ON THE STABILITY OF TYPE I BLOW UP FOR THE ENERGY SUPER CRITICAL HEAT EQUATION

CHARLES COLLOT, PIERRE RAPHAËL, AND JEREMIE SZEFTEL

Abstract. We consider the energy super critical semilinear heat equation
\[ \partial_t u = \Delta u + u^p, \quad x \in \mathbb{R}^3, \quad p > 5. \]
We first revisit the construction of radially symmetric self similar solutions performed through an ode approach in [51], [2], and propose a bifurcation type argument suggested in [3] which allows for a sharp control of the spectrum of the corresponding linearized operator in suitable weighted spaces. We then show how the sole knowledge of this spectral gap in weighted spaces implies the finite codimensional non radial stability of these solutions for smooth well localized initial data using energy bounds. The whole scheme draws a route map for the derivation of the existence and stability of self similar blow up in non radial energy super critical settings.

1. Introduction

1.1. Setting of the problem. We consider the focusing nonlinear heat equation
\[ \begin{cases} \partial_t u = \Delta u + |u|^{p-1}u, & (t, x) \in \mathbb{R} \times \mathbb{R}^d, \\ u|_{t=0} = u_0, \end{cases} \tag{1.1} \]
where \( p > 1 \). This model dissipates the total energy
\[ E(u) = \frac{1}{2} \int |\nabla u|^2 - \frac{1}{p+1} \int u^{p+1}, \quad \frac{1}{2} \frac{dE}{dt} = - \int (\partial_t u)^2 < 0 \tag{1.2} \]
and admits a scaling invariance: if \( u(t, x) \) is a solution, then so is
\[ u_\lambda(t, x) = \lambda^{\frac{2}{p-1}} u(\lambda^2 t, \lambda x), \quad \lambda > 0. \tag{1.3} \]
This transformation is an isometry on the homogeneous Sobolev space
\[ \|u_\lambda(t, \cdot)\|_{\dot{H}^{s_c}} = \|u(t, \cdot)\|_{\dot{H}^{s_c}} \text{ for } s_c = \frac{d}{2} - \frac{2}{p-1}. \]
We address in this paper the question of the existence and stability of blow up dynamics in the energy super critical range \( s_c > 1 \) emerging from well localized initial data.

1.2. Type I and type II blow up. There is a large litterature devoted to the question of the description of blow up solutions for (1.1) and we recall some key facts related to our analysis.

Type I blow-up. The universal scaling lower bound on blow up rate
\[ \|u(t, \cdot)\|_{L^\infty} \gtrsim \frac{1}{(T-t)^{\frac{1}{p-1}}} \]
is saturated by Type I singularities:

$$\|u(t, \cdot)\|_{L^\infty} \sim \frac{1}{(T-t)^{\frac{1}{p-1}}}.$$  

These solutions concentrate to leading order a blow up profile

$$u(t, x) \sim \frac{1}{\lambda(t)^{\frac{1}{p-1}}} v\left(\frac{x}{\lambda(t)}\right), \quad \lambda(t) = \sqrt{T-t},$$

which solves the non linear elliptic equation

$$\Delta v - \frac{1}{2} \Lambda v + v^p = 0, \quad \Lambda v = \frac{2}{p-1} v + y \cdot \nabla v. \quad (1.4)$$

There are three known classes of radial solutions to (1.4):

- the constant solution
  
  $$\kappa = \left(\frac{1}{p-1}\right)^{\frac{1}{p-1}}$$

  which generates the stable ODE type blow up [20, 21, 22, 23, 39, 40];

- the singular at the origin homogeneous self similar solution
  
  $$\Phi^* = \frac{c_\infty}{|x|^{\frac{2}{p-1}}}, \quad c_\infty = \left(\frac{2}{p-1} \left(d - \frac{2}{p-1}\right)^{\frac{1}{p-1}}\right)$$

  for

  $$1 + \frac{4}{d-2} < p < p_{JL} = \begin{cases} +\infty & \text{for } d \leq 10, \\ 1 + \frac{\frac{4}{d-4-2\sqrt{d-1}}}{d} & \text{for } d \geq 11, \end{cases} \quad (1.5)$$

  where $p_{JL}$ is the so called Joseph-Lundgren exponent, there exists a quantized sequence of smooth radially symmetric solutions $\Phi_n$ to (1.4) which behave like

  $$\Phi_n(r) \sim \frac{c_n}{r^{\frac{p}{p-1}}} \text{ as } r \to +\infty.$$  

  These solutions have been constructed using global Lyapunov functionals based ODE methods, [30, 51, 1, 2], and a sharp condition for their existence in the radial positive class is given in [44].

Note that all these profiles have infinite energy and it is not clear how they may participate in singularity formation emerging from smooth well localized initial data. In the radially symmetric setting, the series of breakthrough works [35, 36] gives partial answers showing the universality of the ODE blow up, and the possibility of threshold dynamics with $\Phi^*$ or $\Phi_n$ regimes depending on the value of $p$. The analysis however is strongly restricted to the radial setting and uses the intersection number Lyapunov functionals based on the maximum principle. In particular this approach does not provide any insight into the direct construction of these blow up profiles and their dynamical stability in the non radial setting.

Type II blow-up. For $p > p_{JL}$, there exist type II blow-up solutions

$$\lim_{t \to T} \|u(t)\|_{L^\infty(T-t)^{\frac{1}{p-1}}} = +\infty.$$  

They appear in the radial setting as threshold dynamics again at the boundary of the ODE blow up set, [37], and dynamical proofs were proposed in [25, 43, 45]. Their construction has been revisited in [42, 6] in the setting of dispersive Schrödinger and wave equations, and in [7] for the non radial heat equation, to produce the full
quantized sequence of smooth type II blow up bubbles. The blow-up profile near the singularity is a stationary profile:

\[ u(t, x) \sim \frac{1}{\lambda(t)^{\frac{2}{p-1}}} Q \left( \frac{x}{\lambda(t)} \right), \quad \lambda(t) \ll \sqrt{T-t} \]

where \( Q \) solves the soliton equation:

\[ \Delta Q + Q^p = 0. \quad (1.7) \]

The heart of the analysis is to control the flow near \( Q \) using suitable energy estimates, hence avoiding maximum principle tools or spectral arguments. Type II is intimately connected to the singular self similar profile (1.5), see [25, 42] for a discussion on this fundamental matter.

1.3. Statement of the result. Our aim in this paper is to propose a robust approach for both the existence and stability of self similar blow up with smooth self similar \( \Phi_n \) like profile. For the sake of simplicity, we restrict ourselves to

\[ d = 3, \quad p > 5, \quad p_{JL} = +\infty. \quad (1.8) \]

We first revisit the construction of self similar blow up solutions of [51, 2] and implement an abstract bifurcation argument which relies on the sole existence of the stationary profile \( Q \) given by (1.7). Note that this kind of argument is classical in the ODE literature, see for example [1, 11, 9], and relies on the oscillatory nature of the eigenfunctions of the linearized operator close to \( \Phi^* \) for \( p < p_{JL} \).

**Proposition 1.1** (Existence and asymptotic of excited self similar solutions). Assume (1.8). For all \( n > N \) large enough, there exist a\(^1\) smooth radially symmetric solution to the self similar equation (1.4) such that

\[ \Delta \Phi_n \text{ vanishes exactly } n \text{ times on } (0, +\infty). \]

Moreover, there exists a small enough constant \( r_0 > 0 \) independent of \( n \) such that:

1. Behavior at infinity:

\[ \lim_{n \to +\infty} \sup_{r \geq r_0} \left( 1 + r^{\frac{2}{p-1}} \right) |\Phi_n(r) - \Phi^*(r)| = 0. \quad (1.9) \]

2. Behaviour at the origin: there exists a sequence \( \mu_n > 0 \) with \( \mu_n \to 0 \) as \( n \to +\infty \) such that

\[ \lim_{n \to +\infty} \sup_{r \leq r_0} \left| \Phi_n(r) - \frac{1}{\mu_n} Q \left( \frac{r}{\mu_n} \right) \right| = 0. \quad (1.10) \]

Hence these solutions realize a connection between the ground state behavior \( Q \) at the origin, and the homogeneous self similar decay \( \Phi^* \) at infinity. We now claim that these solutions are the blow up profile of a class of finite energy initial data leaving on a non radial \( n \) codimensional manifold.

**Theorem 1.2** (Finite codimensional stability of \( \Phi_n \)). Assume (1.8). Let \( n > N \) large enough. There exists a Lipschitz codimension \( n \) manifold\(^2\) of non radial initial data with finite energy

\[ u_0 = \chi_{A_0} \Phi_n + w_0 \]

where \( A_0 \gg 1 \) is large enough and \( w_0 \) is small enough\(^3\)

\[ \|w_0\|_{H^2} + \|\Delta w_0\|_{L^2} + \|w_0\|_{H^\infty} \ll 1, \quad (1.11) \]

\(^1\)locally unique in some suitable space

\(^2\)see Proposition 4.9 for a precise statement of the Lipschitz regularity.

\(^3\)See (1.18) and below for the definition of the weighted Sobolev space \( H^2_\rho \).
such that the corresponding solution to (1.1) blows up in finite time $0 < T < +\infty$ with a decomposition

$$u(t, x) = \frac{1}{\lambda(t)^{p-1}} (\Phi_n + v) \left( t, \frac{x - x(t)}{\lambda(t)} \right)$$

where:

1. Control of the geometrical parameters: the blow up speed is self similar

$$\lambda(t) = \sqrt{(2 + o(1))(T - t)} \quad \text{as} \quad t \to T$$

and the blow up point converges

$$x(t) \to x(T) \quad \text{as} \quad t \to T.$$  \hfill (1.12)

2. Behaviour of Sobolev norms: there holds the asymptotic stability of the self similar profile above scaling

$$\lim_{t \to T} \|v(t)\|_{H^s} = 0 \quad \text{for} \quad s_c < s \leq 2,$$  \hfill (1.13)

the boundedness of norms below scaling

$$\limsup_{t \to T} \|u(t)\|_{H^s} < +\infty \quad \text{for} \quad 1 \leq s < s_c,$$  \hfill (1.14)

and the logarithmic growth of the critical norm

$$\|u(t)\|_{H^{s_c}} = c_n(1 + o_{t \to T}(1))\sqrt{\log(T - t)}, \quad c_n \neq 0.$$  \hfill (1.15)

Comments on the results.

1. On the construction of self similar solutions. The construction of self similar solutions has been performed in [51, 2] using a global Lyapounov functional ode approach. A very interesting variational approach has also been developed in [5, 19] in the setting of the related wave map problem. But there are many classical problems which lack both the variational structure and the monotonicity formulas, hence the need for a more systematic approach typically connected in a way or another to a bifurcation argument, which is the method we are implementing here. This procedure has been applied in various settings, see for example [1, 11]. One advantage is that the proof further allows for a control of the linearized operator near the bifurcated object. The prize to pay however is that we only get the bifurcated family locally near the bifurcation point, and not the whole branch, in particular not the fundamental mode. A closely related theorem is the construction [29] for the KdV equation near the critical exponent.

2. Stability of self similar blow up. There is an important literature devoted to the stability of self similar solutions for both parabolic and dispersive problems. We aim at developing a robust approach which will extend to more complicated systems. Hence we avoid on purpose maximum principle like tools. In [13, 14, 15, 16], this kind of question has also been addressed for the radially symmetric supercritical wave map problem, Yang-Mills, wave equation and Yang-Mills heat flow. In those works, the analysis requires a detailed description of the complex spectrum of the linearized operator in suitable spaces which is a delicate matter, and seems to rely heavily on the fact that in the cases under consideration, the self similar solution has an explicit formula. Our approach is different: once we know the spectral gap estimate with exponential weight which is an elementary consequence of either the

\footnote{unless one works for $p_{JL} - \varepsilon < p < p_{JL}$ in which case the whole family could be bifurcated along the same lines as for the supercritical gKdV equation performed in [29].}
variational characterization of the self similar solution as in [5], or the construction of the solution by bifurcation as in the setting of Proposition 1.1, then the control of the nonlinear flow follows by adapting the general strategy based on energy bounds of [48, 42]. In fact, the exponential decay bounds behind (1.13) considerably simplify the analysis with respect to the study of type II blow up. The connexion with type II blow up has been made in [26] using exponential weights again, and the analysis is indeed intrinsically more involved. This energy method in weighted spaces also draws a natural connexion with the analysis of ODE type I blow up for both the heat and the wave equation [21, 39, 41]. Note also that we assume (1.8) for the sake of simplicity only\(^5\). The solutions of Theorem 1.2 will be obtained using first a by now classical Brouwer like topological argument [10, 42], which is then complemented by a local uniqueness statement to construct the Lipschitz manifold as in [6, 33, 28, 17].

3. The flow near the ground state. The question of the classification of the flow near the special class of stationary solutions \(Q\) has attracted a considerable attention in the past ten years in connection with the construction of the unstable manifold [46], or the complete classification of the flow near \(Q\) in energy subcritical [46, 34] and critical settings [8]. The corresponding instabilities are central in the derivation of unstable type II blow up bubbles, [42]. From (1.10), the self similar solution resembles the solitary wave \(Q\) up to scaling near the origin, and hence the stability Theorem 1.2 can be viewed as describing one instability of the solitary wave solution in a suitable function space. Here a fundamental issue is that the linearized operator \(H = -\Delta - pQ^{p-1}\) is unbounded from below in the sense of quadratic forms for \(p < p_{JL}\). This is a major difference with respect to the case \(p > p_{JL}\) where \(H > 0\). Our analysis in this paper shows how the nonlinear bifurcated solution \(\Phi_n\) precisely allows for the suitable modification of the linearized operator which fixes this unboundedness from below of \(H\). One also observes the same behaviour of Sobolev norms (1.13), (1.14) as in [42] which illustrates the deep non trivial structure in space of the associated blow up scenario\(^6\). Let us also stress that the nature of our energy like non linear estimates goes far beyond the stability issues of specific dynamics, and has allowed in [34] in a dispersive setting and [8] in the parabolic setting for a complete description of the flow near the ground states.

This paper and [48, 42, 6, 7] hence display a deep unity and design a route map based on robust energy estimates for the proof of the existence and stability of type I or type II blow up bubbles in both radial and non radial settings.

Acknowledgements. All three authors are supported by the ERC-2014-CoG 646650 SingWave. P.R. is a junior member of the Institut Universitaire de France.

Notations. From now on and for the rest of this paper we fix \[d = 3, \quad p > 5.\]

The ground state expansion. We let \(\Phi_*\) given by (1.5) be the unique radial homogeneous self similar solution to (1.4). We let \(Q(r)\) denote the unique radially symmetric solution to

\[
\begin{cases} 
Q'' + \frac{2}{r}Q' + Q^p = 0, \\
Q(0) = 1, \quad Q'(0) = 0,
\end{cases}
\]

\(^5\)Raising dimensions causes the nonlinearity to become non smooth since \(\lim_{d \to +\infty} p_{JL} = 1\) and hence would lead to additional but manageable technical difficulties.

\(^6\)And hence its relevance in particular for more geometric problems like the harmonic heat flow of surfaces.
which asymptotic behavior at infinity is from standard ODE argument\(^7\) given by
\[
Q(r) = (1 + o_{r \to +\infty}(1))\Phi_*(r).
\]
The next term is this expansion relates to the \(p_{IL}\) exponent (1.6) which is infinite in dimension \(d = 3\). Hence the quadratic polynomial
\[
\gamma^2 - \gamma + PC_{\infty}^{p-1} = 0
\]
has complex roots
\[
\gamma = \frac{1}{2} \pm i \omega, \quad \Delta := 1 - 4PC_{\infty}^{p-1} < 0, \quad \omega := \frac{\sqrt{-\Delta}}{2}
\]
and the asymptotic behavior of \(Q\) may be precised\(^8\):
\[
Q(r) = \Phi_*(r) + \frac{c_1 \sin (\omega \log(r) + c_2)}{r^{\frac{1}{2}}} + o\left(\frac{1}{r^{\frac{1}{2}}}\right) \text{ as } r \to +\infty
\]
where \(c_1 \neq 0\) and \(c_2 \in \mathbb{R}\). Note that
\[
\frac{1}{2} - \frac{2}{p-1} = s_{c} - 1 > 0
\]
so that the second term in the expansion of \(Q\) is indeed a correction term.

**Weighted spaces.** We define the derivation operator
\[
D^k := \begin{cases} 
\Delta^m & \text{for } m = 2k, \\
\nabla \Delta^k & \text{for } m = 2k + 1.
\end{cases}
\]
We define the scalar product
\[
(f, g)_\rho = \int_{\mathbb{R}^3} f(x)g(x)\rho dx, \quad \rho = e^{-\frac{|x|^2}{2}}
\]
and let \(L^2_\rho\) be the corresponded weighted \(L^2\) space. We let \(H^k_\rho\) be the completion of \(C^\infty_c(\mathbb{R}^d)\) for the norm
\[
\|u\|_{H^k_\rho} = \left\langle \sum_{j=0}^k \|D^j u\|_{L^2_\rho} \right\rangle.
\]

**Linearized operators.** The scaling semi-group on functions \(u : \mathbb{R}^d \to \mathbb{R}\):
\[
u_\lambda(x) := \lambda^{\frac{2}{p-1}} u(\lambda x)
\]
has for infinitesimal generator the linear operator
\[
\Lambda u := \frac{2}{p-1} u + x. \nabla u = \frac{\partial}{\partial \lambda} (u_\lambda)_{|\lambda = 1}.
\]
We define the linearized operator corresponding to (1.4) around respectively \(\Phi_*\) and \(\Phi_n\) by
\[
L_\infty := -\Delta + \Lambda - \frac{PC_\infty^{p-1}}{r^2}, \quad L_n := -\Delta + \Lambda - p\Phi_n^{p-1}
\]
and their projection onto spherical harmonics:
\[
L_{\infty,m} := -\partial_{rr} - \frac{2}{r} \partial_r + \frac{2}{p-1} + r \partial_r + \frac{m(m+1)}{r^2} - p\Phi_*^{p-1}, \quad m \in \mathbb{N},
\]
\[
L_{n,m} := -\partial_{rr} - \frac{2}{r} \partial_r + \frac{2}{p-1} + r \partial_r + \frac{m(m+1)}{r^2} - p\Phi_n^{p-1}, \quad m \in \mathbb{N}.
\]
\(^7\)see [12, 27, 31].
\(^8\)see [12, 27, 31].
Note that $L_\infty$ is formally self adjoint for the $L_\rho^2$ scalar product but (1.16) implies that the associated quadratic form is not bounded from below\(^9\) on $H_\rho^1$. We similarly define the linearized operator corresponding to (1.7) around $Q$:

$$H := -\Delta - pQ^{p-1}$$
$$H_m := -\partial_{rr} - \frac{2}{r}\partial_r + \frac{m(m+1)}{r^2} - pQ^{p-1}, \ m \in \mathbb{N}.$$  

and again $H$ is not bounded from below on $\dot{H}^1$.

**General notation.** We let $\chi(x)$ denote a smooth radially symmetric function with

$$\chi(x) := \begin{cases} 
1 & \text{for } |x| \leq \frac{1}{4}, \\
0 & \text{for } |x| \geq \frac{1}{2},
\end{cases}$$

and for $A > 0$ (note the difference with (1.19)),

$$\chi_A(x) = \chi\left(\frac{x}{A}\right).$$

**Organization of the paper.** This paper is organized as follows. In section 2, we construct the family of self similar solutions $\Phi_n$ using a nonlinear matching argument. The argument is classical, but requires a careful track of various estimates to obtain the sharp bounds (1.9), (1.10). In section 3, we show how these bounds coupled with Sturm-Liouville like arguments allow for a sharp counting of the number of instabilities of the linearized operator close to $\Phi_n$ which is self adjoint against the confining measure $\rho(y)dy$, Proposition 3.1. In section 4, we turn to the heart of the dynamical argument and show how the spectral estimates in the weighted space coupled with the control of the super critical $\dot{H}^2$ norm design a stability zone for well localized initial data.

## 2. Construction of self-similar profiles

Our aim in this section is to construct radially symmetric solutions to the self similar equation

$$\Delta v - \Lambda v + v^p = 0,$$

(2.1) by using the classical strategy of gluing solutions which behave like $\Phi_*$ at infinity, and like $Q$ at the origin. As in [3, 11, 1], the matching is made possible by the oscillatory behaviour (1.17) for $p < p_{JL}$. The strength of this approach it that it relies on the implicit function theorem and not on fine monotonicity properties, and in this sense it goes far beyond the scalar parabolic setting, see for example [29] for a deeply related approach. The sharp control of the obtained solution (1.9), (1.10) will allow us to control the eigenvalues of the associated linearized operator in suitable exponentially weighted spaces, see Proposition 3.1.

### 2.1. Exterior solutions.** Recall that $\Phi_*$ given by (1.5) is a solution to (2.1) on $(0, +\infty)$. Our aim in this section is to construct the full family of solutions to (2.1) on $[r_0, +\infty)$ for some small $r_0 > 0$ with the suitable behaviour at infinity. The argument is a simple application of the implicit function theorem and continuity properties of the resolvent of $L_\infty$ in suitable weighted spaces.

---

\(^9\)This is a limit point circle case as $r \to 0$, [49].
Given $0 < r_0 < 1$, we define $X_{r_0}$ as the space of functions on $(r_0, +\infty)$ such that the following norm is finite

$$
\|w\|_{X_{r_0}} = \sup_{r_0 \leq r \leq 1} r^{\frac{3}{2}}|w| + \sup_{r \geq 1} r^2 |w|.
$$

**Lemma 2.1** (Outer resolvent of $\mathcal{L}_\infty$). 1. Basis of fundamental solutions: there exists two solutions $\psi_1$ and $\psi_2$ of

$$
\mathcal{L}_\infty(\psi_j) = 0 \text{ for } j = 1, 2 \text{ on } (0, +\infty)
$$

with the following asymptotic behavior:

$$
\psi_1 = \frac{1}{r^{\frac{3}{2}}} \left(1 + O\left(\frac{1}{r^{\frac{3}{2}}}\right)\right), \quad \psi_2 = r^2 \left(1 + O\left(\frac{1}{r^{\frac{3}{2}}}\right)\right), \quad \text{as } r \to +\infty
$$

and

$$
\psi_1 = \frac{c_3 \sin(\omega \log(r) + c_4)}{r^{\frac{3}{2}}} + O\left(\frac{1}{r^{\frac{3}{2}}}\right), \quad \psi_2 = \frac{c_5 \sin(\omega \log(r) + c_6)}{r^{\frac{3}{2}}} + O\left(\frac{1}{r^{\frac{3}{2}}}\right) \quad \text{as } r \to 0
$$

where $c_3, c_5 \neq 0$ and $c_4, c_6 \in \mathbb{R}$. Moreover, there exists $c \neq 0$ such that

$$
\Lambda \psi_1 = \frac{c}{r^{\frac{3}{2}}} \left(1 + O\left(\frac{1}{r^{\frac{3}{2}}}\right)\right) \text{ as } r \to +\infty.
$$

2. Continuity of the resolvent: let the inverse

$$
\mathcal{T}(f) = \left(\int_1^{+\infty} f \psi_2 r^2 e^{-\frac{r^2}{2}} \, dr'\right) \psi_1 - \left(\int_{r}^{+\infty} f \psi_1 r^2 e^{-\frac{r^2}{2}} \, dr'\right) \psi_2,
$$

then

$$
\mathcal{L}_\infty(\mathcal{T}(f)) = f
$$

and

$$
\|\mathcal{T}(f)\|_{X_{r_0}} \lesssim \int_{r_0}^{1} |f| r^2 \, dr' + \sup_{r \geq 1} r^2 |f|.
$$

**Proof.** The proof is classical and we sketch the details for the reader’s convenience.

**step 1** Basis of homogeneous solutions. Recall (1.16). Let the change of variable and unknown

$$
\psi(r) = \frac{1}{y^2} \phi(y), \quad y = r^2,
$$

then

$$
\partial_r = 2 r \partial_y, \quad \partial_r^2 = 4 r \partial_y (r \partial_y) = 4 r^2 \partial_y^2 + 4 r \partial_y (r \partial_y) = 4 y \partial_y^2 + 2 \partial_y, \quad r \partial_r = 2 y \partial_y.
$$

This yields

$$
\mathcal{L}_\infty(\psi) = \left(-4 y \partial_y^2 - 2 \partial_y - 4 \partial_y + \frac{2}{p - 1} + 2 y \partial_y - \frac{p c_{\infty}^{p - 1}}{y}\right) \left(\frac{1}{y^2} \phi(y)\right).
$$

Since

$$
\begin{align*}
\partial_y \left(\frac{1}{y^2} \phi(y)\right) &= \frac{1}{y^2} \phi'(y) - \frac{\gamma}{2 y^{\frac{3}{2} + 1}} \phi(y), \\
\partial_y^2 \left(\frac{1}{y^2} \phi(y)\right) &= \frac{1}{y^2} \phi''(y) - \frac{\gamma}{y^{\frac{3}{2} + 1}} \phi'(y) + \frac{\gamma}{2} \left(\frac{\gamma}{2} + 1\right) \frac{1}{y^{\frac{3}{2} + 2}} \phi(y),
\end{align*}
$$

then

$$
\begin{align*}
\mathcal{L}_\infty(\psi) &= \left(-4 y \partial_y^2 - 2 \partial_y - 4 \partial_y + \frac{2}{p - 1} + 2 y \partial_y - \frac{p c_{\infty}^{p - 1}}{y}\right) \left(\frac{1}{y^2} \phi(y)\right).
\end{align*}
$$
we infer

\[ \mathcal{L}_\infty(\psi) = \left\{ -4y \left( \frac{1}{y^2} \phi''(y) - \frac{\gamma}{y^{2+1}} \phi'(y) + \frac{\gamma}{2} \left( \frac{\gamma}{2} + 1 \right) \frac{1}{y^{2+2}} \phi(y) \right) + \left( -6 + 2y \right) \left( \frac{1}{y^2} \phi'(y) - \frac{\gamma}{2y^{2+1}} \phi(y) \right) + \left( \frac{2}{p-1} - \frac{pc^{p-1}_\infty}{y} \right) \frac{1}{y^2} \phi(y) \right\} \]

Thus, \( \mathcal{L}_\infty(\psi) = \left\{ -4y \phi''(y) + \left( 4\gamma - 6 + 2y \right) \phi'(y) + \left( \frac{2}{p-1} - \gamma + \left( 3\gamma - \gamma \left( \gamma + 2 \right) - p \gamma^{p-1}_\infty \right) \frac{1}{y} \right) \phi(y) \right\}. \]

Since \( \gamma \) satisfies

\[ \gamma^2 - \gamma + p \gamma^{p-1}_\infty = 0, \]

we infer

\[ \mathcal{L}_\infty(\psi) = -\frac{4}{y^2} \left\{ y \phi''(y) + \left( -\gamma + \frac{3}{2} - \frac{y}{2} \right) \phi'(y) + \frac{1}{4} \left( -\frac{2}{p-1} + \gamma \right) \phi(y) \right\}. \]

We change again variable by setting

\[ \phi(y) = w(z), \quad z = \frac{y}{2}. \]

We have

\[ \phi'(y) = \frac{1}{2} w'(z), \quad \phi''(y) = \frac{1}{4} w''(z) \]

and obtain

\[ \mathcal{L}_\infty(\psi) = -\frac{2}{y^2} \left( zw''(z) + \left( -\gamma + \frac{3}{2} - z \right) w'(z) - \left( \frac{1}{p-1} - \frac{\gamma}{2} \right) w(z) \right). \]

Thus, \( \mathcal{L}_\infty(\psi) = 0 \) if and only if

\[ z \frac{d^2w}{dz^2} + \left( b - z \right) \frac{dw}{dz} - aw = 0 \]  

(2.7)

where we have used the notations

\[ a = \frac{1}{p-1} - \frac{\gamma}{2}, \quad b = -\gamma + \frac{3}{2}. \]  

(2.8)

(2.7) is known as Kummer’s equation. As long as \( a \) is not a negative integer - which holds in particular for our choice of \( a \) in (2.8) -, a basis of solutions to Kummer’s equation consists of the Kummer’s function \( M(a, b, z) \) and the Tricomi function \( U(a, b, z) \). These special functions have the following asymptotic behavior for \( z \geq 0 \) (see for example [47])

\[ M(a, b, z) = \frac{\Gamma(b)}{\Gamma(a)} z^{a-b} e^z (1 + O(z^{-1})) \], \( U(a, b, z) = z^{-a} (1 + O(z^{-1})) \) as \( z \to +\infty, \)  

(2.9)

\[ M(a, b, z) = 1 + O(z) \text{ as } z \to 0, \]  

(2.10)

and for \( 1 \leq \Re(b) < 2 \) with \( b \neq 1, \)

\[ U(a, b, z) = \frac{\Gamma(b-1)}{\Gamma(a)} z^{1-b} + \frac{\Gamma(1-b)}{\Gamma(a-b+1)} + O(z^{2-\Re(b)}) \text{ as } z \to 0. \]  

(2.11)

\[ ^{10}\text{Note that our choice of } b \text{ in (2.8) is such that } \Re(b) = 1 \text{ and } b \neq 1. \]
Since $w$ is a linear combination of $M(a,b,z)$ and $U(a,b,z)$, we immediately infer from (2.9), (2.10) and (2.11) the asymptotic of $w$ both as $z \to +\infty$ and $z \to 0_+$. Finally, since

$$\psi(r) = \frac{1}{r^\gamma} w \left(\frac{r^2}{2}\right),$$

we infer from the asymptotic of $w$ the claimed asymptotic for $\psi$ both as $r \to +\infty$ and $r \to 0_+$. This concludes the proof of (2.3), (2.4).

**Step 2** Estimate on the resolvent. The Wronskian

$$W := \psi_1 \psi_2 - \psi_2 \psi_1,$$

satisfies

$$W' = \left(-\frac{2}{r} + r\right) W, \quad W = C r^2 e^{\frac{r^2}{T}}$$

where we may without loss of generality assume $C = 1$. We then solve

$$\mathcal{L}_\infty(w) = f$$

using the variation of constants which yields

$$w = \left(a_1 + \int_r^{+\infty} f \psi_2 r^2 e^{\frac{r^2}{T}} dr'\right) \psi_1 + \left(a_2 - \int_r^{+\infty} f \psi_1 r^2 e^{\frac{r^2}{T}} dr'\right) \psi_2. \quad (2.12)$$

In particular, $\mathcal{T}(f)$ corresponds to the choice $a_1 = a_2 = 0$ and thus satisfies

$$\mathcal{L}_\infty(\mathcal{T}(f)) = f.$$

Next, we estimate $\mathcal{T}(f)$ using the asymptotic behavior (2.3) and (2.4) of $\psi_1$ and $\psi_2$ as $r \to 0_+$ and $r \to +\infty$. For $r \geq 1$, we have

$$\sup_{r \geq 1} \frac{r^2}{p-1+2} |f(r)|$$

Also, for $r_0 \leq r \leq 1$, we have

$$\sup_{r \geq 1} \frac{r^2}{p-1+2} |f(r)|$$

and (2.6) is proved.

**Step 3** Refined control of $\psi_1$. We now turn to the proof of (2.5). We decompose

$$\psi_1 = \frac{1}{r^{\frac{2}{p-1}}} + \tilde{\psi}_1. \quad (2.13)$$

Since $\mathcal{L}_\infty(\psi_1) = 0$, we infer

$$\mathcal{L}_\infty(\tilde{\psi}_1) = f.$$
where \( f \) is given by
\[
 f = - \mathcal{L}_\infty \left( \frac{1}{r^{p-1}} \right) = \partial_r^2 \left( \frac{1}{r^{p-1}} \right) + \frac{2}{r} \partial_r \left( \frac{1}{r^{p-1}} \right) + \frac{pc_{p-1}}{r^{2}} \frac{1}{r^{p-1}} \\
= \frac{2(p-3)}{p-1} \frac{1}{r^{2}}.
\]
In view of (2.12), we infer
\[
\tilde{\psi}_1 = \left( a_1 + \frac{2(p-3)}{p-1} \int_r^{+\infty} \psi_2 e^{\frac{-r^2}{2}} dr' \right) \psi_1 + \left( a_2 - \frac{2(p-3)}{p-1} \int_r^{+\infty} \psi_1 e^{\frac{-r^2}{2}} dr' \right) \psi_2.
\]
On the other hand, we deduce from the asymptotic behavior of \( \psi_1 \) and \( \psi_2 \) as \( r \to +\infty \), this forces \( a_1 = a_2 = 0 \) and hence
\[
\tilde{\psi}_1 = o \left( \frac{1}{r^{p-1}} \right) \text{ as } r \to +\infty.
\]
In view of the asymptotic behavior of \( \psi_1 \) and \( \psi_2 \) as \( r \to +\infty \), this forces \( a_1 = a_2 = 0 \) and hence
\[
\tilde{\psi}_1 = \frac{2(p-3)}{p-1} \left( \int_r^{+\infty} \psi_2 e^{\frac{-r^2}{2}} dr' \right) \psi_1 - \frac{2(p-3)}{p-1} \left( \int_r^{+\infty} \psi_1 e^{\frac{-r^2}{2}} dr' \right) \psi_2.
\]
Then, applying \( \Lambda \) to both sides, and using the asymptotic behavior of \( \psi_1 \) and \( \psi_2 \) as \( r \to +\infty \) yields
\[
\Lambda \tilde{\psi}_1 = c \left( \frac{1}{r^{p-1}} \right) \left( 1 + O \left( \frac{1}{r^2} \right) \right) \text{ as } r \to +\infty
\]
for some constant\(^{11} \) \( c \neq 0 \). Injecting this into (2.13) yields
\[
\Lambda \psi_1 = \Lambda \tilde{\psi}_1 = c \left( \frac{1}{r^{p-1}} \right) \left( 1 + O \left( \frac{1}{r^2} \right) \right) \text{ as } r \to +\infty
\]
for some constant \( c \neq 0 \) and concludes the proof of Lemma 2.1. \( \square \)

We are now in position to construct the family of outer self similar solutions as a classical consequence of the implicit function theorem.

