Using saturated controls for the stabilization of the wave and KdV equations

Christophe PRIEUR
CNRS, Gipsa-lab, Grenoble, France

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Given a PDE, there exists now a large variety on methods to design linear controllers. It is well known that saturation can reduce the performance or even destabilize the system, even for finite-dimensional systems.
More precisely, even if

$$\dot{z} = Az + BKz$$

is asymp. stable, it may hold that

$$\dot{z} = Az + \mathtt{sat}(BKz)$$

is not globally asymptotically stable.
It may exist new equilibrium, new limit cycles...
See e.g. [Khalil; 1996] and [Tarbouriech, et al; 2011]
Goal of this talk:
What happens for the linear wave equation?
For the nonlinear KdV equation?
In presence of bounded and unbounded control operators?

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For the nonlinear KdV equation?
In presence of bounded and unbounded control operators?

Two objectives

- Well-posedness
- Stability

of the wave and KdV equations in presence of a distributed/boundary saturating control.

Outline

1. Well-posedness and stability of linear wave equation with a saturated in-domain control

   Lyapunov method, LaSalle invariance principle

2. Well-posedness and stability of linear wave equation with a saturated boundary control

   Lyapunov method, local sector condition

3. Well-posedness and stability of nonlinear KdV equation with a saturated in-domain control

   Local sector condition, contradiction argument

4. Numerical simulations on nonlinear KdV equation and nonlinear KdV equation + sat

5. Conclusion

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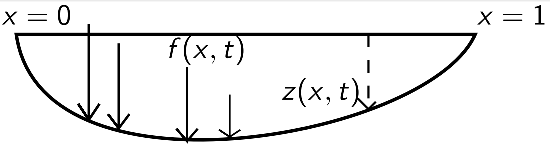

1D wave equation with bounded control operator.

Dynamics of the vibration:

$$z_{tt}(x,t) = z_{xx}(x,t) + f(x,t),\ \forall x \in (0,1), t \geq 0, \qquad (1)$$

Boundary conditions, $\forall t \geq 0$,

$$\begin{aligned} z(0,t) &= 0\ , \\ z(1,t) &= 0\ , \end{aligned} \qquad (2)$$

and with the following initial condition, $\forall x \in (0,1)$,

$$\begin{aligned} z(x,0) &= z^0(x)\ , \\ z_t(x,0) &= z^1(x)\ , \end{aligned} \qquad (3)$$

where z^0 and z^1 stand respectively for the initial deflection and the initial deflection speed.

When closing the loop with a linear control

Let us define the linear control by

$$f(x,t) = -az_t(x,t), x \in (0,1), \ \forall t \geq 0, \tag{4}$$

and consider

$$V_1 = \frac{1}{2}\int (z_x^2 + z_t^2)dx.$$

Formal computation. Along the solutions to (1), (2) and (4):

$$\begin{aligned} \dot{V}_1 &= \int_0^1 (z_x z_{xt} - az_t^2 + z_t z_{xx})dx \\ &= -\int_0^1 az_t^2 dx + [z_t z_x]_{x=0}^{x=1} \\ &= -\int_0^1 az_t^2 dx \end{aligned}$$

Thus, it $a > 0$, V_1 is a (non strict) Lyapunov function.

Using standard technics (Lumer-Philipps thereom (for the well-posedness) and Huang-Prüss theorem (for the exp. stability)):

Proposition

$\forall a > 0$, $\forall (z^0, z^1)$ in $H_0^1(0,1) \times L^2(0,1)$,
$\exists$! solution z: $[0, \infty) \to H_0^1(0,1) \times L^2(0,1)$ to (1)-(4). Moreover, $\exists$ C, $\mu > 0$, such that, for any initial condition $H_0^1(0,1) \times L^2(0,1)$, it holds, $\forall t \geq 0$,

$$\|z\|_{H_0^1(0,1)} + \|z_t\|_{L^2(0,1)} \leq Ce^{-\mu t}(\|z^0\|_{H_0^1(0,1)} + \|z^1\|_{L^2(0,1)}).$$

In the previous proposition:

- stability
- attractivity of the equilibrium
- with an exponential speed

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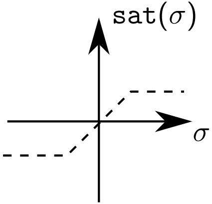
When closing the loop with a saturating control

