Observability from measurable sets and bang-bang property of time optimal controls

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Observability inequality from measurable set for the heat equation

To present our motivations, we begin with the simplest situation. Let $T > 0$ and Ω be a bounded Lipschitz domain in $\mathbb{R}^n$. Consider the heat equation

$$\begin{cases} \partial_t u - \Delta u = 0, & \text{in } \Omega \times (0, T), \\ u = 0, & \text{on } \partial\Omega \times (0, T), \\ u(\cdot, 0) = u_0 \in L^2(\Omega). \end{cases}$$

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In this talk, the solution of the heat equation will be treated as either a function from $[0, T]$ to $L^2(\Omega)$ or a function of two variables x and t.

Two important a priori estimates for the above equation are as follows:

$$\|u(T)\|_{L^2(\Omega)} \leq N(\Omega, T, \mathcal{D}) \int_{\mathcal{D}} |u(x,t)|\, dxdt, \ \forall\ u_0 \in L^2(\Omega), \qquad (1)$$

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where $\mathcal{D}$ is a measurable subset of $\Omega \times (0, T)$, and

$$\|u(T)\|_{L^2(\Omega)} \leq N(\Omega, T, \mathcal{J}) \int_{\mathcal{J}} |\tfrac{\partial}{\partial \nu} u(x,t)|\, d\sigma dt, \ \forall\ u_0 \in L^2(\Omega), \qquad (2)$$

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Such a priori estimates are called observability inequalities in Control Theory. Our aim is to build up estimates (1) and (2) when $\mathcal{D}$ and $\mathcal{J}$ are subsets of positive measure and positive surface measure, respectively.

Bang-bang property of time optimal controls

For simplicity, we introduce the time optimal control problem for the heat equation. Let $\omega \subset \Omega$ be an open subset with its characteristic function χ_ω. Consider the time optimal control problem:

$$(TP)^M : \qquad T(M) \triangleq \inf_{u \in \mathcal{U}^M} \big\{ t > 0 : \, y(t; u) = 0 \big\},$$

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where $\mathcal{U}^M$ is the control constraint given by

$$\mathcal{U}^M = \{u \in L^\infty(\Omega \times \mathbb{R}^+) \mid |u(x,t)| \leq M \ \text{ for a.e. } \ (x,t) \in \Omega \times \mathbb{R}^+\},$$

and $y(\cdot; u)$ solves the equation

$$\begin{cases} y_t - \Delta y = \chi_\omega u & \text{in } \ \Omega \times \mathbb{R}^+, \\ y = 0 & \text{on } \ \partial\Omega \times \mathbb{R}^+, \\ y(x,0) = y_0(x) & \text{in } \ \Omega. \end{cases}$$

• We call $T(M)$ the optimal time (if it exists) and u^* a time optimal control if $y(T(M); u^*) = 0$. In fact, the existence of optimal time and time optimal controls are guaranteed by the observability inequality and the decay of energy of the heat equation.

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• In the state space $L^2(\Omega)$, since the set $\left\{y(T(M); u) : u \in \mathcal{U}^M\right\}$ has no interior point in $L^2(\Omega)$, to the best of our knowledge, we do not know how to separate this set from the target set $\{0\}$ by a hyperplane in $L^2(\Omega)$. Thus, we do not know how to get the Pontryagin maximum principle for Problem $(TP)^M$ by the way used in the case of O.D.E..

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• It is natural to ask if the bang-bang property (for simplicity B-B-P) holds for $(TP)^M$: Any time optimal control u^* for $(TP)^M$ verifies $|u^*(x,t)| = M$ for a.e. $(x,t) \in \omega \times (0, T(M))$.

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The argument is very simple. Assume u^* and v^* are two time optimal controls. Since

$$\left|\frac{u^*(x,t)-v^*(x,t)}{2}\right|^2=\frac{|u^*(x,t)|^2+|v^*(x,t)|^2}{2}-\left|\frac{u^*(x,t)+v^*(x,t)}{2}\right|^2,$$

and $(u^*+v^*)/2$ is also a time optimal control, by the B-B-P, we get that $u^*=v^*$ a.e. in $\omega\times(0,T(M))$.

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- The equivalence between the time optimal control problem and norm optimal control problem, and so provide a necessary and sufficient condition for the time optimal control problem.

Existing results for the heat equation

Case 1: $\mathcal{D} = \omega \times (0, T)$ *and* $\mathcal{J} = \Gamma \times (0, T)$, *where* ω *and* Γ *are nonempty open subsets of* Ω *and* $\partial\Omega$, *respectively.*

In this case, both observability inequalities (1) and (2) (where Ω is smooth) were first established by Lebeau-Robbiano and Fursikov-Imanuvilov in 1995.

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Case 2 : $\mathcal{D} = \omega \times E$, where $\omega \subset \Omega$ is an open subset and $E \subset (0, T)$ is of positive measure.

In this case, the interior estimate (2) was build up through an indirect way by G.Wang in 2008, i.e., the author first proved the null controllability for controls restricted over $\omega \times E$, and then made use of the equivalence of the null controllability and observability estimate.

