ON THE TRANSVERSAL STABILITY OF SELF SIMILAR BLOW UP FOR THE ENERGY SUPER CRITICAL HEAT EQUATION

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Abstract. We consider the energy super critical 4 dimensional semilinear heat equation
\[ \partial_t u = \Delta u + |u|^{p-1}u, \quad x \in \mathbb{R}^4, \quad p > 5. \]
Let \( \Phi(r) \) be a three dimensional radial self similar solution as exhibited and stabilized in [6]. We show the finite codimensional transversal stability of the corresponding blow up solution by exhibiting a manifold of finite energy blow up solutions of the four dimensional problem with cylindrical symmetry which blows up as
\[ u(t,x) \sim \frac{1}{(T-t)^{\frac{p}{p-1}}} U(t,Y), \quad Y = \frac{x}{\sqrt{T-t}} \]
with the profile \( U \) given to leading order by
\[ U(t,Y) \sim \frac{1}{(1 + b(t)z^2)^{\frac{1}{p-1}}} \Phi \left( \frac{r}{\sqrt{1 + b(t)z^2}} \right), \quad Y = (r,z), \quad b(t) = \frac{c}{|\log(T-t)|} \]
corresponding to a constant profile \( \Phi(r) \) in the \( z \) direction reconnected to zero along the moving free boundary \( |z(t)| \sim \frac{1}{c} \sim \sqrt{|\log(T-t)|} \). Our analysis revisits the stability analysis of the self similar ODE blow up [1, 20, 21] and combines it with the study of the Type I self similar blow up [6] to produce an elementary dynamical approach purely based on energy estimates.

1. Introduction

1.1. Type I and type II blow up. Let us consider the focusing nonlinear heat equation
\[ \begin{cases} \partial_t u = \Delta u + |u|^{p-1}u, \quad (t,x) \in \mathbb{R} \times \mathbb{R}^d, \\ u|_{t=0} = u_0, \end{cases} \tag{1.1} \]
where \( p > 1 \). This model dissipates the total energy
\[ E(u) = \frac{1}{2} \int |\nabla u|^2 - \frac{1}{p+1} \int |u|^{p+1}, \quad \frac{dE}{dt} = - \int (\partial_t u)^2 < 0 \tag{1.2} \]
and admits a scaling invariance: if \( u(t,x) \) is a solution, then so is
\[ u_\lambda(t,x) = \lambda^{\frac{2}{p-1}} u(\lambda^2 t, \lambda x), \quad \lambda > 0. \tag{1.3} \]
This transformation is an isometry on the homogeneous Sobolev space
\[ \|u_\lambda(t,\cdot)\|_{\dot{H}^{s_c}} = \|u(t,\cdot)\|_{\dot{H}^{s_c}} \quad \text{for} \quad s_c = \frac{d}{2} - \frac{2}{p-1}. \]
We address in this paper the question of the existence and stability of blow up dynamics in the energy super critical range \( s_c > 1 \) emerging from well localized initial data. There is an important literature devoted to the question of the description of blow up solutions for (1.1) and we recall some key facts related to our analysis.
Type I blow-up. Type I singularities blow up with the self similar speed

$$\|u(t, \cdot)\|_{L^\infty} \sim \frac{1}{(T-t)^{\frac{1}{p-1}}}.$$  

These solutions concentrate to leading order at a point

$$u(t, x) \sim \frac{1}{\lambda(t)^{\frac{1}{p-1}}} v\left(\frac{x}{\lambda(t)}\right), \quad \lambda(t) = \sqrt{T-t},$$

where the blow up profile $v$ solves the non linear elliptic equation

$$\Delta v - \frac{1}{2} \left(\frac{2}{p-1} v + y \cdot \nabla v\right) + |v|^{p-1}v = 0. \quad (1.4)$$

The ODE blow up corresponds to the special solution to (1.4)

$$v = \left(\frac{1}{p-1}\right)^{\frac{1}{p-1}},$$

and the existence and stability of the associated blow up dynamics has been studied in the series of papers [8, 9, 10, 11, 1, 20, 21]. There also exist radial solutions to (1.4) for $p < p_{JL}$ with

$$\lim_{r \to +\infty} v(r) \to 0,$$

see [13, 26, 2, 3, 6], which can be shown to generate a stable self similar blow up to a finite number of instabilities, [6].

Type II blow-up. Type II singularities are slower than self similar

$$\lim_{t \to T} (T-t)^{\frac{1}{p-1}} \|u(t, \cdot)\|_{L^\infty} = +\infty.$$  

Such dynamics have been ruled out in the radial class for $p < p_{JL}$ in [16, 17] where $p_{JL}$ denotes the Joseph-Lundgren exponent

$$p_{JL} := \begin{cases} +\infty & \text{for } d \leq 10, \\ 1 + \frac{4-2\sqrt{2}}{d-1} & \text{for } d \geq 11, \end{cases} \quad (1.5)$$

and the result is sharp since type II blow up solutions can be constructed for $p > p_{JL}$, [12, 22, 5] in connection with the general approach developed in [23, 19, 24].