Let us consider now the nonlinear control

$$f(x,t) = -\mathtt{sat}(az_t(x,t)),\ x \in (0,1),\ \forall t \geq 0, \tag{5}$$

where sat is the localized saturated map:



$$\mathtt{sat}(\sigma) = \begin{cases} \sigma & \text{if } |\sigma| < 1 \\ \mathtt{sign}(\sigma) & \text{else} \end{cases}$$

Equation (1) in closed loop with the control (5) becomes

$$z_{tt} = z_{xx} - \mathtt{sat}(az_t) \tag{6}$$

A formal computation gives, along the solutions to (6) and (2),

$$\dot{V}_1 = -\int_0^1 z_t \mathtt{sat}(az_t) dx$$

which asks to handle the nonlinearity $z_t \mathtt{sat}(az_t)$.

Remark: Choice of the saturation map

[Slemrod, 1989] and [Lasiecka and Seidman, 2003] deal with L^2 saturation:

Given $\sigma : [0,1] \to \mathbb{R}$, $\mathtt{sat}_2(\sigma)$ is the function defined by

$$\mathtt{sat}_2(\sigma)(x) = \begin{cases} \sigma(x) & \text{if } \|\sigma\|_{L^2(0,1)} < 1 \\ \frac{\sigma(x)}{\|\sigma\|_{L^2(0,1)}} & \text{else} \end{cases}$$

Here we consider **localized** saturation which is more physically relevant:

$$\mathtt{sat}(\sigma(x)) = \begin{cases} \sigma(x) & \text{if } |\sigma(x)| < 1 \\ \mathtt{sign}(\sigma(x)) & \text{else} \end{cases}$$

Well-posedness of this nonlinear PDE

Theorem 1 [CP, Tarbouriech, Gomes da Silva Jr; 2015]

$\forall a \geq 0$, for all (z^0, z^1) in $(H^2(0,1) \cap H_0^1(0,1)) \times H_0^1(0,1)$, there exists a unique solution z: $[0, \infty) \to H^2(0,1) \cap H_0^1(0,1)$ to (6) with the boundary conditions (2) and the initial condition (3).

Consider

$$A_1 \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} v \\ u_{xx} - \mathtt{sat}(av) \end{pmatrix}$$

with the domain $D(A_1) = (H^2(0,1) \cap H^1_0(0,1)) \times H^1_0(0,1)$.
Let us use a generalization of Lumer-Phillips theorem which is the so-called Crandall-Liggett theorem, as given in [Barbu; 1976]. See also [Brezis; 1973] and [Miyadera; 1992].
Again two conditions

1. A_1 is dissipative, that is

$$\Re\left(\langle A_1 \begin{pmatrix} u \\ v \end{pmatrix} - A_1 \begin{pmatrix} \tilde{u} \\ \tilde{v} \end{pmatrix}, \begin{pmatrix} u \\ v \end{pmatrix} - \begin{pmatrix} \tilde{u} \\ \tilde{v} \end{pmatrix}\rangle\right) \leq 0$$

2. For all $\lambda > 0$, $D(A_1) \subset \mathtt{Ran}(I - \lambda A_1)$

First item: Easy step!

Instead of proving

$\Re\left(\langle A_1 \begin{pmatrix} u \\ v \end{pmatrix} - A_1 \begin{pmatrix} \tilde{u} \\ \tilde{v} \end{pmatrix}, \begin{pmatrix} u \\ v \end{pmatrix} - \begin{pmatrix} \tilde{u} \\ \tilde{v} \end{pmatrix}\rangle\right) \leq 0$, let us check, for all $\begin{pmatrix} u \\ v \end{pmatrix} \in H_0^1(0,1) \times L^2(0,1)$:

$$\Re\left(\langle A_1 \begin{pmatrix} u \\ v \end{pmatrix}, \begin{pmatrix} u \\ v \end{pmatrix}\rangle\right) \leq 0$$

To do that, using the definition of A_1, and of the scalar product in $H_0^1(0,1) \times L^2(0,1)$, it is equal to:

$$\begin{aligned} &\int_0^1 v_x(x)\overline{u_x(x)}dx + \int_0^1 (u_{xx}(x) - \mathtt{sat}(a\,v(x)))\overline{v(x)}dx \ , \\ &= \int_0^1 v_x(x)\overline{u_x(x)}dx + \int_0^1 u_{xx}(x)\overline{v(x)}dx - \int_0^1 \mathtt{sat}(a\,v(x))\overline{v(x)}dx \\ &= [u_x(x)\overline{v(x)}]_{x=0}^{x=1} - \int_0^1 \mathtt{sat}(a\,v(x))\overline{v(x)}dx \leq 0 \end{aligned}$$

due to the boundary and since $a \geq 0$.