Case 3 : $\mathcal{D} = \omega \times (0, T)$ *and* $\mathcal{J} = \Gamma \times (0, T)$ *with* ω *and* Γ *subsets of positive measure and positive surface measure in* Ω *and* $\partial\Omega$*, respectively.*

In this case, both (1) and (2) were built up by J.Apraiz and L.Escauriaza in 2013, with the help of Lebeau-Robbiano spectral inequality (which will be introduced later), as well as a propagation of smallness estimate from measurable sets for real-analytic functions given by S.Vessella in 1999. Since this kind of propagation of estimate plays a crucial role in our arguments, we introduce it as follows.

Propagation of estimate for real-analytic functions

Theorem

Assume that $f : B_{2R} \subset \mathbb{R}^n \longrightarrow \mathbb{R}$ *is real-analytic in* B_{2R} *verifying*

$$|\partial_x^\alpha f(x)| \leq \frac{M|\alpha|!}{(\rho R)^{|\alpha|}}, \text{ when } x \in B_{2R},\ \alpha \in \mathbb{N}^n,$$

for some $M > 0$ *and* $0 < \rho \leq 1$. *Let* $E \subset B_R$ *be a measurable set with positive measure. Then, there are positive constants* $N = N(\rho, |E|/|B_R|)$ *and* $\theta = \theta(\rho, |E|/|B_R|)$ *such that*

$$\|f\|_{L^\infty(B_R)} \leq N \left(\fint_E |f|\, dx \right)^\theta M^{1-\theta}.$$

S. Vessella, A continuous dependence result in the analytic continuation problem, Forum Math., 11 (1999), 695–703.

Existing results

Case 4 : $\mathcal{D} = \omega \times E$, *with* ω *and* E *subsets of positive measure in* Ω *and* $(0, T)$, *respectively.*

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Note. It is far from clear here for the estimate (2) where $\mathcal{D}$ is a subset of $\Omega \times (0, T)$ with a positive measure. The reason is as: there does exist a subset in $\Omega \times (0, T)$ with a positive measure, which does not contain any product of subsets $F_1 \subseteq \Omega$ and $F_2 \subseteq (0, T)$ with positive measures.

LR Spectral inequality

- For each $\lambda > 0$, we define the projection operator

$$\mathcal{E}_\lambda f = \sum_{\lambda_j \leq \lambda} \langle f, e_j \rangle \, e_j, \quad \forall f \in L^2(\Omega),$$

where $\{\lambda_j\}$ and $\{e_j\}$ are accordingly the sets of eigenvalues and of normalized eigenfunctions of $-\Delta$ with Dirichlet boundary condition.

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where $\{\lambda_j\}$ and $\{e_j\}$ are accordingly the sets of eigenvalues and of normalized eigenfunctions of $-\Delta$ with Dirichlet boundary condition.

- The LR spectral inequality is stated as follows:

For each $0 < R \leq 1$, *there is* $N = N(\Omega, R)$, *such that the inequality*

$$\|\mathcal{E}_\lambda f\|_{L^2(\Omega)} \leq N e^{N\sqrt{\lambda}} \|\mathcal{E}_\lambda f\|_{L^2(B_R(x_0))} \tag{3}$$

holds, when $B_{4R}(x_0) \subset \Omega$, $f \in L^2(\Omega)$ *and* $\lambda > 0$.

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holds, when $B_{4R}(x_0) \subset \Omega$, $f \in L^2(\Omega)$ and $\lambda > 0$.

- To our best knowledge, the above-mentioned spectral inequality has been proved under the condition that Ω is at least C^2-smooth by G.Lebeau, L.Robbiano, E.Zuazua, Q.Lü, etc.

- A Lipschitz domain Ω in $\mathbb{R}^n$ is called locally star-shaped if for each $p \in \partial\Omega$ there are x_p in Ω and $r_p > 0$ such that

$$|p - x_p| < r_p \text{ and } B_{r_p}(x_p) \cap \Omega \text{ is star-shaped with center } x_p.$$

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In particular, it holds the "sign condition" on each partial boundary

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• It is proved that any bounded C^1-smooth domains, polygons in the plane, Lipschitz polyhedrons in $\mathbb{R}^n$, with $n \geq 3$, and bounded convex domains in $\mathbb{R}^n$ are always locally star-shaped.

Main results

Theorem 1 (Spectral inequality)

Let Ω be a bounded and locally star-shaped domain in $\mathbb{R}^n$. Then, Ω verifies the spectral inequality (3).

J. Apraiz, L. Escauriaza, G. Wang and C. Zhang, Observability inequalities and measurable sets. J. Eur. Math. Soc., (2014).

Based on Theorem 1, we have

Theorem 2 (Interior observability)

Let Ω verify the spectral inequality (3). Let $x_0 \in \Omega$ and $R \in (0, 1]$ be such that $B_{4R}(x_0) \subset \Omega$. Then, for each measurable set $\mathcal{D} \subset B_R(x_0) \times (0, T)$ with $|\mathcal{D}| > 0$, there is a positive constant $N = N(\Omega, T, R, \mathcal{D})$ such that

$$\|e^{T\Delta} f\|_{L^2(\Omega)} \leq N \int_{\mathcal{D}} |e^{t\Delta} f(x)|\, dxdt, \ \ \forall f \in L^2(\Omega).$$

Before presenting the main idea of Theorem 2, we begin with several facts.