1.2. Statement of the result. We focus in this paper for sake of simplicity onto the four dimensional focusing semilinear heat equation

$$\partial_t u = \Delta u + |u|^{p-1}u, \quad x \in \mathbb{R}^4 \quad (1.6)$$

in the energy super critical zone

$$p > 5.$$  

We will work in a symmetry class. We decompose $x \in \mathbb{R}^4$ as

$$x = (x', z) \in \mathbb{R}^3 \times \mathbb{R}, \quad r = |x'|$$

and we consider functions $f$ on $\mathbb{R}^4$ which have cylindrical symmetry and are even with respect to $z$, i.e.

$$f(x) = f(r, z), \quad f(r, -z) = f(r, z).$$

We call this symmetry even cylindrical symmetry. Since for any rotation matrix $R$ of $\mathbb{R}^3$ the transformations $u(t, x', z) \to u(t, Rx', z)$ and $u(t, x', z) \to u(t, x', -z)$ map a solution to (1.6) onto another solution to (1.6), uniqueness provided by the Cauchy theory ensures that the even cylindrical symmetry is propagated by the flow.
Since \( p > 5 \), \( p_{LL} = +\infty \) and the only known blow up bubbles correspond to type I blow up bubbles with either the ODE or non trivial self similar profile. Let us consider \( \Phi(r) \) such a 3-dimensional self similar solution, i.e. a solution to the shooting problem

\[
\begin{align*}
\Phi'' + \frac{2}{r} \Phi' - \frac{1}{2} \left( \frac{2}{p-1} \Phi + r \Phi' \right) + \Phi^p &= 0, \\
\Phi'(0) &= 0, \\
\lim_{r \to +\infty} \Phi(r) &= 0.
\end{align*}
\]

(1.7)

A large class of such solutions has been constructed using either a direct Lyapunov functional approach [13, 26, 2, 3] or a bifurcation argument [4, 6], and these solutions satisfy

\[
\begin{align*}
\Phi(r) &\gtrsim \frac{1}{(r)^{p-1}}, \\
\Phi &\in C^\infty(0, +\infty), \quad |\partial_r^k \Phi| \lesssim_k \frac{1}{(r)^{p-1}}, \quad k \in \mathbb{N},
\end{align*}
\]

(1.8)

where \( (r) = \sqrt{1 + r^2} \).

Let \( Y = (y, z) \in \mathbb{R}^3 \times \mathbb{R}, \ r = |y| \), then the linearized operator close to \( \Phi \) for the four dimensional self similar equation corresponding to (1.6) is given by

\[
L_Y = -\Delta_Y + \frac{1}{2} \left( \frac{2}{p-1} + Y \cdot \nabla \right) - p\Phi^{p-1}.
\]

From standard argument, this operator is self adjoint for the measure \( e^{-\frac{|Y|^2}{4}} dY \) with compact resolvent and hence diagonalizable with a finite number of non positive eigenvalues. Note that \((z^2 - 2)\Lambda \Phi(r)\) belongs to the kernel of \( L_Y \), see Remark 2.4.

We shall assume the following non degeneracy condition which we expect is the generic case

\[
\text{Ker}(L_Y) = \langle (z^2 - 2)\Lambda \Phi(r) \rangle \quad \text{for functions with even cylindrical symmetry}. \quad (1.9)
\]

We shall comment on (1.9) in section 2.2, Remark 2.6.

Our main result in this paper is that such a 3-dimensional self similar solution is transversally stable modulo a finite number of instability directions.

**Theorem 1.1 (Finite codimensional transversal stability of self similar blow up).** Let \( \Phi(r) \) solve (1.7), (1.8) and assume that the non degeneracy condition (1.9) is fulfilled. Then, there exists a finite codimensional smooth manifold of initial data \( u_0 \) with even cylindrical symmetry and finite energy satisfying (3.15) (3.16) such that the corresponding solution to (1.6) blows up in finite time \( T < +\infty \) with the following sharp description of the singularity. For \( t \) close enough to \( T \), the solution decomposes in self similar variables

\[
u(t, x) = \frac{1}{(T - t)^\frac{2}{p-1}} U(t, Y), \quad Y = \frac{x}{\sqrt{T - t}}
\]

as

\[
U(t, Y) = \frac{1}{(1 + b(t)z^2)^\frac{p-1}{p}} \Phi \left( \frac{r}{\sqrt{1 + b(t)z^2}} \right) + v(t, Y), \quad Y = (r, z)
\]

with

\[
\lim_{t \to T} ||v(t, \cdot)||_{L^\infty} = 0,
\]
and the free boundary moves at the speed
\[ \frac{1}{\sqrt{b(t)}} = c^*(1 + o(1)) \sqrt{\log(T - t)}, \quad c^* = c^*(\Phi) > 0. \] (1.10)

Comments on the result

1. Moving free boundary. The main feature of Theorem 1.1 is to exhibit blow up solutions with strongly anisotropic blow up profiles. In particular the solution is almost constant in the boundary layer \( |z(t)| \lesssim \sqrt{\log(T - t)} \), and the heart of the proof is to precisely compute the boundary. In particular, the singularity still occurs at a point and not along the full \( z \) axis. The free boundary is computed by constructing the reconnecting profile which generalizes the construction in [1, 20] for the ODE profile, and showing its stability. Here the generic spectral assumption is important as it is likely that more zero modes in the kernel could generate a different boundary behavior.