**Proposition 2.2** (Exterior solutions). Let \( 0 < r_0 < 1 \) a small enough universal constant. For all
\[
0 < \varepsilon \ll r_0^{\kappa_c - 1},
\]
there exists a solution \( u \) to
\[
\Delta u - \Delta u + u^p = 0 \quad \text{on} \quad (r_0, +\infty)
\]
of the form
\[
u = \phi_* + \varepsilon \psi_1 + \varepsilon w
\]
with the bounds:
\[
\|w\|_{X_{r_0}} \lesssim \varepsilon r_0^{1-\kappa_c}, \quad \|\Lambda w\|_{X_{r_0}} \lesssim \varepsilon r_0^{1-\kappa_c}.
\]
Furthermore,
\[
w|_{r_0} = 0 \quad \text{and} \quad \|\partial_\varepsilon w|_{r_0}\|_{X_{r_0}} \lesssim \varepsilon r_0^{1-\kappa_c}.
\]
\(^{11}\)Actually, \( c \) is explicitly given by
\[
c = -\frac{p-3}{p-1} \neq 0.
\]
Proof. This a classical consequence of Lemma 2.1.

step 1. Setting up the Banach fixed point. Let \( v \) such that
\[
  u = \Phi^* + \varepsilon v,
\]
then \( u \) solves (2.15) iff:
\[
  \mathcal{L}_\infty(v) = \varepsilon^{p(p-1)/2} \Phi^{p-2}v^2 + \varepsilon F(\Phi^*, v, \varepsilon) \text{ on } r > r_0,
\]
where
\[
  F(\Phi^*, v, \varepsilon) = \frac{1}{\varepsilon^2} \left( (\Phi^* + \varepsilon v)^p - \Phi^p - p\Phi^{p-1}\varepsilon v - \frac{p(p-1)}{2} \Phi^{p-2}\varepsilon^2 v^2 \right).
\]
Furthermore, we decompose
\[
  v = \psi_1 + w
\]
and hence, using in particular the fact that \( \mathcal{L}_\infty(\psi_1) = 0 \), \( w \) is a solution to
\[
  \mathcal{L}_\infty(w) = p(p-1)\varepsilon G[\Phi^*, \psi_1, \varepsilon]w \text{ on } r > r_0
\]
where we defined the map:
\[
  G[\Phi^*, \psi_1, \varepsilon]w = \left( \int_0^1 (1-s)(\Phi^* + s\varepsilon(\psi_1 + w))^{p-2}ds \right) (\psi_1 + w)^2.
\]
We claim the non linear bounds: assume that
\[
  \| w \|_{X_{r_0}} \leq 1,
\]
then
\[
  \int_{r_0}^1 |G[\Phi^*, \psi_1, \varepsilon]w| r^\frac{3}{2} dr' + \sup_{r \geq 1} r^\frac{3}{2} |G[\Phi^*, \psi_1, \varepsilon]w| \lesssim r_0^{1-s_c} \quad (2.17)
\]
and
\[
  \int_{r_0}^1 |G[\Phi^*, \psi_1, \varepsilon]w_1 - G[\Phi^*, \psi_1, \varepsilon]w_2| r^\frac{3}{2} dr' + \sup_{r \geq 1} r^\frac{3}{2} |G[\Phi^*, \psi_1, \varepsilon]w_1 - G[\Phi^*, \psi_1, \varepsilon]w_2| \lesssim r_0^{1-s_c} \| w_1 - w_2 \|_{X_{r_0}}. \quad (2.18)
\]
Assume (2.17), (2.18), then we look for \( w \) as the solution of the following fixed point
\[
  w = \varepsilon p(p-1)\mathcal{T} \left( G[\Phi^*, \psi_1, \varepsilon]w \right), \; w \in X_{r_0}. \quad (2.19)
\]
In view of the assumption \( \varepsilon r_0^{1-s_c} \ll 1 \), the continuity estimate on the resolvent (2.6) and the nonlinear estimates (2.17), (2.18), the Banach fixed point theorem applies and yields a unique solution \( w \) to (2.19) with
\[
  \| w \|_{X_{r_0}} \lesssim \varepsilon r_0^{1-s_c}.
\]
Differentiating (2.19) in space, we immediately infer
\[
  \| \Lambda w \|_{X_{r_0}} \lesssim \varepsilon r_0^{1-s_c}.
\]
Finally, we compute \( w |_{t=0} \) and \( \partial_t w |_{t=0} \). In view of (2.19), we have
\[
  w |_{t=0} = 0.
\]
Also, we have
\[
  \partial_t w = p(p-1)\mathcal{T} \left( G[\Phi^*, \psi_1, \varepsilon]w \right) + \varepsilon p(p-1)\mathcal{T} \left( \partial_t G[\Phi^*, \psi_1, \varepsilon]w \right)
\]
and hence
\[ \partial_{x} w|_{x=0} = p(p - 1) \mathcal{T} \left( G[\Phi_{*}, \psi_{1}, \varepsilon] w \right) \big|_{x=0}. \]

We have
\[ G[\Phi_{*}, \psi_{1}, \varepsilon] w|_{x=0} = \left( \int_{0}^{1} (1 - s) \Phi_{*}^{p-2} ds \right) \psi_{1}^{2} = \frac{1}{2} \Phi_{*}^{p-2} \psi_{1}^{2} \]
which yields
\[ \partial_{x} w|_{x=0} = \frac{p(p - 1)}{2} \mathcal{T} \left( \Phi_{*}^{p-2} \psi_{1}^{2} \right). \]

The continuity estimate (2.6) and the asymptotic behavior of \( \psi_{1} \) (2.3) (2.4) yield
\[ \| \partial_{x} w|_{x=0} \|_{X_{0}} \lesssim r_{0}^{1-s_{c}}. \]

**Step 2** Proof of the nonlinear estimates (2.17), (2.18). Note first that in view of Lemma 2.1 and the definition of \( \cdot \) and hence, our choice of \( \varepsilon \) and hence
\[ W_{r} \ni \int_{r}^{2r} ds, \]
while for \( \varepsilon \geq 1 \), we have
\[ |w(r)| + |\psi_{1}(r)| \lesssim |\Phi_{*}(r)|, \]
and hence, our choice of \( \varepsilon \) yields for all \( r \geq r_{0} \)
\[ \varepsilon |\psi_{1}(r)| + \varepsilon |w(r)| \lesssim |\Phi_{*}(r)|. \]

Next, we estimate \( G[\Phi_{*}, \psi_{1}, \varepsilon] w \). For \( r_{0} \leq r \leq 1 \), we have
\[ |G[\Phi_{*}, \psi_{1}, \varepsilon] w| \leq (|\Phi_{*}(r)| + \varepsilon (|\psi_{1}(r)| + |w(r)|))^{p-2} (|\psi_{1}(r)| + |w(r)|)^{2} \lesssim |\Phi_{*}(r)|^{p-2} (|\psi_{1}(r)| + |w(r)|)^{2} \lesssim \left( \frac{1}{r^{p/\tau}} \right)^{2} \left( \frac{1}{r^{2\tau}} \right)^{2} \lesssim \frac{2}{r^{p/\tau}} \]
and hence
\[ \int_{r_{0}}^{1} |G[\Phi_{*}, \psi_{1}, \varepsilon] w|r^{2} dr \lesssim \left( \int_{r_{0}}^{1} r^{-s_{c}} dr \right) \lesssim r_{0}^{1-s_{c}}. \]

Also, for \( r \geq 1 \), we have
\[ |G[\Phi_{*}, \psi_{1}, \varepsilon] w| \leq \left( |\Phi_{*}(r)| + \varepsilon (|\psi_{1}(r)| + |w(r)|))^{p-2} (|\psi_{1}(r)| + |w(r)|)^{2} \lesssim \left( \frac{1}{r^{p/\tau}} \right)^{p} (1 + \|w\|_{X_{0}})^{2} \lesssim \frac{1}{r^{2{\tau}/p - 1}} \]
and hence
\[ \sup_{r \geq 1} r^{2{\tau}/p - 1} |G[\Phi_{*}, \psi_{1}, \varepsilon] w| \lesssim 1. \]
and (2.17) is proved. We now prove the contraction estimate:

\[
G[\Phi_*, \psi_1, \varepsilon]w_1 - G[\Phi_*, \psi_1, \varepsilon]w_2 \\
= \left( \int_0^1 (1 - s)(\Phi_* + s\varepsilon(\psi_1 + w_1))^{p-2} ds \right) (\psi_1 + w_1)^2 \\
- \left( \int_0^1 (1 - s)(\Phi_* + s\varepsilon(\psi_1 + w_2))^{p-2} ds \right) (\psi_1 + w_2)^2 \\
= \left( \int_0^1 (1 - s)(\Phi_* + s\varepsilon(\psi_1 + w_1))^{p-2} ds \right) ( (\psi_1 + w_1)^2 - (\psi_1 + w_2)^2 ) \\
+ \left( \int_0^1 (1 - s)(\Phi_* + s\varepsilon(\psi_1 + w_1))^{p-2} ds \right) (\psi_1 + w_2)^2 \\
- \left( \int_0^1 (1 - s)(\Phi_* + s\varepsilon(\psi_1 + w_2))^{p-2} ds \right) (\psi_1 + w_2)^2 \\
= \left( \int_0^1 (1 - s)(\Phi_* + s\varepsilon(\psi_1 + w_1))^{p-2} ds \right) (2\psi_1 + w_1 + w_2)(w_1 - w_2) \\
+ (p-2) \left( \int_0^1 s(1 - s) \int_0^1 (\Phi_* + s\varepsilon(\psi_1 + w_1) + s\varepsilon(\psi_2 - w_1))^{p-3} d\sigma ds \right) \\
\times (\psi_1 + w_2)^2 \varepsilon(w_1 - w_2)
\]

and hence

\[
|G[\Phi_*, \psi_1, \varepsilon]w_1 - G[\Phi_*, \psi_1, \varepsilon]w_2| \\
\lesssim (|\Phi_*(r)| + \varepsilon(|\psi_1(r)| + |w_1(r)|))^{p-2} (|\psi_1(r)| + |w_1(r)| + |w_2(r)|)^2 |w_1(r) - w_2(r)| \\
+ (|\Phi_*(r)| + \varepsilon(|\psi_1(r)| + |w_1(r)|))^{p-3} (|\psi_1(r)| + |w_2(r)|)^2 \varepsilon|w_1(r) - w_2(r)| \\
\lesssim \{ |\Phi_*(r)|^{p-2} (|\psi_1(r)| + |w_1(r)| + |w_2(r)|)^2 + \varepsilon|\Phi_*(r)|^{p-3} (|\psi_1(r)| + |w_2(r)|)^2 \} |w_1(r) - w_2(r)|.
\]

For \( r_0 \leq r \leq 1 \), we have

\[
|G[\Phi_*, \psi_1, \varepsilon]w_1 - G[\Phi_*, \psi_1, \varepsilon]w_2| \\
\lesssim \left( \frac{1}{r^{p-1}} \right)^{p-2} \left( \frac{1}{r^2} \right)^{2} (1 + \|w_1\|_{X_{r_0}} + \|w_2\|_{X_{r_0}}) \|w_1 - w_2\|_{X_{r_0}} \\
+ \varepsilon \left( \frac{1}{r^{p-1}} \right)^{p-3} \left( \frac{1}{r^2} \right)^{3} (1 + \|w_1\|_{X_{r_0}} + \|w_2\|_{X_{r_0}})^2 \|w_1 - w_2\|_{X_{r_0}} \\
\lesssim \left( \frac{1}{r^{p-1}} + \varepsilon \frac{1}{r^2} \right) \|w_1 - w_2\|_{X_{r_0}}
\]

and hence

\[
\int_{r_0}^{1} |G[\Phi_*, \psi_1, \varepsilon]w_1 - G[\Phi_*, \psi_1, \varepsilon]w_2|^{\frac{2}{p'}} dr' \\
\lesssim \left( \int_{r_0}^{1} r'^{-s_0} dr' + \varepsilon \int_{r_0}^{1} r'^{1-2s_0} dr' \right) \|w_1 - w_2\|_{X_{r_0}} \\
\lesssim \frac{1}{r_0^{-s_0}} (1 + \varepsilon r_0^{1-s_0}) \|w_1 - w_2\|_{X_{r_0}} \lesssim \frac{1}{r_0^{-s_0}} \|w_1 - w_2\|_{X_{r_0}}.
\]
Similarly, for \( r \geq 1 \),
\[
\begin{align*}
|G[\Phi, \psi_1, \varepsilon]w_1 - G[\Phi, \psi_1, \varepsilon]w_2| & \lesssim \left( \frac{1}{r^{\frac{2}{p-1}}} \right)^p (1 + \|w_1\|_{X_0} + \|w_2\|_{X_0})^3 \|w_1 - w_2\|_{X_0} \\
& \lesssim \frac{1}{r^{2+p-1}} \|w_1 - w_2\|_{X_0}
\end{align*}
\]
and hence
\[
\sup_{r \geq 1} r^{\frac{2}{p-1}} |G[\Phi, \psi_1, \varepsilon]w_1 - G[\Phi, \psi_1, \varepsilon]w_2| \lesssim \|w_1 - w_2\|_{X_0}.
\]
This concludes the proof of (2.17), (2.18) and of Proposition 2.2. □

2.2. **Constructing interior self-similar solutions.** We now construct the family of inner solutions to (2.1) in \([0, r_0]\) which after renormalization bifurcate from the stationary equation and the ground state solution \( Q \).

We start with the continuity of the resolvent of the linearized operator \( H \) close to \( Q \) in suitable weighted spaces. Given \( r_1 \gg 1 \), we define \( Y_{r_1} \) as the space of functions on \((0, r_1)\) such that the following norm is finite
\[
\|w\|_{Y_{r_1}} = \sup_{0 \leq r \leq r_1} (1 + r)^{-\frac{3}{2}} (|w| + r |\partial_r w|).
\]

**Lemma 2.3** (Interior resolvent of \( H \)). 1. Basis of fundamental solutions: we have
\[
H(\Lambda Q) = 0, \quad H \rho = 0
\]
with the following asymptotic behavior as \( r \to +\infty \)
\[
\Lambda Q(r) = \frac{c_7}{r^{\frac{1}{2}}} \sin \left( \omega \log(r) + c_8 \right) + O \left( \frac{1}{r^{\frac{3}{2} - \frac{1}{2}}} \right),
\]
\[
\rho(r) = \frac{c_9}{r^{\frac{1}{2}}} \sin \left( \omega \log(r) + c_{10} \right) + O \left( \frac{1}{r^{\frac{3}{2} - \frac{1}{2}}} \right),
\]
where \( c_7, c_9 \neq 0, c_8, c_{10} \in \mathbb{R} \).
2. Continuity of the resolvent: let the inverse
\[
S(f) = \left( \int_0^r f \rho r'^2 dr' \right) \Lambda Q - \left( \int_0^r f \Lambda Q r'^2 dr' \right) \rho
\]
then
\[
\|S(f)\|_{Y_{r_1}} \lesssim \sup_{0 \leq r \leq r_1} (1 + r)^{\frac{3}{2}} |f|.
\]

**Proof.** step 1 Fundamental solutions. Define
\[
Q_\lambda(r) = \lambda^{\frac{2}{p-1}} \tilde{Q}(\lambda r), \quad \lambda > 0,
\]
then
\[
\Delta Q_\lambda + Q_\lambda^p = 0 \text{ for all } \lambda > 0
\]
and differentiating w.r.t. \( \lambda \) and evaluating at \( \lambda = 1 \) yields
\[
H(\Lambda Q) = 0.
\]
Let \( \rho \) be another solution to \( H(\rho) = 0 \) which does not depend linearly on \( \Lambda Q \), we aim at deriving the asymptotic of both \( \Lambda Q \) and \( \rho \) as \( r \to +\infty \).

**Limiting problem** We first solve
\[
-\partial_r^2 \varphi - \frac{2}{r} \partial_r \varphi - \frac{pr^{p-1}}{r^2} \varphi = f.
\]
The homogeneous problem admits the explicit basis of solutions

\[ \varphi_1 = \frac{\sin(\omega \log(r))}{r^{\frac{3}{2}}}, \quad \varphi_2 = \frac{\cos(\omega \log(r))}{r^{\frac{3}{2}}}, \quad (2.22) \]

and the corresponding Wronskian is given by

\[ W(r) = \varphi_1'(r)\varphi_2(r) - \varphi_2'(r)\varphi_1(r) = \frac{\omega}{r^2}. \]

Using the variation of constants, the solutions to (2.21) are given by

\[ \varphi(r) = \left( a_{1,0} + \int_{r}^{+\infty} f \varphi_2 \frac{r'^2}{\omega} dr' \right) \varphi_1 + \left( a_{2,0} - \int_{r}^{+\infty} f \varphi_1 \frac{r'^2}{\omega} dr' \right) \varphi_2. \]

Inverting \( H \), we now claim that all solutions to \( H(\phi) = 0 \) admit an expansion

\[ \phi(r) = a_{1,0} \varphi_1 + a_{2,0} \varphi_2 + O \left( \frac{1}{r^{s_c - \frac{1}{2}}} \right) \quad \text{as} \ r \to +\infty. \quad (2.23) \]

Indeed, we rewrite the equation

\[ -\partial_r^2 \phi - \frac{2}{r} \partial_r \phi - \frac{p c_{q-1}^p}{r^2} \phi = f, \quad f = p \left( Q^{p-1}(r) - \frac{c_{q-1}^p}{r^2} \right) \phi(r), \]

and hence

\[ \phi = a_{1,0} \varphi_1 + a_{2,0} \varphi_2 + \tilde{\phi}, \quad \tilde{\phi} = F(\tilde{\phi}) \quad (2.24) \]

where

\[ F(\tilde{\phi})(r) = - \left( \int_{r}^{+\infty} p \left( Q^{p-1}(r') - \frac{c_{q-1}^p}{r'^2} \right) \left( a_{1,0} \varphi_1 + a_{2,0} \varphi_2 + \tilde{\phi} \right) (r') \varphi_2 \frac{r'^2}{\omega} dr' \right) \varphi_1 \]

\[ + \left( \int_{r}^{+\infty} p \left( Q^{p-1}(r') - \frac{c_{q-1}^p}{r'^2} \right) \left( a_{1,0} \varphi_1 + a_{2,0} \varphi_2 + \tilde{\phi} \right) (r') \varphi_1 \frac{r'^2}{\omega} dr' \right) \varphi_2. \]

Recall that

\[ Q(r) = \frac{c_{q-1}^p}{r^{p-1}} + O \left( \frac{1}{r^2} \right) \quad \text{as} \ r \to +\infty \]

so that

\[ \left| p \left( Q^{p-1}(r) - \frac{c_{q-1}^p}{r^2} \right) \right| \lesssim \frac{1}{r^{1+s_c}} \quad \text{for} \ r \geq 1. \]

We infer for \( r \geq 1 \)

\[ \left| F(\tilde{\phi})(r) \right| \lesssim \frac{1}{r^{s_c - \frac{1}{2}}} \left( \int_{r}^{+\infty} \left( \frac{1}{r^{s_c}} + \frac{1}{r^{s_c - \frac{1}{2}}} \right) \left| \tilde{\phi} \right|(r') dr' \right) \]

\[ \lesssim \frac{1}{r^{s_c - \frac{1}{2}}} + \frac{1}{r^{s_c - \frac{1}{2}}} \left( \int_{r}^{+\infty} \frac{1}{r^{s_c - \frac{1}{2}}} \left| \tilde{\phi} \right|(r') dr' \right) \]

and

\[ \left| F(\tilde{\phi}_1)(r) - F(\tilde{\phi}_2)(r) \right| \lesssim \frac{1}{r^{s_c - \frac{1}{2}}} \left( \int_{r}^{+\infty} \frac{1}{r^{s_c - \frac{1}{2}}} \left| \tilde{\phi}_1 - \tilde{\phi}_2 \right|(r') dr' \right). \]

Thus, for \( R \geq 1 \) large enough, the Banach fixed point theorem applies in the space corresponding to the norm

\[ \sup_{r \geq R} r^{s_c - \frac{1}{2}} \left| \tilde{\phi} \right|(r) \]
and yields a unique solution \( \tilde{\varphi} \) to (2.24) with
\[
\sup_{r \geq R} r^{s - \frac{1}{2}} \left| \tilde{\varphi} \right| (r) \leq 1,
\]
and (2.23) is proved.
In particular, in view of the explicit formula (2.22) for \( \varphi_1 \) and \( \varphi_2 \), and in view of the fact that \( H(\Lambda Q) = 0 \) and \( H(\rho) = 0 \), we infer as \( r \to +\infty \)
\[
\Lambda Q(r) = c_7 \sin \left( \omega \log(r) + c_8 \right) + O \left( \frac{1}{r^{s - \frac{1}{2}}} \right), \quad \rho = c_9 \sin \left( \omega \log(r) + c_{10} \right) + O \left( \frac{1}{r^{s - \frac{1}{2}}} \right)
\]
where \( c_7, c_9 \neq 0, c_8, c_{10} \in \mathbb{R} \).

**step 2** Continuity of the resolvent. We compute
\[
W := \Lambda Q' \rho - \rho' \Lambda Q, \quad W' = -\frac{2}{r} W, \quad W = -\frac{1}{r^2},
\]
without loss of generality. Still without loss of generality for \( R_0 > 0 \) small enough such that \( \Lambda Q > 0 \) on \([0, R_0]\) the integration of the Wronskian law yields
\[
\rho = -\Lambda Q \int_r^{R_0} \frac{1}{(\Lambda Q)^2 r'^2} dr'
\]
on \([0, R_0]\) which ensures
\[
|\rho(r)| \lesssim \frac{1}{r}, \quad |\partial_r \rho(r)| \lesssim \frac{1}{r^2} \text{ as } r \to 0.
\]

We now solve
\[
H(w) = f,
\]
using the variation of constants which yields
\[
w = \left( a_1 + \int_0^r f pr'^2 dr' \right) \Lambda Q + \left( a_2 - \int_0^r f \Lambda Q r'^2 dr' \right) \rho.
\]
In particular, \( S(f) \) corresponds to the choice \( a_1 = a_2 = 0 \) and thus
\[
H(S(f)) = f.
\]
Finally, using the estimates (2.25), (2.26), we estimate for \( 0 \leq r \leq 1 \):
\[
|S(f)| = \left| \left( \int_0^r f pr'^2 dr' \right) \Lambda Q - \left( \int_0^r f \Lambda Q r'^2 dr' \right) \rho \right| \lesssim \left( \int_0^r r' dr' + \frac{1}{r} \int_0^r r'^2 dr' \right) \sup_{0 \leq r \leq 1} |f| \lesssim \sup_{0 \leq r \leq r_1} (1 + r)^{\frac{3}{2}} |f|,
\]
\[
|r \partial_r S(f)| = \left| \left( \int_0^r f pr'^2 dr' \right) r \partial_r \Lambda Q - \left( \int_0^r f \Lambda Q r'^2 dr' \right) r \partial_r \rho \right| \lesssim \left( r^2 \int_0^r r' dr' + \frac{1}{r} \int_0^r r'^2 dr' \right) \sup_{0 \leq r \leq 1} |f| \lesssim \sup_{0 \leq r \leq r_1} (1 + r)^{\frac{3}{2}} |f|,
\]
and for $1 \leq r \leq r_1$:

\[
(1 + r)^{-\frac{3}{2}} |S(f)| = (1 + r)^{-\frac{3}{2}} \left| \left( \int_0^r f \rho r'^2 dr' \right) \Lambda Q - \left( \int_0^r f \Lambda Q r'^2 dr' \right) \rho \right|
\]

\[
\lesssim (1 + r)^{-2} \left( \int_0^r f (1 + r')^\frac{3}{2} dr' \right) \lesssim (1 + r)^{-2} \left( \int_0^r (1 + r') dr' \right) \sup_{0 \leq r \leq r_1} (1 + r)^\frac{1}{2} |f|
\]

\[
\lesssim \sup_{0 \leq r \leq r_1} (1 + r)^\frac{3}{2} |f|
\]

\[
(1 + r)^{-\frac{3}{2}} \left| \partial_r S(f) \right| = (1 + r)^{-\frac{3}{2}} \left| \left( \int_0^r f \rho r'^2 dr' \right) r \partial_r \Lambda Q - \left( \int_0^r f \Lambda Q r'^2 dr' \right) r \partial_r \rho \right|
\]

\[
\lesssim (1 + r)^{-2} \left( \int_0^r f (1 + r')^\frac{3}{2} dr' \right) \lesssim (1 + r)^{-2} \left( \int_0^r (1 + r') dr' \right) \sup_{0 \leq r \leq r_1} (1 + r)^\frac{1}{2} |f|
\]

\[
\lesssim \sup_{0 \leq r \leq r_1} (1 + r)^\frac{3}{2} |f|
\]

which concludes the proof of (2.20) and Lemma 2.3.

We are now in position to build the family of interior solutions:

**Proposition 2.4** (Construction of the interior solution). Let $r_0 > 0$ small enough and let $0 < \lambda \leq r_0$. Then, there exists a solution $u$ to

\[\Delta u - \Lambda u + u^p = 0 \text{ on } 0 \leq r \leq r_0\]

of the form

\[u = \frac{1}{\lambda^{\frac{p-1}{2}}} (Q + \lambda^2 T_1) \left( \frac{r}{\lambda} \right)\]

with

\[\|T_1\|_{Y_{\frac{p}{2}}} + \|\Lambda T_1\|_{Y_{\frac{p}{2}}} + \|\Lambda^2 T_1\|_{Y_{\frac{p}{2}}} \lesssim 1. \tag{2.27} \]

**Proof.** This is again a classical consequence of Lemma 2.3.

**step 1** Setting up the Banach fixed point. We look for $u$ of the form

\[u = \frac{1}{\lambda^{\frac{p-1}{2}}} (Q + \lambda^2 T_1) \left( \frac{r}{\lambda} \right)\]

so that $u$ solves $\Delta u - \Lambda u + u^p = 0$ on $[0, r_0]$ if and only if

\[H(T_1) = J[Q, \lambda^2] T_1 \text{ on } 0 \leq r \leq r_1\]

where

\[r_1 = \frac{r_0}{\lambda} \geq 1\]

so that

\[\lambda^2 r_1^2 = r_0^2 \ll 1\]

and with

\[J[Q, \lambda^2] T_1 = -\Lambda Q - \lambda^2 \Lambda T_1 + p(p-1) \lambda^2 \left( \int_0^1 (1-s)(Q + s\lambda^2 T_1)^{p-2} ds \right) T_1^2.\]

We claim the nonlinear estimates: assume $\|w\|_{Y_{r_1}} \lesssim 1$, then

\[\sup_{0 \leq r \leq r_1} (1 + r)^\frac{1}{2} |J[Q, \lambda^2] w| \lesssim 1, \tag{2.28}\]

\[\sup_{0 \leq r \leq r_1} (1 + r)^\frac{1}{2} |J[Q, \lambda^2] w_1 - J[Q, \lambda^2] w_2| \lesssim r_1^2 \lambda^2 \|w_1 - w_2\|_{Y_{r_1}}. \tag{2.29}\]
Assume (2.28), (2.29), we then look for $T_1$ as the solution to the fixed point

$$T_1 = S(J[Q, \lambda^2]T_1).$$  \hfill (2.30)

In view of the bound $\lambda^2 r_1^2 \ll 1$, the resolvent estimate (2.20) and the nonlinear estimates (2.28), (2.29), the Banach fixed point theorem applies and yields a unique solution $T_1$ to (2.30) which furthermore satisfies:

$$\|T_1\|_{Y_{r_0}} \lesssim 1.$$  

**step 2** Proof of (2.28), (2.29). Note first that for $0 \leq r \leq r_1$, we have

$$|w(r)| \lesssim (1 + r)^{\frac{3}{2}} = r_1^{\frac{3}{2}}(1 + r)^{-\frac{1}{2}} \lesssim r_1^2|Q(r)|.$$  

Thus, we infer for all $0 \leq r \leq r_1$

$$\lambda^2|w(r)| \lesssim \lambda^2 r_1^2|Q(r)|$$  

and hence, our choice of $\lambda$ yields for all $0 \leq r \leq r_1$

$$\lambda^2|w(r)| \lesssim |Q(r)|.$$  

Next, we estimate $J[Q, \lambda^2]w$. For $0 \leq r \leq r_1$, we have

$$|J[Q, \lambda^2]w| \leq |\Lambda Q| + p(p-1)\lambda^2(|Q| + \lambda^2|w|)^{p-2}|w|^2 + \lambda^2 \frac{1}{2}|w + r \partial_rw|$$  

$$\lesssim |\Lambda Q| + \lambda^2|Q|^{p-2}|w|^2 + \lambda^2 \frac{1}{2}|w + r \partial_rw|$$  

$$\lesssim (1 + r)^{-\frac{1}{2}} + \lambda^2(1 + r)^{-\frac{2(p-2)}{p-1}}(1 + r)^3\|w\|_{Y_{r_1}}^2 + \lambda^2(1 + r)^{\frac{3}{2}}\|w\|_{Y_{r_1}}^2$$  

$$\lesssim (1 + r)^{-\frac{1}{2}} \left(1 + \lambda^2(1 + r)^{-\frac{2(p-2)}{p-1} + \frac{3}{2}} + \lambda^2(1 + r)^2\right)$$  

$$\lesssim (1 + r)^{-\frac{1}{2}} \left(1 + \lambda^2(1 + r)^{-s_0+3} + \lambda^2 r_1^2\right) \lesssim (1 + r)^{-\frac{1}{2}} \left(1 + \lambda^2 r_1^2\right) \lesssim (1 + r)^{-\frac{1}{2}}$$  

and hence

$$\sup_{0 \leq r \leq r_1} (1 + r)^{\frac{1}{2}} |J[Q, \lambda^2]w| \lesssim 1.$$  

Next, we estimate $|J[Q, \lambda^2]w_1 - J[Q, \lambda^2]w_2|$. We have

\begin{align*}
J[Q, \lambda^2]w_1 - J[Q, \lambda^2]w_2 &= p(p-1)\lambda^2 \left( \int_0^1 (1-s)(Q + s\lambda^2 w_1)^{p-2}ds \right) w_1^2 - p(p-1)\lambda^2 \left( \int_0^1 (1-s)(Q + s\lambda^2 w_2)^{p-2}ds \right) w_2^2 \\
&\quad + \lambda^2 \left( \frac{1}{2}(w_1 - w_2) + r(\partial_t w_1 - \partial_t w_2) \right) w \\
&= p(p-1)\lambda^2 \left( \int_0^1 (1-s)(Q + s\lambda^2 w_1)^{p-2}ds \right) (w_1^2 - w_2^2) \\
&\quad + p(p-1)\lambda^2 \left( \int_0^1 (1-s)(Q + s\lambda^2 w_1)^{p-2}ds - \int_0^1 (1-s)(Q + s\lambda^2 w_2)^{p-2}ds \right) w_2^2 \\
&\quad + \lambda^2 \left( \frac{1}{2}(w_1 - w_2) + r(\partial_t w_1 - \partial_t w_2) \right) w \\
&= p(p-1)\lambda^2 \left( \int_0^1 (1-s)(Q + s\lambda^2 w_1)^{p-2}ds \right) (w_1 + w_2)(w_1 - w_2) \\
&\quad + p(p-1)(p-2)\lambda^2 \left( \int_0^1 s(1-s) \int_0^1 (Q + s\lambda^2 w_1 + \sigma s\lambda^2 (w_2 - w_1))^{p-3}d\sigma ds \right) w_2^2(w_1 - w_2) \\
&\quad + \lambda^2 \left( \frac{1}{2}(w_1 - w_2) + r(\partial_t w_1 - \partial_t w_2) \right) w
\end{align*}

and hence

\begin{align*}
|J[Q, \lambda^2]w_1 - J[Q, \lambda^2]w_2| &\lesssim \lambda^2(|Q(r)| + \lambda^2|w_1(r)|)^{p-2}(|w_1(r)| + |w_2(r)|)|w_1(r) - w_2(r)| \\
&\quad + \lambda^2(|Q(r)| + \lambda^2|w_1(r)| + \lambda^2|w_2(r)|)^{p-3}|w_2(r)|^2|w_1(r) - w_2(r)| \\
&\quad + \lambda^2 \left( \frac{1}{2}(w_1 - w_2) + r(\partial_t w_1 - \partial_t w_2) \right) w \\
&\lesssim \lambda^2|Q(r)|^{p-2}(|w_1(r)| + |w_2(r)|)|w_1(r) - w_2(r)| + \lambda^2|Q(r)|^{p-3}|w_2(r)|^2|w_1(r) - w_2(r)| \\
&\quad + \lambda^2 \left( \frac{1}{2}(w_1 - w_2) + r(\partial_t w_1 - \partial_t w_2) \right) w.
\end{align*}

This yields

\begin{align*}
|J[Q, \lambda^2]w_1 - J[Q, \lambda^2]w_2| &\lesssim \lambda^2(1 + r)^{-\frac{2(p-2)}{p-1}}(1 + r)^3(||w_1||_{Y_1} + ||w_2||_{Y_1})||w_1 - w_2||_{Y_1} \\
&\quad + \lambda^2(1 + r)^{-\frac{2(p-3)}{p-1}}(1 + r)^3 \frac{3}{2} ||w_2||_{Y_1}^2 ||w_1 - w_2||_{Y_1} + \frac{3}{2} \lambda^2(1 + r)^3 ||w_1 - w_2||_{Y_1} \\
&\lesssim \lambda^2(1 + r)^{-\frac{3}{2}} \left( (1 + r)^{-\frac{2}{3} + \frac{3}{2}} + \lambda^2(1 + r)^{\frac{4}{3} + 3} + (1 + r)^3 \right) ||w_1 - w_2||_{Y_1} \\
&\lesssim \lambda^2(1 + r)^{-\frac{1}{2}} \left( (1 + r)^{-s + 6} + \lambda^2(1 + r)^{-2s + 6} + (1 + r)^3 \right) ||w_1 - w_2||_{Y_1} \\
&\lesssim r^2 \lambda^2(1 + r)^{-\frac{1}{2}} \left( (1 + \lambda^2 r^2) \right) ||w_1 - w_2||_{Y_1} \lesssim r^2 \lambda^2(1 + r)^{-\frac{1}{2}} ||w_1 - w_2||_{Y_1}
\end{align*}

which concludes the proof of (2.29) and Proposition 2.4.

\begin{flushright}
\Box
\end{flushright}

2.3. The matching. We now construct a solution to (2.1) by matching the exterior solution to (2.1) constructed in section 2.1 on $[r_0, +\infty)$ to the interior solution to (2.1) constructed in section 2.2 on $[0, r_0]$. The oscillations (1.17) allow to perform the matching at $r_0$ for a quantized sequence of the small parameter $\varepsilon$ introduced in Proposition 2.2.
Proposition 2.5 (Existence of a countable number of smooth selfsimilar profiles). There exists $N \in \mathbb{N}$ large enough so that for all $n \geq N$, there exists a smooth solution $\Phi_n$ to (2.1) such that $\Lambda \Phi_n$ vanishes exactly $n$ times.

**Proof.**

**step 1** Initialization. Since 

$$
\psi_1(r) = \frac{c_3 \sin(\omega \log(r) + c_4)}{r^{\frac{3}{2}}} + O \left( \frac{r^{\frac{3}{2}}}{r^{\frac{7}{2}}} \right) \quad \text{as } r \to 0, \quad c_3 \neq 0
$$

we compute

$$
\Lambda \psi_1(r) = \frac{c_3 (1 - s_c) \sin(\omega \log(r) + c_4) + \omega \cos(\omega \log(r) + c_4)}{r^{\frac{3}{2}}} + O \left( \frac{r^{\frac{3}{2}}}{r^{\frac{7}{2}}} \right) \quad \text{as } r \to 0.
$$

We may therefore choose $0 < r_0 \ll 1$ such that

$$
\psi_1(r_0) = \frac{c_3}{r_0^{\frac{3}{2}}} + O \left( \frac{r_0^{\frac{3}{2}}}{r_0^{\frac{7}{2}}} \right), \quad \Lambda \psi_1(r_0) = \frac{c_3 (1 - s_c)}{r_0^{\frac{3}{2}}} + O \left( \frac{r_0^{\frac{3}{2}}}{r_0^{\frac{7}{2}}} \right), \quad (2.31)
$$

and Proposition 2.2 and Proposition 2.4 apply. We therefore choose $\varepsilon$ and $\lambda$ such that

$$
0 < \varepsilon \ll r_0^{s_c - 1}, \quad 0 < \lambda \leq r_0,
$$

and have from Proposition 2.2 an exterior solution $u_{ext}$ to

$$
-\Delta u_{ext} + \Lambda u_{ext} - u_{ext}^p, \quad r \geq r_0
$$

such that

$$
u_{ext}[\varepsilon] = \Phi_\varepsilon + \varepsilon \psi_1 + \varepsilon w
$$

and

$$
\|w\|_{X_{r_0}} \lesssim \varepsilon r_0^{1 - s_c}, \quad ||\Lambda w||_{X_{r_0}} \lesssim \varepsilon r_0^{1 - s_c}. \quad (2.32)
$$

We also have from Proposition 2.4 an interior solution $u_{int}$ to

$$
-\Delta u_{int} + \Lambda u_{int} - u_{int}^p, \quad 0 \leq r \leq r_0
$$

such that

$$
u_{int}[\lambda] = \frac{1}{\lambda^{p - 1}} (Q + \lambda^2 T_1) \left( \frac{r}{\lambda} \right),
$$

with

$$
||T_1||_{Y_{2, \mathbb{R}}} \lesssim 1. \quad (2.33)
$$

We now would like to match the two solutions at $r = r_0$ which is equivalent to requiring that

$$
u_{ext}(r_0) - u_{int}(r_0) = 0 \quad \text{and} \quad u'_{ext}(r_0) - u'_{int}(r_0).
$$

**step 2** Matching the functions. We introduce the map

$$
\mathcal{F}[r_0](\varepsilon, \lambda) := u_{ext}[\varepsilon](r_0) - u_{int}[\lambda](r_0).
$$

We compute

$$
\partial_\varepsilon \mathcal{F}[r_0](\varepsilon, \lambda) = \partial_\varepsilon u_{ext}[\varepsilon](r_0) = \psi_1(r_0) + w(r_0) + \varepsilon \partial_\varepsilon w(r_0).
$$

In particular, since $w_{\varepsilon=0} = 0$ and $||\partial_\varepsilon w_{\varepsilon=0}||_{X_{r_0}} \lesssim r_0^{1 - s_c}$ in view of Proposition 2.2, we have

$$
\partial_\varepsilon \mathcal{F}[r_0](0, 0) = \psi_1(r_0) \neq 0
$$

since we assumed that $\psi_1(r_0) \neq 0$. Also, in view of the asymptotic behavior of $Q$ at infinity, we have as $\lambda \to 0_+$

$$
\left| \frac{1}{\lambda^{p - 1}} (Q - \Phi_\varepsilon + \lambda^2 T_1) \left( \frac{r_0}{\lambda} \right) \right| \lesssim \frac{1}{\lambda^{p - 1}} \left( \frac{1}{r^{\frac{3}{2}}} + \lambda^2 r_0^{\frac{3}{2}} \right) \left( \frac{r_0}{\lambda} \right) \lesssim \frac{\lambda^{s_c - 1}}{r_0^{\frac{3}{2}}} \lesssim \frac{\lambda^{s_c - 1}}{r_0^{\frac{3}{2}}}.
$$
Indeed, by construction which is equivalent to

\[ \lim_{\lambda \to 0^+} \frac{1}{\lambda^{\frac{2}{p+1}}} (Q - \Phi_* + \lambda^2 T_1) \left( \frac{r_0}{\lambda} \right) = 0. \]

Since

\[ \frac{1}{\lambda^{\frac{2}{p+1}}} \Phi_* \left( \frac{r_0}{\lambda} \right) = \Phi_*(r_0), \]

this yields

\[ \mathcal{F}[r_0](0, 0) = \Phi_*(r_0) - \Phi_*(r_0) = 0. \]

We may thus apply the implicit function theorem which yields the existence of \( \lambda_0 > 0 \) and a \( C^{\min(1, (s_c-1)\ldots)} \) function \( \varepsilon(\lambda) \) defined on \([0, \lambda_0)\) such that \( \mathcal{F}(\varepsilon(\lambda), \lambda) = 0 \) and hence

\[ u_{\text{ext}}[\varepsilon(\lambda)](r_0) = u_{\text{int}}[\lambda](r_0) \text{ on } [0, \lambda_0). \]

**step 3** Control of \( \varepsilon(\lambda) \). We claim for \( \lambda \in [0, \lambda_0) \)

\[ \varepsilon(\lambda) = \frac{1}{\psi_1(r_0)\lambda^{\frac{2}{p+1}}} (Q - \Phi_* + \lambda^2 T_1) \left( \frac{r_0}{\lambda} \right) + O \left[ \lambda^{s_c-1} \left( r_0^2 + \lambda^{s_c-1} r_0^{1-s_c} \right) \right]. \]  

(2.34)

Indeed, by construction

\[ u_{\text{ext}}[\varepsilon(\lambda)](r_0) = u_{\text{int}}[\lambda](r_0) \]

which is equivalent to

\[ \varepsilon(\lambda) \psi_1(r_0) + \varepsilon(\lambda) w(r_0) = \frac{1}{\lambda^{\frac{2}{p+1}}} (Q - \Phi_* + \lambda^2 T_1) \left( \frac{r_0}{\lambda} \right). \]  

(2.35)

We infer from (2.31), (2.32), (2.33) and the asymptotic of \( Q \):

\[ \varepsilon(\lambda) \psi_1(r_0) + \varepsilon(\lambda) w(r_0) = \varepsilon(\lambda) \frac{r_0^2}{r_0^2} \left( 1 + O(r_0^2) + O(\varepsilon(\lambda)r_0^{1-s_c}) \right), \]

\[ \frac{1}{\lambda^{\frac{2}{p+1}}} \left| Q - \Phi_* + \lambda^2 T_1 \right| \left( \frac{r_0}{\lambda} \right) \lesssim \frac{\lambda^{s_c-1}}{r_0^2} (1 + O(r_0^2)). \]

This first yields using (2.14)

\[ |\varepsilon(\lambda)| \lesssim \lambda^{s_c-1}. \]  

(2.36)

which reinjected into (2.35) yields (2.34).