Second item asks to deal with a nonlinear ODE.

Let $\begin{pmatrix} u \\ v \end{pmatrix} \in H_0^1(0,1) \times L^2(0,1)$ we have to find $\begin{pmatrix} \tilde{u} \\ \tilde{v} \end{pmatrix} \in D(A_1)$ such that

$$(I - \lambda A_1) \begin{pmatrix} \tilde{u} \\ \tilde{v} \end{pmatrix} = \begin{pmatrix} u \\ v \end{pmatrix}$$

that is

$$\left\{ \begin{array}{c} \tilde{u} - \lambda \tilde{v} = u \ , \\ \tilde{v} - \lambda(\tilde{u}_{xx} - \mathtt{sat}(a\,\tilde{v})) = v \ , \end{array} \right.$$

In particular, we have to find $\tilde{u}$ such that

$$\begin{array}{l} \tilde{u}_{xx} - \frac{1}{\lambda^2}\tilde{u} - \mathtt{sat}(\frac{a}{\lambda}(\tilde{u} - u)) = -\frac{1}{\lambda}v - \frac{1}{\lambda^2}u \\ \tilde{u}(0) = \tilde{u}(1) = 0 \end{array}$$

holds.

Nonhomogeneous nonlinear ODE with two boundary conditions

Lemma

If a is nonnegative and λ is positive, then there exists $\tilde{u}$ solution to

$$\begin{array}{c} \tilde{u}_{xx} - \frac{1}{\lambda^2}\tilde{u} - \mathtt{sat}(\frac{a}{\lambda}(\tilde{u}-u)) = -\frac{1}{\lambda}v - \frac{1}{\lambda^2}u \\ \tilde{u}(0) = \tilde{u}(1) = 0 \end{array} \qquad (7)$$

To prove this lemma, let us introduce the following map

$$\begin{array}{rccl} \mathcal{T}: & L^2(0,1) & \to & L^2(0,1)\ , \\ & y & \mapsto & z = \mathcal{T}(y)\ , \end{array}$$

where $z = \mathcal{T}(y)$ is the unique solution to

$$\begin{array}{c} z_{xx} - \frac{1}{\lambda^2}z = -\frac{1}{\lambda}v - \frac{1}{\lambda^2}u + \mathtt{sat}(\frac{a}{\lambda}(y-u))\ , \\ z(0) = z(1) = 0\ . \end{array}$$

Prove that $\mathcal{T}$ is well defined and apply the Schauder fixed-point theorem (see e.g., [Coron; 2007]), to deduce that there exists y such that $\mathcal{T}(y) = y$

$\tilde{u} = y$ solves (7)

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Theorem 2

$\forall a > 0$, for all (z^0, z^1) in $(H^2(0,1) \cap H_0^1(0,1)) \times H_0^1(0,1)$, the solution to (6) with the boundary conditions (2) and the initial condition (3) satisfies the following stability property, $\forall t \geq 0$,

$$\|z(.,t)\|_{H_0^1(0,1)} + \|z_t(.,t)\|_{L^2(0,1)} \leq \|z^0\|_{H_0^1(0,1)} + \|z^1\|_{L^2(0,1)} ,$$

together with the attractivity property

$$\|z(.,t)\|_{H_0^1(0,1)} + \|z_t(.,t)\|_{L^2(0,1)} \to 0, \quad \text{as } t \to \infty .$$

Due to Theorem 1, the formal computation

$$\dot{V}_1 = -\int_0^1 z_t \mathtt{sat}(a z_t) dx$$

makes sense. This is only a weak Lyapunov function $\dot{V}_1 \leq 0$
(the state is (z, z_t), and there is no $-z^2$).
To be able to apply LaSalle's Invariance Principle, we have to check that the trajectories are precompact
(see e.g. [Dafermos, Slemrod; 1973], [d'Andréa-Novel *et al*; 1994]).
It comes from:

Lemma

The canonical embedding from $D(A_1)$, equipped with the graph norm, into $H_0^1(0,1) \times L^2(0,1)$ is compact.