2. \( L^\infty \) bounds. The main difficulty of the analysis is to control the perturbation in \( L^\infty \) in order to deal with the nonlinear term. Such estimates were derived for the ODE problem in [1] using explicit resolvent estimates for the linearized flow near the constant self similar solution, and in [21] using general Liouville type classification theorem. These approaches are not obvious to implement here due to the super critical nature of the problem, and the fact that there is no explicit formula for \( \Phi \). We will overcome this using new elementary \( W^{1,q} \) energy estimates, and a by product of our analysis is another proof of the stability of the ODE type I blow up using purely energy estimates. Let us also stress that the case of higher dimensions \( d \geq 4 \) as well as other models like the energy super critical heat flow could be treated along very similar lines.

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Notations. We let \( Y = (y, z) \in \mathbb{R}^3 \times \mathbb{R}, \quad r = |y| \) be the renormalized space variable. We let
\[ \Delta_r = \partial_r^2 + \frac{2}{r} \partial_r, \quad \Delta_Y = \Delta_r + \partial_z^2, \]
and the generator of scalings be
\[ \Lambda_r = \frac{2}{p - 1} \partial_r, \quad \Lambda_Y = \frac{2}{p - 1} + Y \cdot \nabla. \]
We define the weights
\[ \rho_r = e^{-r^2 \frac{2}{p - 1}}, \quad \rho_Y = e^{-\frac{|Y|^2}{4}}, \quad \rho_z = e^{-z^2 \frac{2}{p - 1}} \]
with associated weighted norm
\[ \|u\|_{L^2_{\rho_r}} = \int_{\mathbb{R}^4} |u|^2 \rho_r dY, \quad \|u\|_{L^2_{\rho_r}} = \int_{\mathbb{R}^4} |u(Y)|^2 \rho_Y dY. \]
We say a function \( u(Y) \) has even cylindrical symmetry if
\[ u(Y) = u(r, z) = u(r, -z) \]
and denote
\[ L^2_{\rho_Y} \]
the associated Hilbert space. We let $\Phi(r)$ be a three dimensional self similar solution
\[ \Delta_r \Phi - \frac{1}{2} \Lambda_r \Phi + \Phi^p = 0 \] (1.11)
satisfying (1.8) as build in [6]. We define for $m \in \mathbb{N}$ the $m$-th one dimensional Hermite polynomial
\[ P_m(z) = \sum_{k=0}^{[m/2]} \frac{m!}{k!(m-2k)!} (-1)^k z^{m-2k} \] (1.12)
which satisfy
\[ \int_{\mathbb{R}} P_m P_{m'} \rho_z dz = \sqrt{\pi} 2^{m+1} m! \delta_{mm'} . \]
We let
\[ s_c = \frac{3}{2} - \frac{2}{p-1}, \quad S_c = 2 - \frac{2}{p-1}, \]
where $s_c$ is the 3d critical exponent and $S_c$ is the 4d critical exponent. Also, we let
\[ \langle x \rangle = \sqrt{1 + |x|^2} . \]

2. Approximate solution in the boundary layer

2.1. Reconnecting profiles. Consider the renormalization
\[ u(t,x) = \frac{1}{\lambda(t)^{2/p-1}} U(s,Y), \quad \frac{ds}{dt} = \frac{1}{\lambda^2}, \quad Y = \frac{x}{\lambda(t)} \]
which maps (1.6) onto
\[ \partial_s U = \Delta_Y U + \frac{\lambda_s}{\lambda} \Lambda_Y U + U^p . \] (2.1)
For the self similar choice
\[ \frac{\lambda_s}{\lambda} = \frac{1}{2}, \]
an exact solution is given by $U(Y) = \Phi(r)$, but this solution does not decay along the $z$ direction. A better approximate solution decaying as $|Y| \to +\infty$ can be constructed by generalizing the approach in [1, 20].

**Lemma 2.1** (Reconnecting profiles). For all $b > 0$,
\[ \Phi_b(r,z) = \frac{1}{\mu_b(z)^{2/p-1}} \Phi \left( \frac{r}{\mu_b(z)} \right) \quad \text{with} \quad \mu_b(z) = \sqrt{1 + bz^2} \] (2.2)
solves
\[ \frac{1}{2} z \partial_z \Phi_b = \Delta_r \Phi_b - \frac{1}{2} \Lambda_r \Phi_b + \Phi_b^p . \] (2.3)

