High Frequency Analysis for the Wave Equation with Potential

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Setting

$$\begin{cases} \partial_t^2 u - \Delta_x u + q(x)u = 0 & \text{in} \quad]0,T[\times\Omega \\ u = 0 & \text{on} \quad]0,T[\times\partial\Omega \\ (u(0), \partial_t u(0)) = (u_0, u_1) \end{cases} \tag{W}$$

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ω open subset of Ω and $T > 0$ (suitable)

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Perform a high frequency (HF) study for system (W) within the framework of two precise problems:

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Important remark: The whole work will be achieved under the **microlocal** condition of geometric control (Bardos-Lebeau-Rauch).

General setting

- The couple (ω, T) satisfies the geometric control condition (G.C), i.e every geodesic of Ω issued at $t = 0$ and travelling with speed 1, enters in ω before the time T.

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- Under (G.C), we get the observability inequality

$$\|(\varphi_0, \varphi_1)\|^2_{L^2 \times H^{-1}} \leq C(q) \int_0^T \int_\Omega a_\omega^2 \, |\varphi|^2 \, dxdt \qquad \text{(Obs)}$$

for every solution φ of system (W), where $a_\omega(x) \approx \mathbf{1}_\omega$ smooth.

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- We will see later that $C(q) \rightarrow C_r$ with $\|q\|_\infty \leq r$.
- The potential $q \in W^{2,\infty}(\Omega)$.
- $m > 0$,

$$W_m^{2,\infty}(\Omega) = \left\{q \in W^{2,\infty}(\Omega), \ \|q\|_{W^{2,\infty}} \leq m\right\}$$

Littlewood-Paley Decomposition

Denote $(e_j, \omega_j^2)_{j\geq 1}$ the spectral elements of Ω.

$$-\Delta e_j = \omega_j^2 e_j \quad \text{in } \Omega, \quad e_j = 0 \quad \text{on } \partial\Omega, \quad \|e_j\|_{L^2} = 1$$

Take $\theta \in C_0^\infty(\mathbb{R})$ and $\psi \in C_0^\infty(\mathbb{R}^*)$ such that

$$\theta(s) + \sum_{k=1}^{\infty} \psi(2^{-k}s) = 1, \qquad s \in \mathbb{R}_+$$

$$\begin{cases} \psi_0(s) = \theta(s) \\ \psi_k(s) = \psi(2^{-k}s), \qquad k \geq 1 \end{cases}$$

Spectral localization operators

$k \in \mathbb{N}$, $\quad u = \sum_j a_j e_j$,

$$\psi_k(D)u = \sum_j \psi(2^{-k}\omega_j^2) a_j e_j$$

$$S_k(D) = \sum_{j=0}^{k-1} \psi_j(D) \qquad k \geq 1$$

$$\eta_k(D) = \sum_{j=k}^{\infty} \psi_j(D) = I - S_k(D)$$

In particular, for $j = -1, 0$,

$$\|\eta_k(D)\|_{\mathcal{L}(H^{j+1}, H^j)} \leq C 2^{-k/2}$$

1. $\psi_k(D)u$ and $S_k(D)u$ are resp. the dyadic rings and blocks of u.
2. $\eta_k(D)u$ are the high frequencies of u.
3. On a compact manifold, these are pseudo-differential operators of order 0.
4. They are self-adjoints and commute with the laplacian.

Digression

$$\begin{cases} \partial_t^2 u - \Delta_x u = \chi_\omega(x) f & \text{in} \quad]0,T[\times\Omega \\ u = 0 & \text{on} \quad]0,T[\times\partial\Omega \\ (u(0), \partial_t u(0)) = (u_0, u_1) \in H_0^1 \times L^2 \end{cases}$$

We look for $f \in L^2(]0,T[\times\Omega)$, supported in ω s.t

$$(u(T), \partial_t u(T)) = (0,0)$$

By HUM and under (G.C),

$$f = \chi_\omega(x) v$$

where

$$\begin{cases} \partial_t^2 v - \Delta_x v = 0 & \text{in} \quad]0,T[\times\Omega \\ v = 0 & \text{on} \quad]0,T[\times\partial\Omega \\ (v(0), \partial_t v(0)) = (v_0, v_1) \in L^2 \times H^{-1} \end{cases}$$

$$\begin{cases} \Lambda : H_0^1 \times L^2 \longrightarrow L^2 \times H^{-1} \\ \\ \left(u_0, u_1\right) \longrightarrow \left(v_0, v_1\right) \end{cases}$$

is the HUM optimal control.