Sketch of the proof of

The canonical embedding from $D(A_1)$, equipped with the graph norm, into $H_0^1(0,1) \times L^2(0,1)$ is compact.

Consider a sequence $\begin{pmatrix} u_n \\ v_n \end{pmatrix}_{n\in\mathbb{N}}$ in $D(A_1)$, which is bounded with the graph norm, that is $\exists M > 0$, $\forall n \in \mathbb{N}$,

$$\begin{aligned} \left\| \begin{pmatrix} u_n \\ v_n \end{pmatrix} \right\|_{D(A_1)}^2 &:= \left\| \begin{pmatrix} u_n \\ v_n \end{pmatrix} \right\|^2 + \left\| A_1 \begin{pmatrix} u_n \\ v_n \end{pmatrix} \right\|^2 , \\ &= \int_0^1 (\left|u_n'\right|^2 + \left|v_n\right|^2 + \left|v_n'\right|^2 \\ &\quad + \left|u_n'' - a\texttt{sat}(v_n)\right|^2) dx < M \end{aligned}$$

From that, we deduce that $\int_0^1 (|v_n|^2 + |v_n'|^2) dx$ and $\int_0^1 (|u_n'|^2 + |u_n''|^2) dx$ are bounded.
Thus there exists a subsequence which converges in $H_0^1(0,1) \times L^2(0,1)$. $\square$

Using the dissipativity of A_1, and previous lemma the trajectory $\begin{pmatrix} z(.,t) \\ z_t(.,t) \end{pmatrix}$ is precompact in $H_0^1(0,1) \times L^2(0,1)$.

Moreover the ω-limit set $\omega \left[\begin{pmatrix} z(.,0) \\ z_t(.,0) \end{pmatrix} \right] \subset D(A_1)$, is not empty and invariant with respect to the nonlinear semigroup $T(t)$ (see [Slemrod; 1989]).

We now use LaSalle's invariance principle to show that
$\omega \left[\begin{pmatrix} z(.,0) \\ z_t(.,0) \end{pmatrix} \right] = \{0\}$.

Therefore the convergence property holds. □

For linear PDE, we have exponential convergence (see Proposition on Slide 8).

Do we have exp. stability for the nonlinear PDE?

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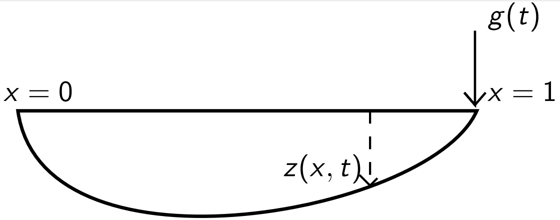

1D wave equation with a boundary control.
Dynamics:

$$z_{tt}(x,t) = z_{xx}(x,t), \ \forall x \in (0,1), t \geq 0, \tag{8}$$

Boundary conditions, $\forall t \geq 0$,

$$\begin{aligned} z(0,t) &= 0 \ , \\ z_x(1,t) &= g(t) \ , \end{aligned} \tag{9}$$

and with the same initial condition, $\forall x \in (0,1)$,

$$\begin{aligned} z(x,0) &= z^0(x) \ , \\ z_t(x,0) &= z^1(x) \ . \end{aligned} \tag{10}$$

When closing the loop with a linear boundary control

Let us define the linear control by

$$g(t) = -bz_t(1,t),\ x \in (0,1),\ \forall t \geq 0 \qquad (11)$$

and consider

$$V_2 = \frac{1}{2}\int (e^{\mu x}(z_t + z_x)^2 dx + \int (e^{-\mu x}(z_t - z_x)^2 dx,$$

Formal computation. Along the solutions to (8), (9) and (11):

$$\dot{V}_2 \quad = \quad -\mu V_2 + \tfrac{1}{2}\left(e^{\mu}(1-b)^2 - e^{-\mu}(1+b)^2\right) z_t^2(1,t)$$

Assuming $b > 0$ and letting $\mu > 0$ such that $e^{\mu}(1-b)^2 \leq e^{-\mu}(1+b)^2$, it holds $\dot{V}_2 \leq -\mu V_2$ and thus V_2 is a strict Lyapunov function and thus (8)-(11) is exponentially stable.