**Proof.** On the one hand, we have
\[ \frac{1}{2} z \partial_z \Phi_b = -\frac{1}{2} \frac{b z^2}{\mu_b^{p-2}} \Lambda_r \Phi \left( \frac{r}{\mu_b} \right) , \]
and on the other hand, we have
\[ \Delta_r \Phi_b - \frac{1}{2} \Lambda_r \Phi_b + \Phi^p_b = \frac{1}{\mu_b^{p+2}} \left( \Delta_r \Phi - \frac{\mu_b^2}{2} \Lambda_r \Phi + \Phi^p \right) \left( \frac{r}{\mu_b} \right) \]
\[ = \frac{1}{\mu_b^{p+2}} \frac{1}{2} (1 - \mu_b^2) \Lambda_r \Phi \left( \frac{r}{\mu_b} \right) \]
\[ = -\frac{1}{2} \frac{b^2}{\mu_b^{p+2}} \Lambda_r \Phi \left( \frac{r}{\mu_b} \right), \]
where we used (1.11). This concludes the proof of the lemma.

2.2. Diagonalization of the linearized operator close to \( \Phi \). Let the 4 dimensional linearized operator close to \( \Phi \):
\[ L_Y = -\Delta Y + \frac{1}{2} \Lambda Y - p \Phi^p - 1, \]
then \( L_Y \) is self adjoint on a domain \( \mathcal{D}(L_Y) \subset L^2_{\rho_Y}(\mathbb{R}^4) \) and with compact resolvent. Let the 3 dimensional radial operator
\[ L_r = -\Delta_r + \frac{1}{2} \Lambda_r - p \Phi^p - 1 \]
which is self adjoint on a domain \( \mathcal{D}(L_r) \subset L^2(r^2 \rho_r dr) \) with compact resolvent and spectrum determined in [6]:

**Lemma 2.2** (Spectrum for \( L_r \) in weighted spaces, [6]). The spectrum of \( L_r \) with domain \( \mathcal{D}(L_r) \subset L^2(r^2 \rho_r dr) \) is given by
\[ \lambda_{-\ell_0} < \cdots < \lambda_{-1} = -1 < 0 < \lambda_0 < \lambda_1 < \cdots \]
for some integer \( \ell_0 \geq 1 \) with
\[ \lambda_j > 0 \text{ for all } j \geq 0 \text{ and } \lim_{j \to +\infty} \lambda_j = +\infty. \] (2.4)
The eigenvalues \( (\lambda_j)_{-\ell_0 \leq j \leq -1} \) are simple and associated to spherically symmetric eigenvectors
\[ \psi_j(r), \quad \|\psi_j\|_{L^2(r^2 \rho_r dr)} = 1, \quad \psi_{-1} = \frac{\Lambda_r \Phi}{\|\Lambda_r \Phi\|_{L^2(r^2 \rho_r dr)}}. \] (2.5)
Moreover, there holds the bound as \( r = |y| \to +\infty \)
\[ |\partial_k \psi_j(r)| \lesssim (1 + r)^{-\frac{2}{p-1} - \lambda_j - k}, \quad -\ell_0 \leq j \leq -1, \quad k \geq 0. \] (2.6)

We may now diagonalize the full operator \( L_Y \) for function with cylindrical symmetry using a standard separation of variables claim and the tensorial structure of \( L_Y \).

**Lemma 2.3** (Spectrum for \( L_Y \) in weighted spaces with cylindrical symmetry). The spectrum of \( L_Y \) restricted to functions of cylindrical symmetry with domain \( \mathcal{D}(L_Y) \subset L^2_{\rho_Y}(\mathbb{R}^4) \) is given by
\[ \mu_{j,m} = \lambda_j + \frac{m}{2}, \quad j \in [-\ell_0, +\infty), \quad m \in \mathbb{N} \]
with eigenfunction
\[ \phi_{j,m}(Y) = \psi_j(r)P_m(z) \] (2.7)
where \( P_m(z) \) is the \( m \)-th one dimensional Hermite polynomial (1.12) and \( \psi_j \) denote the eigenvectors of \( \mathcal{L}_r \). In particular, for \(-\ell_0 \leq j \leq -1\), let \( m(j) \) be the smallest integer such that
\[
\frac{m(j) + 1}{2} + \lambda_j > 0,
\]
then there holds the spectral gap estimate: \( \forall \varepsilon \in H^1_{\rho Y}, \)
\[
(\mathcal{L}_Y \varepsilon, \varepsilon)_{L^2_{\rho Y}} \geq c \| \varepsilon \|^2_{H^1_{\rho Y}} - \sum_{j=-\ell_0}^{-1} \sum_{m=0}^{m(j)} (\varepsilon, \phi_{j,m})^2_{L^2_{\rho Y}}
\]
for some universal constant \( c > 0 \).