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Theorem

(D-Lebeau)
In the setting above and under (G.C),
a) For all $s \geq 0$,

$$\Lambda : H^{s+1} \times H^s \to H^s \times H^{s-1}$$

is an isomorphism.
b)

$$\|\Lambda\psi_k(D) - \psi_k(D)\Lambda\| \leq C2^{-k/2}$$

The $\psi_k(D)$ almost commute to the HUM control operator.

Data assimilation

$$\begin{cases} \Box \Phi[Q] + Q(x)\Phi[Q] = 0 & \text{in} \quad]0,T[\times\Omega \\ \Phi[Q] = 0 & \text{on} \quad]0,T[\times\partial\Omega \\ (\Phi[Q](0), \partial_t \Phi[Q](0)) = (\Phi_0, \Phi_1) \in L^2 \times H^{-1} \end{cases} \tag{1}$$

Question: Reconstruct the initial data

$$(\Phi[Q](0), \partial_t \Phi[Q](0)) = (\Phi_0, \Phi_1)$$

from the measurement

$$a_\omega \Phi \quad \text{in} \quad]0,T[\times\omega$$

Classical case: Q is a known potential.
Minimize over $(\varphi_0, \varphi_1) \in L^2 \times H^{-1}$, the functional

$$J[Q](\varphi_0, \varphi_1) = \frac{1}{2} \int_0^T \int_\Omega a_\omega^2 \left| \varphi[Q] - \Phi[Q] \right|^2 dxdt$$

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Less classical: Q is not known (precisely).
Here, we propose instead the minimizing over $(\varphi_0, \varphi_1) \in L^2 \times H^{-1}$ of the functional

$$J[q](\varphi_0, \varphi_1) = \frac{1}{2} \int_0^T \int_\Omega a_\omega^2 \left|\varphi[q] - \Phi[Q]\right|^2 dxdt \tag{2}$$

$$\begin{cases} \Box\varphi[q] + q(x)\varphi[q] = 0 & \text{in} \quad]0, T[\times\Omega \\ \varphi[q] = 0 & \text{on} \quad]0, T[\times\partial\Omega \\ (\varphi[q](0), \partial_t\varphi[q](0)) = (\varphi_0, \varphi_1) \in L^2 \times H^{-1} \end{cases} \tag{3}$$

Here, the potential q is an **approximation** of Q.

Take

$\rightarrow$ $m > 0$ and $q, Q \in W_m^{2,\infty}(\Omega)$,

$\rightarrow$ $(\Phi_0, \Phi_1) \in L^2 \times H^{-1}$ and $\Phi[Q]$ the associated solution of system (1).

$\rightarrow$ $(\Phi_0[q], \Phi_1[q])$ the minimizer of $J[q]$ in (2).

Theorem

There exists a constant $C > 0$ independent of (Φ_0, Φ_1) such that for every $k \geq 1$,

$$\begin{aligned} &\|\eta_k(D)((\Phi_0, \Phi_1) - (\Phi_0[q], \Phi_1[q]))\|_{L^2 \times H^{-1}} \\ &\leq C 2^{-k/4} \|a_\omega \Phi[Q]\|_{L^2(L^2)} \|q - Q\|_{W^{2,\infty}} \end{aligned} \tag{DA}$$

Remark: Good approximation of the HF of the reconstructed initial data, even if the potential is not well known.

Exact controllability

We still work under condition **(G.C)**.

Find $u \in L^2(]0,T[\times\Omega)$ s.t. the solution of system

$$\begin{cases} \Box y + q(x)y = a_\omega u & \text{in} \quad]0,T[\times\Omega \\ y = 0 & \text{on} \quad]0,T[\times\partial\Omega \\ (y(0),\partial_t y(0)) = (y_0,y_1) \in H_0^1 \times L^2 \end{cases}$$

satisfies

$$(y(T),\partial_t y(T)) = (0,0)$$

Actually, we get the optimal control u, i.e the one having the minimal norm in $L^2(]0,T[\times\Omega)$.