Spectral estimate on measurable sets

The propagation of smallness estimate for real-analytic functions and the spectral condition (3) imply the following:

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Theorem

Assume that Ω *verifies* (3), ω *is a subset of positive measure such that* $\omega \subset B_R(x_0)$, *with* $B_{4R}(x_0) \subset \Omega$, *for some* $R \in (0, 1]$. *Then, there is a positive constant* $N = N(\Omega, R, |\omega|/|B_R|)$ *such that*

$$\|\mathcal{E}_\lambda f\|_{L^2(\Omega)} \leq Ne^{N\sqrt{\lambda}} \|\mathcal{E}_\lambda f\|_{L^1(\omega)}, \ \forall f \in L^2(\Omega) \ \text{and} \ \lambda > 0. \quad (4)$$

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Main idea. Define

$$u(x,y) = \sum_{\lambda_j \leq \lambda} \langle f, e_j \rangle e^{\sqrt{\lambda_j} y} e_j.$$

Then $u(x,0) = \mathcal{E}_\lambda f$ and

$$\Delta_x u + \partial_y^2 u = 0 \text{ in } B_{4R}(0,0) \subset \Omega \times \mathbb{R}.$$

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By the analyticity estimate for harmonic functions in $\mathbb{R}^{n+1}$, there are $N = N(n)$ and $\rho = \rho(n)$ such that

$$\|\partial_x^\alpha \partial_y^\beta u\|_{L^\infty(B_{2R}(0,0))} \leq \frac{N(|\alpha| + \beta)!}{(R\rho)^{|\alpha|+\beta}} \left(\fint_{B_{4R}(0,0)} |u|^2 \, dxdy \right)^{\frac{1}{2}},$$

when $\alpha \in \mathbb{N}^n, \beta \geq 0.$

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when $\alpha \in \mathbb{N}^n, \beta \ge 0$. Thus

$$\|\partial_x^\alpha \mathcal{E}_\lambda f\|_{L^\infty(B_{2R})} \le N|\alpha|!(R\rho)^{-|\alpha|}\|u\|_{L^\infty(\Omega\times(-4,4))},$$

for $\alpha \in \mathbb{N}^n$. Next, applying the propagation of smallness estimate for real-analytic functions $\mathcal{E}_\lambda f$, we can conclude the desired estimate.

Global three-ball inequality

By the exponential decay of energy of heat equation, we can show that the above-mentioned spectral inequality is indeed equivalent to the following interpolation estimate, which is a quantitative version of strong unique continuation of heat equations.

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Theorem

Let Ω verify the spectral inequality (4), and x_0, R, ω be as above. Then, there are $N = N(\Omega, R, |\omega|/|B_R|)$ and $\theta = \theta(\Omega, R, |\omega|/|B_R|)$ in $(0, 1)$, such that

$$\|e^{t\Delta} f\|_{L^2(\Omega)} \leq \left(N e^{\frac{N}{t-s}} \|e^{t\Delta} f\|_{L^1(\omega)} \right)^{\theta} \|e^{s\Delta} f\|_{L^2(\Omega)}^{1-\theta},$$

when $0 \leq s < t$ and $f \in L^2(\Omega)$.

A version of Fubini's theorem

Let $B_R(x_0) \subset \Omega$ and $\mathcal{D} \subset B_R(x_0) \times (0, T)$ be a subset of positive measure. Set

$$\mathcal{D}_t = \{x \in \Omega : (x, t) \in \mathcal{D}\}, \; a.e. \; t \in (0, T),$$
$$E = \{t \in (0, T) : |\mathcal{D}_t| \geq |\mathcal{D}|/(2T)\}.$$

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Then, $\mathcal{D}_t \subset \Omega$ is measurable for a.e. $t \in (0, T)$, E is measurable in $(0, T)$,

$$|E| \geq |\mathcal{D}|/2|B_R|,$$

$$\chi_E(t)\chi_{\mathcal{D}_t}(x) \leq \chi_{\mathcal{D}}(x, t), \text{ in } \Omega \times (0, T).$$

Global Two-ball and One-cylinder inequality

As an immediate consequence of "global three-ball inequality", we have

Theorem

For each $\eta \in (0,1)$*, there are* $N = N(\Omega, R, |\mathcal{D}|/\left(T|B_R|\right), \eta)$ *and* $\theta = \theta(\Omega, R, |\mathcal{D}|/\left(T|B_R|\right), \eta) \in \theta \in (0,1)$ *such that for any* $f \in L^2(\Omega)$

$$\|e^{t_2\Delta}f\|_{L^2(\Omega)} \leq$$

$$\left(Ne^{N/(t_2-t_1)}\int_{t_1}^{t_2}\chi_E(s)\|e^{s\Delta}f\|_{L^1(\mathcal{D}_s)}\,ds\right)^{\theta}\|e^{t_1\Delta}f\|_{L^2(\Omega)}^{1-\theta},$$

when $0 \leq t_1 < t_2 \leq T$, $|E \cap (t_1, t_2)| \geq \eta(t_2 - t_1)$.