**Remark 2.4.** In particular \( \mu_{-1,2} = -1 + \frac{2}{2} = 0 \), and hence there is always a zero eigenmode. In view of (2.7) for \( j = -1 \) and \( m = 2 \), formula (2.5) for \( \psi_{-1} \) and formula (1.12) for \( P_2 \), the corresponding eigenvector is given by
\[
(z^2 - 2)\lambda \Phi.
\]

**Proof.** This is a standard claim based on separation of variables. We compute
\[
\mathcal{L}_Y(\psi(r)P_m(z)) = P_m(z) \left[ -\Delta_r + \frac{1}{2} \Lambda_r - \rho \Phi^{-1} \right] \psi(r) + \psi(r) \left[ -\partial_{zz} + \frac{1}{2} z \partial_z \right] P_m(z)
\]
and hence for an eigenfunction \( \mathcal{L}_r \psi_j = \lambda_j \psi_j: \)
\[
\mathcal{L}_Y(\psi_j(r)P_m(z)) = \psi_j(r) \left[ -\partial_{zz} + \frac{1}{2} z \partial_z + \lambda_j \right] P_m(z) = \psi_j(r) \left[ \frac{m}{2} + \lambda_j \right] P_m(z),
\]
where we used the fact that the one dimensional harmonic oscillator
\[
-\partial_z^2 + \frac{1}{2} z \partial_z
\]
has spectrum \( \frac{m}{2}, m \in \mathbb{N} \) on \( L^2_{\rho Y} \), with eigenfunctions given by the \( m \)-th Hermite polynomial \( P_m(z) \). It remains to observe that \( \psi_j(r)P_m(z) \) is a dense family of the cylindrically symmetric functions of \( L^2_{\rho Y} (\mathbb{R}^4) \) from standard tensorial claims to conclude that it forms a Hilbertian basis of eigenvectors. The spectral gap estimate (2.9) then follows by decomposition of the self adjoint operator \( \mathcal{L}_Y \) in the Hilbertian basis \( \phi_{j,m} \).

Under the additional assumption of even cylindrical symmetry and the fact that \( P_{2m} \) is an even polynomial while \( P_{2m+1} \) is an odd polynomial for all \( m \in \mathbb{N} \) from (1.12), we obtain as a direct consequence of Lemma 2.3:

**Lemma 2.5** (Spectrum for \( \mathcal{L}_Y \) in weighted spaces with even cylindrical symmetry). The spectrum of \( \mathcal{L}_Y \) with domain \( \mathcal{D}(\mathcal{L}_Y) \subset L^2_{\rho Y} (\mathbb{R}^4) \) is given by
\[
\mu_{j,2M} = \lambda_j + M, \quad j \in [-\ell_0, +\infty), \quad M \in \mathbb{N}
\]
with eigenfunction
\[
\phi_{j,2M}(Y) = \psi_j(y)P_{2M}(z)
\]
where \( P_m(z) \) is the \( m \)-th one dimensional Hermite polynomial (1.12). In particular, for \(-\ell_0 \leq j \leq -1\), let \( M(j) \) be the smallest integer such that
\[
M(j) + 1 + \lambda_j > 0,
\]
then there holds the spectral gap estimate: \( \forall \varepsilon \in H_{\rho Y}^{1,e}, \)

\[
(\mathcal{L}\varepsilon, \varepsilon)_{L^2_{\rho Y}} \geq c\|\varepsilon\|_{H_{\rho Y}^{1,e}}^2 - \frac{1}{c} \sum_{j=-\ell_0}^{M(\beta)} \sum_{M=0}^{(\beta)} (\varepsilon, \phi_j, M)_{L^2_{\rho Y}}^2
\]

for some universal constant \( c > 0. \)

**Remark 2.6.** In view of (2.9), when restricted to even cylindrical functions, the non degeneracy condition (1.9) is equivalent to

\[
\forall j \in \{-\ell_0, \ldots, -2\}, \quad \lambda_j \notin \mathbb{Z}.
\]

Note that such a condition would typically be fulfilled for the minimizing self similar solution of the energy super critical heat flow (for which \( \lambda_{-1} = -1 \) is the bottom of the spectrum of \( \mathcal{L}_r \)). More generally, this condition can be easily checked numerically by computing \( \lambda_j \) for \( j = -\ell_0, \ldots, -2 \) as in [4] and we expect it to be generic.

### 2.3. The high order approximate solution in the boundary layer.

Let us consider again the renormalized flow (2.1). The choice

\[
\left( \lambda_s = -\frac{1}{2}, U(Y) = \Phi_b(Y), b(s) = b > 0 \right)
\]

yields an \( O(b) \) approximate solution in the boundary layer \(|z| \lesssim \frac{1}{\sqrt{b}} \). We aim at improving this error and construct a high order approximate solution for \(|z| \ll \frac{1}{\sqrt{b}} \), which will be the key to the control of the flow in \( L^\infty \).