**step 4** Computation of the spatial derivatives. We consider the difference of spatial derivatives at \( r_0 \) for \( \lambda \in [0, \lambda_0) \)

\[ \mathcal{G}[r_0](\lambda) := u_{\text{ext}}[\varepsilon(\lambda)]'(r_0) - u_{\text{int}}[\lambda]'(r_0) \]

and claim the leading order expansion:

\[ \mathcal{G}[r_0](\lambda) = \lambda^{s_c-1} \left[ \frac{c_1 c_3 \omega}{\psi_1(r_0) r_0^2} \sin(-\omega \log(\lambda) + c_2 - c_4) \right. \]

\[ + \left. O \left( r_0^{-s_c-\frac{1}{2}} \lambda^{s_c-1} + r_0^\frac{1}{2} \right) \right]. \]  

(2.37)

\[ ^{12} \text{We actually apply the implicit function theorem to} \]  

\[ \tilde{\mathcal{F}}[r_0](\varepsilon, \mu) := \mathcal{F}(\varepsilon, \mu^{\frac{1}{s_c-1}}) \]

for any \( 0 < \delta < s_c - 1 \) so that \( \tilde{\mathcal{F}} \in C^1 \). This yields the existence of \( \tilde{\varepsilon} \in C^1 \) and we choose \( \varepsilon(\lambda) = \tilde{\varepsilon}(\lambda^{s_c-1-\delta}) \) so that \( \varepsilon \) belongs indeed to \( C^{\min(1, (s_c-1)\ldots)} \).
Indeed,
\[ G[r_0](\lambda) = \varepsilon(\lambda)\psi'(r_0) + \varepsilon(\lambda)w'(r_0) - \frac{1}{\lambda^{s-1}} (Q' - \Phi'_s) \left( \frac{r_0}{\lambda} \right). \]

From (2.36), (2.16):
\[ |\varepsilon(\lambda)w'(r_0)| \lesssim \lambda^{s-1}|w'(r_0)| \lesssim \lambda^{2(s-1)}r_0^{-\frac{1}{2}}. \]

and from (2.27)
\[ \left| \frac{1}{\lambda^{s-1}} \lambda^{2T'_1} \left( \frac{r_0}{\lambda} \right) \right| \lesssim r_0^{\frac{1}{2}} \lambda^{s-1}. \]

and hence using (2.34), (2.31):
\[
G[r_0](\lambda) = \varepsilon(\lambda)\psi'(r_0) - \lambda^{s-1} \frac{1}{\lambda^{\frac{1}{2}}} (Q' - \Phi'_s) \left( \frac{r_0}{\lambda} \right) + O \left( \left( r_0^{-\frac{3}{2}} \lambda^{s-1} + r_0^{\frac{3}{2}} \lambda^{s-1} \right) \right)
\]
\[
= \lambda^{s-1} \left( \frac{1}{\lambda^{\frac{1}{2}}} \psi'(r_0) \right) \left( Q - \Phi_s \right) \left( \frac{r_0}{\lambda} \right) + O \left( \left( r_0^{-\frac{3}{2}} \lambda^{s-1} + r_0^{\frac{3}{2}} \lambda^{s-1} \right) \right)
\]
\[
= \lambda^{s-1} \left( \frac{1}{\lambda^{\frac{1}{2}}} \psi'(r_0) \right) \left( Q - \Phi_s \right) \left( \frac{r_0}{\lambda} \right) + O \left( \left( r_0^{-\frac{3}{2}} \lambda^{s-1} + r_0^{\frac{3}{2}} \lambda^{s-1} \right) \right).
\]

Recall that
\[
\psi_1(r) = \frac{c_3 \sin(\omega \log(r) + c_4)}{r^{\frac{3}{2}}} + O \left( r^{\frac{1}{2}} \right) \text{ as } r \to 0,
\]
\[
\psi'_1(r) = -\frac{c_3 \sin(\omega \log(r) + c_4)}{2r^{\frac{3}{2}}} + \frac{c_3 \omega \cos(\omega \log(r) + c_4)}{r^{\frac{3}{2}}} + O \left( r^{\frac{3}{2}} \right) \text{ as } r \to 0,
\]
\[
Q(r) - \Phi_s(r) = \frac{c_1 \sin(\omega \log(r) + c_2)}{r^{\frac{3}{2}}} + O \left( \frac{1}{r^{s-1}} \right) \text{ as } r \to +\infty,
\]
\[
Q'(r) - \Phi'_s(r) = -\frac{c_1 \sin(\omega \log(r) + c_2)}{2r^{\frac{3}{2}}} + \frac{c_1 \omega \cos(\omega \log(r) + c_2)}{r^{\frac{3}{2}}} + O \left( \frac{1}{r^{s-1}} \right) \text{ as } r \to +\infty.
\]
and hence:
\[
\left(\frac{r_0}{\lambda}\right)^2 (Q - \Phi_\ast) \left(\frac{r_0}{\lambda}\right) \psi'_k(r_0) - \left(\frac{r_0}{\lambda}\right)^2 (Q' - \Phi'_\ast) \left(\frac{r_0}{\lambda}\right) \psi'_k(r_0) = \frac{c_1 c_3}{r_0^2} \left(\sin(\omega \log(r_0) - \omega \log(\lambda) + c_2) \left(-\frac{\sin(\omega \log(r_0) + c_4)}{2} + \omega \cos(\omega \log(r_0) + c_4)\right) \right) - \left(-\frac{\sin(\omega \log(r_0) - \omega \log(\lambda) + c_2) - \omega \cos(\omega \log(r_0) - \omega \log(\lambda) + c_2)}{2} \right) \sin(\omega \log(r_0) + c_4) + O\left(\frac{1}{r_0^2} + \lambda^{s_c-1} r_0^{-s_c-\frac{1}{2}}\right) \right)
\]

\[
= \frac{c_1 c_3 \omega}{r_0^2} \sin(\omega \log(r_0) - \omega \log(\lambda) + c_2) \cos(\omega \log(r_0) + c_4) - \cos(\omega \log(r_0) - \omega \log(\lambda) + c_2) \sin(\omega \log(r_0) + c_4) + O\left(\frac{1}{r_0^2} + \lambda^{s_c-1} r_0^{-s_c-\frac{1}{2}}\right) \right)
\]

\[
= \frac{c_1 c_3 \omega}{r_0^2} \sin(-\omega \log(\lambda) + c_2 - c_4) + O\left(\frac{1}{r_0^2} + \lambda^{s_c-1} r_0^{-s_c-\frac{1}{2}}\right) \right)
\]

The collection of above bounds and (2.31) yields (2.37).

**step 5** Discrete matching. For \(\delta > 0\) a small enough universal constant such that \(\delta \geq r_0\) to be chosen later, we consider

\[
\lambda_{k,+} = \exp\left(-\frac{k \pi - c_4 + c_2 + \delta_0}{\omega}\right) , \quad \lambda_{k,-} = \exp\left(-\frac{k \pi - c_4 + c_2 - \delta_0}{\omega}\right) . \quad (2.38)
\]

From

\[
\lim_{k \to +\infty} \lambda_{k,\pm} = 0,
\]

there holds for \(k \geq k_0\) large enough:

\[
0 < \cdots < \lambda_{k,+} < \lambda_{k,-} < \cdots < \lambda_{k_0,+} < \lambda_{k_0,-} \leq \lambda_0
\]

With the above definition of \(\lambda_{k,\pm}\), we have for all \(k \geq k_0\)

\[
\sin(-\omega \log(\lambda_{k,+}) + c_2 - c_4) = (-1)^k \sin(\delta_0), \quad \sin(-\omega \log(\lambda_{k,-}) + c_2 - c_4) = (-1)^k \sin(\delta_0),
\]

and hence

\[
G[r_0](\lambda_{k,\pm}) = \pm(-1)^k \lambda^{-s_c-1}_{k,\pm} \left(\frac{c_1 c_3 \omega}{\psi'_1(r_0) r_0^2} \sin(\delta_0) + O\left(\frac{1}{r_0^2} + \lambda^{s_c-1} r_0^{-s_c-\frac{1}{2}}\right) \right).
\]

Since \(\delta_0 \geq r_0\), this yields for \(r_0\) small enough and for any \(k \geq k_0\) large enough:

\[
G[r_0](\lambda_{k,-})G[r_0](\lambda_{k,+}) < 0.
\]

Since the function \(\lambda \to G[r_0](\lambda)\) is continuous, we infer from the mean value theorem applied to the intervals \([\lambda_{k,+}, \lambda_{k,-}]\) the existence of \(\mu_k\) such that

\[
\lambda_{k,+} < \mu_k < \lambda_{k,-} \quad \text{and} \quad G[r_0](\mu_k) = 0 \quad \text{for all} \quad k \geq k_0.
\]

Finally, for \(k \geq k_0\), we have

\[
F[r_0](\varepsilon(\mu_k), \mu_k) = 0 \quad \text{and} \quad G[r_0](\mu_k) = 0
\]

which yields

\[
u_{ext}[\varepsilon(\mu_k)](r_0) = u_{int}[\mu_k](r_0) \quad \text{and} \quad u_{ext}[\varepsilon(\mu_k)](r_0) = u_{int}[\mu_k]'(r_0).
\]
and hence the function
\[ u_k(r) := \begin{cases} u_{\text{int}}[\mu_k](r) & \text{for } 0 \leq r \leq r_0, \\ u_{\text{ext}}[\varepsilon(\mu_k)](r) & \text{for } r > r_0 \end{cases} \]
is smooth and satisfies (2.1).

The rest of the proof is devoted to counting the number of zeroes of \( \Lambda u_k \) and showing that this number is an unambiguous way of counting the number of self similar solutions \( u_k \) as \( k \to +\infty \).

**Step 6** Zeros of \( \Lambda u_{\text{ext}}[\varepsilon] \). We claim that
\[ \Lambda u_{\text{ext}}[\varepsilon] \text{ has as many zeros as } \Lambda \psi_1 \text{ on } r \geq r_0. \] (2.39)
Indeed, \( \Lambda \psi_1 + \Lambda w \) does not vanish on \([R_0, +\infty)\) for \( R_0 \) large enough from (2.5) and the uniform bound (2.16). Moreover, \( \Lambda \psi_1(r_0) \neq 0 \) from the normalization (2.31), and the absolute derivative of \( \Lambda \psi_1 \) at any of its zeroes is uniformly lower bounded using (2.2), (2.4), and hence the uniform smallness (2.16)
\[ \|\Lambda w\|_{X_{r_0}} \lesssim \varepsilon r_0^{1-s_c} \ll 1 \]
yields the claim.

**Step 7** Zeros of \( \Lambda u_{\text{int}}[\mu_k] \). We now claim that
\[ \Lambda u_{\text{int}}[\mu_k] \text{ has as many zeros as } \Lambda Q \text{ on } 0 \leq r \leq r_0/\mu_k. \] (2.40)
Indeed, recall that
\[ \Lambda u_{\text{int}}[\mu_k](r) = \frac{1}{\mu_k^{p-1}}(\Lambda Q + \mu_k^2 \Lambda T_1) \left( \frac{r}{\mu_k} \right). \]
We now claim
\[ \left( \frac{r_0}{\mu_k} \right)^{\frac{1}{2}} \left| \Lambda Q \left( \frac{r_0}{\mu_k} \right) \right| \gtrsim 1. \] (2.41)
Assume (2.41), then since the zeros of \( \Lambda Q \) are simple, since we have
\[ \Lambda Q(r) = \frac{c_7 \sin(\omega \log(r) + c_8)}{r^2} + O \left( \frac{1}{r^{3-c_7}} \right) \text{ as } r \to +\infty, \]
since
\[ \|\Lambda T_1\|_{Y_{r_0}} = \sup_{0 \leq r \leq \frac{r_0}{\mu_k}} (1 + r)^{-\frac{3}{2}} |\Lambda T_1| \lesssim 1 \]
so that
\[ \sup_{0 \leq r \leq \frac{r_0}{\mu_k}} (1 + r)^{\frac{1}{2}} |\mu_k^2 \Lambda T_1| \lesssim r_0^2, \]
and similarly for \( \Lambda^2 T_1 \), and since
\[ \Lambda Q(0) = \frac{2}{p-1} \neq 0, \]
we conclude that \( \Lambda Q + \mu_k^2 \Lambda T_1 \) has as many zeros as \( \Lambda Q \) on \( 0 \leq r \leq r_0/\mu_k \).

We deduce that on \( 0 \leq r \leq r_0 \), \( \Lambda u_{\text{int}}[\mu_k] \) has as many zeros as \( \Lambda Q \) on \( 0 \leq r \leq r_0/\mu_k \).

**Proof of (2.41):** Recall that
\[ u_{\text{ext}}[\varepsilon(\mu_k)](r_0) = u_{\text{int}}[\mu_k](r_0) \text{ and } u_{\text{ext}}[\varepsilon(\mu_k)]'(r_0) = u_{\text{int}}[\mu_k]'(r_0), \]
which implies
\[ \Lambda u_{\text{ext}}[\varepsilon(\mu_k)](r_0) = \Lambda u_{\text{int}}[\mu_k](r_0). \]
In view of (2.31) which we recall below and differentiating (2.35):

\[
\varepsilon(\mu_k) = \frac{1}{\psi_1(r_0)\mu_k^\frac{s_c-1}{2}} (Q - \Phi_*) \left( \frac{r_0}{\mu_k} \right) + O\left( \mu_k^{s_c-1} r_0^{s_c-1} + r_0^2 \right)
\]

and differentiating (2.35):

\[
\frac{\varepsilon(\mu_k)}{\mu_k^{s_c-1}} = \frac{1}{\Lambda \psi_1(r_0)\mu_k^\frac{s_c}{2}} \Lambda Q \left( \frac{r_0}{\mu_k} \right) + O\left( \mu_k^{s_c-1} r_0^{s_c-1} + r_0^2 \right).
\]

We infer

\[
\frac{1}{\psi_1(r_0)\mu_k^\frac{s_c}{2}} (Q - \Phi_*) \left( \frac{r_0}{\mu_k} \right) = \frac{1}{\Lambda \psi_1(r_0)\mu_k^\frac{s_c}{2}} \Lambda Q \left( \frac{r_0}{\mu_k} \right) + O\left( \mu_k^{s_c-1} r_0^{s_c-1} + r_0^2 \right).
\]

In view of (2.31) which we recall below

\[
\psi_1(r_0) = \frac{c_3}{r_0^\frac{s_c}{2}} + O\left( r_0^2 \right), \quad \Lambda \psi_1(r_0) = c_3(1 - s_c) + O\left( r_0^{\frac{s_c}{2}} \right),
\]

this yields

\[
\left| \left( \frac{r_0}{\mu_k} \right)^{\frac{1}{2}} (Q - \Phi_*) \left( \frac{r_0}{\mu_k} \right) \right| \leq \frac{2}{s_c - 1} \left| \left( \frac{r_0}{\mu_k} \right)^{\frac{1}{2}} \Lambda Q \left( \frac{r_0}{\mu_k} \right) \right| + O\left( \mu_k^{s_c-1} r_0^{s_c-1} + r_0^2 \right). \tag{2.42}
\]

On the other hand,

\[
Q(r) - \Phi_*(r) = \frac{c_1 \sin \left( \omega \log(r) + c_2 \right)}{r^{\frac{s_c}{2}}} + O\left( \frac{1}{r^{s_c-\frac{1}{2}}} \right) \quad \text{as } r \to +\infty \tag{2.43}
\]

and hence as \( r \to +\infty \)

\[
\Lambda Q(r) = \frac{c_1}{r^{\frac{s_c}{2}}} (1 - s_c) \sin \omega \log(r) + c_2 + \omega \cos \omega \log(r) + c_2 + O\left( \frac{1}{r^{s_c-\frac{1}{2}}} \right)
\]

\[
= c_1 \sqrt{(s_c - 1)^2 + \omega^2} \sin \omega \log(r) + c_2 + c_0 \frac{1}{r^{s_c-\frac{1}{2}}} \tag{2.44}
\]

where

\[
\cos(c_0) = \frac{1 - s_c}{\sqrt{(s_c - 1)^2 + \omega^2}}, \quad \sin(c_0) = \frac{\omega}{\sqrt{(s_c - 1)^2 + \omega^2}}, \quad c_0 \in \left( \frac{\pi}{2}, \pi \right).
\]

Thus there exists \( r_2 > 0 \) sufficiently small and a constant \( \delta_1 > 0 \) sufficiently small only depending on \( \omega \) and \( s_c - 1 \) such that for \( 0 < r < r_2 \), we have

\[
\text{dist} \left( \omega \log(r) + c_2 + c_0, \pi \mathbb{Z} \right) < \delta_1 \Rightarrow r^{\frac{1}{2}} |Q(r) - \Phi_*(r)| \geq \frac{4}{s_c - 1} r^{\frac{1}{2}} |\Lambda Q(r)| + \frac{c_1 \sin(c_0)}{2}. \tag{2.45}
\]

In view of (2.42), we infer for \( k \geq k_1 \) large enough

\[
\text{dist} \left( \omega \log \left( \frac{r_0}{\mu_k} \right) + c_2 + c_0, \pi \mathbb{Z} \right) \geq \delta_1
\]

and (2.41) is proved.

**step 8 Counting.** We have so far obtained

\[
\# \{ r \geq 0 \text{ such that } \Lambda u_k(r) = 0 \} = \# \left\{ 0 \leq r \leq \frac{r_0}{\mu_k} \text{ such that } \Lambda Q(r) = 0 \right\} + \# \{ r > r_0 \text{ such that } \Lambda \psi_1(r) = 0 \}.
\]
which implies
\[
\# \{ r \geq 0 \text{ such that } \Lambda u_{k+1}(r) = 0 \} = \# \{ r \geq 0 \text{ such that } \Lambda u_k(r) = 0 \} + \# A_k,
\]
with
\[
A_k := \left\{ \frac{r_0}{\mu_k} < r \leq \frac{r_0}{\mu_{k+1}} \text{ such that } \Lambda Q(r) = 0 \right\}.
\]
We claim for \( k \geq k_0 \) large enough:
\[
\# A_k = 1 \quad (2.46)
\]
which by possibly shifting the numerotation by a fixed amount ensures that \( \Lambda u_k \) vanishes exactly \( k \) times.

**Upper bound.** We first claim
\[
\# A_k \leq 1 \quad (2.47)
\]
Recall that
\[
\Lambda Q(r) = \frac{c_7 \sin (\omega \log(r) + c_8)}{r^\frac{1}{2}} + O \left( \frac{1}{r^{s-\frac{1}{2}}} \right) \text{ as } r \to +\infty, \quad (2.48)
\]
so that there exists \( R \geq 1 \) large enough such that
\[
\{ r \geq R / \Lambda Q(r) = 0 \} = \{ r_q, \ q \geq q_1 \}, \ \omega \log(r_q) + c_8 = q\pi + O \left( \frac{1}{r_q^{s-1}} \right). \quad (2.49)
\]
In view of (2.44) and (2.48), we have
\[
c_2 + \alpha_0 = c_8
\]
and hence, together with (2.45) and (2.49), we infer
\[
\inf_{q \geq q_1, k \geq k_1} \left| \log \left( \frac{r_0}{\mu_k} \right) - \log(r_q) \right| \geq \frac{\delta_1}{2\omega}. \quad (2.50)
\]
This implies for \( k \geq k_1 \)
\[
A_k = \left\{ q \geq q_1 \text{ such that } r_q \in \left( \frac{r_0}{\mu_k}, \frac{r_0}{\mu_{k+1}} \right) \right\} \quad (2.51)
\]
\[
\subset \left\{ q \geq q_1 \text{ such that } \log \left( \frac{r_0}{\mu_k} \right) + \frac{\delta_1}{2\omega} \leq \log(r_q) \leq \log \left( \frac{r_0}{\mu_{k+1}} \right) - \frac{\delta_1}{2\omega} \right\}.
\]
Since \( \lambda_{k,+} < \mu_k < \lambda_{k,-} \) with \( \lambda_{k,\pm} \) given by (2.38), we have for \( k \geq k_1 \)
\[
\log \left( \frac{r_0}{\mu_{k+1}} \right) - \frac{\delta_1}{2\omega} - \left( \log \left( \frac{r_0}{\mu_k} \right) + \frac{\delta_1}{2\omega} \right) = \log(\mu_k) - \log(\mu_{k+1}) - \frac{\delta_1}{\omega}
\]
\[
\leq \log(\lambda_{k,+}) - \log(\lambda_{k+1,-}) - \frac{\delta_1}{\omega} \leq \frac{\pi + 2\delta_0 - \delta_1}{\omega}.
\]
Also, we have for \( q \geq q_1 \)
\[
\log(r_{q+1}) - \log(r_q) = \frac{\pi}{\omega} + O \left( \frac{1}{r_q^{s-1}} \right).
\]
We now choose \( \delta_0 \) such that
\[
0 < \delta_0 < \frac{\delta_1}{4}. \quad (2.52)
\]
Then, we infer for \( k \geq k_1 \)
\[
\log \left( \frac{r_0}{\mu_{k+1}} \right) - \frac{\delta_1}{2\omega} - \left( \log \left( \frac{r_0}{\mu_k} \right) + \frac{\delta_1}{2\omega} \right) \leq \frac{\pi}{\omega} - \frac{\delta_1}{2\omega}
\]
and hence for $k \geq k_1$ and $q \geq q_1$, we have
\[ \log(r_{q+1}) - \log(r_q) > \log \left( \frac{r_0}{\mu_{k+1}} \right) - \frac{\delta_1}{2\omega} - \left( \log \left( \frac{r_0}{\mu_k} \right) + \frac{\delta_1}{2\omega} \right) \]
which in view of (2.51) implies (2.47).

Lower bound. We now prove (2.46) and assume by contradiction:
\[ \# A_{k_2} = 0. \]
Then, let $q_2 \geq q_1$ such that
\[ r_{q_2} < \frac{r_0}{\mu_{k_2}} < r_{q_2+1}. \]
We infer from (2.50):
\[ \log(r_{q_2}) \leq \log \left( \frac{r_0}{\mu_{k_2}} \right) - \frac{\delta_1}{2\omega} < \log \left( \frac{r_0}{\mu_{k_2+1}} \right) + \frac{\delta_1}{2\omega} \leq \log(r_{q_2+1}). \tag{2.53} \]
However, we have for $k \geq k_1$
\[ \log \left( \frac{r_0}{\mu_{k_2+1}} \right) + \frac{\delta_1}{2\omega} - \left( \log \left( \frac{r_0}{\mu_{k_2}} \right) - \frac{\delta_1}{2\omega} \right) = \log(\mu_{k_2}) - \log(\mu_{k_2+1}) + \frac{\delta_1}{\omega} \geq \log(\lambda_{k_2,-}) - \log(\lambda_{k_2+1,+}) + \frac{\delta_1}{\omega} \geq \frac{\pi - 2\delta_0 + \delta_1}{\omega} \geq \frac{\pi}{\omega} + \frac{\delta_1}{2\omega} \]
in view of our choice (2.52). Hence, we infer
\[ \log \left( \frac{r_0}{\mu_{k_2+1}} \right) + \frac{\delta_1}{2\omega} - \left( \log \left( \frac{r_0}{\mu_{k_2}} \right) - \frac{\delta_1}{2\omega} \right) > \log(r_{q_2+1}) - \log(r_{q_2}) \]
which contradicts (2.53).
This concludes the proof of Proposition 2.5. \hfill \Box

We now collect final estimates on the constructed solution $\Phi_n$ which conclude the proof of Proposition 1.1.

**Corollary 2.6.** Let $\Phi_n$ the solution to (2.1) constructed in Proposition 2.5. Then there exists a small enough constant $r_0 > 0$ independent of $n$ such that:
1. Convergence to $\Phi_*$ as $n \to +\infty$:
\[ \lim_{n \to +\infty} \sup_{r \geq r_0} \left| \Phi_n(r) - \Phi_*(r) \right| = 0. \tag{2.54} \]
2. Convergence to $Q$ at the origin: there holds for some $\mu_n \to 0$ as $n \to +\infty$:
\[ \lim_{n \to +\infty} \sup_{r \leq 0} \left| \Phi_n(r) - \frac{1}{\mu_n} Q \left( \frac{r}{\mu_n} \right) \right| = 0. \tag{2.55} \]
3. Last zeroes: let $r_{0,n} < r_0$ denote the last zero of $\Lambda \Phi_n$ before $r_0$. Then, for $n \geq N$ large enough, we have
\[ e^{-\frac{2\pi}{\omega}} r_0 < r_{0,n} < r_0. \]
Let $r_{AQ,n} < r_0/\mu_n$ denote the last zero of $\Lambda Q$ before $r_0/\mu_n$, then
\[ r_{0,n} = \mu_n r_{AQ,n} (1 + O(r_0^2)). \]
Proof. We choose $r_0 > 0$ small enough as in the proof of Proposition 2.5. We start with the proof of the first claim. Recall from the proof of Proposition 2.5 that we have for $r \geq r_0$

$$\Phi_n(r) = \Phi_\ast(r) + \varepsilon(\mu_n) \psi_1(r) + \varepsilon(\mu_n) w(r)$$

where we have in particular

$$\sup_{r_0 \leq r \leq 1} r^{\frac{1}{2}} (|\psi_1| + |w|) + \sup_{r \geq 1} r^{\frac{2}{p-1}} (|\psi_1| + |w|) \lesssim 1$$

and

$$\lim_{n \to +\infty} \varepsilon(\mu_n) = 0.$$  

We infer

$$\sup_{r \geq r_0} \left(1 + r^{\frac{2}{p-1}}\right) |\Phi_n(r) - \Phi_\ast(r)| \lesssim \varepsilon(\mu_n) \left(\sup_{s \geq r_0} (|\psi_1(s)| + |w(s)|) + \sup_{r \geq 1} r^{\frac{2}{p-1}} (|\psi_1(r)| + |w(r)|)\right)$$

and hence

$$\lim_{n \to +\infty} \sup_{r \geq r_0} \left(1 + r^{\frac{2}{p-1}}\right) |\Phi_n(r) - \Phi_\ast(r)| = 0.$$  

Next, recall from the proof of Proposition 2.5 that we have for $r \leq r_0$

$$\Phi_n(r) = \frac{1}{\mu_n^\frac{2}{p-1}} (Q + \mu_n^2 T_1) \left(\frac{r}{\mu_n}\right)$$

with

$$\sup_{0 \leq r \leq \frac{r_0}{\mu_n}} (1 + r)^{-\frac{3}{2}} |T_1| \lesssim 1.$$  

We infer for $r \leq r_0$

$$\left|\Phi_n(r) - \frac{1}{\mu_n^\frac{2}{p-1}} Q \left(\frac{r}{\mu_n}\right)\right| \lesssim \frac{2^{-\frac{2}{p-1}}}{\mu_n^{\frac{2}{p-1}}} |T_1| \left(\frac{r}{\mu_n}\right) \lesssim \mu_n^{s_n-1}.$$  

and hence

$$\sup_{r \leq r_0} \left|\Phi_n(r) - \frac{1}{\mu_n^\frac{2}{p-1}} Q \left(\frac{r}{\mu_n}\right)\right| \lesssim \mu_n^{s_n-1}. \quad (2.56)$$

and since $\mu_n \to 0$ as $n \to +\infty$, (2.55) is proved.

We now estimate the localization of the last zeroes of $\Phi_n$ and $\Lambda Q$ before $r_0$. Recall that

$$\Lambda Q(r) \sim c_7 \sin(\omega \log(r) + c_8) r^{\frac{1}{2}}$$

as $r \to +\infty$.

Since $\sin(\omega \log(r) + c_8)$ changes sign on the interval

$$e^{-\frac{3\pi}{2}} \frac{r_0}{\mu_n} \leq r \leq \frac{r_0}{\mu_n},$$

and since $r \gg 1$ on this interval, we infer by the mean value theorem that $\Lambda Q(r)$ has a zero on this interval. In particular, this yields

$$e^{-\frac{3\pi}{2}} \frac{r_0}{\mu_n} \leq r_{\Lambda Q, n} \leq \frac{r_0}{\mu_n}.$$
Also, recall from the proof of Proposition 2.5 that we have for \( r \leq r_0 \)
\[
\Lambda \Phi_n(r) = \frac{1}{\mu_n^{\frac{1}{2}}} \left( \Lambda Q + \mu_n^2 \Lambda T_1 \right) \left( \frac{r}{\mu_n} \right),
\]
Since
\[
\Lambda Q(r) \sim \frac{c_7 \sin(\omega \log(r) + c_8)}{r^{\frac{1}{2}}} \quad \text{as} \quad r \to +\infty,
\]
and
\[
\sup_{0 \leq r \leq \frac{r_0}{\mu_n}} (1 + r) \frac{3}{2} |\Lambda T_1| \lesssim 1,
\]
and since
\[
e^{-\frac{2\pi}{\omega} r_0} \leq r \leq r_0,
\]
we have \( r/\mu_n \sim r_0/\mu_n \gg 1 \) for \( n \geq N \) large enough, we infer
\[
\Lambda \Phi_n(r) \sim \frac{c_7 \sin(\omega \log(r) - \omega \log(\mu_n) + c_8) + O(r_0^2)}{\mu_n^{\frac{1}{2}}} \left( \frac{r}{\mu_n} \right)^{\frac{1}{2}}.
\]
This yields
\[
\left| \omega \log(r_{0,n}) - \omega \log(\mu_n) + c_8 - (\omega \log(\Lambda Q_n) + c_8) \right| \lesssim r_0^2
\]
and hence
\[
r_{0,n} = \mu_n r_{\Lambda Q,n} e^{O(r_0^2)} = \mu_n r_{\Lambda Q,n} (1 + O(r_0^2)).
\]
Furthermore, since we have
\[
e^{-\frac{2\pi}{\omega} \frac{r_0}{\mu_n}} \leq r_{\Lambda Q,n} \leq \frac{r_0}{\mu_n},
\]
we deduce
\[
e^{-\frac{2\pi}{\omega} r_0} \leq r_{0,n} \leq r_0.
\]
This concludes the proof of the corollary. \( \square \)

3. Spectral gap in weighted norms

Our aim in this section is to produce a spectral gap for the linearized operator corresponding to (1.4) around \( \Phi_n \):
\[
\mathcal{L}_n := -\Delta + \Lambda - p \Phi_n^{p-1}.
\]
Recall (1.18), then \( L_n \) is self adjoint for the \( L_\rho^2 \) scalar product. Moreover, from (A.1) and the local compactness of the Sobolev embeddings \( H^1(|x| \leq R) \hookrightarrow L^2(|x| \leq R) \),
and the fact that \( \Phi_n \in L^\infty \), the selfadjoint operator \( \mathcal{L}_n + M_n \) for the measure \( p \rho dx \) is for \( M_n \geq 1 \) large enough invertible with compact resolvent. Hence \( \mathcal{L}_n \) is diagonalizable in a Hilbert basis of \( L_\rho^2 \), and we claim the following sharp spectral gap estimate:

**Proposition 3.1** (Spectral gap for \( \mathcal{L}_n \)). Let \( n > N \) with \( N \gg 1 \) large enough, then the following holds:

1. **Eigenvalues.** The spectrum of \( \mathcal{L}_n \) is given by
\[
-\mu_{n+1,n} < \cdots < -\mu_{2,n} < -\mu_{1,n} = -2 < -\mu_{-1,n} = -1 < 0 < \lambda_{0,n} < \lambda_{1,n} < \cdots
\]
(3.2)
with
\[ \lambda_{j,n} > 0 \text{ for all } j \geq 0 \text{ and } \lim_{j \to +\infty} \lambda_{j,n} = +\infty. \] (3.3)

The eigenvalues \((-\mu_{j,n})_{1 \leq j \leq n+1}\) are simple and associated to spherically symmetric eigenvectors
\[ \psi_{j,n}, \quad \|\psi_{j,n}\|_{L^2} = 1, \quad \psi_{1,n} = \frac{\Lambda \Phi_n}{\|\Lambda \Phi_n\|}, \]
and the eigenspace for \(\mu_{-1,n}\) is spanned by
\[ \psi_{-1,n}^k = \frac{\partial_k \Phi_n}{\|\partial_k \Phi_n\|}, \quad 1 \leq k \leq 3. \] (3.4)

Moreover, there holds as \(r \to +\infty\)
\[ |\partial_k \psi_{j,n}(r)| \lesssim (1 + r)^{-\frac{1}{p+1} - \mu_{j,n} - k}, \quad 1 \leq j \leq n + 1, \quad k \geq 0. \] (3.5)

2. Spectral gap. There holds for some constant \(c_n > 0\):
\[ \forall \varepsilon \in H^1_\rho, \quad (\mathcal{L}_n \varepsilon, \varepsilon)_\rho \geq c_n \|\varepsilon\|_{H^1_\rho}^2 - \frac{1}{c_n} \left[ \sum_{j=1}^{n+1} (\varepsilon, \psi_{j,n})_\rho^2 + \sum_{k=1}^{3} (\varepsilon, \psi_{0,n}^k)_\rho^2 \right]. \] (3.6)

In other words, \(\mathcal{L}_n\) admits \(n + 1\) instability directions when \(\Lambda \Phi_n\) vanishes \(n\) times, and 0 is never in the spectrum. Moreover, there are no additional non radial instabilities apart from the trivial translation invariance (3.4).

The rest of this section is devoted to preparing the proof of Proposition 3.1 which is completed in section 3.4.

3.1. Decomposition in spherical harmonics. We first recall some basic facts about spherical harmonics. Spherical harmonics are the eigenfunctions of the Laplace-Beltrami operator on the sphere \(S^2\). The spectrum of this self-adjoint operator with compact resolvent is
\[ \{-m(m+1), \ m \in \mathbb{N}\}. \]

For each \(m \in \mathbb{N}\) the eigenvalue \(m(m+1)\) has geometric multiplicity \(2m+1\). We then denote the associated orthonormal family of eigenfunctions by \((Y^{(m,k)})_{m \in \mathbb{N}, \ -m \leq k \leq m}\) so that we have
\[ L^2(S^2) = \bigoplus_{m=0}^{+\infty} \text{Span} \left\{ Y^{(m,k)}, \ -m \leq k \leq m \right\} \]
and
\[ -\Delta_{S^2} Y^{(m,k)} = m(m+1)Y^{(m,k)}, \quad \int_{S^2} Y^{(m,k)} Y^{(m',k')} d\sigma_{S^2} = \delta_{(m,k),(m',k')}. \] (3.7)

In particular, \(u \in H^1_\rho\) is decomposed as
\[ u = \sum_{m=0}^{+\infty} \sum_{k=-m}^{m} u_{m,k} Y^{(m,k)} \]
where \(u_{m,k}\) are radial functions satisfying the Parseval formula
\[ \|u\|_{H^1_\rho}^2 = \sum_{m=0}^{+\infty} \sum_{k=-m}^{m} \|u_{m,k}\|_{H^1_\rho}^2. \]

This allows us to write
\[ (\mathcal{L}_n(u), u)_\rho = \sum_{m=0}^{+\infty} \sum_{k=-m}^{m} (\mathcal{L}_n(u_{m,k}), u_{m,k})_\rho \] (3.8)
where we recall
\[
\mathcal{L}_{n,m} := -\partial_{rr} - \frac{2}{r} \partial_r + \frac{2}{p-1} + r \partial_r + \frac{m(m+1)}{r^2} - p \Phi_n^{p-1}.
\]

We also recall for further use the definition of the operators:
\[
\mathcal{L}_{\infty,m} := -\partial_{rr} - \frac{2}{r} \partial_r + \frac{2}{p-1} + r \partial_r + \frac{m(m+1)}{r^2} - p \Phi_*^{p-1},
\]
\[
H_m := -\partial_{rr} - \frac{2}{r} \partial_r + \frac{m(m+1)}{r^2} - pQ^{p-1}.
\]

3.2. Linear ODE analysis. We compute in this section the fundamental solutions of \(\mathcal{L}_{n,m}, H_m\) and we recall the behavior of the eigenvalues of \(\mathcal{L}_\infty\). The claims are standard and follow from a classical ODE perturbation analysis using in an essential way the uniform bound (1.10).