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$$\|e^{t_2\Delta} f\|_{L^2(\Omega)} \leq \left(N e^{N/(t_2-t_1)} \int_{t_1}^{t_2} \chi_E(s) \|e^{s\Delta} f\|_{L^1(\mathcal{D}_s)}\, ds \right)^{\theta} \|e^{t_1\Delta} f\|_{L^2(\Omega)}^{1-\theta},$$

when $0 \leq t_1 < t_2 \leq T$, $|E \cap (t_1, t_2)| \geq \eta(t_2 - t_1)$. *Moreover,*

$$e^{-\frac{N+1-\theta}{t_2-t_1}} \|e^{t_2\Delta} f\|_{L^2(\Omega)} - e^{-\frac{N+1-\theta}{q(t_2-t_1)}} \|e^{t_1\Delta} f\|_{L^2(\Omega)} \leq N \int_{t_1}^{t_2} \chi_E(s) \|e^{s\Delta} f\|_{L^1(\mathcal{D}_s)}\, ds, \quad \text{when} \quad q \geq \frac{N+1-\theta}{N+1}. \tag{5}$$

Back to the main idea of Theorem 2

Let $l > 0$ be a density point in E. Then there exists a monotone decreasing sequence $\{l_m\} \to l$ such that

$$l_{m+1} - l_{m+2} = q(l_m - l_{m+1}), q \in (0, 1)$$

and

$$l_m - l_{m+1} \geq \frac{|E \cap (l_{m+1}, l_m)|}{3}.$$

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By (5), we see

$$e^{-\frac{N+1-\theta}{l_m - l_{m+1}}} \|e^{l_m \Delta} f\|_{L^2(\Omega)} - e^{-\frac{N+1-\theta}{(l_{m+1} - l_{m+2})}} \|e^{l_{m+1}\Delta} f\|_{L^2(\Omega)}$$
$$\leq N \int_{l_{m+1}}^{l_m} \chi_E(s) \|e^{s\Delta} f\|_{L^1(\mathcal{D}_s)} \, ds.$$

The addition of the telescoping series in above inequality leads to the desired interior observability inequality.

Boundary observability inequality

Theorem 3

Suppose that a bounded domain Ω verifies the spectral condition (3). Let $q_0 \in \partial\Omega$ and $R \in (0,1]$ be such that $\triangle_{4R}(q_0)$ is real-analytic. Then, for each measurable set $\mathcal{J} \subset \triangle_R(q_0) \times (0,T)$ with $|\mathcal{J}| > 0$, there is a positive constant $N = N(\Omega, T, R, \mathcal{J})$ such that

$$\|e^{T\Delta} f\|_{L^2(\Omega)} \leq N \int_{\mathcal{J}} |\tfrac{\partial}{\partial\nu}\, e^{t\Delta} f(x)|\, d\sigma dt,$$

when $f \in L^2(\Omega)$.

Here, $\triangle_R(q_0)$ denotes $B_R(q_0) \cap \partial\Omega$.

- Recall the definition of real-analytic boundary

Definition

Let $q_0 \in \partial\Omega$ and $0 < R \leq 1$. We say that $\triangle_{4R}(q_0)$ is real-analytic with constants ϱ and δ if for each $q \in \triangle_{4R}(q_0)$, there are a new rectangular coordinate system where $q = 0$, and a real-analytic function $\phi : B'_\varrho \subset \mathbb{R}^{n-1} \to \mathbb{R}$ verifying

$$\begin{cases} \phi(0') = 0, \ \ |\partial^\alpha \phi(x')| \leq |\alpha|!\delta^{-|\alpha|-1}, \ \text{ when } \ x' \in B'_\varrho, \ \alpha \in \mathbb{N}^{n-1}, \\ B_\varrho \cap \Omega = B_\varrho \cap \{(x', x_n) : x' \in B'_\varrho, \ \ x_n > \phi(x')\}, \\ B_\varrho \cap \partial\Omega = B_\varrho \cap \{(x', x_n) : x' \in B'_\varrho, \ \ x_n = \phi(x')\}. \end{cases}$$

Here, B'_ϱ denotes the open ball of radius ϱ and with center at $0'$ in $\mathbb{R}^{n-1}$.

- Recall the definition of real-analytic boundary

Definition

Let $q_0 \in \partial\Omega$ and $0 < R \leq 1$. We say that $\triangle_{4R}(q_0)$ is real-analytic with constants ϱ and δ if for each $q \in \triangle_{4R}(q_0)$, there are a new rectangular coordinate system where $q = 0$, and a real-analytic function $\phi : B'_\varrho \subset \mathbb{R}^{n-1} \to \mathbb{R}$ verifying

$$\begin{cases} \phi(0') = 0, \ |\partial^\alpha \phi(x')| \leq |\alpha|!\delta^{-|\alpha|-1}, \ \text{when} \ x' \in B'_\varrho, \ \alpha \in \mathbb{N}^{n-1}, \\ B_\varrho \cap \Omega = B_\varrho \cap \{(x', x_n) : x' \in B'_\varrho, \ x_n > \phi(x')\}, \\ B_\varrho \cap \partial\Omega = B_\varrho \cap \{(x', x_n) : x' \in B'_\varrho, \ x_n = \phi(x')\}. \end{cases}$$

Here, B'_ϱ denotes the open ball of radius ϱ and with center at $0'$ in $\mathbb{R}^{n-1}$.