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When closing the loop with a saturating control

Let us consider now the nonlinear control $g(t) = -\mathtt{sat}(bz_t(1,t)), \forall t \geq 0$. The boundary conditions become:

$$z(0,t) = 0\ , \quad z_x(1,t) = -\mathtt{sat}(bz_t(1,t))\ . \qquad (12)$$

Theorem 3

$\forall b > 0$, for all (z^0, z^1) in $\{(u,v),\ (u,v) \in H^2(0,1) \times H^1_{(0)}(0,1),\ u_x(1) + \mathtt{sat}(bv(1)) = 0,\ u(0) = 0\}$, the solution to (8) with the boundary conditions (12) and the initial condition (3) satisfies the following stability property, $\forall t \geq 0$,

$$\|z(.,t)\|_{H^1_{(0)}(0,1)} + \|z_t(.,t)\|_{L^2(0,1)} \leq \|z^0\|_{H^1_{(0)}(0,1)} + \|z^1\|_{L^2(0,1)}\ ,$$

together with the attractivity property

$$\|z(.,t)\|_{H^1_{(0)}(0,1)} + \|z_t(.,t)\|_{L^2(0,1)} \to 0, \quad \text{as } t \to \infty\ .$$

To prove the well-posedness of the Cauchy problem we prove that A_2 defined by

$$A_2 \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} v \\ u'' \end{pmatrix}$$

with the domain $D(A_2) = \{(u, v),\ (u, v) \in H^2(0,1) \times H^1_{(0)}(0,1),\ u'(1) + \mathtt{sat}(bv(1)) = 0,\ u(0) = 0\}$ is a semigroup of contraction.

The global stability property comes directly from the dissipativity of A_2.

The global attractivity property comes from the following lemma:

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The global attractivity property comes from the following lemma:

Lemma (semi-global exponential stability)

For all $r > 0$, there exists $\mu > 0$ such that, for all initial condition satisfying

$$\|z^{0\prime\prime}\|^2_{L^2(0,1)} + \|z^1\|^2_{H^1_{(0)}(0,1)} \leq r^2 \ , \tag{13}$$

it holds

$$\dot{V}_2 \leq -\mu V_2$$

along the solution to (8) with the boundary conditions (12).

Sketch of the proof of this lemma

First note that by dissipativity of A_2, it holds that

$$t \mapsto \left\| A_2 \left(\begin{array}{c} z(.,t) \\ z_t(.,t) \end{array} \right) \right\|$$

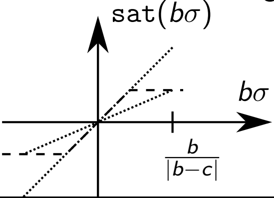
is a non-increasing function. Thus, for all $t \geq 0$,

$$|z_t(1,t)| \leq \left\| A_2 \left(\begin{array}{c} z(.,0) \\ z_t(.,0) \end{array} \right) \right\| . \tag{14}$$

Now for all initial conditions satisfying (13), there exists $c \neq b$ such that, for all $t \geq 0$,

$$(b-c)|z_t(1,t)| \leq 1$$

and thus the following local sector condition holds:



Letting $\sigma = z_t(1,t)$, it holds
$(\mathrm{sat}(b\sigma) - b\sigma)(\mathrm{sat}(b\sigma) - (b-c)\sigma) \leq 0$

We come back to the Lyapunov function candidate V_2. Given $b > 0$, using the previous inequality, we compute

$$\begin{aligned} \dot{V}_2 &= -\mu V_2 + e^{\mu}(\sigma - \mathtt{sat}(b\sigma))^2 - e^{-\mu}(\sigma + \mathtt{sat}(b\sigma))^2 \\ &\leq -\mu V_2 + \begin{pmatrix} \sigma \\ \mathtt{sat}(b\sigma) \end{pmatrix}^\top \begin{pmatrix} e^{\mu} - e^{-\mu} - b^2(b-c) & -e^{\mu} - e^{-\mu} + b + b(b-c) \\ -e^{\mu} - e^{-\mu} + b + b(b-c) & -1 + e^{\mu} - e^{-\mu} \end{pmatrix} \\ &\qquad \times \begin{pmatrix} \sigma \\ \mathtt{sat}(b\sigma) \end{pmatrix} \\ &\leq -\mu V_2 \end{aligned}$$

with a suitable choice of constant values μ and c.
The semi-global exponential stability follows. □