**Lemma 3.2** (Fundamental solution for \(\mathcal{L}_{n,m}, H_m\)). Let \(m \geq 1\). Let \(\Delta_m > 0\) be given by (C.1).

1. Basis for \(\mathcal{L}_{n,m}\). Let \(\phi_{n,m}\) be the solution to \(\mathcal{L}_{n,m} \phi_{n,m} = 0\) with the behaviour at the origin
\[
\varphi_{n,m} = r^m [1 + O(r^2)] \quad \text{as} \quad r \to 0,
\]
then
\[
\varphi_{n,m} \sim c_1 \frac{1}{r^{\frac{1}{p-1}}} + c_2 r^{\frac{2}{p-1} - 3} e^{r} \quad \text{as} \quad r \to +\infty, \quad (c_1, c_2) \neq (0,0).
\]

2. Basis for \(H_1\): let \(m = 1\), then there exists a fundamental basis \((\nu_1, \phi_1)\) with
\[
\nu_1(r) = \frac{Q'(r)}{Q''(0)} = \frac{r [1 + O(r^2)]}{\frac{1}{1 + \sqrt{\Delta_1}}} \quad \text{as} \quad r \to 0
\]
and
\[
\phi_1(r) \sim \frac{1}{r^{\frac{1}{2}}} [1 + O(r^2)] \quad \text{as} \quad r \to 0
\]
\[
\sim \frac{c_{1,-}}{r^{\frac{1}{1 - \sqrt{\Delta_1}}} \quad \text{as} \quad r \to +\infty, \quad c_{1,-} \neq 0.
\]

2. Basis for \(H_m\): let \(m \geq 2\), then there exists a fundamental basis \((\nu_m, \phi_m)\) with
\[
\nu_m \sim \frac{r^m [1 + O(r^2)]}{\frac{1}{1 + \sqrt{\Delta_m}}} \quad \text{as} \quad r \to +\infty, \quad c_{m,-} > 0
\]
and
\[
\phi_m(r) \sim \frac{1}{r^{\frac{1}{2}}} [1 + O(r^2)] \quad \text{as} \quad r \to 0
\]
\[
\sim \frac{c_{m,+}}{r^{\frac{1}{1 + \sqrt{\Delta_m}}} \quad \text{as} \quad r \to +\infty, \quad c_{m,+} \neq 0.
\]

4. Positivity:
\[
\nu_m(r) > 0 \quad \text{on} \quad (0, +\infty).
\]

5. Uniform closeness: Fix \(m \geq 1\). There exists a sequence\(^{13}\) \(\mu_n \to 0\) as \(n \to +\infty\) such that for \(n \geq N\) large enough
\[
\sup_{0 \leq r \leq r_0} \left| \nu_m \left( \frac{r}{\mu_n} \right) \right| + \sup_{0 \leq r \leq r_0} \left| \nu_m' \left( \frac{r}{\mu_n} \right) \right| \lesssim r_0^2.\quad (3.16)
\]

\(^{13}\)(\(\mu_n\))\(_{n \geq N}\) is the same sequence of scales as in (1.10) in Proposition 1.1 and Corollary 2.6.
The uniform in $n$ bound (3.16) follows from the uniform control (1.10) using a standard ODE analysis. We provide a detailed proof of Lemma 3.2 in Appendix C for the sake of completeness.

We now detail the structure of the smooth zero of $L_{0,n}$ which is the key to the counting of non positive eigenvalues. Let $\varphi_{0,n}$ be the solution to

$$L_{0,n}(\varphi_{0,n}) = 0, \quad \varphi_{0,n}(0) = 1, \quad \varphi'_{0,n}(0) = 0.$$  \hspace{1cm} (3.17)

We recall that $r_{0,n} < r_0$ denotes the last zero of $\Lambda \Phi_n$ before $r_0$, and we let $r_{1,n} < r_0$ denote the last zero of $\varphi_{0,n}$ before $r_0$. We claim:

**Lemma 3.3 (Zeroes of $\Phi_{0,n}$).** There holds

$$\sup_{0 \leq r \leq r_0} \left( 1 + \frac{r}{\mu n} \right)^{-\frac{1}{2}} \left| \varphi_{0,n}(r) - \frac{p - 1}{2} \Lambda Q \left( \frac{r}{\mu n} \right) \right| \lesssim r_0^2$$  \hspace{1cm} (3.18)

and

$$r_{1,n} = r_{0,n} + O(r_0^3), \quad e^{-\frac{2\pi}{r_0}} r_0 \leq r_{1,n} \leq r_0.$$  \hspace{1cm} (3.19)

This is again a simple perturbative analysis which proof is detailed in Appendix D.

We now claim the following classical result which relies on the standard analysis of explicit special functions:

**Lemma 3.4 (Special functions lemma).** Let $\lambda \in \mathbb{R}$. The solutions to

$$L_{\infty}(\psi) = \lambda \psi, \quad \psi \in H_{\rho}^1(1, +\infty)$$

behaves for $r \to +\infty$ as

$$\psi \sim r^{-2p-1+\lambda}$$

and for $r \to 0^+$ as

$$\psi = \frac{1}{r^{\frac{3}{2}}} \cos(\omega \log(r) - \Phi(\lambda)) + O\left( r^{\frac{1}{2}} \right)$$  \hspace{1cm} (3.20)

where

$$\Phi(\lambda) = \arg \left( \frac{2^{\frac{\omega}{2}} \Gamma(i\omega)}{\Gamma \left( \frac{1}{p-1} - \frac{\lambda}{2} - \frac{1}{4} + \frac{\omega}{2} \right)} \right).$$

**Proof.** We consider the solution $\psi$ to

$$L_{\infty}(\psi) = \lambda \psi.$$  

The change of variable and unknown

$$\psi(r) = \frac{1}{(2z)^{\frac{3}{2}}} w(z), \quad z = \frac{r^2}{2}$$

leads to

$$L_{\infty}(\psi) - \lambda \psi = \frac{2}{(2z)^{\frac{3}{2}}} \left( zw''(z) + \left( -\gamma + \frac{3}{2} - z \right) w'(z) - \left( \frac{1}{p-1} - \frac{\lambda}{2} - \frac{\gamma}{2} \right) w(z) \right)$$

and thus $L_{\infty}(\psi) = \lambda \psi$ if and only if

$$z \frac{d^2w}{dz^2} + (b - z) \frac{dw}{dz} - aw = 0$$

with

$$a = \frac{1}{p-1} - \frac{\lambda}{2} - \frac{\gamma}{2}, \quad b = -\gamma + \frac{3}{2}.$$  \hspace{1cm} (3.21)
Hence \( w \) is a linear combination of the special functions \( M(a, b, z) \) and \( U(a, b, z) \) whose asymptotic at infinity is given by (2.9):

\[
M(a, b, z) \sim \frac{\Gamma(b)}{\Gamma(a)} z^{a-b} e^z, \quad U(a, b, z) \sim z^{-a} \quad \text{as} \quad z \to +\infty,
\]

In particular, a non zero contribution of \( M(a, b, z) \) to \( w \) would yield for \( \psi(r) \) the following asymptotic

\[
\psi(r) \sim r^{\frac{2}{p-1}-3-\lambda} e^{\frac{r^2}{2}} \quad \text{as} \quad r \to +\infty,
\]

which contradicts \( \psi \in H^1_\rho(1, +\infty) \). Hence

\[
w(z) = U(a, b, z).
\]

In view of the asymptotic of \( U \) recalled in (2.9), we have

\[
w(z) \sim z^{-a} \quad \text{as} \quad z \to +\infty.
\]

Since

\[
\psi(r) = \frac{1}{r^\gamma} w \left( \frac{r^2}{2} \right),
\]

this yields

\[
\psi \sim r^{\frac{2}{p-1}+\lambda} \quad \text{as} \quad r \to +\infty.
\]

Also, in view of the asymptotic of \( U \) recalled in (2.11), we have

\[
w(z) = \frac{\Gamma(-i\omega)}{\Gamma \left( \frac{1}{p-1} - \frac{\lambda}{2} - \frac{1}{4} + \frac{i\omega}{2} \right)} z^{-\omega} + \frac{\Gamma(i\omega)}{\Gamma \left( \frac{1}{p-1} - \frac{\lambda}{2} - \frac{1}{4} + \frac{i\omega}{2} \right)} + O(z) \quad \text{as} \quad z \to 0,
\]

which in view of (3.21) and the fact that \( \gamma = 1/2 + i\omega \) yields

\[
w(z) = \frac{\Gamma(-i\omega)}{\Gamma \left( \frac{1}{p-1} - \frac{\lambda}{2} - \frac{1}{4} - i\omega \right)} z^{i\omega} + \frac{\Gamma(i\omega)}{\Gamma \left( \frac{1}{p-1} - \frac{\lambda}{2} - \frac{1}{4} + i\omega \right)} + O(z) \quad \text{as} \quad z \to 0.
\]

Since

\[
\psi(r) = \frac{1}{r^\gamma} w \left( \frac{r^2}{2} \right),
\]

this yields

\[
\psi(r) = 2^{-i\frac{\omega}{2}} r^{\frac{2}{p-1}} \left( \frac{2^{-i\frac{\omega}{2}} \Gamma(-i\omega)}{\Gamma \left( \frac{1}{p-1} - \frac{\lambda}{2} - \frac{1}{4} - i\omega \right)} r^{i\omega} + \frac{2^{i\frac{\omega}{2}} \Gamma(i\omega)}{\Gamma \left( \frac{1}{p-1} - \frac{\lambda}{2} - \frac{1}{4} + i\omega \right)} r^{-i\omega} \right)
\]

\[+ O \left( r^{\frac{3}{2}} \right) \quad \text{as} \quad r \to 0,
\]

and since \( \psi \) is real valued, we infer\(^{14}\)

\[
\psi(r) = \frac{\cos(\omega \log(r) - \Phi(\lambda))}{r^{\frac{3}{2}}} + O \left( r^{\frac{3}{2}} \right) \quad \text{as} \quad r \to 0,
\]

where

\[
\Phi(\lambda) = \arg \left( \frac{2^{-i\frac{\omega}{2}} \Gamma(i\omega)}{\Gamma \left( \frac{1}{p-1} - \frac{\lambda}{2} - \frac{1}{4} + i\omega \right)} \right).
\]

This concludes the proof of the lemma. \(\square\)

\(^{14}\)Note in particular that \( \Gamma \) satisfies \( \overline{\Gamma(z)} = \Gamma(\overline{z}) \) for all \( z \in \mathbb{C} \).
3.3. Perturbative spectral analysis. We now prove elementary spectral analysis perturbation results based on the uniform bounds (1.9), (1.10) which allow us to precisely count the number of instabilities of $L_{n,0}$.

**Lemma 3.5** (Control of the outside spectrum). Let $r_0 > 0$ and let $r_{n,2}$ such that $r_{n,2} > e^{-\frac{2\pi}{\omega}} r_0$. Let us define the operators

$$
\begin{align*}
A_n[r_{n,2}](f) &= L_{n,0}(f) \text{ on } r > r_{n,2}, \quad f(r_{n,2}) = 0, \\
A_{\infty}[r_{n,2}](f) &= L_{\infty}(f) \text{ on } r > r_{n,2}, \quad f(r_{n,2}) = 0,
\end{align*}
$$

(3.22)

then

$$
\sup_{\lambda \in \text{Spec}(A_n[r_{n,2}])} \inf_{\mu \in \text{Spec}(A_{\infty}[r_{n,2}])} |\lambda - \mu| + \sup_{\mu \in \text{Spec}(A_{\infty}[r_{n,2}])} \inf_{\lambda \in \text{Spec}(A_n[r_{n,2}])} |\lambda - \mu| \to 0
$$

as $n \to +\infty$.

**Proof.** In view of (A.1), the local compactness of the Sobolev embeddings

$$
H^1(|x| \leq R) \hookrightarrow L^2(|x| \leq R) \text{ for all } 1 \leq R < +\infty,
$$

and the fact that $\Phi_n \in L^\infty$ and the selfadjoint operators $A_n[r_{n,2}] + M_n$ for the measure $pdx$ are for $M_n \geq 1$ large enough invertible with compact resolvent, and $A_n[r_{n,2}]$ is diagonalizable. Since $\Phi_n \in L^\infty(r > r_0)$, we deduce similarly that $A_{\infty}[r_{n,2}]$ is diagonalizable. Let then $\lambda_n$ be an eigenvalue of $A_n[r_{n,2}]$ with normalized eigenvector $w_n$:

$$
L_n(w_n) = 0 \text{ on } r > r_{n,2}, \quad w_n(r_{n,2}) = 0, \quad \|w_n\|_{L^p_\rho(r > r_{n,2})} = 1.
$$

Since $A_{\infty}[r_{n,2}]$ is diagonalizable in a Hilbert basis of $L^p_\rho$, we have

$$
\|A_{\infty}[r_{n,2}](w_n) - \lambda_n w_n\|_{L^p_\rho(r > r_{n,2})} \geq \text{dist}(\lambda_n, \text{spec}(A_{\infty}[r_{n,2}])) \|w_n\|_{L^p_\rho(r > r_{n,2})} = \text{dist}(\lambda_n, \text{spec}(A_{\infty}[r_{n,2}])),
$$

On the other hand,

$$
\|A_{\infty}[r_{n,2}](w_n) - \lambda_n w_n\|_{L^p_\rho(r > r_{n,2})} = \|(A_{\infty}[r_{n,2}] - A_n[r_{n,2}])(w_n)\|_{L^p_\rho(r > r_{n,2})}
$$

from which:

$$
\text{dist}(\lambda_n, \text{spec}(A_{\infty}[r_{n,2}])) \leq \|(A_{\infty}[r_{n,2}] - A_n[r_{n,2}])(w_n)\|_{L^p_\rho(r > r_{n,2})}
$$

$$
\leq \left( \sup_{r \geq r_{n,2}} \left( p|\Phi_n(r) - \Phi_\ast(r)|^{p-1} \right) \right)^{\frac{1}{p}} \|w_n\|_{L^p_\rho(r > r_{n,2})} \leq \left( \sup_{r \geq r_{n,2}} \left( p|\Phi_n(r) - \Phi_\ast(r)|^{p-1} \right) \right)^{\frac{1}{p}}
$$

$$
\to 0 \quad \text{as } n \to +\infty
$$

from (1.9). (3.23) follows by exchanging the role $A_n[r_{n,2}]$ and $A_{\infty}[r_{n,2}]$. $\square$

**Lemma 3.6** (Local continuity of the spectrum). Let $r_0 > 0$ and let $r_1$ and $r_2$ such that

$$
e^{-\frac{2\pi}{\omega}} r_0 \leq r_1, r_2 \leq r_0
$$

and

$$
r_1 = r_2 + O(r_0^3).
$$

Then, for any eigenvalue $\lambda_1$ of $A_{\infty}[r_2]$ such that $\lambda_1 \in [-3,1]$, we have

$$
\text{dist}(\lambda_1, \text{Spec}(A_{\infty}[r_2])) \lesssim r_0^{\frac{3}{2}}.
$$

(3.24)
Proof. Recall from the proof of Lemma 3.5 that both $A_{\infty}[r_1]$ and $A_{\infty}[r_2]$ are diagonalizable. Furthermore, by Sturm-Liouville, their eigenvalues are simple. Let $\lambda_1$ be an eigenvalue of $A_{\infty}[r_1]$. We claim the existence of a nearby eigenvalue $\lambda_2$ of $A_{\infty}[r_2]$ using a classical Lyapunov Schmidt procedure. Let $\varphi_1$ the normalized eigenfunction of $A_{\infty}[r_1]$ associated to $\lambda_1$ so that

$$A_{\infty}[r_1](\varphi_1) = \lambda_1 \varphi_1, \quad ||\varphi_1||_\rho = 1.$$  

The eigenvalue equation

$$A_{\infty}[r_2](\varphi_2) = \lambda_2 \varphi_2$$

is equivalent to

$$A_{\infty}[r_1](g) = \lambda_2 g + hg + (r_2 - r_1)\partial_r g$$

where

$$g(r) = \varphi_2(r + r_2 - r_1), \quad h(r) = \frac{pc_{\infty}^{p-1}}{(r + r_2 - r_1)^2} - \frac{pc_{\infty}^{p-1}}{r^2}.$$  

We decompose

$$g = \varphi_1 + r_0 \tilde{g}, \quad \lambda_2 = \lambda_1 + cr_0$$

where the constant $c$ will be chosen later. Then, $g$ satisfies (3.25) if and only if $\tilde{g}$ satisfies

$$(A_{\infty}[r_1] - \lambda_1)(\tilde{g}) = c\varphi_1 + cr_0 \tilde{g} + \frac{h}{r_0}(\varphi_1 + r_0 \tilde{g}) + \frac{r_2 - r_1}{r_0} \partial_r (\varphi_1 + r_0 \tilde{g}),$$

(3.26)

We choose $c$ such that

$$c(\varphi_1, r_0, \tilde{g}) := -\frac{1}{1 + r_0(\tilde{g}, \varphi_1)_\rho} \left( \frac{h}{r_0}(\varphi_1 + r_0 \tilde{g}) + \frac{r_2 - r_1}{r_0} \partial_r (\varphi_1 + r_0 \tilde{g}), \varphi_1 \right)_\rho.$$  

Then, the right-hand side of (3.26) is orthogonal to $\varphi_1$ and hence to the kernel of $A_{\infty}[r_1] - \lambda_1$ since $\lambda_1$ is a simple eigenvalue. Thus, we infer

$$\tilde{g} = \mathcal{F}(\tilde{g})$$

(3.27)

where

$$\mathcal{F}(\tilde{g}) := B_{\infty}[r_1, \lambda_1]^{-1}\left( c(\varphi_1, r_0, \tilde{g})(\varphi_1 + r_0 \tilde{g}) + \frac{h}{r_0}(\varphi_1 + r_0 \tilde{g}) + \frac{r_2 - r_1}{r_0} \partial_r (\varphi_1 + r_0 \tilde{g}) \right)$$

with the operator $B_{\infty}[r_1, \lambda_1]$ being the restriction of $A_{\infty}[r_1] - \lambda_1$ to the orthogonal complement of the kernel of $A_{\infty}[r_1] - \lambda_1$, i.e.

$$B_{\infty}[r_1, \lambda_1] = (A_{\infty}[r_1] - \lambda_1)_{|r_1^\perp}.$$  

Since $\lambda_1$ is an eigenvalue of $A_{\infty}[r_1]$, from the explicit behavior (3.20) of the eigenfunctions of $\mathcal{L}_{\infty}$ and the boundary condition (3.22) at $r_1$ one deduces that there exists $k \in \mathbb{Z}$ such that

$$\omega \log(r_1) - \Phi(\lambda_1) = k\pi + \frac{\pi}{2} + O(r_0^2).$$  

Let $\lambda'_1$ be the smallest eigenvalue of $A_{\infty}[r_1]$ greater than $\lambda_1$. It then satisfies:

$$\omega \log(r_1) - \Phi(\lambda'_1) = k\pi + \frac{\pi}{2} \pm \pi + O(r_0^2)$$

and so

$$|\Phi(\lambda_1) - \Phi(\lambda'_1)| = \pi + O(r_0^2) \geq \frac{\pi}{2}.$$  

As $\Phi$ is a continuous function we deduce that there exists $c > 0$ independent of $r_0$ such that $\lambda'_1 \geq \lambda_1 + c$ and we infer

$$\inf\{|\lambda - \lambda_1|, \lambda \in \text{Spec}(A_{\infty}[r_1]), \lambda > \lambda_1\} \geq c.$$
Similarly,

$$\inf\{ |\lambda - \lambda_1|, \lambda \in \text{Spec}(A_{\infty}[r_1]) \}, \lambda < \lambda_1 \} \geq c', \ c' > 0$$

and we conclude that

$$\|B_{\infty}[r_1, \lambda_1]^{-1}\|_{L^2(L^2, H^2_0)} \lesssim 1$$

with a bound that does not depend on $r_0$. Also, note that

$$\frac{h}{r_0} = \frac{pe_{\infty}^{-1}}{r_0 r^2} \left( \frac{1}{(1 + \frac{r_2 - r_1}{r_2})^2} - 1 \right) = \frac{h_1(r)}{r^2}$$

where

$$h_1(r) = -pe_{\infty}^{-1} \frac{(r_2 - r_1) + (r_1 - r_2)}{r_2 - r_1} \left( 1 + \frac{r_2 - r_1}{r} \right)^2.$$ 

Since

$$e^{-\frac{2r}{\rho}} r_0 \leq r_1 \leq r_0$$

and $r_1 = r_2 + O(r_0^3)$,

we infer

$$\|h_1\|_{L^\infty(r > r_1)} \lesssim \|h_1\|_{L^\infty(r > e^{-\frac{2r}{\rho} r_0})} \lesssim r_0.$$ 

Moreover,

$$\|r^{-2}\|_{L^2(r_1 < r < 1)} \lesssim \left( \int_{r_1}^{1} \frac{dr}{r^2} \right)^{\frac{1}{2}} \lesssim \frac{1}{r_1^{\frac{1}{2}}} \lesssim \frac{1}{r_0^{\frac{1}{2}}}.$$ 

Collecting the previous estimates, we infer

$$\|F(\tilde{g})\|_{H^2_0(r > r_1)} \lesssim \|B_{\infty}[r_1, \lambda_1]^{-1}\|_{L^2(L^2, H^2_0)} \left| c(\varphi_1, r_0, \tilde{g})(\varphi_1 + r_0 \tilde{g}) + \frac{h}{r_0} (\varphi_1 + r_0 \tilde{g}) + \frac{r_2 - r_1}{r_0} \partial_r (\varphi_1 + r_0 \tilde{g}) \right|_{L^2_0}$$

$$\lesssim \left| c(\varphi_1, r_0, \tilde{g}) \right| (1 + r_0 \|\tilde{g}\|_{L^2_0}) + r_0 \|\tilde{g}\|_{H^1_0}$$

$$\|h_1\|_{L^\infty(r > r_1)} (1 + r_0 \|\tilde{g}\|_{L^2_0}) + \|\varphi_1 + r_0 \tilde{g}\|_{L^\infty(r_1 < r < 1)} \|r^{-2}\|_{L^2(r_1 < r < 1)}$$

$$\lesssim \frac{r_0^{\frac{1}{2}}}{1 - r_0 \|\tilde{g}\|_{L^2_0}} (1 + r_0 \|\tilde{g}\|_{L^2_0}) + r_0 \|\tilde{g}\|_{H^1_0}$$

and

$$\|F(\tilde{g}_1) - F(\tilde{g}_2)\|_{H^2_0(r > r_1)} \lesssim \frac{r_0^{\frac{3}{2}}}{1 - r_0 \|\tilde{g}\|_{L^2_0}} (1 + r_0 \|\tilde{g}\|_{L^2_0}) \|\tilde{g}_1 - \tilde{g}_2\|_{L^2_0} + r_0 \|\tilde{g}_1 - \tilde{g}_2\|_{H^1_0}.$$ 

Thus, for $r_0 > 0$ small enough, the Banach fixed point theorem applies in the space $H^2_0(r > r_1)$ and yields a unique solution $\tilde{g}$ to (3.27) with

$$\|\tilde{g}\|_{H^2_0(r > r_1)} \lesssim \frac{1}{r_0^{\frac{1}{2}}}.$$ 

Hence, $\varphi_2$ with

$$\varphi_2(r) = g(r + r_1 - r_2), \quad g = \varphi_1 + r_0 \tilde{g}$$

satisfies

$$A_{\infty}[r_2](\varphi_2) = \lambda_2 \varphi_2$$
where

\[ \lambda_2 = \lambda_1 + c(\varphi_1, r_0, \tilde{g}) r_0 \]

\[ = \lambda_1 - \frac{r_0}{1 + r_0(\tilde{g}, \varphi_1)_\rho} \left( \frac{h}{r_0} (\varphi_1 + r_0 \tilde{g}) + \frac{r_2 - r_1}{r_0} \partial_r (\varphi_1 + r_0 \tilde{g}), \varphi_1 \right)_\rho. \]

Thus, \( \lambda_2 \) belongs to the spectrum of \( A_\infty[r_2] \) and hence

\[ \text{dist}(\lambda_1, \text{Spec}(A_\infty[r_2])) \leq |\lambda_2 - \lambda_1| \]

\[ \leq \left| \frac{r_0}{1 + r_0(\tilde{g}, \varphi_1)_\rho} \left( \frac{h}{r_0} (\varphi_1 + r_0 \tilde{g}) + \frac{r_2 - r_1}{r_0} \partial_r (\varphi_1 + r_0 \tilde{g}), \varphi_1 \right)_\rho \right|. \]

In view of the previous estimates, we infer

\[ \text{dist}(\lambda_1, \text{Spec}(A_\infty[r_2])) \lesssim \frac{r_0^3}{1 - r_0\|\tilde{g}\|_{L^2_\rho}} (1 + r_0\|\tilde{g}\|_{L^2_\rho}) \lesssim r_0^3. \]

and (3.24) is proved. \( \square \)

### 3.4. Proof of Proposition 3.1

Recall that \( L_n \) is diagonalizable in a Hilbertian basis of \( L^2_\rho \), and hence the spectral gap estimate (3.6) follows from the explicit distribution of eigenvalues (3.2) which we now prove. Observe that the symmetry group of dilations and translations generates the explicit eigenmodes

\[ L_n \Delta \Phi_n = -2 \Lambda \Phi_n, \quad L_n \nabla \Phi_n = -\nabla \Phi_n. \quad \quad (3.28) \]

Using the decomposition into spherical harmonics (3.8), the further study of the quadratic form \( (L_n(u), u)_\rho \) reduces to the study of the quadratic form \( (L_{n,m}(u), u)_\rho \) for \( m \geq 0 \) for which classical Strum Liouville arguments are now at hand.

#### step 1 The case \( m = 1 \)

Let \( \varphi_{n,1} \) be defined in Lemma 3.2. In particular, \( \varphi_{n,1} \) satisfies

\[ L_{n,1}(\varphi_{n,1}) = 0, \quad \varphi_{n,1}(0) = 0, \quad \varphi'_{n,1}(0) = 1. \]

Then from standard Sturm Liouville oscillation argument for central potentials, [49], the number of zeros of \( \varphi_{n,1} \) in \( r > 0 \) correspond to the number of strictly negative eigenvalues of \( L_{n,1} \).

Since we have

\[ \nabla \Phi_n(x) = \Phi'_n(r) \frac{x}{r} = \Phi'_n(r)(Y^{(1,-1)}, Y^{(1,1)}, Y^{(1,0)}) \]

and hence

\[ L_n(\nabla \Phi_n) = -\nabla \Phi_n \quad \text{implies} \quad L_{n,1}(\Phi'_n) = -\Phi'_n. \]

Thus, \( L_{n,1} \) has at least one strictly negative eigenvalue, and hence \( \varphi_{n,1} \) has at least one zero which we denote by \( r_{n,1} > 0 \). On \( [0, r_0] \), we have by (3.16):

\[ \sup_{0 \leq r \leq r_0} \left| \frac{\mu_n^{-1} \varphi_{n,1}(r) - \nu_1 \left( \frac{x}{\mu_n} \right)}{\nu_1 \left( \frac{x}{\mu_n} \right)} \right| \lesssim r_0^2 \]

Since \( \nu_1(r) > 0 \) for all \( r > 0 \), we infer that \( \varphi_{n,1} \) can not vanish on \([0, r_0] \). Hence, \( r_{n,1} \geq r_0 \).
No other zero. Assume by contradiction that there exists a second zero \( r_{n,2} > r_{n,1} \).
Let \( f_{n,1} \) being given as
\[
f_{n,1} := \begin{cases} 
\varphi_{n,1} & \text{on } r_{n,1} < r < r_{n,2}, \\
0 & \text{on } r < r_{n,1}, \\
0 & \text{on } r > r_{n,2}.
\end{cases}
\]
Then, we have \( f_{n,1} \in H_\rho^1 \) and
\[
(\mathcal{L}_{n,1}(f_{n,1}), f_{n,1}) = 0.
\] (3.29)
On the other hand, using (1.5):
\[
(\mathcal{L}_{\infty,1}(u), u)_\rho = \|u'\|_\rho^2 + \int_0^{+\infty} \left( \frac{2 - \rho c_{\infty}^{p-1}}{r^2} u^2 r^2 \rho dr \right) \\
= \|u'\|_\rho^2 + \frac{2(p+1)}{p-1} \left( \int_0^{+\infty} \frac{u^2}{r^2} \rho dr \right) \geq \|u\|_{L_\rho^2}^2. \tag{3.30}
\]
We now estimate from (1.10)
\[
\sup_{r \geq r_0} r^2 |\Phi_n^{p-1} - (\Phi_*)^{p-1}| = o_{n \to +\infty}(1) \tag{3.31}
\]
and hence for \( u \) supported in \((r_0, +\infty)\):
\[
|\mathcal{L}_{\infty,1}(u), u)_\rho - (\mathcal{L}_{n,1}(u), u)_\rho| \lesssim \int_{r_0}^{+\infty} |\Phi_n^{p-1} - \Phi_*^{p-1}| u^2 r^2 \rho(r) dr \\
\leq o_{n \to +\infty}(1) \|u\|_{L_\rho^2}^2. \tag{3.32}
\]
Since \( f_{n,1} \) is supported in \((r_{n,1}, r_{n,2}) \subset (r_0, +\infty), (3.30), (3.32) \) applied to \( f_{n,1} \) and (3.29) yield a contradiction for \( n \geq N \) large enough. Thus, \( r_{n,2} \) can not exist, and hence \( \varphi_{n,1} \) vanishes only once.

\( \varphi_{n,1} \) is not an eigenstate. Since \( \varphi_{n,1} \) vanishes only once, \( \mathcal{L}_{n,1} \) has exactly one strictly negative eigenvalue. It remains to check the \( \varphi_{n,1} \notin L_\rho^2 \) i.e. \( \varphi_{n,1} \) is not an eigenvector associated to the eigenvalue 0. To this end, note that \( \varphi_{n,1} \) is strictly positive on \((0, r_{n,1})\) from (3.9) and strictly negative on \((r_{n,1}, +\infty)\). In particular, we have

\[
\varphi'_{n,1}(r_{n,1}) < 0.
\]
Since \( \mathcal{L}_{n,1}(\varphi_{n,1}) = 0 \), we have
\[
(r^2 \rho \varphi'_{n,1})' = r^2 \rho \left[ \frac{2}{p - 1} + \frac{2 - p \rho c_{\infty}^{p-1}}{r^2} \right] \varphi_{n,1}
\]
and from (3.32) for \( r \geq r_{n,1} \geq r_0 \):
\[
2 - r^2 p \Phi_n^{p-1} = 2 - p c_{\infty}^{p-1} + p c_{\infty}^{p-1} - r^2 p \Phi_n^{p-1} \geq \frac{2(p + 1)}{p - 1} + o(1) > 0. \tag{3.33}
\]
Since \( \varphi_{n,1} \) is strictly negative on \((r_{n,1}, +\infty)\), we deduce
\[
r^2 \rho \varphi'_{n,1}(r) \leq r_{n,1}^2 \rho(r_{n,1}) \varphi'_{n,1}(r_{n,1}) = c_1 < 0 \text{ on } (r_{n,1}, +\infty)
\]
which implies
\[
\int_{r_{n,1}}^{+\infty} |\varphi'_{n,1}(r)|^2 r^2 dr \geq \int_{r_{n,1}}^{+\infty} \frac{dr}{r^2 \rho} = +\infty
\]
and hence \( \varphi_{n,1} \notin H_\rho^1 \) and is therefore not an eigenvector.
Conclusion. We conclude that $-1$ is the only negative eigenvalue of $L_{n,1}$, and is associated to the single eigenvector $\Phi'_n$. Hence, there exists a constant $c_n > 0$ such that for all $u \in H^1_\rho$:

\[
(L_{n,1}(u), u)_{\rho} \geq c_n \|u\|_{L^2_\rho}^2 - \frac{1}{c_n} (u, \Phi'_n)_{\rho}^2.
\] (3.34)

step 2 The case $m \geq 2$. Let $\varphi_{n,m}$ be defined in Lemma 3.2. In particular, $\varphi_{n,m}$ satisfies

\[
L_{n,m}(\varphi_{n,m}) = 0 \quad \text{and} \quad \varphi_{n,m} = r^m(1 + O(r^2)) \quad \text{as} \quad r \to 0_+.
\]

Then, the number of zeros of $\varphi_{n,m}$ in $r > 0$ corresponds to the number of strictly negative eigenvalues of $L_{n,m}$. On $[0, r_0]$, we have by Lemma 3.2.

\[
\sup_{0 \leq r \leq r_0} \frac{|\mu_n^{-m} \varphi_{n,m}(r) - \nu_m \left( \frac{r}{\rho_n} \right)|}{|\nu_m \left( \frac{r}{\rho_n} \right)|} \lesssim \frac{r^2}{r_0^2}
\]

and $\nu_m(r) > 0$ for all $r > 0$, and hence $\varphi_{n,m}$ cannot vanish on $[0, r_0]$:

\[
\varphi_{n,m}(r) > 0 \quad \text{on} \quad [0, r_0].
\]

Next, we investigate the sign of $\varphi'_{n,m}(r_0)$. Recall (3.13):

\[
\nu_m(r) \sim c_{m,-} \frac{r^{m-1}}{1 - \sqrt{\Delta_m}} \quad \text{as} \quad r \to +\infty
\]

and hence

\[
\nu'_m(r) \sim c_{m,-} \frac{(\sqrt{\Delta_m} - 1)}{r^{1 - \sqrt{\Delta_m}}} \quad \text{as} \quad r \to +\infty.
\]

We infer for $n \geq N$ large enough

\[
\varphi_{n,m}(r_0) = \frac{c_{m,-}(1 + O(r_0^2))\mu_n^m}{(\frac{r_0}{\mu_n})^{1 - \sqrt{\Delta_m}}}
\]

and

\[
\varphi'_{n,m}(r_0) = \frac{c_{m,-}((\sqrt{\Delta_m} - 1)(1 + O(r_0^2))\mu_n^{m-1}}{(\frac{r_0}{\mu_n})^{1 - \sqrt{\Delta_m}}}.
\]

Thus, taking also into account that $\varphi_{n,m}(r) > 0$ on $[0, r_0]$, we infer from the identity for $\varphi_{n,m}(r_0)$ that

\[
c_{m,-} > 0.
\]

Since $\sqrt{\Delta_m} \geq \sqrt{\Delta_1} = \frac{p+3}{p-1} > 1$, we conclude:

\[
\phi_{n,m}(r_0) > 0, \quad \phi'_{n,m}(r_0) > 0.
\] (3.35)

Since $L_{n,m}(\varphi_{n,m}) = 0$, we have

\[
(r^2 \rho \varphi'_{n,m})'' = r^2 \rho \left[ \frac{2}{p-1} + \frac{m(m+1) - pr^2 \Phi_{n,m}^{-1}}{r^2} \right] \varphi_{n,m}
\] (3.36)

which together with (3.35), (3.33) and the fact that $m \geq 2$, and an elementary continuity argument ensures

\[
\phi'_{m,n}(r) > 0, \quad \phi_{n,m}(r) \geq \phi_{n,m}(r_0) > 0 \quad \text{for} \quad r \geq r_0.
\]

Hence $\phi_{n,m}$ does not vanish on $(0, +\infty)$ and using (3.36):

\[
r^2 \phi'_{n,m}(r) \geq r_0^2 \phi'_{n,m}(r_0) = c_0 > 0
\]
which implies
\[ \int_{r_0}^{+\infty} (\phi'_{n,m})^2 r^2 dr \geq \int_{r_0}^{+\infty} \frac{dr}{r^2 \rho} = +\infty \]
and hence \( \phi_{n,m} \) is not eigenvector. We finally conclude that for \( m = 2 \) and all \( n \geq N \) large enough, \( \mathcal{L}_{n,2} \) has a spectral gap and there exists a constant \( c_n > 0 \) such that we have for all \( u \in H^1_\rho \)
\[ (\mathcal{L}_{n,2}(u), u)_{\rho} \geq c_n \|u\|_{L^2_\rho}^2. \]
Since we have for all \( m \geq 2 \)
\[ (\mathcal{L}_{n,m}(u), u)_{\rho} \geq (\mathcal{L}_{n,2}(u), u)_{\rho}, \]
we infer for all \( m \geq 2 \) and for all \( u \in H^1_\rho \)
\[ (\mathcal{L}_{n,m}(u), u)_{\rho} \geq c_n \|u\|_{L^2_\rho}^2. \] (3.37)

**step 3.** The case \( m = 0 \). We now focus onto \( \mathcal{L}_{n,0} \) which is the most delicate case, and we claim that \( \mathcal{L}_{n,0} \) has exactly \( n + 1 \) strictly negative eigenvalues, and that 0 is not in the spectrum. The key is to combine the uniform bounds (1.9) with the explicit knowledge of the limiting outer spectrum, Lemma 3.4, as nicely suggested at the formal level in [3].

Let \( \varphi_{n,0} \) be the solution to (3.17) so that the number of strictly negative eigenvalues of \( \mathcal{L}_{n,0} \) coincides with the numbers of zeroes of \( \varphi_{n,0} \). We count the number of zeros of \( \varphi_{n,0} \) by comparing them with the number of zeros of \( \Lambda \Phi_n \).