Minimize over $(\varphi_0, \varphi_1) \in L^2 \times H^{-1}$, the functional

$$K[q](\varphi_0, \varphi_1) = \frac{1}{2}\int_0^T \int_\Omega a_\omega^2 |\varphi[q]|^2 \, dxdt + \langle (\varphi_0, \varphi_1), (y_0, y_1) \rangle$$

where

$$\langle (\varphi_0, \varphi_1), (y_0, y_1) \rangle = \int_\Omega \varphi_0 y_1 - \int_\Omega \nabla (-\Delta)^{-1} \varphi_1 . \nabla y_1$$

If $(\Phi_0[q], \Phi_1[q])$ is the minimizer of $K[q]$, the control $u[q]$ is then given by

$$u[q] = a_\omega \Phi[q]$$

where $\Phi[q]$ is the solution of (3).

- $m > 0$ and two potentials $q^a, q^b \in W_m^{2,\infty}(\Omega)$
- $(y_0, y_1) \in H_0^1 \times L^2$
- $(\Phi_0[q^{a,b}], \Phi_1[q^{a,b}])$ the respective minimizers of $K[q^{a,b}]$.

Theorem

There exists a constant $C > 0$, independent of (y_0, y_1) s.t. for every $k \geq 1$,

$$\left\| \eta_k(D) a_\omega (\Phi[q^a] - \Phi[q^b]) \right\|_{L^2(L^2)}$$

$$\leq C 2^{-k/4} \|(y_0, y_1)\|_{H_0^1 \times L^2} \left\| q^a - q^b \right\|_{W^{2,\infty}}$$

Moreover,

$$\left\| \eta_k(D) (\Phi_0[q^a] - \Phi_0[q^b], \Phi_1[q^a] - \Phi_1[q^b]) \right\|_{L^2 \times H^{-1}}$$

$$\leq C 2^{-k/4} \|(y_0, y_1)\|_{H_0^1 \times L^2} \left\| q^a - q^b \right\|_{W^{2,\infty}}$$

Comments

- One can weaken the potential regularity:
  $q \in H^2$ ($d = 1$), $W^{2,p}$ ($p > 2$, $d = 2$), $W^{1,\infty} \cap W^{2,d}$ ($d \geq 3$).

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- Time dependent potential: $q = q(t, x)$.
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- Boundary observation (control): open problem.
- State of the art

  * Bukhgeim-Klibanov (81')
  * Puel-Yamamoto (96'), Yamamoto (99'), Imanuvilov-Yamamoto (03'), Baudouin-Mercado-Osses (07')
  * Zhang (00')
  * Baudouin-Buhan-Ervedoza (11')
  *Stefanov-Uhlmann (11')

Estimates for the bracket

(D-Lebeau precised)

Theorem

There exists a constant $C > 0$ s.t. for every $k \geq 1$ and every $q \in W^{2,\infty}$,

$$\left\| [\psi_k(D), q] \right\|_{\mathcal{L}(L^2)} \leq C 2^{-k/2} \left\| q \right\|_{W^{2,\infty}}$$

$$\left\| [\psi_k(D), q] \right\|_{\mathcal{L}(H_0^1)} \leq C 2^{-k/2} \left\| q \right\|_{W^{2,\infty}}$$

$$\left\| [\psi_k(D), q] \right\|_{\mathcal{L}(H^{-1})} \leq C 2^{-k/2} \left\| q \right\|_{W^{2,\infty}}$$

$$\left\| [\psi_k(D), q] \right\|_{\mathcal{L}(L^2, H^{-1})} \leq C 2^{-k} \left\| q \right\|_{W^{2,\infty}}$$

$$\left\| [\psi_k(D), q] \right\|_{\mathcal{L}(H_0^1, L^2)} \leq C 2^{-k} \left\| q \right\|_{W^{2,\infty}}$$

Remark: Similar estimates hold for high frequency brackets: $[\eta_k(D), q]$.

Theorem

Precised observability estimate

Assume that (ω, T) satisfies the (G.C) condition. Then for every $r > 0$, there exists a constant $C_r > 0$ s.t. for every $q \in L^\infty(\Omega)$ satisfying $\|q\|_{L^\infty} \leq r$, the following estimate

$$\|(\varphi(0), \partial_t \varphi(0))\|^2_{L^2 \times H^{-1}} \leq C_r \int_0^T \int_\Omega a^2_\omega |\varphi|^2 \, dxdt$$

holds true for every solution of the system

$$\Box \varphi + q(x)\varphi = 0 \quad in \quad]0, T[\times\Omega, \qquad \varphi = 0 \quad on \quad]0, T[\times\partial\Omega$$

Remark: This estimate is well known when ω satisfies the geometric $\Gamma-$condition of J-L.Lions (using Carleman inequalities). Here, we deal with a microlocal condition.

