A Source Reconstruction Formula for the Heat Equation Using a Family of Null Controls

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Séminaire GdT Contrôle - LJLL-Paris 6, le 4 Août 2013
Outline

1. Motivation
2. Source recovering and null controls
3. Our reconstruction formula
4. Numerical Results
Motivation

Source inverse problem

**Inverse problem:** To identify (uniqueness, stability, reconstruction) the source \( f(x, t) \) in the heat equation

\[
\begin{aligned}
  & u_t - \Delta u = f(x, t) \quad \text{in } \Omega \times (0, T) \\
  & u(0) = 0 \quad \text{in } \Omega \\
  & u = 0 \quad \text{on } \Gamma \times (0, T)
\end{aligned}
\]

from local measurements of the solution (boundary or internal values).

**Counterexample:** \( u = a(t)\phi(x), \phi \in C_c^\infty(\omega), \omega \) strictly included in \( \Omega \), then \( u_t - \Delta u = f = a'\phi - a\Delta\phi \in C_c^\infty(\omega) \) for each \( t \), so we obtain null data outside \( \omega \).
Yamamoto 1995: \( f = \sigma(t)f(x) \) reconstruction of \( f(x) \) for wave equation using exact controls (exact controllability, Volterra eqns).


Hettlich-Rundell 2001: \( f = \chi_D \) identification of spatially discontinuous sources (domain derivative method).

Yamatani-Ohnaka 1997, El Badia-Ha Duong 2002: \( f = \sum_{j=1}^{N} p_j \delta_{x,j,t_j} \) identify pointwise sources in heat eqn (asymptotic behavior of backward heat solutions/exact controllability).

El Badia-Ha Duong-Hamdi 2005: 1-d heat-transport from internal up-downstream strategic pointwise data (strategic from control theory).

Ikehata 2006. \( f(x_0,T_0) \neq 0 \) when and where the source fistly appeared (plane solutions for the backward heat equation, indicator fncs).
Some applications: (pointwise, distributed sources, stationnary, time varying sources):


- Optimal network monitoring design/observation systems c.f. Rodgers 2000, O., Faundez, Gallardo 2013


Motivation


Observations: Ibuki-GOSAT satellite (2009)
Motivation

Regional: nuclear/mining emissions, when and where

Observations: OMI satellite (2007)
Aura/OMI - Average column for 20070101-20071231
Motivation

City scale: air-quality forecasting (Santiago, Chile)

\[ f(\mathbf{x}) \]
(space variation)

\[ \sigma(t) \]
(time variation)

Fig. 1. Left panel: Official EI averaged in time (logarithmic scale, units in \( \mu g/m^2/s \)), topograph over Santiago basin and location of air quality monitoring stations. 1: Providencia, 2: Independencia, 3: La Florida, 4: Las Condes, 5: Parque O'Higgins, 6: Pudahuel, 7: Cerrillos, 8: El Bosque. Right panel: Normalized emission diurnal cycles for different zones of the city.
Motivation

City scale: air-quality forecasting (Santiago, Chile)

diffusion dominated (morning)
transport dominated (afternoon)

Fig. 5. Concentration (µg/m$^3$) and wind (m/s) fields at the first level of the dispersion model. Panels (a) and (b) are mean fields for 8 a.m. and 16 p.m. LT respectively. Point filling colors represent observation mean values in each station at the same hour. For topography values and station number refer to Fig. 1. Concentration scale is logarithmic.
Motivation
City scale: air-quality forecasting (Santiago, Chile)

Fig. 11. Time average difference between the a priori (official) inventory and the optimized inventory assimilating real observations. Left panel: morning, right panel: afternoon. Note that values over 0 (red) represent a decrease in the emissions and negative values (blue) represent increase in emissions with respect to the official inventory. Units in μg/m²/s.
Source recovering and null controls

Statement of our inverse problem

Our inverse problem: Given an (small) observatory \( \mathcal{O} \subset \Omega, \ T > 0, \ \sigma(t) \) known, to identify the source \( f(x) \) in

\[
\begin{cases}
  u_t - \Delta u = \sigma(t)f(x) & \text{in } \Omega \times (0, T) \\
  u(0) = 0 & \text{in } \Omega \\
  u = 0 & \text{on } \Gamma \times (0, T)
\end{cases}
\]

from local measurements of the heat solution \( u|_{\mathcal{O} \times (0,T)} \).