**Lower bound.** First, since \( \Lambda \Phi_n \) is an eigenvector of \( \mathcal{L}_{n,0} \) corresponding to the eigenvalue \( -2 \) and since \( \Lambda \Phi_n \) vanishes \( n \) times from Proposition 2.5, we infer from Sturm Liouville
\[ \#\text{Spec}(\mathcal{L}_{n,0} + 2) \cap (-\infty, 0] = n + 1. \]
In particular, since the number of strictly negative eigenvalues of \( \mathcal{L}_{n,0} \) coincides with the number of zeroes of \( \varphi_{n,0} \), we infer
\[ \#\{ r \geq 0 \text{ such that } \varphi_{n,0}(r) = 0 \} \geq n + 1. \]

**Upper bound.** Recall (3.18):
\[ \sup_{0 \leq r \leq r_0} \left( 1 + \frac{r}{\mu_n} \right)^\frac{1}{2} |\varphi_{n,0}(r) - \frac{p - 1}{2} \Lambda Q \left( \frac{r}{\mu_n} \right) | \lesssim r_0^2. \]
Also, we have \( \Lambda Q(0) \neq 0 \) and from (2.41):
\[ \left( \frac{r_0}{\mu_n} \right)^\frac{1}{2} |\Lambda Q \left( \frac{r_0}{\mu_n} \right) | \geq c > 0 \]
for some constant \( c > 0 \) independent of \( n \). Hence \( \varphi_{n,0} \) and \( \Lambda Q \) vanish the same number of times on \( [0, r_0] \). Since on the other hand \( \Lambda Q \) and \( \Lambda \Phi_n \) vanish the same number of times on \( [0, r_0] \) from (2.40), \( \varphi_{n,0} \) and \( \Lambda \Phi_n \) vanish the same number of times of \( [0, r_0] \).

Let now \( r_{n,0} \) to be the last zero of \( \Lambda \Phi_n \) before \( r_0 \). In view of Corollary 2.6, we have
\[ e^{-2\pi r_0} r_0 \leq r_{n,0} \leq r_0. \]
Let us now consider the operators (3.22):
\[ A_n[r_{n,0}](f) = \mathcal{L}_{n,0}(f) \text{ on } r > r_{n,0}, \quad f(r_{n,0}) = 0, \]
\[ A_\infty[r_{n,0}](f) = \mathcal{L}_\infty(f) \text{ on } r > r_{n,0}, \quad f(r_{n,0}) = 0, \]
then
\[ \mathcal{L}_{n,0}(\Lambda \Phi_n) = -2\Lambda \Phi_n \text{ and } \Lambda \Phi_n(r_{n,0}) = 0, \]
implies
\[ A_n[r_{n,0}](\Lambda \Phi_n) = -2\Lambda \Phi_n. \]
In particular, \(-2\) belongs to the spectrum of \(A_n[r_{n,0}]\). In view of Lemma 3.5, we deduce for \(n \geq N\) large enough that there exists an eigenvalue \(\lambda_0\) of \(A_\infty[r_{n,0}]\) such that \(\lambda_0 = -2 + o(1)\). On the other hand, in view of Lemma 3.4, the solutions to
\[ \mathcal{L}_\infty(f) = \lambda f \]
with \(f \in H^1_\rho\) are completely explicit and behave for \(r \to 0\) as
\[ f \sim \frac{1}{r^2} \cos(\omega \log(r) - \Phi(\lambda)) \]
with
\[ \Phi(\lambda) = \arg \left( \frac{2^{2\frac{3}{4}} \Gamma(i\omega)}{\Gamma \left( \frac{1}{2p-1} - \frac{\lambda}{2} - \frac{1}{4} + \frac{i\omega}{2} \right)} \right). \]
In order for \(f\) to be an eigenfunction of \(A_\infty[r_{n,0}]\), we need \(f(r_{n,0}) = 0\) and hence there should exist \(k \in \mathbb{Z}\) such that
\[ \omega \log(r_{n,0}) - \Phi(\lambda) \sim \frac{\pi}{2} + k\pi. \]
Recall that \(\lambda_0 = -2 + o(1)\) is an eigenvalue of \(A_\infty[r_{n,0}]\), and let \(\lambda_1 > \lambda_0\) be the next eigenvalue of \(A_\infty[r_{n,0}]\). Then, there exists \(k_0 \in \mathbb{R}\) such that
\[ \omega \log(r_{n,0}) - \Phi(\lambda_0) \sim \frac{\pi}{2} + k_0\pi, \quad \omega \log(r_{n,0}) - \Phi(\lambda_1) \sim \frac{\pi}{2} + (k_0 - 1)\pi \]
and hence
\[ \Phi(\lambda_1) = \Phi(-2) + \pi + o(1). \quad (3.38) \]
Now, by numerical check, we have\(^{15}\)
\[ \sup_{5 \leq p < +\infty} \sup_{-2 \leq \lambda \leq 0.5} (\Phi(\lambda) - \Phi(-2) - \pi) \sim -0.5945 < 0, \]
and hence, the solution \(\lambda_1\) to (3.38) satisfies
\[ \inf_{5 \leq p < +\infty} \lambda_1 \geq 0.5 > 0. \]
We infer that \(A_\infty[r_{n,0}]\) has no eigenvalue between \(\lambda_0 = -2 + o(1)\) and \(\lambda_1 \geq 0.5\). Hence, using again Lemma 3.5, \(A_n[r_{n,0}]\) has no eigenvalue between \(-2\) and \(\lambda_1 + o(1) \geq 0.25\). Thus, we have
\[ \# \text{Spec}(A_n[r_{n,0}]) \cap (-\infty, 0] = \# \text{Spec}(A_n[r_{n,0}] + 2) \cap (-\infty, 0]. \]
On the other hand, we have
\[ \# \text{Spec}(A_n[r_{n,0}] + 2) \cap (-\infty, 0] = \# \{ r > r_{n,0} \text{ such that } \Lambda \Phi_n(r) = 0 \} + 1 \]
\(^{15}\)Notice that \(\Phi(\lambda)\) has a well defined limit as \(p \to +\infty\) given by
\[ \Phi_\infty(\lambda) = \arg \left( \frac{2^{2\frac{3}{4}} \Gamma(i\lambda)}{\Gamma \left( \frac{1}{2p-1} - \frac{\lambda}{2} - \frac{1}{4} + \frac{i\lambda}{2} \right)} \right). \]
Our numerics are carried out using Matlab and indicate that \(\Phi_p(\lambda)\) is increasing on \([-2, 0.5]\) for all \(p \geq 5\) so that the maximum on \([-2, 0.5]\) is achieved at \(\lambda = 0.5\). Also, this maximum appears to be a growing function of \(p\) so that the maximum in \(p\) is given by \(\Phi_\infty(0.5) - \Phi_\infty(-2) - \pi \sim -0.5945\). See [3] for a similar numerical computation.
since $\Lambda \Phi_n$ is in the kernel of $A_n[r_{n,0}] + 2$, and hence

$$\# \text{Spec}(A_n) \cap (-\infty, 0] = \# \{ r > r_{n,0} \text{ such that } \Lambda \Phi_n(r) = 0 \} + 1.$$  

Also, since $\varphi_{n,0}$ can not be an eigenvector of $A_n$\footnote{Indeed, $\varphi_{n,0}$ would be an eigenvector for the eigenvalue 0, but 0 is not in the spectrum of $A_n$ as seen above.}, we have

$$\# \text{Spec}(A_n[r_{n,0}]) \cap (-\infty, 0] = \# \{ r > r_{n,0} \text{ such that } \varphi_{n,0}(r) = 0 \}.$$  

We infer

$$\# \{ r > r_{n,0} \text{ such that } \varphi_{n,0}(r) = 0 \} = \# \{ r > r_{n,0} \text{ such that } \Lambda \Phi_n(r) = 0 \} + 1.$$  

But since $r_{n,0}$ has been chosen to be the last zero of $\Lambda \Phi_n$ before $r_0$, we have

$$\# \{ r > r_{n,0} \text{ such that } \Lambda \Phi_n(r) = 0 \} = \# \{ r > r_0 \text{ such that } \Lambda \Phi_n(r) = 0 \}$$

and hence

$$\# \{ r > r_{n,0} \text{ such that } \varphi_{n,0}(r) = 0 \} = \# \{ r > r_0 \text{ such that } \Lambda \Phi_n(r) = 0 \} + 1.$$  

Next, together with the fact that $\varphi_{n,0}$ and $\Lambda \Phi_n$ vanish the same number of times of $[0, r_0]$, we infer

$$\# \{ r > 0 \text{ such that } \varphi_{n,0}(r) = 0 \} \leq \# \{ 0 \leq r \leq r_0 \text{ such that } \varphi_{n,0}(r) = 0 \} + \# \{ r > r_{n,0} \text{ such that } \varphi_{n,0}(r) = 0 \}$$

$$= \# \{ 0 \leq r \leq r_0 \text{ such that } \Lambda \Phi_n(r) = 0 \} + \# \{ r > r_0 \text{ such that } \Lambda \Phi_n(r) = 0 \} + 1$$

$$= \# \{ r > 0 \text{ such that } \Lambda \Phi_n(r) = 0 \} + 1$$

$$= n + 1$$

and since

$$\# \{ r \geq 0 \text{ such that } \varphi_{n,0}(r) = 0 \} \geq n + 1.$$  

$\phi_{n,0}$ is not an eigenstate. We conclude that

$$\# \{ r \geq 0 \text{ such that } \varphi_{n,0}(r) = 0 \} = n + 1.$$  

Assume now by contradiction that $\varphi_{n,0}$ is in the kernel of $\mathcal{L}_{n,0}$. Recall that $r_{0,n} < r_0$ is the last 0 of $\Lambda \Phi_n$ and let $r_{1,n} < r_0$ be the last 0 of $\varphi_{n,0}$. In particular, we have from Lemma 3.3:

$$e^{-\frac{2\pi}{\omega}} r_0 \leq r_{0,n}, r_{1,n} \leq r_0$$

and $r_{1,n} = r_{0,n} + O(r_0^3)$. Also, since $\varphi_{n,0}$ is in the kernel of $\mathcal{L}_{n,0}$ and $\varphi_{n,0}(r_{1,n}) = 0$, we infer that 0 is in the spectrum of $A_n[r_{1,n}]$, and hence applying Lemma 3.5 twice as well as Lemma 3.6, we obtain that

$$\text{dist}(\text{Spec}(A_n[r_{0,n}]), 0) \lesssim r_0^3 + o(1)$$

as $n \to +\infty$. In particular, we have for $r_0 > 0$ small enough and $n \geq N$ large enough

$$\text{dist}(\text{Spec}(A_n[r_{0,n}]), 0) \leq 0.2.$$  

On the other hand, we have proved above that $A_n[r_{n,0}]$ has no eigenvalue between $-2$ and $\lambda_1 + o(1) \geq 0.25$ so that

$$\text{dist}(\text{Spec}(A_n[r_{0,n}]), 0) \geq 0.25$$

which is a contradiction. Hence $\varphi_{0,n}$ is not in the kernel of $\mathcal{L}_{n,0}$.
Conclusion. We conclude that $L_{n,0}$ has exactly $n + 1$ strictly negative eigenvalues. On the other hand, since $\Lambda\Phi_n$ is an eigenvector of $L_{n,0}$ corresponding to the eigenvalue $-2$ and since $\Lambda\Phi_n$ vanishes $n$ times, we infer

$$\#\text{Spec}(L_{n,0} + 2) \cap (-\infty, 0] = n + 1,$$

and hence $L_{n,0}$ has exactly $n + 1$ negative eigenvalues and the largest negative eigenvalue is $-2$. We denote these eigenvalues by $-\mu_{n+1,n} < \cdots < -\mu_{2,n} < -\mu_{1,n} = -2$.

By Sturm Liouville, these eigenvalues are simple and associated to eigenvectors

$$\psi_{j,n}, \quad \|\psi_{j,n}\|_{L^2_\rho} = 1, \quad \psi_{1,n} = \frac{\Lambda\Phi_n}{\|\Lambda\Phi_n\|_\rho}.$$

Also, there holds for some constant $c_n > 0$ and for all $u \in H^1_\rho$

$$\left( L_{n,0}(u), u \right)_\rho \geq c_n \|u\|_{L^2_\rho}^2 - \frac{1}{c_n} \left[ \sum_{j=1}^{n+1} (u, \psi_{j,n})^2_\rho + 3 \sum_{k=1}^3 (u_1, \Phi_n')^2_\rho \right]. \quad (3.39)$$

The behavior as $r \to +\infty$ of the eigenstates (3.5) follows from the asymptotic in Lemma 3.4 and a standard ODE argument using the variation of constants formula, this is left to the reader.

**step 4** Conclusion. We decompose $u \in H^1_\rho$ as

$$u = \sum_{m=0}^{+\infty} \sum_{k=-m}^m u_{m,k} Y^{(m,k)}$$

where $u_{m,k}$ are radial functions satisfying

$$\|u\|_{L^2_\rho}^2 = \sum_{m=0}^{+\infty} \sum_{k=-m}^m \|u_{m,k}\|_{L^2_\rho}^2.$$

We have

$$\left( L_n(u), u \right)_\rho = \sum_{m=0}^{+\infty} \sum_{k=-m}^m \left( L_{n,m}(u_{m,k}), u_{m,k} \right)_\rho.$$

Together with (3.34), (3.37) and (3.39), we infer for all $u \in H^1_\rho$

$$\left( L_n(u), u \right)_\rho = \left( L_{n,0}(u_{0,0}), u_{0,0} \right)_\rho + \sum_{k=-1}^1 \left( L_{n,1}(u_{1,k}), u_{1,k} \right)_\rho + \sum_{m=2}^{+\infty} \sum_{k=-m}^m \left( L_{n,m}(u_{m,k}), u_{m,k} \right)_\rho$$

$$\geq c_n \|u\|_{L^2_\rho}^2 - \frac{1}{c_n} \left[ \sum_{j=1}^{n+1} (u_{0,0}, \psi_{j,n})^2_\rho + 3 \sum_{k=1}^3 (u_1, \Phi_n')^2_\rho \right].$$

Since $\psi_{j,n}$ are all radial, we have

$$\left( u_{0,0}, \psi_{j,n} \right)_\rho = \left( u, \psi_{j,n} \right)_\rho.$$ 

Also, since

$$\nabla \Phi_n(x) = \Phi'_n(r) \frac{x}{r} = \Phi'_n(r)(Y^{(1,-1)}, Y^{(1,1)}, Y^{(1,0)}),$$

we infer

$$\sum_{k=1}^3 \left( u_{1,k}, \Phi'_n \right)_\rho^2 = \sum_{k=1}^3 \left( u, \partial_k \Phi_n \right)_\rho^2.$$
Finally, there holds for some constant $c_n > 0$ and for all $u \in H^1_{\rho}$
\[
(\mathcal{L}_n u, u)_\rho \geq c_n \|u\|_{H^1_{\rho}}^2 - \frac{1}{c_n} \left[ \sum_{j=0}^{n} (u, \psi_{j,n})_{\rho}^2 + \sum_{k=1}^{3} (u, \partial_k \Phi_n)_{\rho}^2 \right].
\]
This concludes the proof of Proposition 3.1.

4. Dynamical control of the flow

We now turn to the question of the stability of the self similar solution, and more precisely the construction of a manifold of finite energy initial data such that the corresponding solution to (1.1) blows up in finite time with $\Phi_n$ profile in the self similar regime described by Theorem 1.2. $n$ is now fixed.

4.1. Setting of the bootstrap. We set up in this section the bootstrap analysis of the flow for a suitable set of finite energy initial data. The solution will be decomposed in a suitable way with standard technique, see [32, 38].

Geometrical decomposition. We start by showing the existence of the suitable decomposition. Recall the spectral Proposition 3.1. To ease notations we now omit the $n$ subscript and write $\psi_j, \mu_j$ and $\lambda_j$ instead.

Define the $L^\infty$ tube around the renormalized versions of $\Phi_n$:
\[
X_\delta = \left\{ u = \frac{1}{\lambda^{p-1}} (\Phi_n + v) \left( \frac{x - y}{\lambda} \right), \ y \in \mathbb{R}^d, \ \lambda > 0, \ \|v\|_{L^\infty} < \delta \right\}
\]

**Lemma 4.1** (Geometrical decomposition). There exists $\delta > 0$ and $C > 0$ such that any $u \in X_\delta$ has a unique decomposition
\[
u = \frac{1}{\lambda^{p-1}} (\Phi_n + \sum_{j=2}^{n+1} a_j \psi_j + \varepsilon) \left( \frac{x - \overline{x}}{\lambda} \right),
\]
where $\varepsilon$ satisfies the orthogonality conditions
\[(\varepsilon, \psi_j)_\rho = (\varepsilon, \partial_k \Phi_n)_\rho = 0, \ 1 \leq j \leq n + 1, \ 1 \leq k \leq 3,
\]
the parameters $\lambda, \overline{x}$ and $a_j$ being Fréchet differentiable on $X_\delta$, and with
\[
\|\varepsilon\|_{L^\infty} + \sum_{j=2}^{n+1} |a_j| \leq C. \tag{4.1}
\]

**Proof.** It is a classical consequence of the implicit function theorem.

**step 1** Decomposition near $\lambda = 1, \overline{x} = 0$. We introduce the smooth maps
\[
F(v, \mu, x, b_1, \ldots, b_n) = \mu^{2-p} (\Phi_n + v) (\mu y + x) - \Phi_n - \sum_{j=2}^{n+1} b_j \psi_j
\]
and
\[
G = \left( (F, \Lambda \Phi_n), (F, \partial_1 \Phi_n), (F, \partial_2 \Phi_n), (F, \partial_3 \Phi_n), (F, \psi_2), \ldots, (F, \psi_{n+1}) \right).
\]
We immediately check that $G(\Phi_n, 1, 0, \ldots, 0) = 0$ and that
\[
\frac{\partial G}{\partial (\mu, x, b_2, \ldots, b_{n+1})} |_{(\Phi_n, 1, 0, \ldots, 0)}
\]
is invertible. In view of the implicit function theorem, for $\kappa > 0$ small enough, for any
\[
\|v\|_{L^\infty} \leq \kappa
\]
there exists \((\mu, z, a_2, \ldots, a_{n+1})\) and 
\[
\varepsilon = F(v, \mu, z, a_2, \ldots, a_{n+1})
\]
such that 
\[
u = \Phi_n + v = \frac{1}{\mu^{\frac{2}{p-1}}} \left( \Phi_n + \sum_{j=2}^{n+1} a_j \psi_j + \varepsilon \right) \left( \frac{x - z}{\mu} \right),
\]
\[(\varepsilon, \psi_j) = (\varepsilon, \partial_k \Phi_n) = 0, \quad 1 \leq j \leq n, \quad 1 \leq k \leq 3,\]
and there exist two universal constants \(K, \tilde{K} > 0\) such that 
\[
\|\varepsilon\|_{L^\infty} + \sum_{j=2}^{n+1} |a_j| + |\mu - 1| + |z| \leq K\|v\|_{L^\infty}
\]
and such that the decomposition is unique under the bound 
\[
\|\varepsilon\|_{L^\infty} + \sum_{j=2}^{n+1} |a_j| + |\mu - 1| + |z| \leq \tilde{K}. \tag{4.2}
\]

**Step 2** Decomposition near any \(\lambda, \varpi\). For any \(\delta > 0\), we take 
\[
C = C(\delta) := K\delta.
\]
Let 
\[
u \in X_\delta \text{ then for some } \lambda' > 0 \text{ and } y \text{ one has }
\]
\[
u(x) = \frac{1}{\lambda' \frac{2}{p-1}} (\Phi_n + v) \left( \frac{x - y}{\lambda'} \right), \quad \|v\|_{L^\infty} < \delta.
\]
The first step then provides the decomposition claimed in the lemma for \(\delta\) small enough via the formulas \(\lambda = \lambda' \mu(v), \varpi = y - \lambda' z(v), a_j = a_j(v)\) and \(\varepsilon = \varepsilon(v)\). We will show in the next step that the decomposition is unique, implying that the parameters are Fréchet differentiable on \(X_\delta\) for those of step 1 are.

**Step 3** Uniqueness of the decomposition. First, from a continuity argument, for any \(\varepsilon > 0\), there exists \(\delta > 0\) such that if 
\[
(\Phi_n + v)(x) = \frac{1}{\mu^{\frac{2}{p-1}}} (\Phi_n + v') \left( \frac{x - y}{\mu} \right), \quad \|v\|_{L^\infty} + \|v'\|_{L^\infty} \leq \delta
\]
then 
\[
|\mu - 1| + |y| \leq \varepsilon.
\]
Now recall that \(C = K\delta\) and assume that we are given a second decomposition for 
\(u \in X_\delta\). In view of step 2, performing a change of variable, this amounts to say that 
\(\Phi_n + v\) admits another decomposition: 
\[
(\Phi_n + v)(x) = \frac{1}{\mu^{\frac{2}{p-1}}} (\Phi_n + \sum_{j=2}^{n+1} \bar{a}_j \bar{\psi}_j + \bar{\varepsilon}) \left( \frac{x - \bar{z}}{\mu} \right)
\]
and the bound (4.1) gives 
\[
\sum_{j=2}^{n+1} |\bar{a}_j| + \|\bar{\varepsilon}\|_{L^\infty} \leq K\delta.
\]
Using the above continuity estimate, one obtains that for \(\delta\) small enough 
\[
|\bar{\varpi}| + |\bar{\mu} - 1| \ll \tilde{K}.
\]
Therefore, for \(\delta\) small enough the second decomposition associated with \(\bar{\varpi}, \bar{\varpi}, \bar{a}_j\) 
and \(\bar{\varepsilon}\) satisfies (4.2), and is therefore the one given by step 2 by uniqueness. \(\square\)
Description of the initial datum. We will now focus on solutions of (1.1) that are a suitable perturbation of \( \Phi_n \) at initial time:

\[
u_0 = \frac{1}{\lambda_0^{p-1}} (\Phi_n + v_0) \left( \frac{x}{\lambda_0} \right)
\]

(4.3)

with

\[
v_0 = \sum_{j=2}^{n+1} a_j \psi_j + \varepsilon_0, \quad (\varepsilon_0, \psi_j)_\rho = (\varepsilon_0, \partial_k \Phi_n)_\rho = 0, \quad 1 \leq j \leq n+1, \quad 1 \leq k \leq 3.
\]

(4.4)

For \( s_0 \gg 1 \) and \( \mu, K_0 > 0 \) three constants to be defined later on, the parameters \( \lambda_0, a_j \) and the profile \( \varepsilon_0 \) satisfy the bounds

- rescaled solution:
  \[
  \lambda_0 = e^{-s_0};
  \]
  (4.5)

- initial control of the unstable modes:
  \[
  \sum_{j=2}^{n+1} |a_j|^2 \leq e^{-2\mu s_0};
  \]
  (4.6)

- smallness of suitable initial norms:
  \[
  \|\varepsilon_0\|_{H^2} + \|\Delta v_0\|_{L^2} + \|w_0\|_{H^s_c} \leq K_0 e^{-\mu s_0};
  \]
  (4.7)

where \( w_0 \) is given by

\[
w_0 = \left( 1 - \frac{1}{\lambda_0} \right) \Phi_n + v_0.
\]

Note that in view of the \( L^\infty \) bound (4.23), the decomposition (4.3) is precisely the one given by Lemma 4.1.

Renormalized flow. As long as the solution \( u(t) \) starting from (4.3) belongs to \( X_\delta \), Lemma 4.1 applies and it can be written

\[
u(t, x) = \frac{1}{\lambda(t)^{p-1}} (\Phi_n + \psi + \varepsilon)(s, z), \quad y = \frac{x - x(t)}{\lambda(t)}
\]

(4.8)

with

\[
\psi = \sum_{j=2}^{n+1} a_j \psi_j, \quad (\varepsilon, \psi_j)_\rho = (\varepsilon, \partial_k \Phi_n)_\rho = 0, \quad 1 \leq j \leq n+1, \quad 1 \leq k \leq 3.
\]

(4.9)

Moreover, as the parameters are Fréchet differentiable in \( L^\infty \), and as \( u \in C^1((0, T), L^\infty) \) from parabolic regularizing effects, the above decomposition is differentiable with respect to time. We also introduce a further decomposition

\[
v = \psi + \varepsilon, \quad \Phi_n + v = \chi_1 \Phi_n + w.
\]

(4.10)

Consider the renormalized time

\[
s(t) = \int_0^t \frac{d\tau}{\lambda^2(\tau)} + s_0.
\]

Injecting (4.8) into (1.1) yields the renormalized equation

\[
\partial_s \varepsilon + \mathcal{L}_n \varepsilon = F - \text{Mod}
\]

(4.11)

with the modulation term

\[
\text{Mod} = \sum_{j=2}^{n+1} [(a_j)_s - \mu_j a_j] \psi_j - \left( \frac{\lambda_s}{\lambda} + 1 \right) (\Lambda \Phi_n + \Lambda \psi) - \frac{x_s}{\lambda} \cdot (\nabla \Phi_n + \nabla \psi)
\]

(4.12)
and the force terms
\[ F = L(\varepsilon) + NL, \quad L(\varepsilon) = \left( \frac{\lambda_s}{\lambda} + 1 \right) \Lambda \varepsilon + \frac{\varepsilon_s}{\lambda} \cdot \nabla \varepsilon \]  
(4.13)

\[ NL = g(\varepsilon + \psi), \quad g(\psi) = (\Phi_n + \psi)^p - \Phi_n^p - p\Phi_n^{p-1}\psi. \]  
(4.14)

We claim the following bootstrap proposition.

**Proposition 4.2 (Bootstrap).** There exist universal constants \(0 < \mu, \eta \ll 1, K \gg 1\) such that for all \(s_0 \geq s_0(K, \mu, \eta) \gg 1\) large enough the following holds. For any \(\lambda_0\) and \(\varepsilon_0\) satisfying (4.5) and (4.4), and

\[ \| (1 - \chi \frac{1}{s_0} ) \Phi_n + \varepsilon_0 \|_{H^{sc}} + \| \varepsilon_0 \|_{H^2} + \| \Delta \varepsilon_0 \|_{L^2} \leq e^{-2\mu s_0}, \]  
(4.15)

there exist \((a_2(0), \ldots, a_{n+1}(0))\) satisfying (4.6) such that the solution starting from \(u_0\) given by (4.3), decomposed according to (4.8) satisfies for all \(s \geq s_0\):

- **control of the scaling:**
  \[ 0 < \lambda(s) < e^{-\mu s}; \]  
  (4.16)

- **control of the unstable modes:**
  \[ \sum_{j=2}^{n+1} |a_j|^2 \leq e^{-2\mu s}; \]  
  (4.17)

- **control of the exponentially weighted norm:**
  \[ \| \varepsilon \|_{H^2} < Ke^{-\mu s}; \]  
  (4.18)

- **control of a Sobolev norm above scaling:**
  \[ \| \Delta \varepsilon \|_{L^2} < Ke^{-\mu s}; \]  
  (4.19)

- **control of the critical norm:**
  \[ \| w \|_{H^{sc}} < \eta. \]  
  (4.20)

Proposition 4.2 is the heart of the analysis, and the corresponding solutions are easily shown to satisfy the conclusions of Theorem 1.2. The strategy of the proof follows [10, 42]: we prove Proposition 4.2 by contradiction using a topological argument à la Brouwer: given \((\varepsilon_0, \lambda_0)\) satisfying (4.5), (4.15) and (4.4), we assume that for all \((a_2(0), \ldots, a_{n+1}(0))\) satisfying (4.6), the exit time

\[ s^* = \sup \{ s \geq s_0 \text{ such that } (4.16), (4.17), (4.18), (4.19), (4.20) \text{ holds on } [s_0, s] \} \]  
(4.21)

is finite

\[ s^* < +\infty \]  
(4.22)

and look for a contradiction for \(0 < \mu, \eta, \frac{1}{K}\) small enough and \(s_0 \geq s_0(K, \mu)\) large enough. From now on, we therefore study the flow on \([s_0, s^*]\) where (4.16), (4.17), (4.18), (4.19) and (4.20) hold. Using a bootstrap method we show that the bounds (4.16), (4.18), (4.19) and (4.20) can be improved, implying that at time \(s^*\) necessarily the unstable modes have grown and (4.17) is violated. Since 0 is a linear repulsive equilibrium for these modes, this would contradict Brouwer fixed point theorem.

From the asymptotic (3.5) of \(\psi_j\) for \(2 \leq j \leq n + 1\), (4.6) and (4.15), one can fix the constant \(K_0\) independently of \((s_0, \mu)\) such that (4.7) holds. Also, note that the bootstrap bounds (4.17), (4.18), (4.19) and (4.20) imply the \(L^\infty\) bound (4.23), and
therefore the decomposition used in the Proposition is well defined since Lemma 4.1 applies.

4.2. \( L^\infty \) bound. We start with the derivations of unweighted \( L^\infty \) and Sobolev bounds on \( v, w \) which will be essential to control nonlinear terms in the sequel and follow from (4.19), (4.20).

Lemma 4.3 \((L^\infty \) smallness\). There holds
\[
\|v\|_{L^\infty} + \|w\|_{L^\infty} \leq e^{-c\mu s} \leq \eta \ll 1
\]  
for some universal constants \( c > 0, 0 < \eta \ll 1 \).

Proof. We compute from (4.10):
\[
w = (1 - \chi_1^\lambda)\Phi_n + v.
\]  
The self similar decay (1.9) and (4.19) yield:
\[
\|w\|_{\dot{H}^2} \lesssim \|v\|_{\dot{H}^2} + \|(1 - \chi_1^\lambda)\Phi_n\|_{\dot{H}^2} \lesssim K\left[e^{-\mu s} + \lambda(s)^{2-s_c}\right] \leq e^{-c\mu s}.
\]
Hence by interpolation using \( s_c = \frac{3}{2} - \frac{2}{p+1} < \frac{3}{2} < 2 \):
\[
\|w\|_{L^\infty} \lesssim \|\hat{w}\|_{L^1} \lesssim \|w\|_{\dot{H}^{1-s_c}}^\alpha \|w\|_{\dot{H}^2}^{1-\alpha}, \quad \alpha = \frac{3}{2} - s_c
\]
which together with (4.20) ensures:
\[
\|w\|_{L^\infty} \lesssim e^{-c\mu s}.
\]
The decay (1.9) and (4.16), (4.24) yield the \( L^\infty \) smallness for \( v \) and conclude the proof. \( \square \)

4.3. Modulation equations. We now compute the modulation equations which
describe the time evolution of the parameters. They are computed in the self-similar zone, and involve the \( \rho \) weighted norm.

Lemma 4.4 \((\text{Modulation equations})\). There holds the bounds
\[
\left|\frac{\lambda_s^1 + 1}{\lambda} + \frac{x_2}{\lambda} + \sum_{j=2}^{n+1} (|a_j| - \mu_j a_j)\right| \lesssim \|\varepsilon\|^2_{H^1} + \|\Delta v\|^2_{L^2} + \sum_{j=2}^{n+1} |a_j|^2.
\]  
\[(4.25)\]

Proof. This lemma is a classical consequence of the choice of orthogonality conditions (4.9), but the control of the nonlinear term relies in an essential way on the \( L^\infty \) smallness (4.23).

Step 1 Law for \( a_j \). Take the \( L^2_\rho \) scalar product of (4.11) with \( \psi_j \) for \( 2 \leq j \leq n + 1 \), then using (4.9) and the orthogonality
\[
(\psi_j, \psi_k)_{\rho} = \delta_{jk}, \quad \psi_1 = \frac{\Lambda \Phi_n}{\|\Lambda \Phi_n\|_{L^2_\rho}},
\]  
we obtain
\[
(a_j)_s - \mu_j a_j = \left(\frac{\lambda_s^1 + 1}{\lambda}\right)(\Lambda \psi, \psi_j)_{\rho} + (F, \psi_j)_{\rho}.
\]

First, from (4.17) one has
\[
|(\Lambda \psi, \psi_j)_{\rho}| \lesssim e^{-\mu s} \ll \eta.
\]
We now estimate the \( F \)-term given by (4.13). We use the bound from \( p > 5 \):
\[
\|1 + z|^p - 1 - pz^{p-1} \| \lesssim |z|^p + |z|^2
\]
to estimate from the $L^\infty$ bound (4.23):
\[
|\text{NL}| \lesssim |\varepsilon + \psi| + \Phi_n \left( |\varepsilon + \psi|^2 \right) \lesssim (\varepsilon + \psi)^2 = v^2.
\] (4.27)

We estimate from the Hardy inequality (A.5):
\[
\int \frac{|\nabla v|^2}{1 + |y|^2} + \frac{|v|^2}{1 + |y|^4} \lesssim \int |\Delta v|^2 + \|v\|_{H^1_0}^2 \lesssim \int |\Delta v|^2 + \|\varepsilon\|_{H^1_0}^2 + \sum_{j=2}^{n+1} |a_j|^2
\] (4.28)
and hence using the polynomial bound (3.5):
\[
|\langle \text{NL}, \psi_j \rangle| \lesssim \int v^2 |\psi_j| \lesssim \int \frac{|v|^2}{1 + |y|^4} \lesssim \int |\Delta v|^2 + \|v\|_{H^1_0}^2 \lesssim \|\varepsilon\|_{H^1_0}^2 + \|\Delta v\|_{L^2}^2 + \sum_{j=2}^{n+1} |a_j|^2.
\]

Next, we integrate by parts and use Cauchy Schwarz and (3.5) to estimate:
\[
\left| \left( \left( \frac{\lambda s}{\lambda} + 1 \right) - \nabla \varepsilon \right) \cdot \nabla \varepsilon, \psi_j \right| \lesssim \left[ \left( \frac{\lambda s}{\lambda} + 1 \right) - \frac{x_s}{\lambda} \right] \|\varepsilon\|_{L^2_0}
\]
and hence the first bound
\[
|\langle a_j \rangle - \mu_j a_j| \lesssim \left( \left( \frac{\lambda s}{\lambda} + 1 \right) - \frac{x_s}{\lambda} \right) \eta + \|\varepsilon\|_{H^1_0}^2 + \sum_{j=2}^{n+1} |a_j|^2 + \|\Delta v\|_{L^2}^2.
\]

**step 2** Law for scaling and translation. We scalarize (4.11) with $\psi_1 = \frac{\lambda \Phi_n}{\|\lambda \Phi_n\|_{L^2}}$ and $\frac{\partial \lambda \Phi_n}{\|\partial \lambda \Phi_n\|_{L^2}}$ and obtain in a completely similar way
\[
\left| \frac{\lambda s}{\lambda} + 1 \right| \lesssim \left( \left( \frac{\lambda s}{\lambda} + 1 \right) - \frac{x_s}{\lambda} \right) \eta + \|\varepsilon\|_{H^1_0}^2 + \sum_{j=2}^{n+1} |a_j|^2 + \|\Delta v\|_{L^2}^2.
\]

Summing the above estimates and using the smallness of $\eta$ yields (4.25).

\[\square\]

4.4. **Energy estimates with exponential weights.** We now turn to the proof of exponential decay which is an elementary consequence of the spectral gap estimate (3.6), the dissipative structure of the flow and the $L^\infty$ bound (4.23) to control the nonlinear term.