Sketch of the proof

Compactness-uniqueness argument.

- Relaxed observability (m.d.m's propagation)

$$\|(\varphi(0), \partial_t \varphi(0))\|^2_{L^2 \times H^{-1}} \leq C_r \int_0^T \int_\Omega a_\omega^2 |\varphi|^2 \, dxdt + C_r \|\varphi\|^2_{L^2(H^{-1})}$$

- Removing the compact term (wave front propagation + unique continuation for Δ).

$X[q] = \{\varphi[q] \in L^2(]0, T[\times\Omega), \ \varphi[q] \text{ solution}, \ \varphi[q] = 0 \text{ in }]0, T[\times\omega\}$

$$X[q] = \{0\}$$

Lemma

$$\begin{cases} \text{If } \|q_n\|_{L^\infty} \leq r, \quad q_n \rightharpoonup Q \text{ in } L^2(\Omega) \\ \text{and } \varphi_n \rightharpoonup \varphi \text{ in } L^2(]0,T[\times\Omega), \end{cases}$$

then

$$q_n\varphi_n \rightharpoonup Q\varphi \text{ in } L^2(]0,T[\times\Omega)$$

Key point: $q_n = q_n(x)$, i.e $\frac{\partial q_n}{\partial t} = 0$.
And $\partial_t^2\varphi_n - \Delta\varphi_n + q_n\varphi_n = 0$.

So the mdm's of (q_n) et (φ_n) are supported on transverse manifolds (see P.Gérard 91').

Estimates for the bracket (Sketch of the proof)

$\psi \in C_0^\infty(\mathbb{R}^*), \quad R = 2^k,$

$$\psi_R(D)(\sum_j a_j e_j) = \sum_j \psi(\frac{\omega_j^2}{R}) a_j e_j$$

For $z \in \mathbb{C} \backslash \mathbb{R}_+$,

$$(z+\Delta)^{-1}(\sum_j a_j e_j) = \sum_j \frac{a_j}{z-\omega_j^2} e_j$$

Then for $z \in K$ compact $\subset \mathbb{C} \backslash \mathbb{R}_+$

$$\left\|(zR+\Delta)^{-1} f\right\|_{H^m} \leq \frac{1}{R|\mathrm{Im}\, z|} \left\|f\right\|_{H^m}, \qquad m = 0, 1$$

$$\left\|(zR+\Delta)^{-1} f\right\|_{H^m} \leq \frac{C_1}{\sqrt{R}|\mathrm{Im}\, z|} \left\|f\right\|_{H^{m-1}}, \qquad m = 0, 1$$

Let $\chi \in C_0^\infty(\mathbb{R})$, $(\alpha_k \to +\infty)$ and

$$\widetilde{\psi}(x+iy) = \sum_{k\geq 0} \frac{\psi^{(k)}(x)}{k!}(iy)^k \chi(\alpha_k y)$$

an almost analytic extension of ψ, i.e $\widetilde{\psi}(x) = \psi(x)$ for x real and

$\overline{\partial}\widetilde{\psi}(z) \in O(|\mathrm{Im}\, z|^\infty)$.

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$\overline{\partial}\widetilde{\psi}(z) \in O(|\mathrm{Im}\, z|^\infty)$.

Helffer-Sjöstrand formula

$$\psi_R(D) = \frac{-1}{\pi}\int_{\mathbb{C}} \frac{\overline{\partial}\widetilde{\psi}(z)}{z+\Delta/R}dz = \frac{-R}{\pi}\int_{\mathbb{C}} \frac{\overline{\partial}\widetilde{\psi}(z)}{zR+\Delta}dz$$

Therefore,

$$[q, \psi_R(D)] = \frac{-R}{\pi}\int_{\mathbb{C}} \overline{\partial}\widetilde{\psi}(z)(zR+\Delta)^{-1}[\Delta, q](zR+\Delta)^{-1}dz$$

Proof of the data assimilation theorem

Recall the setting

$$\begin{cases} \Box\Phi[Q] + Q(x)\Phi[Q] = 0 \qquad \text{in} \quad]0,T[\times\Omega \\ (\Phi[Q](0), \partial_t\Phi[Q](0)) = (\Phi_0, \Phi_1) \in L^2 \times H^{-1} \end{cases}$$

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$\rightarrow$The potential Q is not (well) known.

