

# Null controllability for a degenerating reaction-diffusion system in electrocardiology

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- Bidomain Model
- Monodomain model
- Control Theory

# Bidomain model

The bidomain model for the propagation of cardiac action potentials was formulated in the late 1970's and is widely used in medical and bioengineering studies to describe the evolution of the electrical potential through the cardiac tissue.

From the mathematical viewpoint, the problem consists of a system of two degenerate parabolic reaction-diffusion equations, coupled with a system of ODEs.

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Consider a bounded subset  $\Omega$  of  $\mathbb{R}^d$  ( $d = 2, 3$ ) representing the myocardium. The bidomain model reads:

$$\left\{ \begin{array}{ll} c_m v_t - \text{Div}(M_i(x)\nabla u_i) + h(v, w) = I_i & \text{in } \Omega \times (0, T), \\ c_m v_t + \text{Div}(M_e(x)\nabla u_e) + h(v, w) = I_e & \text{in } \Omega \times (0, T), \\ w_t + g(v, w) = 0 & \text{in } \Omega \times (0, T) \\ M_i(x)\nabla u_i \cdot \nu = 0, M_e(x)\nabla u_e \cdot \nu = 0 & \text{in } \partial\Omega \times (0, T), \\ v = u_i - u_e & \text{in } \Omega \times (0, T), \\ v(0) = v_0, w(0) = w_0 & \text{in } \Omega. \end{array} \right. \quad (1)$$

Here, the unknowns are the functions  $u_i(t, x) \in \mathbb{R}$ ,  $u_e(t, x) \in \mathbb{R}$  and  $w(t, x) \in \mathbb{R}^m (m \geq 1)$ , which are respectively the intra- and extra-cellular potentials and some ionic variables (currents, gating variables, concentrations, etc.). The variable  $v$  denotes the transmembrane potential. The surface capacitance of the membrane is represented by the constant  $c_m > 0$ . Naturally,  $\nu$  denotes the unit normal to  $\partial\Omega$  outward of  $\Omega$ .

The other data:  $M_{i,e}(x)$ , are conductivity matrices;  $h : \mathbb{R} \times \mathbb{R}^m \rightarrow \mathbb{R}$  and  $g : \mathbb{R} \times \mathbb{R}^m \rightarrow \mathbb{R}^m$  are functions representing the ionic activity in the myocardium and  $I_{i,e} : (0, \infty) \times \Omega \rightarrow \mathbb{R}$  are external applied current sources.

**Remark 3:** The *ionic equation* is related to the electrical behavior of the myocardium cells membrane, in terms of the (vector) variable  $w$  representing the averaged ion concentrations and gating states.



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The bidomain model is sometimes too complicated to work with...

# Monodomain model

If  $M_i = \mu M_e$ , for some  $\mu \in \mathbb{R}$ , then the bidomain is equivalent to:

$$\left\{ \begin{array}{ll} c_m v_t - \frac{\mu}{\mu+1} \operatorname{Div}(M_e(x) \nabla v) + h(v, w) = I & \text{in } \Omega \times (0, T), \\ -\operatorname{Div}(M(x) \nabla u_e) = \operatorname{Div}(M_i(x) \nabla v) & \text{in } \Omega \times (0, T) \\ w_t + g(v, w) = 0 & \text{in } \Omega \times (0, T) \\ M_i(x) \nabla v \cdot \nu = 0, M_e(x) \nabla u_e \cdot \nu = 0 & \text{in } \partial\Omega \times (0, T), \\ v(0) = v_0, w(0) = w_0 & \text{in } \Omega. \end{array} \right. \quad (2)$$

which, nevertheless, conserves some essential features of the full bidomain model as excitability phenomena. For this reason, this system has deserved noticeable attention for itself.

Here,  $M = M_i + M_e$ .

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FROM NOW ON WE FORGET THE ODE'S...

It is natural to ask if the controllability of the monodomain model can be seen as a limit process of the controllability of a family of parabolic systems.

We approximate the Monodomain model by the following parabolic system:

$$\left\{ \begin{array}{ll} c_m v_t^\epsilon - \frac{\mu}{\mu+1} \operatorname{Div}(M_e(x) \nabla v^\epsilon) + h(v^\epsilon) = I^\epsilon & \text{in } \Omega \times (0, T), \\ \epsilon u_{e,t}^\epsilon - \operatorname{Div}(M(x) \nabla u_e^\epsilon) = \operatorname{Div}(M_i(x) \nabla v^\epsilon) & \text{in } \Omega \times (0, T), \\ v^\epsilon(0) = v_0, u_e^\epsilon = u_{e,0} & \text{in } \Omega. \end{array} \right. \quad (3)$$

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**Question:** *If, for each  $\epsilon > 0$ , there exists a control  $I^\epsilon = f^\epsilon \chi$  that drives the solution  $(v^\epsilon, u_e^\epsilon)$  of (3) to zero at time  $t = T$ , i.e.*

$$v^\epsilon(T) = u_e^\epsilon(T) = 0,$$

*is it true that when  $\epsilon \rightarrow 0$  the control sequence  $f^\epsilon$  converges to a function  $f = f \chi$ , that drives the solution  $(v, u_e)$  of (2) (No ODE's) to zero at time  $t = T$ ?*

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Since the bidomain model is a system of two coupled parabolic equations and the monodomain model is a system of parabolic-elliptic type, these two systems have, at least a priori, different control properties.