- The proof of Theorem 3 is based on the arguments as those for the interior observability inequality, as well as the following new space-time analyticity estimate for solutions to the heat equation.

Space-time analyticity estimate

Theorem

Assume that $\triangle_{4R}(q_0)$ *is analytic with constants* ϱ *and* δ. *Then, there are* $N = N(\varrho, \delta)$, $\rho = \rho(\varrho, \delta)$, *with* $0 < \rho \leq 1$, *such that*

$$|\partial_x^\alpha \partial_t^\beta e^{t\Delta} f(x)| \leq \frac{N\,(t-s)^{-\frac{n}{4}}\; e^{8R^2/(t-s)} |\alpha|!\,\beta!}{(R\rho)^{|\alpha|}\,((t-s)\,/4)^{\beta}} \|e^{s\Delta} f\|_{L^2(\Omega)},$$

when $x \in B_{2R}(q_0) \cap \overline{\Omega}$, $0 \leq s < t$, $\alpha \in \mathbb{N}^n$ *and* $\beta \geq 0$.

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when $x \in B_{2R}(q_0) \cap \overline{\Omega}$, $0 \le s < t$, $\alpha \in \mathbb{N}^n$ and $\beta \ge 0$.

Main idea. Let $f = \sum_{j \ge 1} a_j e_j$. Set

$$u(x,t,y) = \sum_{j=1}^{+\infty} a_j e^{-\lambda_j t + \sqrt{\lambda_j} y} e_j(x), \quad x \in \overline{\Omega}, t > 0, y \in \mathbb{R},$$

$$u(x,t,0) = \sum_{j=1}^{+\infty} a_j e^{-\lambda_j t} e_j(x) = u(x,t), x \in \overline{\Omega}, t > 0.$$

Hence, for each $t > 0$,

$$\begin{cases} \Delta_x \partial_t^\beta u + \partial_y^2 \partial_t^\beta u = 0, & \text{in } \Omega \times \mathbb{R}, \\ \partial_t^\beta u = 0, & \text{on } \partial\Omega \times \mathbb{R}, \end{cases}$$

$$\partial_t^\beta u(x,t,y) = (-1)^\beta \sum_{j \geq 1} a_j \lambda_j^\beta e^{-\lambda_j t + \sqrt{\lambda_j} y} e_j(x).$$

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Because $\triangle_{4R}(q_0)$ is real-analytic, by the analyticity estimate up to the boundary for solutions to elliptic differential equations, there are constants $N = N(\varrho, \delta)$ and $\rho = \rho(\varrho, \delta)$ such that

$$\|\partial_x^\alpha \partial_t^\beta u(\cdot, t, \cdot)\|_{L^\infty(B_{2R}(q_0,0)\cap\Omega\times\mathbb{R})} \leq \frac{N|\alpha|!}{(R\rho)^{|\alpha|}} \left(\fint_{B_{4R}(q_0,0)\cap\Omega\times\mathbb{R}} |\partial_t^\beta u(x,t,y)|^2 \, dxdy \right)^{\frac{1}{2}},$$

when $\alpha \in \mathbb{N}^n, \beta \geq 0$. This implies the desired estimate.

Abstract evolution equations

We start with introducing the evolution equation

$$\frac{du}{dt} = Au, \quad t > 0, \quad u(0) = u_0 \in X,$$

where X is a Hilbert space and $A : D(A) \subset X \to X$ is the infinitesimal generator of a C_0 semigroup $\{S(t); t \geq 0\}$ in X.

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- It is an admissible observation operator for $\{S(t); t \geq 0\}$: For each $\tau > 0$, there exists a positive constant $C(\tau)$ such that

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• (A, B) verifies the observability inequality from time intervals: *There are two positive constants d and k such that for any $L \in (0, 1]$,*

$$\|S(L)u_0\|_X^2 \leq e^{\frac{d}{L^k}} \int_0^L \|BS(t)u_0\|_U^2 \, dt \ \textit{ for all } \ u_0 \in D(A).$$

Observability from a measurable set in time

Theorem 4

Let A generate an analytic semigroup $\{S(t); t \geq 0\}$ in X. Then, given any $T > 0$ and subset $E \subset (0, T)$ of positive measure, there exists a positive constant $C = C(E, T, d, k, \|B\|_{\mathcal{L}(D(A),U)})$ such that

$$\|S(T)u_0\|_X \leq C \int_E \|BS(t)u_0\|_U \, dt \quad \textit{for all} \;\; u_0 \in D(A).$$

G. Wang and C. Zhang. Observability estimate from measurable sets in time for some evolution equations, 1406.3422v1.

Examples: Parabolic equations associated with second order elliptic operators; Stokes system

By same arguments as those in the heat equation, the observability inequalities (1) and (2) from measurable sets are first extended to some second-order systems, as well as higher-order parabolic evolutions, with real-analytic coefficients not depending on time variable.