**Lemma 4.5** (Lyapunov control of exponentially weighed norms). There holds the differential bound
\[
\frac{d}{ds} \|\varepsilon\|_{L^2_0}^2 + c_n \|\varepsilon\|_{H^1_0}^2 \lesssim \sum_{j=2}^{n+1} |a_j|^4 + \|\Delta v\|_{L^2}^2 + \|v\|_{L^\infty}^2 \left[ \|\Delta v\|_{L^2}^2 + \sum_{j=2}^{n+1} |a_j|^2 \right],
\] (4.29)
\[
\frac{d}{ds} \|L_\varepsilon \|_{L^2_0}^2 + c_n \|L_\varepsilon \|_{H^1_0}^2 \lesssim \|\varepsilon\|_{H^1_0}^2 + \|\Delta v\|_{L^2}^2 + \sum_{j=2}^{n+1} |a_j|^4 + \|\Delta v\|_{L^2}^2
\] (4.30)
\[
+ \|v\|_{L^\infty}^2 \left[ \|\Delta v\|_{L^2}^2 + \|\varepsilon\|_{H^1_0}^2 + \sum_{j=2}^{n+1} |a_j|^2 \right],
\]
with $c_n > 0$ given by (3.6).
Proof. step 1 \(L^2\) weighted bound. We compute from (4.11):

\[
\frac{1}{2} \frac{d}{ds} \|\varepsilon\|^2_{L^2} = (\varepsilon, \partial_s \varepsilon)_\rho = -(L_n \varepsilon, \varepsilon)_\rho + (F - \text{Mod}, \varepsilon)_\rho. \tag{4.31}
\]

From (4.12), (4.25):

\[
|\langle \varepsilon, \text{Mod} \rangle_\rho| \lesssim \|\varepsilon\|_{L^2_\rho} \|\text{Mod}\|_{L^2_\rho} \lesssim \|\varepsilon\|_{L^2_\rho}^2 + \sum_{j=2}^{n+1} |a_j|^2 + \|\nabla \varepsilon\|^2_{L^2_\rho} \\
\lesssim \delta \|\varepsilon\|^2_{L^2_\rho} + C_\delta \left( \|\varepsilon\|^2_{H^1_\rho} + \sum_{j=2}^{n+1} |a_j|^4 + \|\Delta \varepsilon\|^2_{L^2_\rho} \right)
\]

for any \(\delta > 0\). Integrating by parts and using (A.1), we estimate

\[
|\langle \varepsilon, \Lambda \varepsilon \rangle_\rho| + |\langle \nabla \varepsilon, \varepsilon \rangle_\rho| \lesssim \int (1 + |y|^2) \varepsilon^2 \rho dy \lesssim \|\varepsilon\|^2_{H^1_\rho} \tag{4.32}
\]

from which using (4.25):

\[
\left| \langle L(\varepsilon), \varepsilon \rangle_\rho \right| \lesssim \|\varepsilon\|_{H^1_\rho}^2 \left( \|\varepsilon\|^2_{L^2_\rho} + \sum_{j=2}^{n+1} |a_j|^2 + \|\Delta \varepsilon\|^2_{L^2_\rho} \right).
\]

Finally using (4.27), (4.28):

\[
|\langle \text{NL}, \varepsilon \rangle_\rho| \lesssim \int |\varepsilon|^2 |\nabla v|^2 \rho dy \leq \delta \int |\varepsilon|^2 \rho + C_\delta \int |\nabla v|^4 \rho dy \\
\leq \delta \int |\varepsilon|^2 \rho + C_\delta \|\nabla v\|^2_{L^\infty} \int \frac{|\nabla v|^2}{1 + |y|^4} dy \\
\leq \delta \|\varepsilon\|^2_{L^2_\rho} + C_\delta \|\nabla v\|^2_{L^\infty} \left[ \int |\Delta \nabla v|^2 + \|\varepsilon\|^2_{H^1_\rho} + \sum_{j=2}^{n+1} |a_j|^2 \right].
\]

Injecting the collection of above bounds into (4.31) and using the spectral gap estimate (3.6) with the choice of orthogonality conditions (4.9) yields

\[
\frac{d}{ds} \|\varepsilon\|^2 \leq -2c_n \|\varepsilon\|_{H^1_\rho} \left( 1 - C(\|\varepsilon\|^2_{H^1_\rho} - \sum_{j=2}^{n+1} |a_j|^2 - \|\nabla \varepsilon\|^2_{L^2_\rho}) - C_\delta - C_\delta \|\varepsilon\|_{H^1_\rho} \right) \\
+ C_\delta \|\varepsilon\|_{L^\infty} \left[ \int |\Delta \nabla v|^2 + \|\varepsilon\|^2_{H^1_\rho} + \sum_{j=2}^{n+1} |a_j|^2 \right]
\]

which using the bootstrap bounds (4.17), (4.18) and (4.19) gives (4.29) for \(s_0\) large enough and \(\delta\) small enough.

step 2 \(H^2\) weighted bound. Let

\[
\varepsilon_2 = L_n \varepsilon,
\]

then \(\varepsilon_2\) satisfies the orthogonality conditions (4.9):

\[
\langle \varepsilon_2, \psi_j \rangle = (\varepsilon_2, \partial_k \Phi_n) = 0, \quad 1 \leq j \leq n + 1, \quad 1 \leq k \leq 3, \tag{4.33}
\]

and the equation from (4.11):

\[
\partial_s \varepsilon_2 + L_n \varepsilon_2 = L_n(F - \text{Mod}).
\]
Hence:
\[
\frac{1}{2} \frac{d}{ds} \|\varepsilon_2\|_{L^2}^2 = -(\mathcal{L}_n \varepsilon_2, \varepsilon_2)_\rho + (\mathcal{L}_n(F-\text{Mod}), \varepsilon_2)_\rho. \quad (4.34)
\]

We estimate from (4.25):
\[
\|\mathcal{L}_n \text{Mod}\|_{L^2} \lesssim \left| \frac{\lambda_n}{\Lambda} - 1 \right| + \left| \frac{\xi_n}{\Lambda} \right| + \sum_{j=2}^{n+1} |(a_j)_s - a_j| \lesssim \|\varepsilon\|_{H^1}^2 + \sum_{j=2}^{n+1} |a_j|^2 + \|\Delta v\|_{L^2}^2.
\]

We now use the commutator relation
\[
[\Delta, \Lambda] = 2\Delta
\]
to compute
\[
[\mathcal{L}_n, \Lambda] = [-\Delta + \Lambda - p\Phi_n^{p-1}, \Lambda] = -2\Delta + p(p-1)\Phi_n^{p-2}r\partial_r, \Phi_n = 2(\mathcal{L}_n - \Lambda + p\Phi_n^{p-1}) + p(p-1)\Phi_n^{p-2}r\partial_r, \Phi_n
\]
from which using (4.32), (A.1):
\[
|\varepsilon_2, [\mathcal{L}_n, \Lambda]_\varepsilon| \rho = |\varepsilon_2, [\mathcal{L}_n, \Lambda]_\varepsilon| + |\varepsilon_2, \Lambda \varepsilon_2|_\rho \lesssim \|\varepsilon\|_{H^1}^2 + |(\varepsilon_2, \Lambda_\varepsilon)|_\rho + |(\varepsilon_2, \Phi_n^{p-1}\varepsilon)|_\rho + |(\varepsilon_2, \Phi_n^{p-2}\Lambda\varepsilon)|_\rho \lesssim \|\varepsilon\|_{H^1}^2 + \|\varepsilon\|_{H^1}^2
\]

and similarly
\[
|\varepsilon_2, [\mathcal{L}_n, \partial_k_\varepsilon]|_\rho \lesssim \|\varepsilon\|_{H^1}^2 + \|\varepsilon\|_{H^1}^2.
\]

Hence from (4.25):
\[
|\varepsilon_2, [\mathcal{L}_n, L(\varepsilon)]|_\rho \lesssim \left( \|\varepsilon\|_{H^1}^2 + \|\varepsilon\|_{H^1}^2 \right) \left( \|\varepsilon\|_{H^1}^2 + \sum_{j=2}^{n+1} |a_j|^2 + \|\Delta v\|_{L^2}^2 \right).
\]

It remains to estimate the nonlinear term. We first integrate by parts since \(\mathcal{L}_n\) is self-adjoint for \((\cdot, \cdot)_\rho\) to estimate using the notation (4.14):
\[
|\mathcal{L}_n \text{NL}, \varepsilon_2|_\rho = \left| (\nabla \text{NL}, \nabla \varepsilon_2)_\rho + \left( \frac{2}{p-1} - \text{NL} - p\Phi_n^{p-1}\text{NL}, \varepsilon_2 \right)_\rho \right| \lesssim |(\nabla g(v), \nabla \varepsilon_2)|_\rho + \left| \left( \frac{2}{p-1} g(v) - p\Phi_n^{p-1} g(v), \varepsilon_2 \right)_\rho \right|.
\]

We now compute explicitly
\[
\nabla g(v) = p\nabla v \left[ (\Phi_n + v)^{p-1} - \Phi_n^{p-1} \right] + p\nabla \Phi_n \left[ (\Phi_n + v)^{p-1} - \Phi_n^{p-1} - (p-1)\Phi_n^{p-2}v \right]. \quad (4.35)
\]

We estimate by homogeneity with the \(L^\infty\) bound (4.23):
\[
|g(v)| \lesssim |v|^2, \quad |\nabla g(v)| \lesssim |\nabla v||v| + |v|^2
\]
and hence the bound using (4.23) again:

\[ |(\nabla g(v), \nabla \varepsilon_2)_\rho| + \left| \left( \frac{2}{p-1} g(v) - p \Phi_n^{p-1} g(v), \varepsilon_2 \right)_\rho \right| \]

\[ \lesssim \int \left[ |v| |\nabla (v)| + |v|^2 \right] |\nabla \varepsilon_2| \rho dy + \int |\varepsilon_2| |v|^2 \rho dy \]

\[ \leq \delta \|\varepsilon_2\|_{H^1_\rho}^2 + C_\delta \left[ \int |v|^2 |\nabla v|^2 \rho dy + \int |v|^4 \rho dy \right] \]

\[ \leq \delta \|\varepsilon_2\|_{L^2_\rho}^2 + C_\delta \|v\|_{L^\infty}^2 \left[ \int \frac{|\nabla v|^2}{1 + |y|^2} dy + \int \frac{|v|^2}{1 + |y|^4} dy \right] \]

\[ \leq \delta \|\varepsilon_2\|_{L^2_\rho}^2 + C_\delta \|v\|_{L^\infty}^2 \left[ \|\varepsilon\|_{H^0_\rho}^2 + \sum_{j=2}^{n+1} |a_j|^2 + \|\Delta v\|_{L^2_\rho}^2 \right]. \]

The collection of above bounds together with the spectral gap estimate (3.6) and the orthogonality conditions (4.33) injected into (4.34) yields (4.30).

\[ \square \]

Remark 4.6. The proof of (4.29) is elementary but requires in an essential way the \( L^\infty \) smallness bound\(^{17} \) (4.23), and in particular the sole control of the \( H^1_\rho \) norm cannot suffice to control the nonlinear term \( \int |\varepsilon|^{p+1} \rho \) due to both the energy supercritical nature of the problem and the exponential weight.

4.5. Outer global \( \dot{H}^2 \) bound. We recall

\[ v = \varepsilon + \psi \]

and now aim at propagating an unweighted global \( \dot{H}^2 \) decay estimate for \( v \). We have

\[ \partial_s v - \Delta v - \frac{\lambda_s}{\lambda} \Delta v - \frac{x_s}{\lambda} \cdot \nabla v = G \]

with

\[ G = \left[ \left( \frac{\lambda_s}{\lambda} + 1 \right) \Delta \Phi_n + \frac{x_s}{\lambda} \cdot \nabla \Phi_n \right] + \bar{\Delta}L, \quad \bar{\Delta}L = (\Phi_n + v)^p - \Phi_n^p. \]

Lemma 4.7 (Global \( \dot{H}^2 \) bound). There holds the Lyapunov type monotonicity formula

\[ \frac{d}{ds} \left[ \frac{1}{\lambda^{4-\delta-2s}} \int |\Delta v|^2 dy \right] + \frac{1}{\lambda^{4-\delta-2s}} \int |\nabla \Delta v|^2 dy \lesssim \frac{1}{\lambda^{4-2s-\delta}} \left[ \|\varepsilon\|_{H^0_\rho}^2 + \sum_{j=2}^{n+1} |a_j|^2 \right] \]

for some universal constant \( 0 < \delta \ll 1 \).

Proof. We compute the \( \dot{H}^2 \) energy identity:

\[ \frac{1}{2} \frac{d}{ds} \int |\Delta v|^2 dy = \int \Delta v \Delta \left[ \Delta v + \frac{\lambda_s}{\lambda} \Delta v + \frac{x_s}{\lambda} \cdot \nabla v + G \right] dy \]

\[ = - \int |\nabla \Delta v|^2 dy + \int \Delta v \Delta \left[ \frac{\lambda_s}{\lambda} \Delta v + \frac{x_s}{\lambda} \cdot \nabla v + G \right] dy \]

and estimate all terms.

\(^{17}\)or anything above or equal scaling in terms of regularity.
step 1 Parameters terms. For any $\mu > 0$, let $v_\mu = \frac{1}{\mu^{1/4}}v\left(\frac{y}{\mu}\right)$, then:

$$\int |\Delta v_{\mu}|^2 dy = \frac{1}{\mu^{4-2s}} \int |\Delta v|^2 dy$$

and hence differentiating and evaluating at $\mu = 1$:

$$-2 \int \Delta v \Delta (A v) dy = -(4 - 2s_c) \int |\Delta v|^2 dy.$$

Hence

$$\frac{\lambda_s}{\lambda} \int \Delta v \Delta (A v) = (2 - s_c) \frac{\lambda_s}{\lambda} \int |\Delta v|^2 dy.$$ 

Also, integrating by parts:

$$\int \Delta v \Delta \left( \frac{x_s}{\lambda} \cdot \nabla v \right) dy = 0.$$

step 2 $G$ terms. Thanks to the decay of the self similar solution from (1.9):

$$\int |\Delta \Phi_n|^2 dy + \int |\Delta \nabla \Phi_n|^2 dy < +\infty,$$

we estimate in brute force using (4.25) the terms induced by the self similar solution:

$$\left| \int \Delta v \Delta \left\{ \left[ \frac{\lambda_s}{\lambda} + 1 \right] \Phi_n + \frac{x_s}{\lambda} \cdot \nabla \Phi_n \right\} \right| \leq \left[ \frac{\lambda_s}{\lambda} + 1 \right] |\Delta v|_{L^2} \leq \delta |\Delta v|_{L^2}^2 + C_\delta \left( \|\varepsilon\|_{H^2}^2 + |\nabla v|^2 + \sum_{j=2}^{n+1} |a_j|^2 \right)^2 \leq \delta |\Delta v|_{L^2}^2 + C_\delta \left( \|\varepsilon\|_{H^2}^2 + \sum_{j=2}^{n+1} |a_j|^2 \right).$$

It remains to estimate the nonlinear term. We estimate by homogeneity:

$$|\Delta \tilde{NL}| = |p \Delta \Phi_n [(\Phi_n + v)^{p-1} - \Phi_n^{p-1}] + p(\Phi_n + v)^{p-1} \Delta v + p(p-1)|\nabla \Phi_n|^2 [(\Phi_n + v)^{p-2} - \Phi_n^{p-2}] + p(p-1)|\nabla v|^2 (\Phi_n + v)^{p-1} + 2p(p-1)(\Phi_n + v)^{p-2} \Delta \Phi_n \cdot \nabla v| \lesssim |\Delta \Phi_n|(|v|^{p-1} + |\Phi_n|^{p-2}|v|) + |\Delta v|(|v|^{p-1} + |\Phi_n|^{p-1}) + |\nabla \Phi_n|^2(|v|^{p-2} + |\Phi_n|^{p-3}|v|) + |\nabla v|^2(|v|^{p-1} + |\Phi_n|^{p-1}) + |\nabla v||\nabla \Phi_n||(|\Phi_n|^{p-2} + |v|^{p-2})$$

and hence using the self similar decay of $\Phi_n$ and the $L^\infty$ smallness (4.23):

$$|\Delta \tilde{NL}| \lesssim \left[ \frac{|\Delta v|}{1 + |y|^2} + \frac{|\nabla v|}{1 + |y|^3} + \frac{|v|}{1 + |y|^4} \right] + \eta \left[ |\Delta v| + \frac{|\nabla v|}{1 + |y|^2} + \frac{|v|}{1 + |y|^2} \right] + |\nabla v|^2(|v|^{p-1} + |\Phi_n|^{p-1}).$$
The linear term is estimated using (A.5):
\[
\int \left| \frac{\Delta v}{1 + |y|^2} + \frac{|\nabla v|}{1 + |y|^2} + \frac{|v|}{1 + |y|^2} \right|^2 \lesssim \frac{1}{A^2} \int_{|y| \geq A} |\Delta v|^2 + C_A \|v\|_{H^2}^2 \\
\leq \delta \int |\Delta v|^2 + C_\delta \left( \|\varepsilon\|_{H^p}^2 + \sum_{j=2}^{n+1} |a_j|^2 \right)
\]
and using (A.5) again:
\[
\int |\nabla v| \left| \frac{\Delta v}{1 + |y|^2} + \frac{|\nabla v|}{1 + |y|^2} + \frac{|v|}{1 + |y|^2} \right|^2 \lesssim \eta \|\Delta v\|_{L^2}^2 + \|\varepsilon\|_{H^p}^2 + \sum_{j=2}^{n+1} |a_j|^2.
\]
To estimate the nonlinear term, we let
\[ q_c = \frac{3(p-1)}{2} \]
so that \( \dot{H}^{q_c} \subset L^{q_c} \).
We estimate using (4.23) with \( 6(p-2) > q_c \) and Sobolev:
\[
\int |\nabla v|^q \left( |v|^{2(p-2)} + \|\Phi_n\|_{L^q}^{2(p-2)} \right) \lesssim \|\nabla v\|_{L^q} \left( \|v\|_{L^q}^{2(p-2)} + \|\Phi_n\|_{L^q}^{2(p-2)} \right) \\
\lesssim \|\Delta v\|_{L^2}^4 \left( \|\Phi_n\|_{L^q}^{2(p-2)} + \|\nabla u\|_{L^q}^{2(p-2)} \right) \lesssim \|\Delta v\|_{L^2}^4 \left[ 1 + \|u\|_{\dot{H}^{q_c}}^{\frac{p+1}{2}} \right] \leq \delta \|\Delta v\|_{L^2}^2.
\]
We have therefore obtained
\[
\int |\Delta \dot{N}L|^2 \leq \delta \|\Delta v\|_{L^2}^2 + C_\delta \left( \|\varepsilon\|_{H^p}^2 + \sum_{j=2}^{n+1} |a_j|^2 \right).
\]
The collection of above bounds and (4.25) yields (4.36).

4.6. Control of the critical norm. We now claim the control of the critical norm of \( w \) (defined by (4.10)).

**Lemma 4.8** (Control of the critical norm). There holds the Lyapunov type control
\[
\frac{d}{ds} \int |\nabla^s w|^2 dy + \int |\nabla^s w|^2 dy \lesssim \|\varepsilon\|_{H^2}^2 + \sum_{j=2}^{n+1} |a_j|^2 + \lambda^{2(s-c)} + \|\Delta v\|_{L^2}^2 \quad (4.37)
\]
for some small enough universal constant \( 0 < \delta = \delta(p) \ll 1 \).

**Proof.** Let
\[
\tilde{\Phi}_n = \chi_{\frac{1}{\lambda}} \Phi_n, \quad (4.38)
\]
we compute the evolution equation of \( w \):
\[
\partial_s w - \Delta w = \frac{\lambda_s}{\lambda} \Delta w + \frac{x_s}{\lambda} \cdot \nabla w + \tilde{G} \quad (4.39)
\]
with
\[
\tilde{G} = \left( \frac{\lambda_s}{\lambda} + 1 \right) \chi_{\frac{1}{\lambda}} \Delta \Phi_n + \frac{x_s}{\lambda} \cdot \nabla \tilde{\Phi}_n + 2 \nabla \chi_{\frac{1}{\lambda}} \cdot \nabla \tilde{\Phi}_n + \Delta \chi_{\frac{1}{\lambda}} \Phi_n - \left( \chi_{\frac{1}{\lambda}} - \chi_{\frac{1}{2}} \right) \Phi_n^p + \tilde{N}L,
\]
\[
\tilde{N}L = (\tilde{\Phi}_n + w)^p - (\tilde{\Phi}_n)^p.
\]
Observe from the space localization of the cut, from the decay of the self similar solution, and from (4.19) and (4.20):
\[
\forall s_c \leq s \leq 2, \quad \|w\|_{H^s} \lesssim \eta. \quad (4.40)
\]
We compute:
\[
\frac{1}{2} \frac{d}{ds} \int |\nabla^{s_c} w|^2 dy = \int \nabla^{s_c} w \cdot \nabla^{s_c} \left[ \Delta w + \frac{\lambda_s}{\lambda} \Delta w + \frac{x_s}{\lambda} \cdot \nabla w + \tilde{G} \right] dy
\]
\[
= - \int |\nabla^{s_c+1} w|^2 + \int \nabla^{s_c} w \cdot \nabla^{s_c} \left[ \frac{\lambda_s}{\lambda} \Delta w + \frac{x_s}{\lambda} \cdot \nabla w + \tilde{G} \right] dy
\]
and estimate all terms.

**step 1** Parameters terms. For any \( \mu > 0 \), let \( w_\mu = \frac{1}{\mu^{s_c}} w \left( \frac{y}{\mu} \right) \), then:
\[
\int |\nabla^{s_c} w_\mu|^2 dy = \int |\nabla^{s_c} w|^2 dy
\]
and hence differentiating at \( \mu = 1 \):
\[
-2 \int \nabla^{s_c} w \cdot \nabla^{s_c} (\Delta w) dy = 0.
\]
Integrating by parts:
\[
\int \nabla^{s_c} w \cdot \nabla^{s_c} \left( \frac{x_s}{\lambda} \cdot \nabla \tilde{G} \right) dy = 0.
\]

**step 2** \( \tilde{G} \) terms. The decay of the self similar solution and the space localization of the cut ensure using \( 1 < s_c < 2 \):
\[
\left\| 2 \nabla \chi_1 \cdot \nabla \Phi_n + \Delta \chi_1 \Phi_n \right\|_{H^{s_c}} \leq \left\| 2 \nabla \chi_1 \cdot \nabla \Phi_n + \Delta \chi_1 \Phi_n \right\|_{H^{2-s_c}}^2 \left\| 2 \nabla \chi_1 \cdot \nabla \Phi_n + \Delta \chi_1 \Phi_n \right\|_{H^{2}}^{s_c-1}
\]
\[
\leq \left( \frac{\lambda^2}{\lambda^{s_c-1}} \right)^{2-s_c} \left( \frac{\lambda^2}{\lambda^{s_c-2}} \right)^{s_c-1} \lesssim \lambda^2,
\]
and similarly
\[
\left\| \left( \chi_1 - \chi_1^p \right) \Phi_n^p \right\|_{H^{s_c}} \lesssim \left\| \left( \chi_1 - \chi_1^p \right) \Phi_n^p \right\|_{H^{2-s_c}} \left\| \left( \chi_1 - \chi_1^p \right) \Phi_n^p \right\|_{H^{2}}^{s_c-1}
\]
\[
\lesssim \left( \lambda^{3-s_c} \right)^{2-s_c} \left( \lambda^{4-s_c} \right)^{s_c-1} \lesssim \lambda^2.
\]
Using (4.25):
\[
\left\| \left( \frac{\lambda_s}{\lambda} + 1 \right) \chi_1 \Phi_n + \frac{x_s}{\lambda} \cdot \nabla (\chi_1 \Phi_n) \right\|_{H^{s_c}} \lesssim \left\| \frac{x_s}{\lambda} + 1 \right\| \lesssim \| \varepsilon \|^2 + \sum_{j=2}^{n+1} |a_j|^2 + \| \Delta v \|^2 + \lambda^2,
\]
We now turn to the control of the nonlinear term and claim the bound:
\[
\| \nabla^{s_c} N L \|_{L^2} \lesssim \| \nabla^{s_c+\alpha} w \|_{L^2}
\]
for some small enough universal constant \( 0 < \alpha = \alpha(p) \ll 1 \). Assume (4.41), we then interpolate with \( \delta = \frac{\alpha}{2-s_c} \) and use (4.24), (4.20) and the decay of the self similar solution to estimate:
\[
\| \nabla^{s_c+\alpha} w \|_{L^2} \lesssim \| \nabla^{s_c} w \|_{L^2}^{1-\delta} \| \Delta w \|_{L^2}^\delta \lesssim \lambda^{\delta (2-s_c)} + \| \Delta v \|_{L^2}^\delta,
\]
and the collection of above bounds yields (4.37).
Proof of (4.41). We compute
\[ \nabla N L = p \nabla (\Phi_n + w)(\Phi_n + w)^{p-1} - p \nabla \Phi_n \nabla \Phi_n^{p-1} \]
\[ = p \nabla \Phi_n \left( (\Phi_n + w)^{p-1} - \Phi_n^{p-1} \right) + p \nabla w (\Phi_n + w)^{p-1} \]
with
\[ g_1 (w) = (\Phi_n + w)^{p-1} - \Phi_n^{p-1}. \]
Hence letting
\[ s_c = 1 + \nu, \quad 0 < \nu = \frac{1}{2} - \frac{2}{p-1} < \frac{1}{2}, \]
we estimate:
\[ \| \nabla s_c N L \|_{L^2} \lesssim \left\| \nabla^\nu (g_1 (w) \nabla (\Phi_n + w)) \right\|_{L^2} + \left\| \nabla^\nu (\Phi_n^{p-1} \nabla w) \right\|_{L^2}. \]  
(4.42)
For the first term, we use the following commutator estimate proved in Appendix B: let
\[ 0 < \nu < 1, \quad 1 < p_1, p_2, p_3, p_4 < +\infty, \quad \frac{1}{2} = \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p_3} + \frac{1}{p_4} \]
then
\[ \| \nabla^\nu (u w) \|_{L^2} \lesssim \| u \|_{\dot{B}^{p_1}_{p_1,2}} \| v \|_{L^{p_2}} + \| u \|_{L^{p_4}} \| v \|_{\dot{B}^{p_3}_{p_3,2}}, \]  
(4.43)
where we use here the standard space formulation of Besov norms for \( 0 < s < 1 \) and \( 1 \leq p < +\infty \):
\[ \| u \|_{\dot{B}^{p_2}_{p_2,2}} \sim \left( \int_0^{+\infty} \left( \sup_{|y| \leq t} \| u(t) - u(y) \|_{L^p} \right) \frac{dt}{t} \right)^{\frac{1}{2}}. \]  
(4.44)
We pick a small enough \( 0 < \alpha \ll 1 \) to be chosen later and
\[ \frac{1}{p_1} = \frac{1}{3} + \frac{\alpha}{3}, \quad \frac{1}{p_2} = \frac{1}{6} - \frac{\alpha}{3}, \quad \frac{1}{p_3} = \frac{1 + \alpha + \nu}{3}, \quad \frac{1}{p_4} = \frac{1 - 2(\alpha + \nu)}{\nu}. \]
Observe that
\[ -\nu + \frac{3}{p_2} = \frac{3}{p_4} \]
and hence from (4.43), the embedding of \( \dot{H}^{s,p} \) in \( \dot{B}^{s}_{p,2} \), and Sobolev\(^{19}\):
\[ \| \nabla^\nu (g_1 (w) \nabla (\Phi_n + w)) \|_{L^2} \lesssim \| \nabla (\Phi_n + w) \|_{L^{p_1}} \| g_1 (w) \|_{\dot{B}^{p_2}_{p_2,2}} + \| \nabla (\Phi_n + w) \|_{\dot{B}^{p_3}_{p_3,2}} \| g_1 (w) \|_{L^{p_4}} \]
\[ \lesssim \| \nabla^{\nu + \frac{3}{p_1}} (\Phi_n + w) \|_{L^2} \| g_1 (w) \|_{\dot{B}^{p_2}_{p_2,2}} + \| \nabla^{\nu + \frac{3}{p_3}} (\Phi_n + w) \|_{L^2} \| \nabla^\nu g_1 (w) \|_{L^{p_2}} \]
\[ \lesssim \| \nabla^\nu (\Phi_n + w) \|_{L^2} \| g_1 (w) \|_{\dot{B}^{p_2}_{p_2,2}}. \]
Since \( s_c = \frac{3}{2} - \frac{2}{p-1} < \frac{3}{2} \), we may pick \( 0 < \alpha \ll 1 \) with \( \frac{3}{2} - \alpha > s_c \) and hence using (4.40) and the decay of the self similar solution:
\[ \| \nabla^\nu (\Phi_n + w) \|_{L^2} \lesssim 1. \]
\(^{18}\)See for example [4].
\(^{19}\)Using \( \frac{3}{2} - \frac{3}{p_3} = \frac{1}{2} - (\alpha + \nu) > 0 \).
Let now
\[ f(z) = (1 + z)^{p-1} - 1 \]
then \( f(0) = 0 \) and
\[ |f(z_2) - f(z_1)| = \left| \int_{z_1}^{z_2} f'(\tau) d\tau \right| \lesssim \int_{z_1}^{z_2} (1 + |\tau|^{p-2}) d\tau \lesssim |z_2 - z_1|(1 + |z_1|^{p-2} + |z_2|^{p-2}) \]
and hence by homogeneity:
\[ |g_1(w_2) - g_1(w_1)| \lesssim |w_2 - w_1|(|\Phi_n|^{p-2} + |w_2|^{p-2} + |w_1|^{p-2}). \]
Using the \( L^\infty \) bound (4.23), (4.44), and Sobolev\(^{20}\)
\[ \|g_1(w)\|_{\dot{B}_{p,2}^{s'}} \lesssim \left( \int_0^{+\infty} \left( \sup_{|y| \leq t} \|g_1(w(\cdot - y)) - g_1(w(\cdot))\|_{L^{p/2}} \right)^2 \frac{dt}{t} \right)^{\frac{1}{2}} \lesssim \left( \int_0^{+\infty} \left( \frac{\sup_{|y| \leq t} \|w(\cdot - y) - w(\cdot)\|_{L^{p/2}}}{{t^\nu}} \right)^2 \frac{dt}{t} \right)^{\frac{1}{2}} \sim \|w\|_{\dot{B}_{p,2}^{s'}} \]
The collection of above bounds yields the control of the first term of (4.42):
\[ \|\nabla^\nu \left[ g_1(w)\nabla(\Phi_n + w) \right]\|_{L^2} \lesssim \|\nabla^{s_c + \alpha} w\|_{L^2}. \]
For the second term in (4.42), we recall the following estimate proved in [42]: let \( 0 < \nu < 1 \) and \( \mu > 0 \) with \( \mu + \nu < \frac{3}{2} \), let \( f \) smooth radially symmetric with
\[ |\partial_r^k f| \lesssim \frac{1}{1 + r^{\mu+k}}, \quad k = 0, 1, \quad (4.45) \]
then there holds the generalized Hardy bound
\[ \|\nabla^\nu (uf)\|_{L^2} \lesssim \|\nabla^{\nu+\mu} f\|_{L^2}. \quad (4.46) \]
We then pick again a small enough \( 0 < \alpha \ll 1 \) and let
\[ \mu = \alpha, \quad \mu + \nu = \nu + \alpha = s_c - 1 + \alpha < \frac{3}{2} \]
for \( 0 < \alpha \ll 1 \) small enough, and \( f = (\chi_{\frac{1}{\lambda}} \Phi_n)^{p-1} \) satisfies
\[ |\partial_r^k f| \lesssim \frac{1}{1 + r^{2+\mu}} \lesssim \frac{1}{1 + r^{\mu+k}}. \]
Hence
\[ \|\nabla^\nu \left( \Phi_n^{-p-1}\nabla w \right)\|_{L^2} \lesssim \|\nabla^{\nu+\mu+1} w\|_{L^2} = \|\nabla^{s_c + \alpha} w\|_{L^2}. \]
This concludes the proof of (4.41). \( \square \)

\(^{20}\)Here we use that \( \dot{B}_{\nu,2}' \) embeds in \( \dot{B}_{p,2}^{s} \) with \( s - 3/2 = t - 3/p \) for \( p \geq 2 \), and \( \dot{B}_{\nu,2}^{s} = \dot{H}^{s} \).
4.7. **Conclusion.** We are now in position to conclude the proof of Proposition 4.2 which then easily implies Theorem 1.2.

**Proof of Proposition 4.2.** We recall that we are arguing by contradiction assuming (4.22). We first show that the bounds (4.16), (4.18), (4.19) and (4.20) can be improved on \([s_0, s^*]\), and then, the existence of the data \((a_j(0))_{2 \leq j \leq n+1}\) follows from a classical topological argument à la Brouwer.

**step 1** Improved scaling control. We estimate from (4.17), (4.18), (4.19), (4.25):

\[
\left| \frac{\lambda_s}{\lambda} + 1 \right| \lesssim K^2 e^{-2\mu s}
\]  
(4.47)

and hence after integration:

\[
\left\| \log \left( \frac{\lambda(s)}{\lambda_0} \right) \right\| + s - s_0 \lesssim \int_{s_0}^{s} K^2 e^{-2\mu \tau} d\tau \lesssim 1 + o(1)
\]

for \(s_0\) large enough, which together with (4.5) implies:

\[
\lambda(s) = (\lambda(s_0)e^0) e^{-s}(1 + o(1)) \quad \text{and hence} \quad \frac{e^{-s}}{2} \leq \lambda(s) \leq 2e^{-s}.
\]  
(4.48)

**step 2** Improved Sobolev bounds.

\(L^2_\rho\) bound. From (4.29), (4.17), (4.19), (4.23):

\[
\frac{d}{ds} \|\varepsilon\|^2_{L^2_\rho} + c_n \|\varepsilon\|^2_{H^1_\rho} \lesssim (1 + K^4)e^{-4\mu s} + K^2 e^{-2\mu s} e^{-2\mu s} \lesssim e^{-(2+c)\mu s}
\]

for \(s \geq s_0\) large enough. From now on, we may fix once and for all the value

\[
\mu = \frac{c_n}{4}
\]  
(4.49)

and hence

\[
\frac{d}{ds} \|\varepsilon\|^2_{L^2_\rho} + 4\mu \|\varepsilon\|^2_{H^1_\rho} \lesssim e^{-(2+c)\mu s}
\]  
(4.50)

which time integration yields using (4.7):

\[
\|\varepsilon(s)\|^2_{L^2_\rho} + 2\mu e^{-2\mu s} \int_{s_0}^{s} e^{2\mu \tau} \|\varepsilon\|^2_{H^1_\rho} d\tau \lesssim \left( e^{2\mu s_0} \|\varepsilon(s_0)\|^2_{L^2_\rho} \right) e^{-2\mu s} + e^{-2\mu s} \int_{s_0}^{s} e^{-\mu c \tau} d\tau
\]

\[
\lesssim K_0^2 e^{-2\mu s}.
\]  
(4.51)

\(H^2_\rho\) bound. We estimate from (4.30) like for the proof of (4.50):

\[
\frac{d}{ds} \|\mathcal{L}_n \varepsilon\|^2_{L^2_\rho} + 4\mu \|\mathcal{L}_n \varepsilon\|^2_{H^1_\rho} \lesssim \|\varepsilon\|^2_{H^1_\rho} + e^{-(2+c)\mu s}
\]

whose time integration with the initial bound (4.7) and the bound (4.51) ensures:

\[
\|\mathcal{L}_n \varepsilon(s)\|^2_{L^2_\rho} \lesssim K_0^2 e^{-2\mu s}.
\]

We recall

\[
(\mathcal{L}_n \varepsilon, \varepsilon)_\rho = \|\nabla \varepsilon\|^2_{L^2_\rho} + \int \left( \frac{2}{p-1} - p \Phi_n^{p-1} \right) |\varepsilon|^2 \rho dy
\]

and hence we first estimate from the spectral bound (3.6), the orthogonality conditions (4.9), and Cauchy-Schwarz:

\[
\|\nabla \varepsilon\|^2_{L^2_\rho} \lesssim (\mathcal{L}_n \varepsilon, \varepsilon)_\rho + C \|\varepsilon\|^2_{L^2_\rho} \lesssim \|\mathcal{L}_n \varepsilon\|^2_{L^2_\rho} + \|\varepsilon\|^2_{L^2_\rho} \lesssim K_0^2 e^{-2\mu s}.
\]  
(4.52)

This yields using (A.2):

\[
\|\varepsilon\|^2_{H^2_\rho} \lesssim \|\mathcal{L}_n \varepsilon\|^2_{L^2_\rho} + \|\varepsilon\|^2_{H^1_\rho}
\]  
(4.53)
and hence the improved bound
\[ \| \varepsilon \|_{L^2}^2 \lesssim K_0^2 e^{-2\mu s}. \] (4.54)

\( \dot{H}^2 \) bound. We rewrite (4.36) using (4.17), (4.25), (4.54)
\[
\frac{d}{ds} \| \Delta v \|_{L^2}^2 + (4 - \delta - 2s_c) \| \Delta v \|_{L^2}^2 \lesssim K_0^2 e^{-2\mu s}.
\]
By possibly diminishing the value of \( c_n \), we may always assume
\[ 4 - \delta - 2s_c > c_n = 4 \mu \]
and hence from (4.7):
\[ \| \Delta v \|_{L^2}^2 \lesssim K_0^2 e^{-4\mu s} e^{A\mu s_0} e^{-2\mu s_0} + e^{-4\mu s} \int_{s_0}^s K_0^2 e^{A\mu \tau} e^{-2\mu \tau} d\tau \lesssim K_0^2 e^{-2\mu s}. \] (4.55)

\( \dot{H}^{sc} \) bound. We now rewrite (4.37) using (4.16)-(4.20):
\[
\frac{d}{ds} \int \| \nabla^{sc} w \|_{L^2}^2 \lesssim e^{-c\mu s}
\]
for some universal constant \( c > 0 \) which time integration using (4.7) ensures:
\[ \| \nabla^{sc} w(s) \|_{L^2}^2 \lesssim \| \nabla^{sc} w(s_0) \|_{L^2}^2 + e^{-c s_0} < \frac{\eta}{2} \] (4.56)
for \( s_0 \) large enough.

**step 3** The Brouwer fixed point argument. We conclude from (4.48), (4.54), (4.55), (4.56), the definition (4.21) of \( s^* \) and a simple continuity argument that the contradiction assumption (4.22) implies from (4.17):
\[ \sum_{j=2}^{n+1} |a_j(s^*)|^2 = e^{-2\mu s^*}. \] (4.57)
Moreover, the vector field is strictly outgoing from (4.25), (4.17), (4.18), (4.19):
\[
\frac{1}{2} \frac{d}{ds} \sum_{j=2}^{n+1} |a_j e^{\mu s}|^2 = \sum_{j=2}^{n+1} a_j e^{2\mu s} ((a_j)_s + \mu a_j) = \sum_{j=2}^{n+1} a_j e^{2\mu s} [ (\mu + \mu_j) a_j + O(K^2 e^{-2\mu s}) ] \geq \mu \sum_{j=2}^{n+1} |a_j e^{\mu s}|^2 + O(K^2 e^{-\mu s})
\]
from which
\[
\left( \frac{d}{ds} \sum_{j=2}^{n+1} |a_j e^{\mu s}|^2 \right) (s^*) > \mu + O(K^2 e^{-\mu s_0}) > 0
\]
for \( s_0 \) large enough. We conclude from standard argument that the map
\[ (a_j(0)e^{\mu s_0})_{2 \leq j \leq n+1} \mapsto (a_j(s^*)e^{\mu s^*})_{2 \leq j \leq n+1} \]
is continuous in the unit ball of \( \mathbb{R}^n \), and the identity on its boundary, a contradiction to Brouwer’s theorem. This concludes the proof of Proposition 4.2. \( \square \)

We are now in position to conclude the proof of Theorem 1.2.
Proof of Theorem 1.2. Let an initial data as in Proposition 4.2, then the corresponding solution $u(s, y)$ admits on $[s_0, +\infty)$ a decomposition (4.8) with the bounds (4.17), (4.23), (4.19), (4.20), (4.48).