We are interested in:

- uniqueness and stability of \( f(x) \) w.r.t. measurements.
- reconstruction algorithm for \( f(x) \) using null controls.
Source recovering and null controls
What are null controls?

A null control is a locally supported function \( v \) that drives the backward heat equation:

\[
\begin{cases}
-\varphi_t - \Delta \varphi = v|_{\mathcal{O} \times (0,\tau)} & \text{in } \Omega \times (0, \tau) \\
\varphi(\tau) = \varphi_0 & \text{in } \Omega \\
\varphi = 0 & \text{on } \Gamma \times (0, \tau).
\end{cases}
\]

exactly to zero in finite time \( \tau > 0 \) from \( \varphi(\tau) = \varphi_0 \) to \( \varphi(0) = 0 \).

It is possible to show (for instance using a global Carleman inequality [Fursikov-Imanuvilov 1996]) that there exists a minimal norm null control \( v^{(\tau)} \) such that

\[
\| v^{(\tau)} \|_{L^2(\mathcal{O} \times (0,\tau))} \leq C \exp \left( \frac{C_1}{\tau} \right) \| \varphi_0 \|_{L^2(\Omega)}.
\]
Source recovering and null controls

Example: null controlled eigenfrequencies

\[ v_{p_1} = 19.75 \]
\[ v_{p_2} = 49.42 \]
\[ v_{p_3} = 79.15 \]
\[ v_{p_8} = 128.81 \]
\[ v_{p_{16}} = 258.45 \]
\[ v_{p_{32}} = 500.63 \]
Source recovering and null controls
Example: null controlled eigenfrequencies
Source recovering and null controls
A much simpler problem: recover “final” conditions

Problem 1: In order to explain the relationship between null controls and the inverse source problem we consider here, let us start with the simpler problem:

\[
\begin{align*}
    u_t - \Delta u &= 0 \quad \text{in } \Omega \times (0, T) \\
    u(0) &= u_0 \quad \text{in } \Omega \\
    u &= 0 \quad \text{on } \Gamma \times (0, T).
\end{align*}
\]

(3)

If we want to recover \( u_0 \in L^2(\Omega) \) from \( u|_{\partial \Omega \times (0, T)} \) without any other a priori information is severily ill posed (see example later).

Nevertheless, recover \( u(T) \) for any \( T > 0 \) from \( u|_{\partial \Omega \times (0, T)} \) and without knowing \( u_0 \) is well posed.
Source recovering and null controls
Null controllability and recovery

Multiplying $u$ by the null controlled solution $\varphi$ we obtain

$$\int_0^T \int_\Omega u^T(\tau) \, dx \, dt = - \int_\Omega u(T) \varphi(T) \, dx + \int_\Omega u_0 \varphi(0) \, dx$$

\[ \text{duality} \]

In this way we derive the following reconstruction formula that has been used in data assimilation problems [Puel 2002, García-Osses-Puel 2010]:

$$\int_\Omega u(T) \varphi_0 \, dx = - \int_0^T \int_\Omega u^T(T) \, dx \, dt .$$

It does not depend on $u_0$ and only in local observations. From Carleman we obtain Lipschitz stability:

$$\| u(T) \|_{L^2(\Omega)} \leq C e^{C_1/T} \| u \|_{L^2(\Omega \times (0,T))} .$$
Problem 2: Let us now consider another very simple source inverse problem. To identify \( f(x) \) in

\[
\begin{cases}
  u_t - \Delta u = f(x) & \text{in } \Omega \times (0, T) \\
  u(0) = 0 & \text{in } \Omega \\
  u = 0 & \text{on } \Gamma \times (0, T)
\end{cases}
\]

from local observations \( u|_{\partial \Omega \times (0, T)} \). We can reduce this problem to the previous one by taking time derivative: \( w = u_t \)

\[
\begin{cases}
  w_t - \Delta w = 0 & \text{in } \Omega \times (0, T) \\
  w(0) = f(x) & \text{in } \Omega \\
  w = 0 & \text{on } \Gamma \times (0, T)
\end{cases}
\]