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The semi-global exponential stability follows. □

3 – KdV equation

Let us consider the following nonlinear PDE

$$\begin{cases} z_t + z_x + z_{xxx} + zz_x + f = 0, \ x \in [0, L], \ t \geq 0, \\ z(0, t) = z(L, t) = z_x(L, t) = 0, \ t \geq 0, \\ z(x, 0) = z^0(x), \ x \in [0, L], \end{cases} \tag{KdV}$$

where z stands for the state and f for the control.

As noted in [Rosier; 1997], if $f = 0$ and

$$L \in \left\{ 2\pi\sqrt{\frac{k^2 + kl + l^2}{3}} \middle/ k, l \in \mathbb{N}^* \right\},$$

then, there exist solutions of the linearized version of (KdV) for which the $L^2(0, L)$-energy does not decay to zero. Consider, for instance, $L = 2\pi$ and $z^0 = 1 - \cos(x)$ for all $x \in [0, L]$.

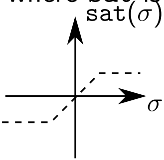
See [Coron, Crépeau; 2004], [Cerpa; 2007] and [Cerpa, Crépeau; 2009] for controllability properties with critical lengths.

Saturating control for KdV

Given $a > 0$, let us consider the KdV equation controlled by a saturated distributed control as follows

$$\begin{cases} z_t + z_x + z_{xxx} + zz_x + \mathtt{sat}(az) = 0, \\ z(0,t) = z(L,t) = 0, \\ z_x(L,t) = 0, \\ z(x,0) = z^0(x). \end{cases} \qquad \text{(KdVsat)}$$

where $\mathtt{sat}$ is the same saturation as before:



$$\mathtt{sat}(\sigma) = \begin{cases} \sigma & \text{if } |\sigma| < 1 \\ \mathtt{sign}(\sigma) & \text{else} \end{cases}$$

In a work in progress with S. Marx, E. Cerpa, and V. Andrieu, we consider also the other saturation $\mathtt{sat}_2$:

Given $\sigma : [0, L] \to \mathbb{R}$, $\mathtt{sat}_2(\sigma)$ is the function defined by

$$\mathtt{sat}_2(\sigma)(x) = \begin{cases} \sigma(x) & \text{if } \|\sigma\|_{L^2(0,L)} < 1 \\ \frac{\sigma(x)}{\|\sigma\|_{L^2(0,L)}} & \text{else} \end{cases} .$$

Theorem 4 [Marx, Cerpa, CP, Andrieu, 2016]: Well posedness

For any initial condition $z^0 \in L^2(0, L)$, there exists a unique solution $z \in C([0, T]; L^2(0, L)) \cap L^2(0, T; H^1(0, L))$ to (KdVsat).

Sketch of proof

First we prove the existence of solution on $[0, T']$ for T' small, as done in [Rosier, Zhang; 2006] and [Chapouly; 2009].
Then we remove the smallness assumption on T' by proving an a priori estimate.

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Theorem 5: Global asymptotic stability.

There exist μ and a class $\mathcal{K}$ function $\alpha : [0, \infty) \to [0, \infty)$ such that for any $z^0 \in L^2(0,1)$, any solution z to (KdVsat) satisfies, for all $t \geq 0$,

$$\|z(t)\|_{L^2(0,1)} \leq \alpha(\|z^0\|_{L^2(0,1)})e^{-\mu t}.$$

In the previous result, we have stability and attractivity properties.