$\rightarrow$ $(\Phi_0[q], \Phi_1[q])$ is the minimizer of the functional $J[q]$.

Step 1: First evaluations

$$\|(\Phi_0, \Phi_1)\|_{L^2\times H^{-1}} + \|\Phi[Q]\|_{C(L^2)\cap C^1(H^{-1})} \leq C\,\|a_\omega \Phi[Q]\|_{L^2(L^2)}$$

$$\|a_\omega \Phi[q]\|_{L^2(L^2)} \leq C\,\|a_\omega \Phi[Q]\|_{L^2(L^2)}$$

$$\|(\Phi_0[q], \Phi_1[q])\|_{L^2\times H^{-1}} + \|\Phi[q]\|_{C(L^2)\cap C^1(H^{-1})} \leq C\,\|a_\omega \Phi[Q]\|_{L^2(L^2)}$$

$$\|\Phi[q] - \Phi[Q]\|_{C(L^2)\cap C^1(H^{-1})} \leq C\,\|q - Q\|_{W^{2,\infty}}\,\|a_\omega \Phi[Q]\|_{L^2(L^2)}$$

Step 2: We estimate

$$w_k = \phi_k - \eta_k^2(D)(\Phi[q] - \Phi[Q])$$

where ϕ_k is the solution of

$$\begin{cases} \Box\phi_k + q(x)\phi_k = 0 & \text{in} \quad]0, T[\times\Omega \\ (\phi_k(0), \partial_t\phi_k(0)) = \eta_k^2(D)(\Phi_0[q] - \Phi_0, \Phi_1[q] - \Phi_1) \end{cases}$$

$$\begin{cases} \Box w_k + q(x)w_k = f_k & \text{in} \quad]0, T[\times\Omega \\ (w_k(0), \partial_t w_k(0)) = (0, 0) \end{cases}$$

$$f_k = [\eta_k^2(D), q](\Phi[q] - \Phi[Q]) + \eta_k^2(D)((q - Q)\Phi[Q])$$

$$\|w_k\|_{C(L^2)\cap C^1(H^{-1})} \leq C2^{-k/2}\|q - Q\|_{W^{2,\infty}}\|a_\omega\Phi[Q]\|_{L^2(L^2)}$$

Last step: We estimate

$$\widetilde{w}_k = \widetilde{\phi}_k - \eta_k(D)(\Phi[q] - \Phi[Q])$$

where $\widetilde{\phi}_k$ solves

$$\begin{cases} \Box\widetilde{\phi}_k + q(x)\widetilde{\phi}_k = 0 & \text{in} \quad]0, T[\times\Omega \\ (\widetilde{\phi}_k(0), \partial_t\widetilde{\phi}_k(0)) = \eta_k(D)(\Phi_0[q] - \Phi_0, \Phi_1[q] - \Phi_1) \end{cases}$$

$$\left\| a_\omega \widetilde{\phi}_k \right\|_{L^2(L^2)} \leq C 2^{-k/4} \left\| q - Q \right\|_{W^{2,\infty}} \left\| a_\omega \Phi[Q] \right\|_{L^2(L^2)}$$

And the result follows.

Proof of the control theorem

Recall the setting

$$\begin{cases} \Box\Phi\left[q^a\right] + q^a(x)\Phi\left[q^a\right] = 0 & \text{in} \quad]0,T[\times\Omega \\ \left(\Phi\left[q^a\right](0), \partial_t\Phi\left[q^a\right](0)\right) = (y_0, y_1) \in H_0^1 \times L^2 \end{cases}$$

$$\begin{cases} \Box\Phi\left[q^b\right] + q^b(x)\Phi\left[q^b\right] = 0 & \text{in} \quad]0,T[\times\Omega \\ \left(\Phi\left[q^b\right](0), \partial_t\Phi\left[q^b\right](0)\right) = (y_0, y_1) \in H_0^1 \times L^2 \end{cases}$$

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$\rightarrow q^a, q^b \in W_m^{2,\infty}$.