Let us indeed pick a smooth mapping \( s \mapsto b(s) \) with \( 0 < b(s) \ll 1 \) and look for a solution to (2.1) of the form

\[
U(s, Y) = \Phi_{b(s)}(Y) + v(s, Y)
\]

which together with (2.3) yields:

\[
\partial_s v + \mathcal{L}_{\rho Y} v = \partial_s^2 \Phi_b - \partial_s \Phi_b + \left( \frac{\lambda_s}{\lambda} + \frac{1}{2} \right) (\Lambda_Y \Phi_b + \Lambda_Y v) + F(v)
\]

where

\[
F(v) = (\Phi_{b} + v)^p - \Phi_{b}^p - p\Phi_{b}^{p-1}v + p(\Phi_{b}^{p-1} - \Phi^{p-1})v.
\]

We shall solve an approximate version of (2.11). First let

\[
Z = \sqrt{b}Z
\]

and

\[
\Phi_b(r, z) = G(r, Z), \quad G(r, Z) = \frac{1}{\mu(Z)^{p-1}} \Phi \left( \frac{r}{\mu(Z)} \right), \quad \mu(Z) = \sqrt{1 + Z^2}.
\]

In order to construct an approximate solution, we anticipate the laws

\[
b_s = -bB(b), \quad \frac{\lambda_s}{\lambda} + \frac{1}{2} = M(b)
\]

and look for a solution of the form

\[
v_{b(s)}(s, r, z) = V_{b(s)}(r, Z)
\]

so that

\[
\partial_s v = -B(b) \left[ b\partial_b + \frac{1}{2} Z\partial_Z \right] V, \quad \partial_s^2 v = b\partial_Z^2 V, \quad z\partial_z v = Z\partial_Z V
\]
and (2.11) becomes:
\[
\left(\mathcal{L}_r + \frac{1}{2} Z \partial_Z\right) V = b \partial_Z^2 (G + V) + B(b) \left(\frac{1}{2} Z \partial_Z G + \frac{1}{2} Z \partial_Z V + b \partial_b V\right) \\
+ M(b)(\Lambda_r + Z \partial_Z)(G + V) + \tilde{F}(V),
\]
where $\tilde{F}(V)$ is defined by
\[
\tilde{F}(V) = (G + V)^p - G^p - p G^{p-1} V + p(G^{p-1} - \Phi^{p-1}) V.
\]
Given $0 < \delta \ll 1$, we let:
\[
\Omega_\delta = \{|Z| \leq \delta\},
\]
and construct an arbitrarily high order approximate solution in $\Omega_\delta$ using an elementary Hilbert expansion.

**Lemma 2.7** (High order approximate solution). *Let $n \in \mathbb{N}^*$ such that $n \geq p$. Then for all $0 < \delta < \delta(n) \ll 1$ and $0 < b < b(n) \ll 1$ small enough, there exist*
\[
V_b(r, Z) = \sum_{i=1}^{n} \sum_{j=0}^{n} b^i V_{i,j}(r) Z^{2j}, \quad B(b) = \sum_{i=1}^{n} c_i b^i, \quad M(b) = \sum_{i=1}^{n} d_i b^i
\]
*(2.15)*

*where*
\[
|\partial^k_r V_{i,j}| \lesssim_{n,k} \frac{1}{(r)^{\frac{1}{p-1} - \frac{1}{2} + k}}, \quad k \in \mathbb{N}
\]
*(2.16)*

*such that*
\[
(V_{i,0}, \Lambda_r \Phi)_{L^2_{r\rho}} = (V_{i,1}, \Lambda_r \Phi)_{L^2_{r\rho}} = 0, \quad 1 \leq i \leq n,
\]
*(2.17)*

*and*
\[
\Psi_b = \left(\mathcal{L}_r + \frac{1}{2} Z \partial_Z\right) V_b - b \partial_Z^2 (G + V_b) - F(V_b) \\
- B(b) \left(\frac{1}{2} Z \partial_Z (G + V_b) + b \partial_b V_b\right) - M(b)(\Lambda_r + Z \partial_Z)(G + V_b)
\]
*satisfies*
\[
\forall Z \in \Omega_\delta, \quad |\partial^j_r \partial^k_Z \Psi_b| \lesssim_{n} \frac{b^{n+1} + b|Z|^{2n+2-k}}{(r)^{\frac{1}{p-1} - \frac{1}{2} + j}} , \quad 0 \leq j + k \leq 2.
\]
*(2.19)*

*Moreover, there holds for the first terms:*
\[
V_{1,0} = 0
\]
*(2.20)*

*and*
\[
c_1 = 2(2 - s_c) + \frac{\|r \Lambda \Phi\|_{L^2_{r\rho}}^2}{2\|\Lambda \Phi\|_{L^2_{r\rho}}^2}, \quad d_1 = 1.
\]
*(2.21)*

**Remark 2.8.** *The law (2.13), (2.21) written in the setting of the ODE type I blow up $\Phi = \left(\frac{1}{p-1}\right)^{\frac{1}{p-1}}$ yields the leading order $b$ law*
\[
b_n + \frac{4p}{p-1} b^2 = 0
\]
*which is the frontier boundary computed in [1, 20].*
**Proof.** The proof follows by a brute force expansion.