To prove this result, we follow the approach of [Rosier, Zhang; 2006] and [Cerpa; 2014].
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To be more specific:

Lemma: semi-global exponential stability

Given $r > 0$, there exist positive values C and μ such that for all initial condition z^0 satisfying $\|z^0\|_{L^2(0,L)} \leq r$, it holds, for all $t \geq 0$,

$$\|z(t)\|_{L^2(0,L)} \leq C\|z^0\|_{L^2(0,L)} e^{-\mu t}.$$

To do that, the key result is:

$\forall T > 0$, $r > 0$, $\exists$ $C > 0$ *such that for any solution* z *to* (KdVsat) *starting from* $z^0 \in L^2(0, L)$ *with* $\|z^0\|_{L^2(0,L)} \leq r$, *it holds*

$$\|z^0\|^2_{L^2(0,L)} \leq C \left(\int_0^T |z_x(t,0)|^2 dt + 2 \int_0^T \int_0^L \mathrm{sat}(az) z dt dx \right). \tag{15}$$

Assume (15) for the time being. Then with

$$\|z(.,T)\|^2_{L^2(0,L)} = \|z^0\|^2_{L^2(0,L)} - \int_0^T |z_x(0,t)|^2 dt \\ -2\int_0^T \int_0^L \mathtt{sat}(az) z dx dt$$

we get

$$\|z(.,kT)\|^2_{L^2(0,L)} \leq \gamma^k \|z^0\|^2_{L^2(0,L)} \qquad \forall k \geq 0$$

where $\gamma \in (0,1)$. From the dissipativity property, we have $\|z(.,t)\|_{L^2(0,L)} \leq \|z(.,kT)\|_{L^2(0,L)}$ for $kT \leq t \leq (k+1)T$. Thus we obtain, for all $t \geq 0$,

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We conclude the proof of the semi-global exponential stability. □

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$$\begin{aligned}\|z(.,T)\|^2_{L^2(0,L)} &= \|z^0\|^2_{L^2(0,L)} - \int_0^T |z_x(0,t)|^2 dt \\ &\quad -2\int_0^T\int_0^L \mathtt{sat}(az)z dx dt\end{aligned}$$

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Proof of the blue equation (15). By contradiction !

Assume that there exists a sequence of solution z^n to (KdVsat) with

$$\|z^n(.,0)\|_{L^2(0,L)} \leq r \tag{16}$$

and such that

$$\lim_{n\to+\infty} \frac{\|z^n\|^2_{L^2(0,T;L^2(0,L))}}{\int_0^T |z^n_x(0,t)|^2 dt + 2\int_0^T \int_0^L \mathtt{sat}(az^n(x,t))z^n(x,t)dtdx} = +\infty. \tag{17}$$

By dissipativity property, $\exists\ \beta > 0$,

$$\sup_{t\in[0,T]} \|z^n(.,t)\|_{L^2(0,L)} \leq r\ , \quad \sup_{x\in[0,L]} \int_0^T |z^n(x,t)|^2 dt \leq \beta. \tag{18}$$

Now let us define $\Omega_i := \left\{ t \in [0,T],\ \sup_{x\in[0,L]} |z(x,t)| > i \right\} \subset [0,T]$.
We have

$$\beta \geq \int_0^T \sup_{x\in[0,L]} |z^n(x,t)|^2 dt \geq \int_{\Omega_i} \sup_{x\in[0,L]} |z^n(x,t)|^2 dt \geq i^2 \nu(\Omega_i),$$

Therefore, we obtain for the complementary set
$\nu(\Omega_i^c) \geq \max\left(T - \frac{\beta}{i^2}, 0\right)$.

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Moreover, using again the local sector condition, we have, for a suitable positive function k and $\forall i \in \mathbb{N}$,

$$\begin{aligned} \int_0^T \int_0^L \mathtt{sat}(az^n)z^n dtdx &= \int_{\Omega_i} \int_0^L \mathtt{sat}(az^n)z^n dtdx + \int_{\Omega_i^c} \int_0^L \dots \\ &\geq 0 + \int_{\Omega_i^c} \int_0^L ak(i)(z^n)^2 dtdx. \end{aligned} \tag{19}$$

Let $\lambda^n := \|z^n\|_{L^2(0,T;L^2(0,L))}$ and $v^n(x,t) = \frac{z^n(x,t)}{\lambda^n}$. Due to (16), up to extracting a subsequence, we may assume that $\lambda^n \to \lambda \geq 0$. Due to (17) and (19), we have, for all $i \in \mathbb{N}$

$$\int_0^T |v_x^n(0,t)|^2 dt + 2\int_{\Omega_i^c}\int_0^L ak(i)(v^n)^2 dtdx \to 0 \qquad (20)$$

Using Aubin-Lions lemma in [Simon, 1987]: $\{v^n\}_{n\in\mathbb{N}}$ converges strongly in $L^2(0,T;L^2(0,L))$. Thus, with (20), we have, for all $i \in \mathbb{N}$

$$v_x(0,t) = 0,\ \forall t \in (0,T) \text{ and } v(x,t) = 0,\ \forall x \in [0,L], \forall t \in \Omega_i^c.$$

We know that $\nu\left(\bigcup_{i\in\mathbb{N}} \Omega_i^c\right) = T$. We get a contradiction with $\|v\|_{L^2(0,T;L^2(0,L))} = 1$. □

To discretize (KdVsat), we follow the approach of [Pazoto, Sepúlveda, Vera Villagrán; 2010], and solve, at each time-step, a fixed point problem.