Since we cannot recover \( w(0) \) from the local measurements \( w|_{\partial \Omega \times (0, T)} = u_t|_{\partial \Omega \times (0, T)} \), our problem is to recover \( f \) appearing in the final condition

\[ w(T) = \Delta u(T) + f(x) \]
Source recovering and null controls
Another recovery formula

$$w(T) = \Delta u(T) + f(x)$$

So, by multiplying the equation of $w$ by the controlled solution $\varphi$ as before and using duality, we obtain the recovery formula:

$$\int_{\Omega} f(x) \varphi_0 \, dx = - \int_{\Omega} \Delta u(T) \varphi_0 \, dx - \int_0^T \int_{\partial \Omega} u_t v^{(T)} \, dx \, dt. \tag{C}$$

An extra annoying local in time measurement of $\Delta u(T)$ appears!!

The corresponding stability inequality is easily obtained from Carleman:

$$\|f\|_{L^2(\Omega)} \leq C\|\Delta u(T)\|_{L^2(\Omega)} + Ce^{C_1/T}\|u_t\|_{L^2(\partial \Omega \times (0, T))}. \tag{C}$$
Source recovering and null controls

The annoying measure $\Delta u(T)$

We cannot drop $\Delta u(T)$ in the previous inequality due to high frequencies. Example: $\mathcal{O} \subset (0, \pi) = \Omega$. Take

$$u^n(x, t) = \frac{1}{n^2} \sqrt{\frac{2}{\pi}} (1 - e^{-n^2 t}) \sin(nx)$$

that satisfies

$$\begin{cases} u^n_t - \Delta u^n = \sqrt{\frac{2}{\pi}} \sin(nx) := f_n(x) & \text{in } (0, \pi) \times (0, T) \\ u^n(0) = 0 & \text{in } (0, \pi), \\ u(0, t) = u(\pi, t) = 0 & \text{on } (0, T). \end{cases}$$

Then

$$\|f_n\|_{L^2(0, \pi)} = 1 \quad \|u_t\|_{L^2(\mathcal{O} \times (0, T))} \leq \frac{1}{n^2 \pi} \to 0, \quad n \to \infty.$$

but $\|\Delta u(T)\|_{L^2(0, \pi)} = (1 - e^{-n^2 T}) \to 1$. 

Source recovering and null controls
Log estimates

This shows that if \( f \) is only \( L^2(\Omega) \) a log estimate

\[
\|f\|_{L^2(\Omega)} \leq C \log \|u_t\|_{L^2(\Omega \times (0,T))}^{-\alpha}
\]

for \( \alpha \in (0,1] \) does not hold.

If \( f \) is more regular, say \( \|f\|_{D((\Delta)\epsilon)} \leq M \) for some \( \epsilon \in (0,1) \), you still have logarithmic conditional stability:

\[
\|f\|_{L^2(\Omega)} \leq C_{M,\epsilon} \log \|u_t\|_{L^2(0,T;L^2(\Omega))}^{-\epsilon \frac{\epsilon}{1-\epsilon}}
\]

as proved by [Garcia-Takahashi, 2011]. The first estimates of this type were obtained by [Li-Yamamoto-Zou 2009] for recovering regular \( u_0 \).
Our reconstruction formula
Return to the problem

Our problem: In order to recover $f(x)$ in

\[
\begin{align*}
\begin{cases}
  u_t - \Delta u &= \sigma(t)f(x) \quad \text{in } \Omega \times (0, T) \\
  u(0) &= 0 \quad \text{in } \Omega \\
  u &= 0 \quad \text{on } \Gamma \times (0, T)
\end{cases}
\end{align*}
\]

(1)

from $u\mid_{\Omega \times (0, T)}$ we reduce the problem by changing variables:

\[
u = \int_0^t \sigma(t - \tau)w(\tau)\,d\tau := Kw \quad \text{(notice } w = u_t \text{ if } \sigma = 1)\]