**step 1** Taylor expansion near in $\Omega_\delta$. Recall the uniform bound

$$\frac{1}{\langle r \rangle^{p+1}} \lesssim \Phi(r) \lesssim \frac{1}{\langle r \rangle^{p+1}}$$

and

$$\forall k \geq 1, \ |\Lambda^k \Phi| \lesssim_k \frac{1}{\langle r \rangle^{p+1} + 2}.$$ \hspace{1cm} (2.22)

Moreover, we compute

$$\partial_Z G = -\frac{\mu'}{\mu} \frac{1}{\langle r \rangle^{p+1}} \Lambda_r \Phi \left( \frac{r}{\mu(Z)} \right)$$

and a simple induction argument based on (2.23) ensures for $k \geq 1$ the bound:

$$\forall |Z| \leq \delta, \ |\partial_Z^k G(r, Z)| \lesssim_k \sum_{j=1}^{2k} |\Lambda_j^2 \Phi| \lesssim \frac{1}{\langle r \rangle^{p+1} + 2}.$$ \hspace{1cm} (2.23)

In particular,

$$\left| \frac{\partial Z^k G}{\Phi} \right| \lesssim_k \frac{1}{\langle r \rangle^{2}}, \ k \geq 1.$$ \hspace{1cm} (2.24)

We may therefore replace $G$ by its Taylor expansion at the origin

$$G(r, Z) = G_n(r, 0) + \frac{Z^{2n+2}}{(2n+1)!} \mathcal{I}_0^1 (1 - \tau)^{2n+1} \partial_Z^{2n+2} G(r, \tau Z) d\tau,$$

$$G_n(r, Z) = \sum_{k=0}^{n} \frac{\partial_Z^k G(r, 0)}{(2k)!} Z^{2k}$$

with for $|Z| \leq \delta$,

$$G_n(r, Z) = \Phi(r) \left[ 1 + \sum_{k=1}^{n} Z^{2k} F_k(r) \right], \quad F_k(r) \lesssim \frac{1}{\langle r \rangle^2}$$ \hspace{1cm} (2.25)

and

$$|G - G_n| \lesssim_n \frac{Z^{2n+2}}{\langle r \rangle^{2} + 2}.$$ \hspace{1cm} (2.26)

Next, let $\mu : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ a smooth cut-off function such that

$$\mu = 1 \text{ on } 0 \leq s \leq 1 \text{ and } \mu = 0 \text{ on } s \geq 2,$$

and let $\mu_b$ be defined by

$$\mu_b(r) = \mu(b^nr).$$

Note that for $|Z| \leq \delta$ and $\delta$ small enough, we have

$$\frac{1}{b(r)^\frac{1}{n}} \frac{|V_b|}{|G_n|} \lesssim \frac{1}{b(r)^\frac{1}{n}} \frac{|V_b|}{\Phi} \lesssim \sum_{i=1}^{n} \sum_{j=0}^{n} b^{i-1} \delta^j \langle r \rangle^{\frac{1}{n} + \frac{2}{p+1}} |V_{i,j}| \lesssim b + \delta$$
where we anticipated on (2.20). For \( b \) and \( \delta \) small enough, we infer
\[
|V_b| |G_n| \leq \frac{1}{2}
\] on the support of \( \mu_b \).

We now Taylor expand the nonlinearity using
\[
(1 + x)^p - 1 - px^{p-1} = \sum_{k=2}^{2n+1} a_k x^k + O(x^{2n+2}), \quad |x| \leq \frac{1}{2}
\]
which yields
\[
\mu_b(r) \left( (G_n + V_b)^p - G_n^p - pG_n^{p-1}V_b \right) = \mu_b(r) \left( \sum_{k=2}^{2n+1} a_k V_b^k G_n^{p-k} + O(V_b^{2n+2} G_n^{-(2n+2)}) \right)
\]

Also, from (2.25): \( \forall \alpha \in \mathbb{Z} \):
\[
G_n^\alpha = \Phi(r)^\alpha \left( 1 + \sum_{j=1}^{n} Z^{2j} F_j(r) \right) = \Phi(r)^\alpha \left[ 1 + \sum_{j=1}^{n} Z^{2j} H_{\alpha,j}(r) + O(Z^{2n+2}) \right]
\]
with
\[
|\partial_r^k H_{\alpha,j}(r)| \lesssim_k \frac{1}{(r)^{2+k}}.
\]

Thus, we decompose
\[
\Psi_b = \Psi_b^{(1)} + \Psi_b^{(2)}
\]
where
\[
\Psi_b^{(1)} = \left( \mathcal{L} + \frac{1}{2} Z \partial_Z \right) V_b - b \partial_Z^2 (G_n + V_b)
\]
\[
- \mu_b(r) \sum_{k=2}^{2n+1} a_k \left( \frac{V_b}{\Phi} \right)^k \Phi^p \left[ 1 + \sum_{j=1}^{n} Z^{2j} H_{p-k,j}(r) \right] - p\Phi^{p-1} V_b \sum_{j=1}^{n} Z^{2j} H_{p-1,j}(r)
\]
\[
- B(b) \left[ \frac{1}{2} Z \partial_Z (G_n + V_b) + b \partial_b V_b \right] - M(b)(\Lambda_r + Z \partial_Z)(G_n + V_b).
\]