Systems of parabolic equations which degenerates into parabolic-elliptic ones arise in many areas, such as biology and models describing gravitational interaction of particles.

We consider the following linearized version of (3):

$$\begin{cases}
 c_m v_t^\epsilon - \frac{\mu}{\mu+1} \operatorname{Div}(M_e(x) \nabla v^\epsilon) + a(t, x) v^\epsilon = f^\epsilon \mathbf{1}_\omega & \text{in } Q, \\
 \epsilon u_{e,t}^\epsilon - \operatorname{Div}(M(x) \nabla u_e^\epsilon) = \operatorname{Div}(M_i(x) \nabla v^\epsilon) & \text{in } Q, \\
 v^\epsilon = 0, \quad u_e^\epsilon = 0 & \text{on } \Sigma, \\
 v^\epsilon(0) = v_0, \quad u_e^\epsilon(0) = u_{e,0} & \text{in } \Omega,
 \end{cases} \quad (4)$$

where  $a$  is a bounded function.

Our objective then will be drive both  $v^\epsilon$  and  $u^\epsilon$ , solution of (4), to zero at time  $T$  by means of a control  $f^\epsilon$  in such a way that the sequence of controls  $f^\epsilon$  remains bounded when  $\epsilon \rightarrow 0$ .

Accordingly, we consider the corresponding adjoint system:

$$\begin{cases}
 -c_m \varphi_t^\epsilon - \frac{\mu}{\mu+1} \operatorname{Div}(M_e(x) \nabla \varphi^\epsilon) + a(t, x) \varphi^\epsilon = \operatorname{Div}(M_i(x) \nabla \varphi_e^\epsilon) & \text{in } Q, \\
 -\epsilon \varphi_{e,t}^\epsilon - \operatorname{Div}(M(x) \nabla \varphi_e^\epsilon) = 0 & \text{in } Q, \\
 \varphi^\epsilon = 0, \varphi_e^\epsilon = 0 & \text{on } \Sigma, \\
 \varphi^\epsilon(T) = \varphi_T, \varphi_e^\epsilon(T) = \varphi_{e,T} & \text{in } \Omega.
 \end{cases} \quad (5)$$

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It is very easy to prove that our task turns out to be equivalent to the following observability inequality:

$$\epsilon \|\varphi_e^\epsilon(0)\|_{L^2(\Omega)}^2 + \|\varphi^\epsilon(0)\|_{L^2(\Omega)}^2 \leq C \iint_{Q_\omega} |\varphi^\epsilon|^2 dx dt, \quad Q_\omega := \omega \times (0, T), \quad (6)$$

for all  $(\varphi_T, \varphi_{e,T}) \in L^2(\Omega)^2$  and a constant  $C = C(\epsilon, \Omega, \omega, T)$  that remains bounded when  $\epsilon \rightarrow 0$ .

### **Proof of (6):**

We consider  $\varphi_T$  and  $\varphi_{e,T}$  smooth enough and define

$$\rho^\epsilon(x, t) = \text{Div}(M(x)\nabla\varphi_e^\epsilon(x, t)).$$

This new function satisfies

$$-\epsilon\rho_t^\epsilon - \text{Div}(M(x)\nabla\rho^\epsilon) = 0 \quad \text{in } Q. \quad (7)$$

We apply a Carleman estimate for non degenerate heat equations to equation (5)<sub>1</sub> and apply a sharp Carleman inequality, with respect to  $\epsilon$ , to (7) .

Combining the two inequalities, **and working a bit more**, we obtain a Carleman type estimate in the form:

$$\iint_Q \beta_1^2 |\varphi^\epsilon|^2 dxdt + \iint_Q \beta_2^2 |\rho^\epsilon|^2 dxdt \leq C\epsilon^{-2} \iint_{\omega \times (0, T)} \beta_3^2 |\varphi|^2 dxdt, \quad (8)$$

for some appropriate weight functions  $\beta_i := \beta_i(x, t)$  (for  $i = 1, 2, 3$ ) and some constant  $C = C(\Omega, \omega, \|a\|_\infty, T) > 0$ .