Then if $w$ solves

\[
\begin{align*}
\begin{cases}
  w_t - \Delta w &= 0 \quad \text{in } \Omega \times (0, T) \\
  w(0) &= f(x) \quad \text{in } \Omega \\
  w &= 0 \quad \text{on } \Gamma \times (0, T)
\end{cases}
\end{align*}
\]

(6)

$u$ solves (1). So this time we will try to recover $f$ from:

\[
\sigma(0)w(T) = \Delta u(T) + \sigma(T)f(x) - \int_0^T \sigma'(T - \tau)w(\tau)\,d\tau
\]
Our reconstruction formula
Volterra equation and duality

The definition of \( K \) naturally leads to solve a Volterra equation of second kind. Given \( v \in L^2(0, T; L^2(\mathcal{O})) \), \( \exists ! \theta \in H^1(0, T; L^2(\mathcal{O})) \) such that \( \theta(T) = 0 \) and

\[
K^* \theta := \sigma(0) \theta_t + \int_t^T \left( \sigma(s - t) \theta(s) + \sigma'(s - t) \theta_t(s) \right) ds = v(t)
\]

with continuous dependence and \( \forall w \in L^2(0, T; L^2(\mathcal{O})) \)

\[
(w, K^* \theta)_{L^2(0, T; L^2(\mathcal{O}))} = (Kw, \theta)_{H^1(0, T; L^2(\mathcal{O}))}.
\]

This duality was previously used by [Yamamoto 1995] to derive a reconstruction source formula for the wave equation.
Our reconstruction formula
Intermediate reconstruction formula

\[ \sigma(0)w(T) = \Delta u(T) + \sigma(T)f(x) - \int_0^T \sigma'(T - \tau)w(\tau)d\tau \]

By introducing the family of null controls \( v^{(\tau)} \) controlling from \( \varphi(\tau) = \varphi_0 \) to \( \varphi(0) = 0 \) and \( K^*\theta^{(\tau)} = v^{(\tau)} \), \( Kw = u \) we have

\[
\sigma(T) \int_\Omega f(x) \varphi_0 \, dx = \\
= -\int_\Omega \Delta u(T) \varphi_0 + \sigma(0) \int_\Omega w(T) \varphi_0 + \int_0^T \sigma'(T - \tau) \int_\Omega w(\tau) \varphi_0 \, d\tau \\
= -\int_\Omega \Delta u(T) \varphi_0 - \sigma(0) \left[ \int_0^T \int_\Omega w v^{(T)} - \int_0^T \sigma'(T - \tau) \int_\Omega w v^{(\tau)} \, dt \, d\tau \right]_{(w,K^*\theta^{(T)})} \\
\]
Theorem 1 Let $\sigma \in W^{1,\infty}(0, T)$, $\sigma(T) \neq 0$ then $\forall \varphi_0 \in L^2(\Omega)$

$$
\int_{\Omega} f \varphi_0 = -\sigma(T)^{-1}(\Delta u(T), \varphi_0)_{L^2(\Omega)} - \sigma(0)\sigma(T)^{-1}(u, \theta^{(T)})_{H^1(L^2(\Omega))}
- \sigma(T)^{-1} \int_0^T \sigma'(T - \tau)(u, \theta^{(\tau)})_{H^1(L^2(\Omega))} d\tau
$$

where $\theta^{(\tau)}$ are the Volterra solutions associated to the null controls $v^{(\tau)}$ for $\tau \in (0, T]$. Moreover, if $\sigma'(t) = 0$ for $t \in (T - \varepsilon, T]$ or $\sigma'(t) = e^{-C_2/(T-t)} \rho(t)$, $\forall t \in (0, T)$, $\rho \in L^\infty(0, T)$ for large $C_2$ then

$$
\|f\|_{L^2(\Omega)} \leq C(\|\Delta u(T)\|_{L^2(\Omega)} + \|u\|_{H^1(0,T;L^2(\Omega))}).
$$

with $C \sim O(e^{C_1/\varepsilon})$ or $C \sim O(1)$. 
Our reconstruction formula

Eliminating $\Delta u(T)$...