(No proof of convergence of the numerical scheme)

Consider the first critical length $L = 2\pi$ and the initial condition $z^0(x) = 1 - \cos(x)$. The energy is constant along the linearized KdV equation.

Consider the control $f(x,t) = \mathtt{sat}(a(x)z(x,t))$ where $a(x) = \mathbf{1}_{[\frac{1}{3}L, \frac{2}{3}L]}(x)$.

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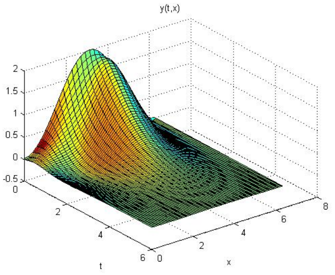

Figure: Solution to (KdV) with linear control

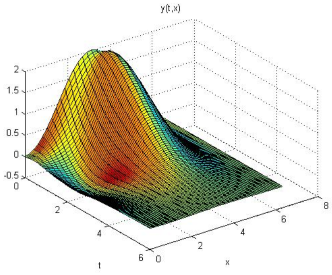

Figure: Solution to (KdVsat) with saturated control

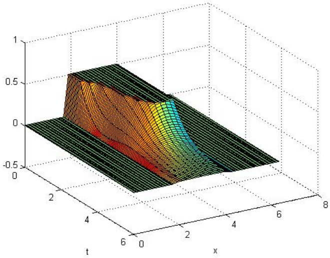

Figure: Control $f = \mathtt{sat}(az)$

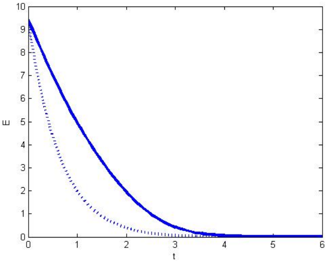

Figure: Time evolution of the energy function $\|z\|^2_{L^2(0,L)}$ for (KdV) ($\cdots$) and for (KdVsat) ($-$)

Conclusion

- Well-posedness and global asymptotic stability for the nonlinear PDEs:

$$\left\{ \begin{array}{l} z_{tt} = z_{xx} - \mathtt{sat}(az_t) \\ z(0,t) = z(1,t) = 0 \end{array} \right. \qquad \left\{ \begin{array}{l} z_{tt} = z_{xx} \\ z(0,t) = 0,\ z_x(1,t) = -\mathtt{sat}(bz(1,t)) \end{array} \right.$$

and also

$$\left\{ \begin{array}{l} z_t + z_x + z_{xxx} + zz_x + \mathtt{sat}(a(x)z) = 0 \\ z(0,t) = z(L,t) = z_x(L,t) = 0 \end{array} \right.$$

- distributed/localized (localized/L^2) saturating control
- Lyapunov function has been used for the wave equation
- non-constructive proof for the KdV equation

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Under actual investigation

Under actual investigation #1

Boundary saturating control for KdV?

$$z_t + z_x + z_{xxx} + zz_x = 0$$
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Under actual investigation #2

In [Marx, Cerpa; 2014] an output feedback control has been computed for the linearized KdV

$$z_t + z_x + z_{xxx} = 0$$
$$y(t) = z_{xx}(L,t) \text{ output}$$
$$z(0,t) = \mathcal{F}(y),\ z(L,t) = z_x(L,t) = 0$$

Could we compute a saturating output feedback control?

Under actual investigation

Under actual investigation #1

Boundary saturating control for KdV?

$$
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$$

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\end{aligned}
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Open questions

Output feedback controller for KdV? Saturating controller?

Exponential convergence for the nonlinear dynamics?

Rapid stabilization, as in [Cerpa, Crépeau; 2009]?