$\rightarrow (\Phi_0[q], \Phi_1[q])$ is the minimizer over $L^2 \times H^{-1}$ of the functional

$$K[q](\varphi_0, \varphi_1) = \frac{1}{2}\int_0^T \int_\Omega a_\omega^2 \left|\varphi\left[q\right]\right|^2 dxdt + \langle(\varphi_0, \varphi_1), (y_0, y_1)\rangle$$

Step 1: First evaluations

$$\left\|(\Phi_0[q],\Phi_1[q])\right\|_{L^2\times H^{-1}}+\left\|\Phi[q]\right\|_{C(L^2)\cap C^1(H^{-1})}\leq C\left\|(y_0,y_1)\right\|_{H_0^1\times L^2}$$

Let $\varphi\left[q^a\right]$ be the solution of

$$\begin{cases}\Box\varphi+q^a(x)\varphi=0 & \text{on} \quad]0,T[\times\Omega\\ (\varphi(0),\partial_t\varphi(0))=(\Phi_0[q^a]-\Phi_0[q^b],\Phi_1[q^a]-\Phi_1[q^b])\end{cases}$$

$$\left\|\varphi\left[q^a\right]-(\Phi[q^a]-\Phi[q^b])\right\|_{C(L^2)\cap C^1(H^{-1})}\leq C\left\|q^a-q^b\right\|_{W^{2,\infty}}\left\|(y_0,y_1)\right\|_{H_0^1\times L^2}$$

$$\left\| a_\omega(\Phi[q^a] - \Phi[q^b]) \right\|_{L^2(L^2)} \le C \left\| q^a - q^b \right\|_{W^{2,\infty}} \|(y_0, y_1)\|_{H_0^1 \times L^2}$$

$$\begin{cases} \left\| (\Phi_0[q^a] - \Phi_0[q^b], \Phi_1[q^a] - \Phi_1[q^b]) \right\|_{L^2 \times H^{-1}} \\ \qquad + \left\| \Phi[q^a] - \Phi[q^b] \right\|_{C(L^2) \cap C^1(H^{-1})} \\ \qquad \le C \left\| q^a - q^b \right\|_{W^{2,\infty}} \|(y_0, y_1)\|_{H_0^1 \times L^2} \end{cases}$$

Step 2: Taking $q \in W^{2,\infty}_m$, we introduce $\phi_k[q]$ the solution of

$$\begin{cases} \Box\phi_k + q(x)\phi_k = 0 \quad \text{in} \quad]0,T[\times\Omega \\ (\phi_k(0), \partial_t\phi_k(0)) = \eta_k^2(D)(\Phi_0[q^a] - \Phi_0[q^b], \Phi_1[q^a] - \Phi_1[q^b]) \end{cases}$$

$$\begin{cases} \left\|\phi_k[q] - \eta_k^2(D)(\Phi[q^a] - \Phi[q^b])\right\|_{C(L^2)\cap C^1(H^{-1})} \\ \quad \leq C2^{-k/2}\left\|q^a - q^b\right\|_{W^{2,\infty}} \|(y_0, y_1)\|_{H_0^1\times L^2} \end{cases}$$

$$\left\|\eta_k(D)a_\omega(\Phi[q^a] - \Phi[q^b])\right\|_{L^2(L^2)} \leq C2^{-k/4}\left\|q^a - q^b\right\|_{W^{2,\infty}} \|(y_0, y_1)\|_{H_0^1\times L^2}$$

Last step: We estimate

$$\widetilde{\phi}_k - \eta_k(D)(\Phi[q^a] - \Phi[q^b])$$

where $\widetilde{\phi}_k$ solves

$$\begin{cases} \Box\widetilde{\phi}_k + q^a(x)\widetilde{\phi}_k = 0 \qquad \text{in} \qquad]0, T[\times\Omega \\ (\widetilde{\phi}_k(0), \partial_t\widetilde{\phi}_k(0)) = \eta_k(D)(\Phi_0[q^a] - \Phi_0[q^b], \Phi_1[q^a] - \Phi_1[q^b]) \end{cases}$$

$$\left\| a_\omega \widetilde{\phi}_k \right\|_{L^2(L^2)} \leq C 2^{-k/4} \left\| q^a - q^b \right\|_{W^{2,\infty}} \|(y_0, y_1)\|_{H_0^1 \times L^2}$$

And the result follows.