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Example

$$\partial_t u - \sum_{i,j=1}^{n} a_{ij}(x)\partial_{ij} u + \sum_{i=1}^{n} b_i(x)\partial_i u + c(x)u.$$

There exists ρ_0, $0 < \rho_0 < 1$ s.t.

$$\sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \geq \rho_0|\xi|^2 \text{ in } \Omega,$$

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$$\sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \geq \rho_0|\xi|^2 \text{ in } \Omega,$$

$$|\partial_x^\gamma a_{ij}(x)| + |\partial_x^\gamma b_i(x)| + |\partial_x^\gamma c(x)| \leq \rho_0^{-1-|\gamma|}|\gamma|! \text{ in } \Omega$$

$\forall \gamma = (\gamma_1, \ldots, \gamma_n) \in \mathbb{N}^n$ and $|\gamma| = \gamma_1 + \cdots + \gamma_n$.

As explained in the boundary observability inequality from measurable sets for the heat equation, in present situation the proofs for observability inequalities (1) and (2) over measurable sets are mainly rely on:

As explained in the boundary observability inequality from measurable sets for the heat equation, in present situation the proofs for observability inequalities (1) and (2) over measurable sets are mainly rely on:

- New quantitative estimates of space-time analyticity of the form

$$|\partial_x^\gamma \partial_t^p u(x,t)| \le e^{1/\rho t^{1/(2m-1)}} \rho^{-|\gamma|-p} \, |\gamma|! \, p! \, t^{-p} \|u_0\|_{L^2(\Omega)},$$

  with some constant $\rho \in (0,1)$, for any $0 < t \le 1$, $\gamma \in \mathbb{N}^n$, $p \ge 0$ and $2m$ is the order of the parabolic problem with time-independent coefficients solved by u.

L. Escauriaza, S. Montaner, C. Zhang. Observation from measurable sets for parabolic analytic evolutions and applications. J. Math. Pures Appl. 104 (2015), 837-867.

Space-time analyticity of parabolic evolutions

The above new quantitative estimates of space-time analyticity are proved by using:

- Analyticity estimates of solutions to elliptic boundary problems with zero Dirichlet data and analytic coefficients (John (1955), Morrey (1966), Morrey-Nirenberg (1957)).

Space-time analyticity of parabolic evolutions

The above new quantitative estimates of space-time analyticity are proved by using:

- Analyticity estimates of solutions to elliptic boundary problems with zero Dirichlet data and analytic coefficients (John (1955), Morrey (1966), Morrey-Nirenberg (1957)).
- Quantification of the reasoning developed by Landis-Oleinik (1974) to reduce the *strong unique continuation within characteristic hyperplanes* for parabolic equations with *time-independent coefficients* to its elliptic counterpart.

Remark for non-symmetric case

Since eigenfunctions may not be available for non-symmetric elliptic operators, to deal with parabolic evolutions associated to non-symmetric elliptic operators with real-analytic time-independent coefficients, we use the following technique lemma:

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Lemma (Landis-Oleinik, 1974)

Let

$$L = \sum_{|\alpha| \leq 2m} a_\alpha(x) \partial_x^\alpha$$

be an elliptic operator of order $2m$ acting on functions $u(x)$ defined on $\Omega \subseteq \mathbb{R}^n$. Then there exists a symmetric elliptic operator P acting on functions $u(x,z)$ defined on $\Omega \times \{|z| < 1\} \subseteq \mathbb{R}^{n+1}$ s.t.

$$Pv \equiv Lv$$

if v is a function independent on z.

Some remarks on the quantitative estimates

The quantitative estimate of space-time real-analyticity

$$|\partial_x^\gamma \partial_t^p u(x,t)| \leq e^{t^{-\frac{1}{2m-1}}} \rho^{-1-|\gamma|-p} |\gamma|!\, p!\, t^{-p} \|u_0\|_{L^2(\Omega)}$$

- yields a positive lower bound ρ for the radius of convergence of the Taylor series in the spatial variables independent of t.

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The quantitative estimate of space-time real-analyticity

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These features of the quantitative estimates of analyticity are essential in our proof of the observability estimate over measurable sets.

Parabolic operators with time dependent coefficients

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So far, we have only dealt with parabolic operators with real-analytic coefficients not depending on time, other techniques are needed for time-dependent coefficients.

More precisely, consider the $2m$-th order operator

$$L = \sum_{|\alpha| \le 2m} a_\alpha(x,t)\partial_x^\alpha,$$

and assume that for some ρ_0, $0 < \rho_0 < 1$,

$$\sum_{|\alpha|=2m} a_\alpha(x,t)\xi^\alpha \ge \rho_0|\xi|^{2m} \ \ \forall \xi \in \mathbb{R}^n, \text{ in } \overline{\Omega} \times [0,T],$$

and

$$|\partial_x^\gamma \partial_t^p a_\alpha(x,t)| \le {\rho_0}^{-1-|\gamma|-p}|\gamma|!p! \ \text{ in } \overline{\Omega} \times [0,T].$$

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Theorem

There is $0 < \rho \leq 1$, $\rho = \rho(\rho_0, n, \partial\Omega)$ *such that* $\forall \alpha \in \mathbb{N}^n, p \in \mathbb{N}$

$$|\partial_x^\gamma \partial_t^p u(x,t)| \leq \rho^{-1-\frac{|\gamma|}{2m}-p} |\gamma|!\ p!\ t^{-\frac{|\gamma|}{2m}-p-\frac{n}{4m}} \|u_0\|_{L^2(\Omega)},$$

in $\overline{\Omega} \times (0, T]$ when u solves

$$\begin{cases} \partial_t u + (-1)^m L u = 0, & \text{in } \Omega \times (0, T], \\ u = Du = \ldots = D^{m-1} u = 0, & \text{in } \partial\Omega \times (0, T], \\ u(\cdot, 0) = u_0, & u_0 \in L^2(\Omega). \end{cases}$$

and $\partial\Omega$ is real-analytic.