**step 1** Self similar time blow up. Using (4.48), the life space of the solution $u$ is finite

$$T = \int_{s_0}^{+\infty} \lambda^2(s)ds \lesssim \int_{s_0}^{+\infty} e^{-2s}ds < +\infty,$$

and hence

$$T - t = \int_{s}^{+\infty} \lambda^2(s)ds \sim e^{-2s}.$$

We may therefore rewrite (4.47):

$$|\lambda \lambda^t + 1| \lesssim (T - t)^\mu$$

and integrating in time using $\lambda(T) = 0$ yields

$$\lambda(t) = \sqrt{(2 + o(1))(T - t)}.$$  \hspace{1cm} (4.58)

Also from (4.25):

$$\int_{0}^{T} |x_t| = \int_{s_0}^{+\infty} |x_s|ds \lesssim \int_{s_0}^{+\infty} e^{-s-2\mu s}ds < +\infty$$

and (1.12) is proved.

**step 2** Asymptotic stability above scaling. We now prove (1.13) and (1.15). We first estimate from (4.24) using the self similar decay of $\Phi_n$:

$$\|w\|_{H^2} \lesssim \|v\|_{H^2} + \|(1 - \chi_{\lambda})\Phi_n\|_{H^2} \lesssim e^{-2\mu s} + \lambda^{2-s_c}(s) \rightarrow 0 \text{ as } t \rightarrow T.$$

Hence from (4.20):

$$\forall s_c < \sigma \leq 2, \lim_{s \rightarrow +\infty} \|w(s)\|_{H^s} = 0$$

which using (4.24) and the self similar decay of $\Phi_n$ again implies

$$\forall s_c < \sigma \leq 2, \lim_{s \rightarrow +\infty} \|v(s)\|_{H^s} = 0,$$

this is (1.13). At the critical level, we have from (4.8), (4.10) and the sharp self similar decay from Proposition 2.2:

$$\|u(t)\|_{H^{s_c}} = \|\chi_{\frac{\lambda}{2}}\Phi_n + w\|_{H^{s_c}} = c_n(1 + o(1))\sqrt{|\log \lambda|}, \quad c_n \neq 0,$$

and (4.58) now yields (1.15).

**step 3** Boundedness below scaling. We now prove (1.14). *Control of the Dirichlet energy.* Recall the notation (4.38) and compute by rescaling using the self similar decay of $\Phi_n$:

$$\lambda^{2(s_c-1)} \left[ \|\nabla \Phi_n\|_{L^2}^2 + \|\Phi_n\|_{L^{p+1}}^{p+1} \right] \lesssim 1.$$  

Hence the dissipation of energy which is translation invariant ensures

$$\lambda^{2(s_c-1)}\|\nabla w\|_{L^2}^2 \lesssim \lambda^{2(s_c-1)} \left[ \|\nabla (\Phi_n + w)\|_{L^2}^2 + \|\nabla \Phi_n\|_{L^2}^2 \right] \lesssim 1 + 2E(u) + \frac{2}{p + 1}\|u\|_{L^{p+1}}^{p+1}$$

$$\lesssim 1 + |E_0| + \lambda^{2(s_c-1)}\|w\|_{L^{p+1}}^{p+1}.$$
We now interpolate using the smallness\footnote{This is the only place in the proof where we use that the critical norm is small, bounded suffices everywhere else.} (4.20)
\[||w||_{L^p}^{p+1} \lesssim ||w||_{H^{s_0}}^{p-1} ||\nabla w||_{L^2}^2 \lesssim \eta ||\nabla w||_{L^2}^2\]
and hence
\[\lambda^{2(s_c-1)} ||\nabla w||_{L^2}^2 \lesssim C(u_0)\]  
(4.59)
and
\[||\nabla u||_{L^2}^2 \lesssim \lambda^{2(s_c-1)} \left[ ||\nabla \Phi_n||_{L^2}^2 + ||\nabla w||_{L^2}^2 \right] \lesssim 1.\]

**Proof of (1.14).** Let now $1 \leq \sigma < s_c$, then using (4.20), (4.59) and interpolation:
\[||\nabla^\sigma u||_{L^2} \lesssim \lambda^{s_c-\sigma} ||\nabla^\sigma \widetilde{\Phi}_n||_{L^2} + \lambda^{s_c-\sigma} ||\nabla^\sigma w||_{L^2} \lesssim 1 + \lambda^{s_c-\sigma} ||\nabla w||_{L^2}^{s_c-\sigma} ||\nabla^{s_c} w||_{L^2}^{s_c-\sigma} \lesssim C(u_0)\]
and (1.14) is proved. This concludes the proof of Theorem 1.2. \hfill \Box

### 4.8. The Lipschitz dependence.

We now state the Lipschitz aspect of the set of solutions constructed in this paper.

**Proposition 4.9** (Lipschitz dependence). Let $s_0 \gg 1$, $\varepsilon^{(1)}_0$ and $\varepsilon^{(2)}_0$ satisfy (4.4) and (4.15), and take $\lambda^{(1)}_0 = \lambda^{(2)}_0 = e^{-s_0}$. Then the parameters $(a^{(1)}_j(0))_{2 \leq j \leq n+1}$ and $(a^{(2)}_j(0))_{2 \leq j \leq n+1}$, associated by Proposition 4.2 to $(\varepsilon^{(1)}, \lambda^{(1)}_0)$ and $(\varepsilon^{(2)}, \lambda^{(2)}_0)$ respectively, satisfy:

\[\sum_{j=2}^{n+1} \left| a^{(1)}_j(0) - a^{(2)}_j(0) \right|^2 \lesssim \left| \varepsilon^{(1)}_0 - \varepsilon^{(2)}_0 \right|^2.\]  
(4.60)

**Proof.** The idea of the proof is classical, see for instance [17]. We study the difference of two solutions, and use the bounds we already derived in the existence result as a priori bounds now. This allows us to control the difference of solutions at a low regularity level which is sufficient to conclude.

We use the superscripts $(i)$, $i = 1, 2$ for all variables associated to the two solutions respectively: $u^{(i)}$ for (4.8), $v^{(i)}$ for (4.10), $\psi^{(i)}$ for (4.9), $\lambda^{(i)}$ for the scales and $x^{(i)}$ for the central points. The differences are denoted by
\[\Delta \varepsilon := \varepsilon^{(1)} - \varepsilon^{(2)}, \quad \Delta a_j := a^{(1)}_j - a^{(2)}_j, \quad \Delta v := v^{(1)} - v^{(2)}.\]

We compare the two renormalized solutions at the same renormalized time $s$. The time evolution for the difference is given by

\[\Delta \varepsilon + \mathcal{L}_n \Delta \varepsilon = \frac{d}{ds} \left[ \log \left( \frac{\lambda^{(1)}}{\lambda^{(2)}} \right) \right] \Lambda(\Phi_n + v^{(2)}) + \left( \frac{a^{(1)}_j}{\lambda^{(1)}} - \frac{a^{(2)}_j}{\lambda^{(2)}} \right) \cdot \nabla(\Phi_n + v^{(2)}) - \sum_{j=2}^{n+1} (\Delta a_{j,s} - \mu_j \Delta a_j) \psi_j + \left( \frac{\lambda^{(1)}_j}{\lambda^{(1)}} + 1 \right) \Lambda \Delta v + \frac{\varepsilon^{(1)}_j}{\lambda^{(1)}} \cdot \nabla \Delta v + \left[ \Phi_n + v^{(1)} - (\Phi_n + v^{(2)}) \right] p - p \Phi_n^{-1} \Delta v.\]  
(4.61)
step 1 Modulation equations. We claim that
\[
\left| \frac{d}{ds} \log \frac{\lambda^{(1)}}{\lambda^{(2)}} \right| + \frac{x_s^{(1)}}{\lambda^{(1)}} - \frac{x_s^{(2)}}{\lambda^{(2)}} + \sum_{j=2}^{n+1} |\Delta a_j| \right| 
\lesssim e^{-c\mu s} \left( \|\Delta \varepsilon\|_{L^2_\rho} + \sum_{j=2}^{n+1} |\Delta a_j| \right). \tag{4.62}
\]
We now show this estimate. Taking the scalar product of (4.61) with \(\psi_1 = \frac{\Lambda \Phi_n}{\|\Lambda \Phi_n\|_{L^2_\rho}}\), using the orthogonality conditions (4.9) and (4.26) and the fact that \(\psi_j\) is radial for \(1 \leq j \leq n + 1\), yields the identity
\[
\left| \frac{d}{ds} \log \frac{\lambda^{(1)}}{\lambda^{(2)}} \right| (\Lambda (\Phi_n + v^{(2)}), \psi_1) = - \left( \frac{x_s^{(1)}}{\lambda^{(1)}} - \frac{x_s^{(2)}}{\lambda^{(2)}} \right) \cdot \nabla v^{(2)}, \psi_1 \right| \rho \tag{4.63}
\]
and we now estimate each term. The coercivity (A.1) and the bounds (4.17) and (4.18) yields
\[
(\Lambda (\Phi_n + v^{(2)}), \psi_1) = 1 + O(e^{-\mu s}),
\]
The modulation estimate (4.25), with (4.17), (4.18) and (4.19) and an integration by parts yields
\[
\left| \frac{\lambda^{(1)}}{\lambda^{(1)}} + 1 \right| (\Lambda \Delta v, \psi_1) - \left( \frac{x_s^{(1)}}{\lambda^{(1)}} \cdot \nabla \varepsilon^{(2)}, \psi_1 \right) \rho \lesssim e^{-\mu s} \left( \frac{x_s^{(1)}}{\lambda^{(1)}} - \frac{x_s^{(2)}}{\lambda^{(2)}} \right).
\]
Eventually, for the difference of the nonlinear terms the nonlinear inequality
\[
|(x + y)^p - (x + z)^p - p(x^{p-1}(y - z)| \leq ((x|^{p-2} + |y|^{p-2} + |z|^{p-2}) |y - z|
\]
for any \(x, y, z\) and the bound (4.23) yields the pointwise estimate
\[
|\Phi_n + v^{(1)}|^p - |\Phi_n + v^{(2)}|^p - p|\Phi_n^{p-1}\Delta v| \lesssim e^{-c\mu s}|\Delta v|, \tag{4.64}
\]
which implies
\[
\left| |\Phi_n + v^{(1)}|^p - |\Phi_n + v^{(2)}|^p - p|\Phi_n^{p-1}\Delta v, \psi_1 \right| \rho \lesssim e^{-\mu s} \left( \|\Delta \varepsilon\|_{L^2_\rho} + \sum_{j=2}^{n+1} |\Delta a_j| \right). \tag{4.65}
\]
The collection of the above bounds, when plugged in (4.63), yields
\[
\left| \frac{d}{ds} \log \frac{\lambda^{(1)}}{\lambda^{(2)}} \right| \lesssim e^{-\mu s} \left( \frac{x_s^{(1)}}{\lambda^{(1)}} - \frac{x_s^{(2)}}{\lambda^{(2)}} \right) + e^{-\mu s} \left( \|\Delta \varepsilon\|_{L^2_\rho} + \sum_{j=2}^{n+1} |\Delta a_j| \right).
With the same techniques, taking the scalar product of (4.61) with $\partial^k \Phi_n$, $k = 1, 2, 3$ implies
\[
\left| \frac{x_s^{(1)}}{\lambda^{(1)}} - \frac{x_s^{(2)}}{\lambda^{(2)}} \right| \lesssim e^{-\mu_s} \frac{d}{ds} \left[ \log \left( \frac{\lambda^{(1)}}{\lambda^{(2)}} \right) \right] + e^{-c\mu_s} \left( \|\Delta \epsilon\|_{L^p} + \sum_{j=2}^{n+1} |\Delta a_j| \right).
\]

The two above equations, when put together, imply the estimate
\[
\left| \frac{d}{ds} \left[ \log \left( \frac{\lambda^{(1)}}{\lambda^{(2)}} \right) \right] + \left| \frac{x_s^{(1)}}{\lambda^{(1)}} - \frac{x_s^{(2)}}{\lambda^{(2)}} \right| \lesssim e^{-c\mu_s} \left( \|\Delta \epsilon\|_{L^p} + \sum_{j=2}^{n+1} |\Delta a_j| \right).
\]

The corresponding estimate for $|\Delta a_{j,s} + \mu_j \Delta a_j|$ follows along the same lines, and therefore (4.62) is proven.

**step 2** Localized energy estimate. We claim the differential bound
\[
\frac{d}{ds} \|\Delta \epsilon\|^2_{L^p} + c_n \|\Delta \epsilon\|^2_{L^p} \lesssim e^{-c\mu_s} \sum_{j=2}^{n+1} |\Delta a_j|^2 \tag{4.66}
\]
which we now prove. From the evolution equation (4.61) and the orthogonality conditions (4.9) one obtains first the identity
\[
\frac{d}{ds} \|\Delta \epsilon\|^2_{L^p} = -\langle \mathcal{L}_n \Delta \epsilon, \Delta \epsilon \rangle + \frac{d}{ds} \left[ \log \left( \frac{\lambda^{(1)}}{\lambda^{(2)}} \right) \right] (\Lambda_v^{(2)} \Delta \epsilon) + \left( \frac{x_s^{(1)}}{\lambda^{(1)}} - \frac{x_s^{(2)}}{\lambda^{(2)}} \right) \cdot \nabla v^{(2)} \cdot (\Delta \epsilon) + (\Lambda \Delta v, \Delta \epsilon) + \left( \|\Phi_n + v^{(1)}\| - \|\Phi_n + v^{(2)}\| - p \Phi_n^{-1} \Delta v, \Delta \epsilon \right)
\]
and we now estimate each term. The spectral gap (3.6) and (4.9) imply
\[
-\langle \mathcal{L}_n \Delta \epsilon, \Delta \epsilon \rangle \leq -c_n \|\Delta \epsilon\|^2_{L^p}.
\]

The modulation estimates (4.62) of step 1 and Cauchy-Schwarz imply
\[
\left| \frac{d}{ds} \left[ \log \left( \frac{\lambda^{(1)}}{\lambda^{(2)}} \right) \right] (\Lambda v^{(2)} \Delta \epsilon) + \left( \frac{x_s^{(1)}}{\lambda^{(1)}} - \frac{x_s^{(2)}}{\lambda^{(2)}} \right) \cdot \nabla v^{(2)} \cdot (\Delta \epsilon) \right| \lesssim \left| \frac{d}{ds} \right[ \log \left( \frac{\lambda^{(1)}}{\lambda^{(2)}} \right) \right] \|\Lambda v^{(2)}\|_{L^p} + \left| \frac{x_s^{(1)}}{\lambda^{(1)}} - \frac{x_s^{(2)}}{\lambda^{(2)}} \right| \|\nabla v^{(2)}\|_{L^p} \|\Delta \epsilon\|_{L^p}
\]
\[
\lesssim \|v^{(2)}\|_{H^2} e^{-c\mu_s} \left( \|\Delta \epsilon\|_{L^p} + \sum_{j=2}^{n+1} |\Delta a_j| \right) \|\Delta \epsilon\|_{L^p}
\]
\[
\lesssim e^{-(1+c)\mu_s} \left( \|\Delta \epsilon\|^2_{L^p} + \sum_{j=2}^{n+1} |\Delta a_j|^2 \right)
\]
where we used (A.1), (4.17) and (4.18) to control $v^{(2)}$. Using the modulation estimate (4.25), with (4.17), (4.18) and (4.19) for $u^{(1)}$, integrating by parts and applying
Cauchy-Schwarz and (A.1) yields
\[
\left| \left( \frac{\lambda_1^{(1)}}{\lambda_1^{(1)}} + 1 \right) (\Lambda \Delta v, \Delta \varepsilon)_\rho + \left( \frac{x_1^{(1)}}{\lambda_1^{(1)}} \nabla \Delta v, \Delta \varepsilon \right)_\rho \right| \lesssim \left( \frac{\lambda_1^{(1)}}{\lambda_1^{(1)}} + 1 + \frac{x_1^{(1)}}{\lambda_1^{(1)}} \right) \left| (\Lambda \Delta \psi, \Delta \varepsilon)_\rho \right| + \left| (\Lambda \Delta \varepsilon, \Delta \varepsilon)_\rho \right| + \left| (\nabla \Delta \psi, \Delta \varepsilon)_\rho \right| + \left| (\nabla \Delta \varepsilon, \Delta \varepsilon)_\rho \right|
\]
\[
\lesssim e^{-2\mu s} \left( \sum_{j=2}^{n+1} |\Delta a_j|^2 + \|\Delta \varepsilon\|^2_{H^1}\right).
\]

Finally, the pointwise estimate (4.64) and Cauchy-Schwarz imply for the nonlinear term
\[
\left| (\Phi_n + v^{(1)})^p - (\Phi_n + v^{(2)})^p - p\Phi_n^{p-1} \Delta v, \Delta \varepsilon \right|_\rho \lesssim e^{-c\mu s} \left( \|\Delta \varepsilon\|^2_{L^p} + \sum_{j=2}^{n+1} |\Delta a_j|^2 \right).
\]

We inject all the above bounds in the identity (4.67), which for \( s_0 \) large enough imply the desired estimate (4.66) since \( 0 < c \leq 1 \).

**step 3** Lipschitz bound by reintegration. We define
\[
A := \sup_{s \geq s_0} \sum_{j=2}^{n+1} |\Delta a_j(s)| e^{\mu s} < +\infty, \quad \mathcal{E} := \sup_{s \geq s_0} \|\Delta \varepsilon\|^2_{L^p} e^{2\mu s} < +\infty, \quad (4.68)
\]
which are finite from (4.17) and (4.18).

**Identity for \( \Delta a_j \).** Fix \( j \) with \( 2 \leq j \leq n+1 \). Reintegrating the modulation equation (4.62) yields
\[
\Delta a_j = \Delta a_j(0) e^{\mu_j(s-s_0)} + e^{\mu_j s} \int_{s_0}^{s} e^{-\mu_j s'} O(e^{-c\mu s'}(\|\Delta \varepsilon\|_{L^p}^2 + \sum_{j=2}^{n+1} |\Delta a_j|)) ds'
\]
\[
= \Delta a_j(0) e^{\mu_j(s-s_0)} + e^{\mu_j s} \int_{s_0}^{s} O(e^{-(\mu_j + (c+1)\mu)s'}(A + \sqrt{E})) ds'
\]
\[
= \left( \Delta a_j(0) e^{-\mu_j s_0} + \int_{s_0}^{+\infty} O(e^{-(\mu_j + (c+1)\mu)s'}(A + \sqrt{E})) ds' \right) e^{\mu_j s}
\]
\[
- e^{\mu_j s} \int_{s}^{+\infty} O(e^{-(\mu_j + (c+1)\mu)s'}(A + \sqrt{E})) ds'. \quad (4.69)
\]
The integral appearing in this identity is indeed convergent and satisfies:
\[
\left| \int_{s}^{+\infty} O(e^{-(\mu_j + (c+1)\mu)s'}(A + \sqrt{E})) ds' \right| \lesssim e^{-(\mu_j + (c+1)\mu)s'(A + \sqrt{E})}.
\]

From (4.68) one gets \( |\Delta a_j| \lesssim e^{-\mu s} \) and from the two above identities one necessarily must have that the parameter in front of the diverging term \( e^{\mu_j s} \) is 0:
\[
\Delta a_j(0) e^{-\mu_j s_0} + \int_{s_0}^{+\infty} O(e^{-(\mu_j + (c+1)\mu)s'}(A + \sqrt{E})) ds' = 0
\]
which gives the first bound
\[
|\Delta a_j(0)| \lesssim e^{-(c+1)\mu s_0}(A + \sqrt{E}), \quad (4.70)
\]
and going back to the identity (4.69) one obtains:

\[ |\Delta a_j| \lesssim e^{-(c+1)\mu s_0}(A + \sqrt{E}) \]

which implies from the definition (4.68) of A the bound

\[ A \lesssim e^{-c\mu s_0} \sqrt{E}. \]  \tag{4.71} \]

Identity for $\Delta \varepsilon$. We reintegrate the energy bound (4.66) to find

\[
\|\Delta \varepsilon\|_{L^2_\rho}^2 \lesssim \|\Delta \varepsilon(0)\|^2_{L^2_\rho} e^{-c_\mu s_0} + e^{-c\mu s} \int_0^s e^{c\mu s'} \sum_{j=2}^{n+1} |\Delta a_j| e^{-\mu cs'} ds' \\
\lesssim \|\Delta \varepsilon(0)\|^2_{L^2_\rho} e^{-c_\mu s_0} + A^2 e^{-(c+2)\mu s}
\]

since $\mu = \frac{c}{4}$ from (4.49) and $0 < c \ll 1$ can be chosen arbitrarily small. Injecting (4.71) in the above identity yields

\[ E \lesssim \|\Delta \varepsilon(0)\|^2_{L^2_\rho} e^{2\mu s_0} \]

so that (4.71) can be rewritten as $A \lesssim \|\Delta \varepsilon(0)\|_{L^2_\rho} e^{(1-c)\mu s_0}$. We inject these two last bounds in (4.70) which finally yields the desired estimate (4.60). \hfill \Box

Appendix A. Coercivity estimates

**Lemma A.1** (Weighted $L^2$ estimate). Let $u, \partial_r u \in L^2_\rho(\mathbb{R}^3)$, then

\[ \|ru\|_{L^2_\rho} \lesssim \|u\|_{H^1_\rho}. \]  \tag{A.1} \]

Moreover,

\[ \|\Delta u\|_{L^2_\rho}^2 \lesssim \| - \Delta u + y \cdot \nabla u \|_{L^2_\rho}^2 + \|u\|_{H^1_\rho}^2. \]  \tag{A.2} \]

**Proof.** We may assume by density $u \in D(\mathbb{R}^3)$.

**step 1** Proof of (A.1). We use $\partial_r \rho = -r \rho$ and integrate by parts to compute:

\[
\int_0^{+\infty} \left( \partial_r u - \frac{1}{2} ru \right)^2 pr^2 dr \\
= \int_0^{+\infty} (\partial_r u)^2 pr^2 dr + \frac{1}{4} \int_0^{+\infty} r^2 u^2 dr - \int_0^{+\infty} ru \partial_r u pr^2 dr \\
= \int_0^{+\infty} (\partial_r u)^2 pr^2 dr + \frac{1}{4} \int_0^{+\infty} r^2 u^2 dr - \frac{1}{2} \left[ \frac{3}{2} r^2 \rho^2 \right]_0^{+\infty} \\
+ \frac{1}{2} \int_0^{+\infty} u^2 (3 - r^2) \rho^2 dr \\
= \int_0^{+\infty} (\partial_r u)^2 pr^2 dr - \frac{1}{4} \int_0^{+\infty} r^2 u^2 dr + \frac{3}{2} \int_0^{+\infty} u^2 \rho^2 dr
\]

and hence

\[
\|ru\|_{L^2_\rho}^2 = \int_0^{+\infty} r^2 u^2 pr^2 dr \leq 4 \int_0^{+\infty} (\partial_r u)^2 pr^2 dr + 6 \int_0^{+\infty} u^2 \rho^2 dr \lesssim \|u\|_{H^1_\rho}^2
\]

which concludes the proof of (A.1).

**step 2.** Proof of (A.2). We compute:

\[
\| - \Delta u + y \cdot \nabla u \|_{L^2_\rho}^2 = \|\Delta u\|_{L^2_\rho}^2 + \|y \cdot \nabla u\|_{L^2_\rho}^2 - 2 \int (\Delta u) y \cdot \nabla u dy.
\]
To compute the crossed term, let \( u_\lambda(y) = u(\lambda y) \), then
\[
\int |\nabla u_\lambda(y)|^2 \rho dy = \frac{1}{\lambda} \int |\nabla u(y)|^2 \rho \left( \frac{y}{\lambda} \right) dy
\]
and hence differentiating in \( \lambda \) and evaluating at \( \lambda = 1 \):
\[
2 \int \nabla u \cdot \nabla (y \cdot \nabla u) \rho dy = \int |\nabla u|^2 (-\rho - y \cdot \nabla \rho) dy
\]
i.e.
\[
2 \int y \cdot \nabla u (\rho \Delta u + \nabla u \cdot \nabla \rho) = \int |\nabla u|^2 (\rho + y \cdot \nabla \rho) dy
\]
which using \( \nabla \rho = -y \rho \) becomes:
\[
-2 \int (\Delta u)y \cdot \nabla u \rho dy = \int |\nabla u|^2 \rho |y|^2 - 2 \int |y \cdot \nabla u|^2 \rho - \int \rho |\nabla u|^2.
\]
Hence:
\[
\| - \Delta u + y \cdot \nabla u \|_{L^2}^2 = \| \Delta u \|_{L^2}^2 + \int \rho (|y|^2 |\nabla u|^2 - |y \cdot \nabla u|^2) - \int \rho |\nabla u|^2
\]
\[
\geq \| \Delta u \|_{L^2}^2 - \| \nabla u \|_{L^2}^2
\]
which concludes the proof of (A.2).

We now turn to the proof of Hardy type inequalities. All proofs are more or less standard and we give the argument for the sake of completeness.

**Lemma A.2** (Radial Hardy with best constants). Let \( u \in C_0^\infty (r > 1) \) and
\[
\gamma \neq -1,
\]
then
\[
\int_1^{+\infty} \frac{(\partial_r u)^2}{r^{\gamma + 1}} dr \geq \left( \frac{\gamma + 1}{2} \right)^2 \int_1^{+\infty} \frac{u^2}{r^{\gamma + 2}} dr.
\]

**Proof.** We integrate by parts:
\[
\int_1^{+\infty} \frac{u^2}{r^{\gamma + 2}} dr = \frac{2}{\gamma + 1} \int_1^{+\infty} \frac{u \partial_r u}{r^{\gamma + 1}} dr \leq \frac{2}{\gamma + 1} \left( \int_1^{+\infty} \frac{u^2}{r^{\gamma + 2}} dr \right)^{\frac{1}{2}} \left( \int_1^{+\infty} \frac{(\partial_r u)^2}{r^{\gamma + 1}} dr \right)^{\frac{1}{2}}
\]
and (A.4) follows.

**Lemma A.3** (Global Hardy for \( \Delta \)). Then there exists \( c > 0 \) such that \( \forall u \in C_0^\infty (|x| > 1) \),
\[
\int |\Delta u|^2 dx \geq c \int \left( \frac{|\nabla u|^2}{|x|^2} + \frac{|u|^2}{|x|^4} \right) dx.
\]

**Proof.** We decompose \( u \) in spherical harmonics and consider
\[
\Delta_m u_m = \partial_r^2 u_m + \frac{2}{r} \partial_r u_m - \frac{m(m+1)}{r^2}, \quad m \in \mathbb{N}.
\]
We claim that for all \( u \in C_0^\infty ((1, +\infty)) \),
\[
\int_1^{+\infty} |\Delta_m u|^2 r^2 dr \geq c \int_1^{+\infty} \left( \frac{|\partial_r u|^2}{r^2} + \frac{(1 + m^4)|u|^2}{r^4} \right) r^2 dr
\]
with \( c \) independent of \( m \). Assume (A.6), then
\[
\int \frac{|\nabla u|^2}{r^2} dx \sim \sum_{m \geq 0} \sum_{k = -m}^m \int \left( \frac{|\partial_r u_{m,k}|^2}{r^2} + \frac{m^2 |u_{m,k}|^2}{r^4} \right) r^2 dr
\]
and hence summing (A.6) ensures (A.5).
To prove (A.6), we factorize the Laplace operator:

\[ \Delta_m = -A^*_m A_m \quad \text{with} \quad A_m = -\partial_r - \frac{\gamma_m}{r} \partial_r, \quad A^*_m = \partial_r + \frac{2-\gamma_m}{r} \partial_r = \frac{1}{r^{2-\gamma_m}} \partial_r (r^{2-\gamma_m}). \]

Hence from (A.4):

\[
\int_1^{+\infty} (\Delta_m v)^2 r^2 dr = \int_1^{+\infty} (A^*_m A_m v)^2 r^2 dr = \int_1^{+\infty} \frac{1}{r^{2-2\gamma_m}} (\partial_r (r^{2-\gamma_m} A_m v))^2 dr
\] 

\[
\geq \left( \frac{2 - 2\gamma_m + 1}{2} \right)^2 \int_1^{+\infty} (A_m v)^2 dr = \left( \frac{2 - 2\gamma_m + 1}{2} \right)^2 \int_1^{+\infty} \frac{1}{r^{2\gamma_m}} (\partial_r (r^{\gamma_m} v))^2 dr
\]

\[
\geq \left( \frac{2 - 2\gamma_m + 1}{2} \right)^2 \left( \frac{2\gamma_m + 1}{2} \right)^2 \int_1^{+\infty} \frac{v^2}{r^2} dr
\]

since \(\gamma_m = -m\) with \(m \in \mathbb{N}\) which ensures that the forbidden value (A.3) is never attained. We conclude that for some universal constant \(\delta > 0\) independent of \(m\):

\[ \int_1^{+\infty} (\Delta_m v)^2 r^2 dr \geq \delta (1 + m^4) \int_1^{+\infty} \frac{v^2}{r^4} r^2 dr. \]

Also, since we have also proved that

\[ \int_1^{+\infty} |A_m v|^2 dr \lesssim \int_1^{+\infty} (\Delta_m v)^2 r^2 dr, \]

we infer

\[ \int_1^{+\infty} \frac{(\partial_r v)^2}{r^2} r^2 dr \lesssim \int_1^{+\infty} |A_m v|^2 dr + \gamma_m \int_1^{+\infty} \frac{v^2}{r^4} r^2 dr \lesssim \int_1^{+\infty} (\Delta_m v)^2 r^2 dr \]

and (A.6) follows. \(\Box\)

\[ \text{Appendix B. Proof of (4.43)} \]

Let

\[ 0 < \nu < 1, \quad 1 < p_1, p_2, p_3, p_4 < +\infty, \quad \frac{1}{2} = \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p_3} + \frac{1}{p_4}. \]
Using (4.44), we have
\[ \| \nabla^\nu (uv) \|_{L^2} \sim \| uv \|_{B^{\nu,2}_{2,2}} \]
\[ \sim \left( \int_0^{+\infty} \left( \frac{\sup_{|y|\leq t} \| uv(\cdot - y) - uv(\cdot) \|_{L^2}}{t^\nu} \right)^2 \frac{dt}{t} \right)^{\frac{1}{2}} \]
\[ \leq \left( \int_0^{+\infty} \left( \frac{\sup_{|y|\leq t} \| u(\cdot - y)(v(\cdot - y) - v(\cdot)) \|_{L^2}}{t^\nu} \right)^2 \frac{dt}{t} \right)^{\frac{1}{2}} \]
\[ \leq \| u \|_{L^p} \left( \int_0^{+\infty} \left( \frac{\sup_{|y|\leq t} \| v(\cdot - y) - v(\cdot) \|_{L^p}}{t^\nu} \right)^2 \frac{dt}{t} \right)^{\frac{1}{2}} \]
\[ + \| v \|_{L^p} \left( \int_0^{+\infty} \left( \frac{\sup_{|y|\leq t} \| u(\cdot - y) - u(\cdot) \|_{L^p}}{t^\nu} \right)^2 \frac{dt}{t} \right)^{\frac{1}{2}} \]
\[ \lesssim \| u \|_{B^{\nu,2}_{p,1,2}} \| v \|_{L^p} + \| u \|_{L^p} \| v \|_{B^{\nu,2}_{p,1,2}} \]
which concludes the proof of (4.43).

**Appendix C. Proof of Lemma 3.2**

The existence and uniqueness of \( \phi_{n,m}, \nu_m \) satisfying (3.9) and (3.13) is well known. Thus, we focus on their behaviour as \( r \to +\infty \).

**Step 1** Inverting \( \mathcal{L}_{m,\infty} \). Let \( \gamma_m \) be the solution to
\[ \gamma_m^2 - \gamma_m + p\nu_{\infty}^{p-1} - m(m+1) = 0, \]
the corresponding discriminant \( \Delta_m \) is given by
\[ \Delta_m := 1 - 4p\nu_{\infty}^{p-1} + 4m(m+1). \quad (C.1) \]
For \( m = 1 \),
\[ \Delta_1 = \left( \frac{p + 3}{p - 1} \right)^2 > 0 \quad (C.2) \]
and hence for all \( m \geq 1 \)
\[ \Delta_m \geq \Delta_1 > 0. \]
Therefore, \( \gamma_m \) is real and we choose the smallest root\(^{22}\) so that \( \gamma_m \) is given by
\[ \gamma_m = \frac{1 - \sqrt{\Delta_m}}{2}. \]
We now solve
\[ \mathcal{L}_{\infty,m}(\psi) = 0 \]
\(^{22}\)This is motivated by the fact that we obtain below the Kummer’s equation with \( b = -\gamma_m + 1/2 \). This is equivalent to \( -b = \pm \sqrt{\Delta_m} \). Since the Kummer function is not defined for \( -b \in \mathbb{N} \), this justifies to consider the smallest root \( \gamma_m \).
through the change of variable and unknown
\[ \psi(r) = \frac{1}{(2z)^{\frac{3}{2}}} w(z), \quad z = \frac{r^2}{2} \]
which leads to
\[ L_{\infty,m}(\psi) = -\frac{2}{(2z)^{\frac{3}{2}}} \left( z w''(z) + \left( -\gamma_m + \frac{3}{2} - z \right) w'(z) - \left( \frac{1}{p - 1} - \frac{\gamma_m}{2} \right) w(z) \right). \]
Thus, \( L_{\infty,m}(\psi) = 0 \) if and only if
\[ zd^2w/dz^2 + (b - z) dw/dz - aw = 0 \]
where we have used the notations
\[ a = \frac{1}{p - 1} - \frac{\gamma_m}{2}, \quad b = -\gamma_m + \frac{3}{2}. \]
Hence \( w \) is a linear combination of two special functions, the Kummer’s function \( M(a,b,z) \) and the Tricomi function \( U(a,b,z) \). These special functions have the following asymptotic behavior at infinity (see for example [47]):
\[ M(a,b,z) \sim \frac{\Gamma(b)}{\Gamma(a)} z^{a-b} e^{z}, \quad U(a,b,z) \sim z^{-a} \text{ as } z \to +\infty. \]
This allows us to infer the asymptotic for \( w \) for \( z \to 0_+ \). Finally, since
\[ \psi(r) = \frac{1}{r^{\frac{3}{2}}} w \left( \frac{r^2}{2} \right), \]
we infer from the asymptotic of \( w \) the following asymptotic behavior for \( \psi_{1,m} \) and \( \psi_{2,m} \)
\[ \psi_{1,m} \sim \frac{1}{r^{\frac{3}{2}}} \quad \text{and} \quad \psi_{2,m} \sim r^{\frac{2}{p-1} - \frac{3}{2}} e^{\frac{r^2}{2}} \text{ as } r \to +\infty. \]
Consider the Wronskian \( W \) which is defined as
\[ W := \psi_{1,m} \psi_{2,m} - \psi_{2,m} \psi_{1,m}, \]
then without loss of generality since \( W' = (r - \frac{3}{2}) W \)
\[ W = \frac{1}{r^{\frac{3}{2}}} e^{\frac{r^2}{2}}. \]
We deduce using the variation of constants that the solution \( w \) to
\[ L_{\infty,m}(u) = f, \]
is given by
\[ u = \left( a_1 + \int_0^{+\infty} f \psi_{2,m} r^2 e^{-\frac{r^2}{2}} dr' \right) \psi_{1,m} + \left( a_2 - \int_0^{+\infty} f \psi_{1,m} r^2 e^{-\frac{r^2}{2}} dr' \right) \psi_{2,m}. \]

**step 2** Basis of \( L_{m,n} \) near \( +\infty \). We now construct a solution to \( L_{n,m}(\varphi) = 0 \) near \( +\infty \) by solving:
\[ L_{\infty,m}(\varphi) = L_{n,m}(\varphi) + p(\Phi_n^{p-1} - \Phi_s^{p-1}) = p(\Phi_n^{p-1} - \Phi_s^{p-1}) \varphi \]

ie
\[ \varphi = \left( a_1 + \int_0^{+\infty} p(\Phi_n^{p-1} - \Phi_s^{p-1}) \varphi \psi_{2,m} r^2 e^{-\frac{r^2}{2}} dr' \right) \psi_{1,m} \]
\[ + \left( a_2 - \int_0^{+\infty} p(\Phi_n^{p-1} - \Phi_s^{p-1}) \varphi \psi_{1,m} r^2 e^{-\frac{r^2}{2}} dr' \right) \psi_{2,m}. \]
To construct the solution $\varphi_1$ with the choice $a_1 = 1$ and $a_2 = 0$ we solve the fixed point equation

$$
\varphi_1 = \psi_{1,m} + \tilde{\varphi}_1, \quad \tilde{\varphi}_1 = G(\tilde{\varphi}_1)
$$

(C.3)

where

$$
G(\tilde{\varphi})(r) = \left( \int_r^{+\infty} p(\Phi_n^p - \Phi_s^p)(\psi_{1,m} + \tilde{\varphi})(r')r^2 e^{-\frac{r^2}{2}} dr' \right) \psi_{1,m} - \left( \int_r^{+\infty} p(\Phi_n^p - \Phi_s^p)(\psi_{1,m} + \tilde{\varphi})(r')r^2 e^{-\frac{r^2}{2}} dr' \right) \psi_{2,m}.
$$

Recall that we have in view of Corollary 2.6

$$
\lim_{n \to +\infty} \sup_{r \geq 1} r^{\frac{2}{p-1}} |\Phi_n(r) - \Phi_s(r)| = 0.
$$

Thus, for $n \geq N$ large enough, we infer

$$
|\Phi_n(r) - \Phi_s(r)| \leq \frac{1}{r^{\frac{p}{p-1}}} \quad \text{for } r \geq 1.
$$

so that

$$
|p(\Phi_n^p - \Phi_s^p)| \lesssim \frac{1}{r^2}.
$$

We infer for $r \geq 1$

$$
|G(\tilde{\varphi})(r)| \lesssim \frac{1}{r^{2+\frac{2}{p-1}}} \left( \int_r^{+\infty} r^{2+\frac{2}{p-1}} \frac{1}{r^{p-1}} + |\tilde{\varphi}(r')| |dr'| \right) + r^{2+\frac{2}{p-1}} e^{\frac{r^2}{2}} \left( \int_r^{+\infty} r^{2+\frac{2}{p-1}} \frac{1}{r^{p-1}} |\tilde{\varphi}(r')| |dr'| \right) \lesssim \frac{1}{r^{2+\frac{2}{p-1}}} + \frac{1}{r^{p-1}} \left( \int_r^{+\infty} r^{2+\frac{2}{p-1}} |\tilde{\varphi}(r')| |dr'| \right) + r^{2+\frac{2}{p-1}} e^{\frac{r^2}{2}} \left( \int_r^{+\infty} r^{2+\frac{2}{p-1}} |\tilde{\varphi}(r')| |dr'| \right)
$$

and

$$
|G(\tilde{\varphi}_1)(r) - G(\tilde{\varphi}_2)(r)| \lesssim \frac{1}{r^{2+\frac{2}{p-1}}} \left( \int_r^{+\infty} r^{2+\frac{2}{p-1}} |\tilde{\varphi}_1(r') - \tilde{\varphi}_2(r')| |dr'| \right) + r^{2+\frac{2}{p-1}} e^{\frac{r^2}{2}} \left( \int_r^{+\infty} r^{2+\frac{2}{p-1}} e^{-\frac{r^2}{2}} |\tilde{\varphi}_1(r') - \tilde{\varphi}_2(r')| |dr'| \right)
$$

Thus, for $R \geq 1$ large enough, the Banach fixed point theorem applies in the space corresponding to the norm

$$
\sup_{r \geq R} r^{1+\frac{2}{p-1}} |\tilde{\varphi}(r)|.
$$

Hence, there exists a unique solution $\tilde{\varphi}_1$ to (C.3) and

$$
\sup_{r \geq R} r^{1+\frac{2}{p-1}} |\tilde{\varphi}_1(r)| \lesssim 1.
$$

Hence, $\varphi_1$ satisfies $L_{n,m}(\varphi_1) = 0$ and

$$
\varphi_1 \sim \frac{1}{r^{\frac{p}{p-1}}}, \quad \text{as } r \to +\infty.
$$
The behaviour of the other solution at infinity is computed using the Wronskian relation
\[ W = \varphi'_1\varphi_2 - \varphi'_2\varphi_1 = \frac{1}{r^2} e^{r^2} \]
and hence
\[ \left( \frac{\varphi_2}{\varphi_1} \right)' = \frac{W}{\varphi_1^2} = \frac{1}{r^2\varphi_1^2} e^{r^2} \]
from which
\[ \varphi_2(r) = \varphi_1(r) \int_1^r \frac{1}{r^2\varphi_1^2(r')} e^{r'^2} dr' \sim r^{2-3} e^{r^2} \text{ as } r \to +\infty \]
and (3.10) is proved.

