for any u_0 and ϵ fixed one has:

$$\lim_{t\to\infty} e^{-tT_\epsilon} u_0 = 0$$

this convergence is not uniform and may not be exponential. Bernard Salvarani , Han-Kwan Leautaud.

3 If a GCC is satisfied then with respect to the support of $\sigma(x)$ then one has for fix ϵ uniform exponential convergence. Han-Kwan Leautaud. What happens for $\epsilon \to 0$ has not to the best of my knowledge yet being considered.

4 The proof of 3 relies on the observation estimate (for ϵ fixed)

$$\|u_\epsilon(x,\omega,0)\|_{L^2(\Omega\times\mathbf{S}^{N-1})} \leq C(\epsilon)\int_0^{T(\epsilon)}\int_{\Omega\times\mathbf{S}^{N-1}}\frac{\sigma_\epsilon}{\epsilon^2}(x)(u_\epsilon-\overline{u}_\epsilon)^2 dxd\omega dt$$

which is not valid without GCC (GCC$\Leftrightarrow C(\epsilon)<\infty$) .
With $C(\epsilon)$ uniformly bounded (Did not check if this is true) and well prepared initial data $u_\epsilon(x,,0)=\rho_0(x)$ this would correspond in the diffusion approximation to the estimate to the estimate:

$$\|\rho_0\|_{L^2(\Omega)} \leq C \lim_{\epsilon\to 0}\int_0^t\int_{A\times\mathbf{S}^{N-1}}\frac{1}{\epsilon^2}\sigma_\epsilon(x)(u_\epsilon-\overline{u}_\epsilon)^2 dxd\omega ds$$
$$=\int_0^t\int_A\frac{1}{N\sigma(x)}|\nabla_x\rho(s,x)|^2 dxds\,.$$