Notice

If u satisfies

$$|\partial_x^\gamma \partial_t^p u(x,t)| \le \rho^{-1-\frac{|\gamma|}{2m}-p} |\gamma|!\ p!\ t^{-\frac{|\gamma|}{2m}-p-\frac{n}{4m}} \|u_0\|_{L^2(\Omega)},$$
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This estimate is useless in our arguments for proving observability inequalities (1) and (2) from measurable sets.

Recently, Escauriaza-Montaner-Zhang have an improvement adapted to show the observability estimate (1) and (2) over measurable sets when the coefficients of L are space-time real-analytic.

Theorem 5

Let $T \in (0,1]$ and $\partial\Omega$ be a real-analytic. There are constants ρ and N s.t. for any $\alpha \in \mathbb{N}^n$ and $p \in \mathbb{N}$

$$|\partial_x^\alpha \partial_t^p u(x,t)| \leq N e^{Nt^{-\frac{1}{2m-1}}} \rho^{-|\alpha|-p} t^{-p} |\alpha|! p! \|u\|_{L^2(\Omega\times(0,T))} \text{ in } \overline{\Omega}\times(0,T],$$

if $u \in C([0,T]; L^2(\Omega))$ solves

$$\begin{cases} \partial_t u + (-1)^m L u = 0, & \text{in } \Omega \times (0,T], \\ u = Du = \ldots = D^{m-1} u = 0 & \text{in } \partial\Omega \times (0,T], \\ u(0) = u_0, & u_0 \in L^2(\Omega). \end{cases}$$

L. Escauriaza, S. Montaner, C. Zhang. Analyticity of solutions to parabolic evolutions and applications. arXiv1509.04053v1.

Main idea: the case of second order parabolic operators

- We mainly prove an interior and weighted L^2-estimate by induction on $|\gamma|$ and p:

$$(1-r)^2\|t^{p+1}e^{-\frac{\theta}{t}}\partial_t^{p+1}\partial_x^\gamma u\|_{L^2(B_r\times(0,T)}$$
$$+\sum_{k=0}^{2}(1-r)^k\,\|t^{p+\frac{k}{2}}e^{-\frac{\theta}{t}}D^k\partial_t^p\partial_x^\gamma u\|_{L^2(B_r\times(0,T))}$$
$$\leq \rho^{-1-|\gamma|-p}\theta^{-\frac{|\gamma|}{2}}(1-r)^{-|\gamma|}|\gamma|!p!\|u\|_{L^2(\Omega\times(0,T))},$$

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The precise form of the weights $t^{p+1}e^{-\frac{\theta}{t}}$ is crucial to obtain:

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The precise form of the weights $t^{p+1} e^{-\frac{\theta}{t}}$ is crucial to obtain:

- the lower bound $\rho\theta^{\frac{1}{2}}(1-r)$ (not depending on t) for the spatial radius of convergence of the Taylor series of u.
- the adequate factor $p!$.

The development of the B-B-P

(i) The B-B-P of time optimal controls for the heat equation with control constraint

$$\mathcal{U}^M = \Big\{ u \in L^\infty(\mathbb{R}^+; L^2(\Omega)) : \|u(t)\| \leq M \Big\},$$

was first studied by H. Fattorini (1964). He obtained this property when $\omega = \Omega$.

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Observability estimates from measurable sets in time $\Longrightarrow$ B-B-P

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Observability estimates from measurable sets in time $\Longrightarrow$ B-B-P

(iii) Observability estimates from measurable sets in time was obtained by G. Wang (2008). The way is based on the LR spectral inequality.

Main result

As an application of Theorem 2, we have

Theorem 6

Problem $(TP)^M$ *has B-B-P, i.e., any time optimal control* u^* *satisfies* $|u^*(x,t)| = M$ *for a.e.* $(x,t) \in \omega \times (0, T(M))$*. Consequently, Problem* $(TP)^M$ *has a unique time optimal control.*

Main idea

By contradiction, we would suppose that there were a constant $\varepsilon \in (0, M)$ and a subset of positive measure $\mathcal{D} \subset \omega \times (0, T(M))$ such that

$$|u^*(x,t)| \leq M - \varepsilon, \quad \forall\, (x,t) \in \mathcal{D}.$$

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This provides a "room" for constructing another control.
Indeed, we can see from Theorem 2 that there exist $\delta \in (0, T(M))$ and $v \in L^\infty(\Omega \times \mathbb{R}^+)$, with $\|v\|_{L^\infty} \leq M$, such that

$$\begin{cases} \partial_t y - \Delta y = \chi_\omega v & \text{in} \quad \Omega \times (0, T^* - \delta), \\ y = 0 & \text{on} \quad \partial\Omega \times (0, T^* - \delta), \\ y(x,0) = y_0(x) & \text{in} \quad \Omega, \\ y(x, T^* - \delta) = 0 & \text{in} \quad \Omega. \end{cases}$$

This leads to a contradiction with the time optimality of $T(M)$.