**step 3** Behaviour of \( \nu_m \) at \(+\infty\). First, consider the solution \( \varphi \) to
\[ -\partial_r^2 \varphi - \frac{2}{r} \partial_r \varphi + \frac{m(m+1)}{r^2} - \frac{p c_{\infty}^{p-1}}{r^2} \varphi = f. \]  
(C.4)

The homogeneous equation admits the basis of solutions
\[ \varphi_+ = \frac{1}{r^{1-\sqrt{2m}}} \quad \varphi_- = \frac{1}{r^{1+\sqrt{2m}}} \]
and the corresponding Wronskian is given by
\[ W(r) = \varphi'_+(r)\varphi_-(r) - \varphi'_-(r)\varphi_+(r) = -\frac{1}{r^2}. \]
Using the variation of constants, the solutions to (C.4) are given by
\[ \varphi(r) = \left( a_1 - \int_r^{+\infty} f \varphi_- r'^2 dr' \right) \varphi_+ + \left( a_2 + \int_r^{+\infty} f \varphi_+ r'^2 dr' \right) \varphi_- \]
Now, the equation \( H_m(\phi) = 0 \) can be written as
\[ -\partial_r^2 \phi - \frac{2}{r} \partial_r \phi + \frac{m(m+1)}{r^2} \phi - \frac{p c_{\infty}^{p-1}}{r^2} \phi = p \left( Q^{p-1}(r) - \frac{c_{\infty}^{p-1}}{r^2} \right) \phi(r), \]
i.e. (C.4) with
\[ f = p \left( Q^{p-1}(r) - \frac{c_{\infty}^{p-1}}{r^2} \right) \phi(r). \]
We construct the solution \( \phi_{m,1} \) to \( H_m(\phi_{m,1}) = 0 \) with the choice \( a_1 = 1 \) and \( a_2 = 0 \)
by solving the fixed point equation
\[ \phi_{m,1} = \varphi_+ + \tilde{\phi}, \quad \tilde{\phi} = F \left( \tilde{\phi} \right) \]
where
\[ F \left( \tilde{\phi} \right) (r) = - \left( \int_r^{+\infty} p \left( Q^{p-1}(r') - \frac{c_{\infty}^{p-1}}{r'^2} \right) \left( \varphi_+ + \tilde{\phi} \right) (r') \varphi_- r'^2 dr' \right) \varphi_+ + \left( \int_r^{+\infty} p \left( Q^{p-1}(r') - \frac{c_{\infty}^{p-1}}{r'^2} \right) \left( \varphi_+ + \tilde{\phi} \right) (r') \varphi_+ r'^2 dr' \right) \varphi_- \]
Recall that
\[ Q(r) = \frac{c_{\infty}}{r^2} + \frac{c_1}{r^{2+\gamma}} \sin \left( \omega \log(r) + c_2 \right) + o \left( \frac{1}{r^2} \right) \text{ as } r \to +\infty. \]
corresponding to the norm and yields a unique solution case \( m \). The other independent solution \( \varphi \) and \( \tilde{\varphi} \) so that

\[
\left| p \left( Q^{p-1}(r) - \frac{e_\infty}{r^2} \right) \right| \lesssim \frac{1}{r^{1+s_c}} \text{ for } r \geq 1.
\]

We infer for \( r \geq 1 \)

\[
|F(\tilde{\varphi})(r)| \lesssim \frac{1}{r^{1 + \sqrt{3}m}} \left( \int_r^{+\infty} \frac{1}{r^{1 + \sqrt{3}m}} \left( \frac{1}{1 + \sqrt{3}m} \phi(r') \right) \frac{1}{1 + \sqrt{3}m} dr' \right)
+ \frac{1}{r^{1 + \sqrt{3}m}} \left( \int_r^{+\infty} \frac{1}{r^{1 + \sqrt{3}m}} \left( \frac{1}{1 + \sqrt{3}m} \phi(r') \right) \frac{1}{1 + \sqrt{3}m} dr' \right)
\]

\[
\lesssim \frac{1}{r^{s_c - 1} + \sqrt{3}m} + \frac{1}{r^{1 + \sqrt{3}m}} \left( \int_r^{+\infty} \frac{1}{r^{s_c - 1} + \sqrt{3}m} - \frac{1}{1 + \sqrt{3}m} \phi(r') dr' \right)
+ \frac{1}{r^{1 - \sqrt{3}m}} \left( \int_r^{+\infty} \frac{1}{r^{s_c - 1} + \sqrt{3}m} - \frac{1}{1 + \sqrt{3}m} \phi(r') dr' \right)
\]

and

\[
|F(\tilde{\varphi}_1)(r) - F(\tilde{\varphi}_2)(r)| \lesssim \frac{1}{r^{1 + \sqrt{3}m}} \left( \int_r^{+\infty} \frac{1}{r^{s_c - 1} + \sqrt{3}m} - \frac{1}{1 + \sqrt{3}m} \phi(r') dr' \right)
+ \frac{1}{r^{1 - \sqrt{3}m}} \left( \int_r^{+\infty} \frac{1}{r^{s_c - 1} + \sqrt{3}m} - \frac{1}{1 + \sqrt{3}m} \phi(r') dr' \right).
\]

Thus, for \( R \geq 1 \) large enough, the Banach fixed point theorem applies in the space corresponding to the norm

\[
\sup_{r \geq R} r^{s_c - 1} + \sqrt{3}m \phi(r)
\]

and yields a unique solution \( \tilde{\varphi} \) to (C.5) with

\[
\sup_{r \geq R} r^{s_c - 1} + \sqrt{3}m \phi(r) \leq 1.
\]

Hence, \( \varphi_{m,1} \) satisfies \( H_m(\varphi_{m,1}) = 0 \) and

\[
\varphi_{m,1} \sim \frac{1}{r^{1 + \sqrt{3}m}}, \text{ as } r \to +\infty.
\]  

(C.6)

The other independent solution \( \varphi_{m,2} \) to \( H_m(\varphi_{m,2}) = 0 \) is computed through the Wronskian relation

\[
W := \varphi'_{m,1} \varphi_{m,2} - \varphi'_{m,2} \varphi_{m,1} = -\frac{1}{r^2}
\]

ie

\[
\varphi_{m,2}(r) = \varphi_{m,1}(r) \int_1^r \frac{1}{r'^2} \varphi_{m,2}^2(r') dr' \sim -\frac{1}{r^{1 + \sqrt{3}m}} \text{ as } r \to +\infty.
\]

Since \( \nu_m \) is a linear combination of \( \varphi_{m,1} \) and \( \varphi_{m,2} \), we infer

\[
\nu_m(r) \sim \frac{c_{m,+}}{r^{1 + \sqrt{3}m}} + \frac{c_{m,-}}{r^{1 - \sqrt{3}m}} \text{ as } r \to +\infty
\]  

(C.7)

for some constant \( c_{m,+} \) and \( c_{m,-} \).

case \( m = 1 \): By translation invariance

\[
H_1(Q') = 0 \text{ and } Q'(r) = Q''(0)r(1 + O(r^2))
\]  

(C.8)
Hence, by uniqueness of \( \nu_1 \), we infer
\[
\nu_1(r) = \frac{Q'(r)}{Q''(0)} < 0 \quad \text{on} \quad (0, +\infty)
\]
where we used from standard ODE arguments \( Q''(0) < 0 \) and
\[
Q' < 0 \quad \text{on} \quad (0, +\infty).
\]  

(C.9)

**case** \( m = 2 \): From (C.8), (C.9) and standard Sturm Liouville oscillation arguments for central potentials \([49]\), the quadratic form \( (H_u, u) \) is positive on \( H^1_{\text{rad}}(0, +\infty) \) and hence for \( m \geq 2 \), \( H_m > H_1 \) is definite positive, and hence \( \nu_m > 0 \) on \( (0, +\infty) \).

Moreover, If \( c_{m,-} = 0 \) in (C.7), then \( \nu_m \in H^1_{\text{rad}} \) satisfies \( (H_m \nu_m, \nu_m) = 0 \) which is a contradiction, hence the leading order behaviour (3.13).

**step 4** Completing the basis.

**case** \( m = 2 \). Let \( \phi_m \) be the solution to \( H_m(\phi_m) = 0 \) constructed above with the behaviour (C.6). At the origin, the equation \( H_m \psi \) reads
\[
A^*_m A_m \psi = V \psi,
\]
with
\[
A_m v = r^m \partial_r \left( \frac{v}{r^m} \right), \quad A^*_m = \frac{v}{r^{m+1}} \partial_r (r^{m+1} v)
\]
and \( V \in L^\infty \) and hence all solutions on \((0, \delta)\) with \( 0 < \delta < 1 \) are of the form
\[
\psi = c_0 r^m + \frac{c_1}{r^{m+1}} + r^m \int_0^\delta \frac{d\tau}{\tau^{2m+1}} \int_0^r \tau^{m+1} V \psi d\tau
\]
through an elementary fixed point argument. Hence
\[
\phi_m = c_1 + O(r^2).
\]  

(C.10)

Assume by contradiction that \( c_1 = 0 \). Then, the fixed point above leads to \( \phi_m = O(r^m) \). Hence \( \phi_m \) is a zero of \( H_m \) in \( H^1_{\text{rad}} \) which is a contradiction. Thus, \( c_1 \neq 0 \) and together with (C.10), we have obtained (3.14).

**case** \( m = 1 \). We let \( \phi_1 \) be given by the Wronskian relation
\[
\phi_1 = \nu_1(r) \int_r^1 \frac{d\tau}{\tau^2 \nu_1^2(\tau)} d\tau \sim \left| \frac{\nu_1}{1 - \nu_1^2} \right| \text{ as } r \to +\infty,
\]
which is (3.12).

**step 5** Proof of (3.16). Let
\[
\kappa_{n,m} := \mu_{n}^{-m} \varphi_{n,m}(\mu_n r).
\]
Then, since \( \varphi_{n,m} \) satisfies \( L_{n,m}(\varphi_{n,m}) = 0 \), we infer
\[
-\partial_r^2 \kappa_{n,m} - \frac{2}{p} \partial_r \kappa_{n,m} + \frac{m(m+1)}{r^2} \kappa_{n,m} - p \left( \frac{r^2}{\mu^2_n} \Phi_n(\mu_n r) \right)^{p-1} \kappa_{n,m} = -\mu_n^2 \Lambda \kappa_{n,m}.
\]
This yields
\[
H_m(\kappa_{n,m}) = f_{n,m} := p \left( \frac{r^2}{\mu^2_n} \Phi_n(\mu_n r) \right)^{p-1} - Q^{p-1}(r) \right) \kappa_{n,m} - \mu_n^2 \Lambda \kappa_{n,m}.
\]
Since \( H_m(\nu_m) = 0 \), we infer
\[
H_m(\kappa_{n,m} - \nu_m) = f_{n,m}.
\]
We let \((\nu_m, \phi_m)\) be the completed fundamental basis for \(H_m\) so that
\[
\kappa_{n,m} - \nu_m = \left( a_1 - \int_0^r f_{n,m} \phi_m r^2 dr' \right) \nu_m + \left( a_2 + \int_0^r f_{n,m} \nu_m r^2 dr' \right) \phi_m.
\]
Since
\[
\nu_m(r) = r^m(1 + O(r^2)) \quad \text{and} \quad \varphi_{n,m}(r) = r^m(1 + O(r^2)) \quad \text{as} \quad r \to 0_+,
\]
we infer for \(0 \leq r \leq 0_+ \),
\[
\kappa_{n,m}(r) - \nu(r) = O(r^{m+2})
\]
and hence (3.12), (3.14) implies \(a_1 = a_2 = 0\) and:
\[
\kappa_{n,m} - \nu_m = - \left( \int_0^r f_{n,m} \phi_m r^2 dr' \right) \nu_m + \left( \int_0^r f_{n,m} \nu_m r^2 dr' \right) \phi_m.
\]
In order to estimate \(f_{n,m}\), recall from Corollary 2.6 that we have
\[
\sup_{r \leq r_0} \left| \nu_m(r) - \frac{1}{\mu_n^{r_0}} Q \left( \frac{r}{\mu_n} \right) \right| \lesssim \mu_n^{s_n-1}
\]
This yields
\[
\sup_{r \leq \frac{r_0}{\mu_n^{p_0}}} \left| p \left( \nu_m(\mu_n r) - Q^{p-1}(r) \right) \right| \lesssim \mu_n^{s_n+1} \frac{r_0^{p-1}}{\mu_n^{p_0}}.
\] (C.11)

Also, we rewrite \(f_{n,m}\) as
\[
f_{n,m} = p \left( \frac{2}{\mu_n^{n-1}} \nu_m(\mu_n r) - Q^{p-1}(r) \right) \nu_m - \mu_n^{2} \Lambda \nu_m + p \left( \frac{2}{\mu_n^{n-1}} \nu_m(\mu_n r) - Q^{p-1}(r) \right) (\kappa_{n,m} - \nu_m) - \mu_n^{2} \Lambda (\kappa_{n,m} - \nu_m).
\] (C.12)

\(0 \leq r \leq 1\). In view of the asymptotic behavior as \(r \to 0_+\) (3.12), (3.14) of the basis of solutions \(\nu_m, \phi_m\), and after integrating by parts the term \(\Lambda (\kappa_{n,m} - \nu_m)\), we have for \(0 \leq r \leq 1\) using (C.11) and (C.12):
\[
|\kappa_{n,m} - \nu_m|(r) \lesssim \mu_n^{2} r^{2m} |\kappa_{n,m} - \nu_m|(r)
\]
\[
+ \left( \frac{2}{\mu_n^{n-1}} + \mu_n^{2} \right) \left( r^{m+2} + r^m \left( \int_0^r |\kappa_{n,m} - \nu_m| r^{1-m} dr' \right) \right)
\]
\[
+ r^{-m-1} \left( \int_0^r |\kappa_{n,m} - \nu_m| r^{m+2} dr' \right).
\]

Using again the asymptotic behavior of \(\nu_m\) as \(r \to 0_+\), we infer for all \(m \geq 1\)
\[
\sup_{0 \leq r \leq 1} \frac{|\kappa_{n,m} - \nu_m|(r)}{|\nu_m(r)|} \lesssim \frac{\mu_n^{s_n+1}}{r_0^{p-1}} + \mu_n^{2}.
\] (C.13)

In particular, this yields
\[
\int_0^1 |f_{n,m}| r^{1-m} dr' + \int_0^1 |f_{n,m}| r^{m+2} dr' \lesssim \frac{\mu_n^{s_n+1}}{r_0^{p-1}} + \mu_n^{2}.
\] (C.14)
Next, we consider the region $r \geq 1$. In view of the asymptotic behavior at infinity \((3.12), (3.14), (3.11), (3.13)\), after integrating by parts the term $\Lambda(\kappa_{n,m} - \nu_m)$ and using also \((C.14)\), we have

$$|\kappa_{n,m} - \nu_m| \lesssim \mu_n^2 r^2 |\kappa_{n,m} - \nu_m|$$

$$+ \frac{1}{1 + r^{\frac{1 + \sqrt{2} \Delta_m}{2}}} \left( \frac{\mu_n^{s+1}}{r_0^{\frac{2 - p}{2}}} + \mu_n^2 + \int_1^r |f_{n,m}| \frac{r^2}{(1 + r')^{\frac{1 - \sqrt{2} \Delta_m}{2}}} dr' \right)$$

$$+ \frac{1}{1 + r^{\frac{1 - \sqrt{2} \Delta_m}{2}}} \left( \frac{\mu_n^{s+1}}{r_0^{\frac{2 - p}{2}}} + \mu_n^2 + \int_1^r |f_{n,m}| \frac{r^2}{(1 + r')^{\frac{1 + \sqrt{2} \Delta_m}{2}}} dr' \right).$$

After integrating by parts the term $\Lambda(\kappa_{n,m} - \nu_m)$, and in view of the asymptotic behavior of $\nu_m$ as $r \to +\infty$ as well as \((C.11)\), we deduce

$$|(\kappa_{n,m} - \nu_m)(r)|$$

$$\lesssim \frac{1}{(1 + r^{\frac{1 + \sqrt{2} \Delta_m}{2}})} \left( \int_1^r \left( \frac{\mu_n^{s+1}}{r_0^{\frac{2 - p}{2}}} |\nu_m| + \mu_n^2 \Lambda |\nu_m| + \left( \frac{\mu_n^{s+1}}{r_0^{\frac{2 - p}{2}}} + \mu_n^2 \right) |\kappa_{n,m} - \nu_m| \right) dr' \right)$$

$$\times \frac{r^2}{(1 + r')^{\frac{1 - \sqrt{2} \Delta_m}{2}}} + \frac{\mu_n^{s+1}}{r_0^{\frac{2 - p}{2}}} + \mu_n^2 \right)$$

$$+ \frac{1}{(1 + r^{\frac{1 - \sqrt{2} \Delta_m}{2}})} \left( \int_1^r \left( \frac{\mu_n^{s+1}}{r_0^{\frac{2 - p}{2}}} |\nu_m| + \mu_n^2 \Lambda |\nu_m| + \left( \frac{\mu_n^{s+1}}{r_0^{\frac{2 - p}{2}}} + \mu_n^2 \right) |\kappa_{n,m} - \nu_m| \right) dr' \right)$$

$$\times \frac{r^2}{(1 + r')^{\frac{1 + \sqrt{2} \Delta_m}{2}}} + \frac{\mu_n^{s+1}}{r_0^{\frac{2 - p}{2}}} + \mu_n^2 \right).$$

\textbf{Case $m \geq 2$:} We estimate from \((3.13)\):

$$\frac{|(\kappa_{n,m} - \nu_m)(r)|}{|\nu_m(r)|}$$

$$\lesssim \frac{1}{(1 + r^{\sqrt{2} \Delta_m}} \left\{ \int_1^r \left( \frac{\mu_n^{s+1}}{r_0^{\frac{2 - p}{2}}} + \mu_n^2 \right) \left( \frac{1}{(1 + r')^{\frac{1 - \sqrt{2} \Delta_m}{2}}} + |\kappa_{n,m} - \nu_m| \right) \right\} dr'$$

$$\times \frac{r^2}{(1 + r')^{\frac{1 - \sqrt{2} \Delta_m}{2}}} + \frac{\mu_n^{s+1}}{r_0^{\frac{2 - p}{2}}} + \mu_n^2 \right)$$

$$+ \int_1^r \left( \frac{\mu_n^{s+1}}{r_0^{\frac{2 - p}{2}}} + \mu_n^2 \right) \left( \frac{1}{(1 + r')^{\frac{1 + \sqrt{2} \Delta_m}{2}}} + |\kappa_{n,m} - \nu_m| \right) \frac{r^2}{(1 + r')^{\frac{1 + \sqrt{2} \Delta_m}{2}}} dr'$$

$$+ \frac{\mu_n^{s+1}}{r_0^{\frac{2 - p}{2}}} + \mu_n^2.$$

This yields

$$\sup_{1 \leq r \leq \frac{\rho_m}{4}} \frac{|(\kappa_{n,m} - \nu_m)(r)|}{|\nu_m(r)|} \lesssim r_0^2 \left( 1 + \frac{\mu_n^{s+1}}{r_0^{\frac{2 - p}{2}}} \right)$$
which together with (C.13) concludes the proof of (3.16) for \( n \geq N \) large enough and \( m \geq 2 \).

**case** \( m = 1 \) We estimate using (3.11), (3.12):

\[
\frac{|(\kappa_{n,1} - \nu_1)(r)|}{|\nu_1(r)|} \leq \frac{\int_1^r \left( \frac{\lambda_{n+1}}{r_0^{2-\frac{p}{p-1}}} + \mu_n^2 \right) \left( \frac{1}{1 + r'} + |\kappa_{n,1} - \nu_1| \right) \frac{r'}{(1 + r')^{1 + \frac{\sqrt{\Delta_1}}{2}}} \, dr'}{\int_1^r \left( \frac{\lambda_{n+1}}{r_0^{2-\frac{p}{p-1}}} + \mu_n^2 \right) \left( \frac{1}{1 + r'} + |\kappa_{n,1} - \nu_1| \right) \frac{r'}{(1 + r')^{1 + \frac{\sqrt{\Delta_1}}{2}}} \, dr'}
\]

\[
+ \frac{\mu_n^{\epsilon_n} + \mu_n^2}{r_0^{2-\frac{p}{p-1}}} + \mu_n^2
\]

\[
+(1 + r')^{\frac{\sqrt{\Delta_1}}{2}} \left( \int_1^r \left( \frac{\lambda_{n+1}}{r_0^{2-\frac{p}{p-1}}} + \mu_n^2 \right) \left( \frac{1}{1 + r'} + |\kappa_{n,1} - \nu_1| \right) \frac{r'}{(1 + r')^{1 + \frac{\sqrt{\Delta_1}}{2}}} \, dr'ight)
\]

This yields\(^{23}\)

\[
\sup_{1 \leq r \leq r_0} \frac{|(\kappa_{n,1} - \nu_1)(r)|}{|\nu_1(r)|} \leq r_0^2 \left( 1 + \frac{\mu_n^{\epsilon_n - 1}}{r_0^{2-\frac{p}{p-1}}} \right) + \left( \frac{r_0}{\mu_n} \right)^{\sqrt{\Delta_1}} \left( \frac{\mu_n^{\epsilon_n + 1}}{r_0^{2-\frac{p}{p-1}}} + \mu_n^2 \right)
\]

and hence, together with (C.13) and the fact that\(^{24}\) \( \sqrt{\Delta_1} < 2 \), we have for \( n \geq N \) large enough

\[
\sup_{0 \leq r \leq \frac{r_0}{\mu_n}} \frac{|(\kappa_{n,1} - \nu_1)(r)|}{|\nu_1(r)|} \lesssim r_0^2.
\]

The corresponding estimates for first order derivatives are obtained in the same way, and (3.16) is proved.

**Appendix D. Proof of Lemma 3.3**

**step 1** Proof of (3.18). Let\( \kappa_n := \varphi_{n,0}(\mu_n r) \).

Then, since \( \varphi_{n,0} \) satisfies \( L_{n,0}(\varphi_{n,0}) = 0 \), we infer

\[
-\partial_r^2 \kappa_n - \frac{2}{r} \partial_r \kappa_n - p \left( \frac{2}{\mu_n} \Phi_n(\mu_n r) \right)^{p-1} \kappa_n = -\mu_n^2 \Lambda \kappa_n.
\]

\(^{23}\)Here, we use the fact that \( \sqrt{\Delta_1} - 1 = \frac{4}{p-1} < 1 \)

since \( p > 5 \), so that

\[
\int_0^r \frac{r'}{(1 + r')^{1 + \sqrt{\Delta_1}}} \lesssim (1 + r')^{2 - \sqrt{\Delta_1}}.
\]

\(^{24}\)Indeed, we have in view of (C.2)

\[
\sqrt{\Delta_1} = \frac{p + 3}{p - 1} = 2 - \frac{p - 5}{p - 1} < 2
\]

since \( p > 5 \).
This yields
\[ H(\kappa_n) = f_n \]
where we have introduced the notation
\[ f_n := \left( \frac{\mu_n}{\mu_n} \phi_n(\mu_nr) \right)^{p-1} - H^{p-1}(r) \kappa_n - \mu_n^2 \Lambda \kappa_n. \]

Since \( H(\Lambda Q) = 0 \), we infer
\[ H \left( \kappa_n - \frac{p-1}{2} \Lambda Q \right) = f_n. \]

Recall the solution \( \rho \) to \( H(\rho) = 0 \) constructed in Lemma 2.3 such that \( (\Lambda Q, \rho) \) forms a basis of solutions of \( H(w) = 0 \), then the solution to \( H(w) = f \) is given by
\[ w = \left( a_1 + \int_0^r f \rho \rho' \, dr' \right) \Lambda Q + \left( a_2 - \int_0^r f \Lambda Q \rho \rho' \, dr' \right) \rho. \]

We infer
\[ \kappa_n - \frac{p-1}{2} \Lambda Q = \left( a_1 + \int_0^r f_n \rho \rho' \, dr' \right) \Lambda Q + \left( a_2 - \int_0^r f_n \Lambda Q \rho \rho' \, dr' \right) \rho. \]

Since \( \Lambda Q \) is a smooth function at \( r = 0 \) with
\[ \Lambda Q(0) = \frac{2}{p-1} \neq 0, \]
we infer from the Wronskian relation that \( \rho \) has the following asymptotic behavior
\[ \rho \sim \frac{c}{r} \quad \text{as} \quad r \to 0_+ \]
for some constant \( c \neq 0 \), and hence, we must have \( a_2 = 0 \). Furthermore, since we have
\[ \left( \kappa_n - \frac{p-1}{2} \Lambda Q \right)(0) = 0, \quad \Lambda Q(0) = \frac{2}{p-1} \neq 0, \]
we infer \( a_1 = 0 \). Hence, we have
\[ \kappa_n - \frac{p-1}{2} \Lambda Q = \left( \int_0^r f_n \rho \rho' \, dr' \right) \Lambda Q - \left( \int_0^r f_n \Lambda Q \rho \rho' \, dr' \right) \rho. \]

In order to estimate \( f_n \), recall from Corollary 2.6 that we have
\[ \sup_{r \leq r_0} \left| \frac{1}{\mu_n^{p-1}} Q \left( \frac{r}{\mu_n} \right) \right| \leq \mu_n^{s_\epsilon - 1}. \]

This yields
\[ \sup_{r \leq r_0} \left| p \left( \frac{\mu_n^{2/p} \phi_n(\mu_nr)}{\mu_n} \right)^{p-1} - H^{p-1}(r) \right| \leq \frac{\mu_n^{2s_\epsilon + 1}}{r_0^{2/p-1}}. \]
Also, we rewrite \( f_n \) as

\[
\begin{align*}
    f_n &= p \left( \mu_n^{\frac{2}{x}} \Phi_n(\mu_n r) \right)^{p-1} - Q^{p-1}(r) \left( p - \frac{1}{2} \Lambda Q - \mu_n^2 \rho \right) \\
    &\quad + p \left( \mu_n^{\frac{2}{x}} \Phi_n(\mu_n r) \right)^{p-1} \left( \kappa_n - p - \frac{1}{2} \Lambda Q \right) - \mu_n^2 \Lambda \left( \kappa_n - p - \frac{1}{2} \Lambda Q \right). 
\end{align*}
\]

We start with the region \( 0 \leq r \leq 1 \). In view of the asymptotic behavior for \( \Lambda Q \) and \( \rho \):

\[
\Lambda Q \sim \frac{2}{p - 1} \quad \text{and} \quad \rho \sim \frac{c}{r} \quad \text{as} \quad r \to 0+, 
\]

we infer

\[
\left| \kappa_n - \frac{p - 1}{2} \Lambda Q \right| \lesssim \int_0^r |f_n| r' dr' + \frac{1}{r} \left( \int_0^r |f_n| r'^2 dr' \right). 
\]

Together with (D.1) and (D.2) and integrating by parts the term \( \Lambda(\kappa_n - (p - 1)/2\Lambda Q) \), we deduce

\[
\left| \kappa_n - \frac{p - 1}{2} \Lambda Q \right| \lesssim \left( \frac{\mu_n^{s+1}}{r_0^{2 - \frac{1}{p-1}}} + \mu_n^2 \right) \left( 1 + \int_0^r \left| \kappa_n - \frac{p - 1}{2} \Lambda Q \right| r' dr' \\
+ \frac{1}{r} \left( \int_0^r \left| \kappa_n - \frac{p - 1}{2} \Lambda Q \right| r'^2 dr' \right) \right). 
\]

We infer

\[
\sup_{0 \leq r \leq 1} \left| \kappa_n - \frac{p - 1}{2} \Lambda Q \right| \lesssim \frac{\mu_n^{s+1}}{r_0^{2 - \frac{1}{p-1}}} + \mu_n^2. 
\]

In particular, this yields

\[
\int_0^1 |f_n| r' dr' + \int_0^1 |f_n| r'^2 dr' \lesssim \frac{\mu_n^{s+1}}{r_0^{2 - \frac{1}{p-1}}} + \mu_n^2. 
\]

Next, we consider the region \( r \geq 1 \). Recall the asymptotic behavior at infinity of \( \Lambda Q \) and \( \rho \) given by Lemma 2.3

\[
\Lambda Q(r) \sim \frac{c_7 \sin(\omega \log(r) + c_8)}{r^{\frac{1}{2}}} \quad \rho(r) \sim \frac{c_9 \sin(\omega \log(r) + c_{10})}{r^{\frac{1}{2}}} \quad \text{as} \quad r \to +\infty, 
\]

where \( c_7, c_9 \neq 0, c_8, c_{10} \in \mathbb{R} \). We infer for \( r \geq 1 \)

\[
\left| \kappa_n - \frac{p - 1}{2} \Lambda Q \right| \lesssim \left( \frac{\mu_n^{s+1}}{r_0^{2 - \frac{1}{p-1}}} + \mu_n^2 + \int_1^r |f_n| \frac{r'^2}{(1 + r')^{\frac{3}{2}}} dr' \right) \frac{1}{(1 + r)^{\frac{3}{2}}}.
\]
After integrating by parts the term \( \Lambda(\kappa_n - (p - 1)/2\Lambda Q) \), and together with (D.1) and (D.2), we deduce

\[
(1 + r)^{\frac{1}{2}} \left| \kappa_n - \frac{p - 1}{2} \Lambda Q \right| \leq \left( \frac{\mu_n^{s_n+1}}{r_0^{2-p}} + \mu_n^2 \right) \left( 1 + \int_1^r \left( \frac{1}{(1 + r')^{\frac{1}{2}}} + \left| \kappa_n - \frac{p - 1}{2} \Lambda Q \right| \frac{r'^2}{(1 + r')^{\frac{1}{2}}} dr' \right)
\]

\[
\leq \left( \frac{\mu_n^{s_n+1}}{r_0^{2-p}} + \mu_n^2 \right) (1 + r)^2 + \left( \frac{\mu_n^{s_n+1}}{r_0^{2-p}} + \mu_n^2 \right) \left( \int_1^r \kappa_n - \frac{p - 1}{2} \Lambda Q \right) \frac{r'^2}{(1 + r')^{\frac{1}{2}}} dr'.
\]

This yields

\[
\sup_{1 \leq r \leq \frac{\mu_n}{r_0}} (1 + r)^{\frac{1}{2}} \left| \kappa_n - \frac{p - 1}{2} \Lambda Q \right| \lesssim r_0^2 \left( 1 + \frac{\mu_n^{s_n-1}}{r_0^{2-p}} \right).
\]

which together with (D.3) implies

\[
\sup_{0 \leq r \leq \frac{\mu_n}{r_0}} (1 + r)^{\frac{1}{2}} \left| \kappa_n - \frac{p - 1}{2} \Lambda Q \right| \lesssim r_0^2 \left( 1 + \frac{\mu_n^{s_n-1}}{r_0^{2-p}} \right).
\]

Hence, we have for \( n \geq N \) large enough

\[
\sup_{0 \leq r \leq \frac{\mu_n}{r_0}} \left( 1 + \frac{r}{\mu_n} \right)^{\frac{1}{2}} \left| \varphi_{n,0}(r) - \frac{p - 1}{2} \Lambda Q \left( \frac{r}{\mu_n} \right) \right| \lesssim r_0^2.
\]

**step 2** Proof of (3.19). Recall from Lemma 3.3 that we have for \( n \geq N \) large enough

\[
\sup_{0 \leq r \leq \frac{\mu_n}{r_0}} \left( 1 + \frac{r}{\mu_n} \right)^{\frac{1}{2}} \left| \varphi_{n,0}(r) - \frac{p - 1}{2} \Lambda Q \left( \frac{r}{\mu_n} \right) \right| \lesssim r_0^2.
\]

Also, recall that

\[
\Lambda Q(r) \sim \frac{c_7 \sin(\omega \log(r) + c_8)}{r^{\frac{1}{2}}} \quad \text{as } r \to +\infty
\]

and that \( r_{\Lambda Q,n} < r_0/\mu_n \) introduced in Corollary 2.6 denotes the last zero of \( \Lambda Q \) before \( r_0/\mu_n \). This yields

\[
\left| \omega \log(r_{1,n}) - \omega \log(\mu_n) + c_8 - (\omega \log(r_{\Lambda Q,n}) + c_8) \right| \lesssim r_0^2
\]

and hence

\[
r_{1,n} = \mu_n r_{\Lambda Q,n} e^{O(r_0^2)} = \mu_n r_{\Lambda Q,n} (1 + O(r_0^2)).
\]

Furthermore, since we have from the proof of Corollary 2.6 that

\[
e^{-\frac{3\pi r_0}{2\mu_n}} \leq r_{\Lambda Q,n} \leq \frac{r_0}{\mu_n},
\]

and

\[
r_{0,n} = \mu_n r_{\Lambda Q,n} (1 + O(r_0^2)),
\]

we deduce

\[
r_{1,n} = r_{0,n} + O(r_0^3)
\]

and

\[
e^{-\frac{2\pi r_0}{r_0}} \leq r_{1,n} \leq r_0.
\]
References

[8] Collot, C.; Merle, F; Raphaël, P., Dynamics near the ground state for the energy critical nonlinear heat equation in large dimension, preprint 2016.