Fortunately, in the particular case that $\varphi_0 = \varphi_k$ the eigenfunctions of the Laplacian

$$\int_{\Omega} f \varphi_k = -\frac{\sigma(T)^{-1}(\Delta u(T), \varphi_k)_{L^2(\Omega)}}{\lambda_k} - \frac{\sigma(0)\sigma(T)^{-1}(u, \theta_k^{(T)})_{H^1(L^2(\Omega))}}{C_{1k}}$$

$$- \sigma(T)^{-1} \int_0^T \sigma'(T - \tau)(u, \theta_k^{(\tau)})_{H^1(L^2(\Omega))} d\tau$$

$$\leq C_{2k}$$

it is possible to replace the annoying first term by using the alternative expression:

$$\int_{\Omega} f \varphi_k = -\frac{(\Delta u(T), \varphi_k)_{L^2(\Omega)}}{\lambda_k \int_0^T e^{-\lambda_k(T-s)} \sigma(s) ds}$$

where $\lambda_k > 0$ are the corresponding eigenvalues.
Our reconstruction formula
Main result: new reconstruction formula

\textbf{Theorem 2} Let $f \in L^2(\Omega)$ and let $\{(\lambda_k, \varphi_k)\}_{k \geq 0}$ be the eigenvalues and $L^2$-orthonormal eigenvectors of the $-\Delta$ in $\Omega$ with homogeneous Dirichlet boundary conditions. Given $\sigma \in W^{1,\infty}(0, T)$, $\sigma(T) \neq 0$ such that

$$a_k := 1 - \frac{\lambda_k}{\sigma(T)} \int_0^T e^{-\lambda_k(T-s)} \sigma(s) ds \neq 0,$$

(1)

for some $k \geq 0$, then we have the reconstruction formula:

$$\int_{\Omega} f \varphi_k = a_k^{-1}(C_{1k} + C_{2k}),$$

(2)

where $C_{1k} = C_1(\varphi_k)$ and $C_{2k} = C_2(\varphi_k)$ only depend on the local observations $(u, u_t)|_{\partial \times (0, T)}$ of the solution of (1).
Our reconstruction formula

Examples of functions for which \( a_k \neq 0 \ \forall k \)

a) \( \sigma = \sigma_0 \) constant \( \) (in this case \( a_k = e^{-\gamma \lambda_k T} > 0 \)),

b) \( \sigma \) a non negative and increasing function \( \) (in this case \( a_k \geq e^{-\gamma \lambda_k T} \)),

c) \( \sigma = \sigma_0 + b \cos(\omega t) \), \( \sigma_0 \) constant, \( b \neq 0 \), and \( \omega \in \mathbb{R} \setminus D \), where \( D \) is some discrete set that results from the intersections of \( \tan(\omega T) \) (solid line) and a sequence of functions associated to each eigenfrequency (dotted lines). There are no intersections in regions where \( \tan(\omega T) \) and \( \cos(\omega T) \) have opposite signs (arrows).
Numerical results: Three different choices for the time dependency of the source $\sigma(t)$

- case a)
- case b)
- case c)
Numerical Results

Main features


- Solution measured in $1/3$ of the spatial domain with 5% noise.

- Source $f^*$ reconstructed from our proposed formula based on null controls for $\sigma(t)$ in cases a) constant, b) increasing and c) oscillating.

- We optimize w.r.t. the cutoff frequency.

- Next fig. d) is an optimization of case b) starting from $f^*$ by modifying only the relevant frequencies ($p = 2$) or sparse ($p = 1$) using proxy algorithms.

$$
\min_{f=\sum_{k=0}^{K} f_k \varphi_k} \| u^{meas} - u(f) \|_{H^1(0,T,L^2(\Omega))}^2 + \beta \| f - f^* \|_{L^p(\Omega)}^2
$$
Numerical Results

\( \Omega = (0, 1)^2, \mathcal{O} = (0, 1) \times (0.3, 0.7). \) 5\% noise. 47 eigen.
Numerical Results
Contribution of each term
Numerical Results
Zoom on “crux” source: control terms
Numerical Results

Zoom on “crux” source: optimized frequencies
Numerical Results

Zoom on “crux” source: optimization
References:

**G. García, A. Osses, M. Tapia**, A heat source reconstruction formula from single internal measurements using a family of null controls, *J. Inverse Ill-Posed Probl.*, Ahead of print, Published Online: January 2013