Boundary control

Consider the time optimal boundary control problems for heat equations:

$$(TP)_b^M \qquad T(M) \triangleq \inf_{v\in\mathcal{W}^M} \big\{t>0:\ y(t;v)=0\big\},$$

where

$$\mathcal{W}^M = \big\{v\in L^\infty(\partial\Omega\times\mathbb{R}^+)\ \big|\ |v(x,t)|\le M \text{ for a.e. } (x,t)\in\partial\Omega\times\mathbb{R}^+\big\}$$

with $M>0$, and $y(\cdot\,;v)$ solves

$$\begin{cases} y_t-\Delta y=0 & \text{in } \Omega\times\mathbb{R}^+,\\ y=\chi_\Gamma v & \text{on } \partial\Omega\times\mathbb{R}^+,\\ y(\cdot,0)=y_0 & \text{in } \Omega. \end{cases}$$

Here Γ is a non-empty open subset of $\partial\Omega$.

As an application of Theorem 3, we get

Theorem 7

When Γ is real-analytic, the bang-bang property for $(TP)_b^M$ holds: Any time optimal control v^* for $(TP)_b^M$ satisfies $|v^*(x,t)| = M$ for a.e. $(x,t) \in \Gamma \times (0, T(M))$.

Question

When the boundary Γ is only smooth, the corresponding bang-bang property is still quite open.

Including remarks

Different kinds of observability inequality from measurable subsets imply different versions of bang-bang property for time optimal control problems, as well as norm optimal control problems.

Including remarks

Different kinds of observability inequality from measurable subsets imply different versions of bang-bang property for time optimal control problems, as well as norm optimal control problems.

When the controlled system is not "time-invariant", we still do not know how to derive the bang-bang property even if we have established the corresponding observability inequality from measurable subsets.

For the progress of B-B-P, we refer the following recent work, which in particular derive, but from another aspect, the bang-bang property of time optimal control problem for the heat equation with a potential $a(x,t) = a(x) + b(t) \in L^\infty$.

G.Wang, Y. Xu, and Y. Zhang. Attainable subspaces and the bang-bang property of time optimal controls for heat equations. SIAM J. Control Optim., 2015.

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Study the bang-bang property of optimal control problems to the general parabolic equation with some nonlinearity terms. On this issue, the bang-bang property for the semilinear heat equation with global Lipschitz nonlinearity is proved. For the general case, it is still open.

K.D. Phung, L. Wang and C. Zhang, Bang-bang property for time optimal control of semilinear heat equation. Annales de l'Institut Henri Poincaré (C) Non Linear Analysis, 31 (2014), 477-499.

Singular/degenerate parabolic equations

Example. Let $T > 0$ and $\Omega \subset \mathbb{R}^n$, $n \geq 3$, be a bounded domain with a C^2 boundary and $0 \in \Omega$. Consider the following heat equation with singular potentials:

$$\begin{cases} y_t - \Delta y - \frac{\mu}{|x|^2} y - \frac{k}{|x|^\alpha} y = 0, & \text{in } \Omega \times (0, T), \\ y = 0 & \text{on } \partial\Omega \times (0, T), \\ y(x, 0) = y_0(x) & \text{in } \Omega, \end{cases}$$

with $\mu \leq \mu^* \triangleq (d-2)^2/4$, $1 \leq \alpha < 2$, $k \in \mathbb{R}$ and $y_0 \in L^2(\Omega)$.

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We believe the following observability estimate from measurable sets holds.

Let $\mathcal{D} \subset \Omega \times (0, T)$ be of positive measure. Then there exists a constant $C = C(\Omega, \mathcal{D}, T, \alpha, k)$ such that

$$\left(\int_\Omega |y(x, T)|^2 \, dx \right)^{\frac{1}{2}} \leq C \int_{\mathcal{D}} |y(x, t)| \, dxdt, \quad \forall y_0 \in L^2(\Omega).$$

Some references

J. Apraiz, L. Escauriaza, Null-control and measurable sets, ESAIM Control Optim. Calc. Var., 19 (2013), 239–254.

H. O. Fattorini, Time-optimal control of solutions of operational differential equations, SIAM J. Control, Ser. A, 2 (1964), 54-59.

A. Fursikov, O.Yu. Imanuvilov, Controllability of Evolution Equations, Seoul National University, Korea, Lecture Notes Series 34, 1996.

G. Lebeau, L. Robbiano, Contrôle exact de l'équation de la chaleur, Comm. Partial Differential Equations, 20 (1995), 335-356.

V. J. Mizel, T. I. Seidman, An abstract "bang-bang principle" and time optimal boundary control of the heat equation, SIAM J. Control Optim., 35 (1997), 1204-1216.

G. Wang, L^∞-Null controllability for the heat equation and its consequences for the time optimal control problem, SIAM J. Control Optim., 47 (2008), 1701–1720.

Observability from measurable sets and bang-bang property of time optimal controls

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Thank you for your attention!