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Next, we choose a weight function  $\beta_4 = \beta_4(t)$  satisfying

$$|(\beta_4)_t(t)| \leq C\beta_2(x, t)$$

for all  $(x, t) \in Q$ .

We are able to show that

$$\iint_Q \beta_4^2 |\rho^\epsilon|^2 dx dt \leq C\epsilon^2 \iint_Q \beta_2^2 |\rho^\epsilon|^2 dx dt. \quad (9)$$

Inequality (9) is proved applying an energy inequality for the heat like equation satisfied by  $\beta_4 \rho^\epsilon$ .

Inequalities (8) and (9) give

$$\iint_Q \beta_4^2 |\rho^\epsilon|^2 dxdt \leq C \iint_{\omega \times (0, T)} \beta_3^2 |\varphi^\epsilon|^2 dxdt \quad (10)$$

and we choose a weight  $\beta_5$  such that

$$\iint_Q \beta_5^2 |\varphi^\epsilon|^2 dxdt \leq C \iint_{\omega \times (0, T)} \beta_3^2 |\varphi^\epsilon|^2 dxdt. \quad (11)$$

Therefore

$$\iint_Q \beta_5^2 |\varphi^\epsilon|^2 dxdt + \iint_Q \beta_4^2 |\rho^\epsilon|^2 dxdt \leq C \iint_{\omega \times (0, T)} \beta_3^2 |\varphi|^2 dxdt. \quad (12)$$

Now it is easy to prove that

$$\|\varphi^\epsilon(0)\|_{L^2(\Omega)}^2 + \epsilon \|\rho^\epsilon(0)\|_{L^2(\Omega)}^2 \leq C \iint_{\omega \times (0, T)} \beta_3 |\varphi^\epsilon|^2 dxdt, \quad (13)$$

for some constant  $C = C(\Omega, \omega, \|a\|_\infty, T) > 0$ .

This gives the observability inequality (6).  $\square$

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Let  $v_0$  and  $u_{e,0}$  be in  $L^2(\Omega)$  and  $q_N$  satisfying

$$q_N \in (2, \infty) \text{ if } N = 1, 2, \quad \frac{N+2}{2} < q_N < 2\frac{N+2}{N-2} \text{ if } N \geq 3. \quad (14)$$

Combining the null controllability for the linear system and an appropriate version of Kakutani's fixed point theorem we can prove the Theorem:

### Theorem

• If  $h$  is  $C^1(\mathbb{R})$ , global lipschitz and satisfies  $h(0) = 0$ . There exist a control  $f^\epsilon \chi_\omega \in L^2(\omega \times (0, T))$  such that the solution  $(v^\epsilon, u_e^\epsilon)$  of (3) satisfies  $t$

$$v^\epsilon(T) = u_e^\epsilon(T) = 0.$$

Besides, the control  $f^\epsilon$  has the estimate

$$\|f^\epsilon \chi_\omega\|_{L^2(Q)} \leq C(\|v_0\|_{L^2(\Omega)} + \epsilon \|u_{e,0}\|_{L^2(\Omega)}). \quad (15)$$

Assuming that

$$h(0) = 0, \quad \frac{h(v_1) - h(v_2)}{v_1 - v_2} \geq -C, \quad \forall v_1 \neq v_2, \quad (16)$$

$$0 < \liminf_{|v| \rightarrow \infty} \frac{h(v)}{v^3} \leq \limsup_{|v| \rightarrow \infty} \frac{h(v)}{v^3} < \infty, \quad (17)$$

we also can prove the following Theorem:

### Theorem

• Let  $h$  be a  $C^1$  function satisfying (16) and (17) and the initial data  $(v_0, u_{e,0}) \in (H_0^1(\Omega) \cap W^{2(1-\frac{1}{q_N}), q_N}(\Omega))^2$ , with  $\|(v_0, u_{e,0})\|_{L^\infty} \leq \gamma$ , for sufficient small  $\gamma$  does not depending on  $\varepsilon$ . There exists a control  $f^\varepsilon \in L^{q_N}(\omega \times (0, T))$  such that the solution  $(v^\varepsilon, u_e^\varepsilon)$  of (3), with  $(v^\varepsilon, u_e^\varepsilon) \in (W_{q_N}^{2,1}(Q))^2$ , satisfies

$$v^\varepsilon(T) = u_e^\varepsilon(T) = 0.$$

Moreover, the control  $f^\varepsilon$  has the estimate

$$\|f^\varepsilon \mathbf{1}_\omega\|_{L^{q_N}(Q)}^2 \leq C (\|v_0\|_{L^2(\Omega)}^2 + \varepsilon \|u_{e,0}\|_{L^2(\Omega)}^2). \quad (18)$$