and
\[
\Psi_b^{(2)} = -b \partial_Z^2 (G - G_n) - (1 - \mu_b(r)) \left( (G + V_b)^p - G^p - pG^{p-1}V_b \right)
\]
\[
- \mu_b(r) \left\{ (G + V_b)^p - G^p - pG^{p-1}V_b - \sum_{k=2}^{2n+1} a_k \left( \frac{V_b}{\Phi} \right)^k \Phi^p \left[ 1 + \sum_{j=1}^{n} Z^{2j} H_{p-k,j}(r) \right] \right\}
\]
\[
- pG^{p-1}V_b + p\Phi^{p-1} V_b \left[ 1 + \sum_{j=1}^{n} Z^{2j} H_{p-1,j}(r) \right]
\]
\[
- B(b) \frac{1}{2} Z \partial_Z (G - G_n) - M(b)(\Lambda_r + Z \partial_Z)(G - G_n).
\]

**Step 2** Solving the approximate problem. We solve (2.31) up to an error of order \( Z^{2n+2} \) or \( b^{p+1} \) by looking for a solution of the form
\[
V_b(r, Z) = \sum_{i=1}^{n} \sum_{j=0}^{n} b^i V_{i,j}(r) Z^{2j}, \quad B(b) = \sum_{i=1}^{n} c_i b^i, \quad M(b) = \sum_{i=1}^{n} d_i b^i.
\]
Since the polynomial dependance in both $b$ and $Z$ is preserved by the RHS of (2.31), we sort the terms in $b^i Z^{2j}$ and obtain a hierarchy of equations of the following form for $1 \leq i \leq n$, $0 \leq j \leq n$

\[
\left[ \mathcal{L}_r + \frac{1}{2} Z \partial Z \right] (V_{i,j}(r) Z^{2j}) = F_{i,j}(r) Z^{2j} + Z^{2j} \frac{d_i \Lambda \Phi}{2(2j-1)!!} \partial_{Z}^{2j} G(r,0) + \frac{d_i}{(2j)!!} [\Lambda_r + 2j] \partial_{Z}^{2j} G(r,0)
\]

or equivalently:

\[
[\mathcal{L}_r + j] V_{i,j}(r) = F_{i,j}(r) + \frac{d_i \Lambda \Phi}{2(2j-1)!!} \partial_{Z}^{2j} G(r,0) + \frac{d_i}{(2j)!!} [\Lambda_r + 2j] \partial_{Z}^{2j} G(r,0)
\]

where $F_{i,j}$ depends only on $V_{i,j}$ with $i' \leq i$, $j' \leq j$ and $(i',j') \neq (i,j)$, and on $d_{i'}$ and $c_{i'}$ with $i' < i$. Moreover, a fundamental observation is that the decay (2.16) is preserved by the forcing term (2.31), i.e.

\[
|\partial_r^k F_{i,j}(r)| \lesssim_n \frac{1}{\langle r \rangle^{\frac{\alpha}{\alpha_r} + k - \frac{1}{2}}}
\]

where we used in particular the fact that for $2 \leq k \leq 2n + 1$, we have

\[
\langle r \rangle^{\frac{2}{\alpha_r} - \frac{1}{\alpha}} \left( \frac{|V_i|}{\Phi} \right)^k \Phi^p \lesssim \langle r \rangle^{\frac{2}{\alpha_r} - \frac{1}{2}} \left( \langle r \rangle \right)^k \left( \frac{1}{\langle r \rangle^{2\gamma}} \right)^p \lesssim \langle r \rangle^{\frac{k-1}{2} - 2} \lesssim 1.
\]

In order to invert (2.33), we will rely on the following lemma which is proved in Appendix B.

**Lemma 2.9.** Let $j \in \mathbb{N}$, and let $u_j(r)$ the solution to

\[(\mathcal{L}_r + j) u = f_j \text{ and } (u_1, \Lambda_r \Phi) = 0 \text{ if } j = 1.
\]

Furthermore, assume that we have in the case $j = 1$

\[(f_1, \Lambda_r \Phi)_{L^2_{\rho_r}} = 0.
\]

Then, for $\eta > 0$ and $k \in \mathbb{N}$, $u$ satisfies the following bound

\[
\sum_{l=0}^{k} \left\| \langle r \rangle^{\frac{2}{\alpha_r} + l - \eta} \partial_r^{l} u_l \right\|_{L^\infty} \lesssim_{k, \eta} \sum_{l=0}^{k} \left\| \langle r \rangle^{\frac{2}{\alpha_r} + l - \eta} \partial_r^{l} f_j \right\|_{L^\infty}.
\]

We may now come back to (2.33). We consider first the case $j = 0$, then $j = 1$, and finally $j \geq 2$.

- We have for $j = 0$

\[
\mathcal{L}_r V_{i,0}(r) = F_{i,0}(r) + d_i \Lambda \Phi
\]

and hence, in view of Lemma 2.9, the exists a unique $V_{i,0}$ which in view of the above estimate for $F_{i,j}$ satisfies

\[
|\partial_r^k V_{i,0}(r)| \lesssim_n \frac{1}{\langle r \rangle^{\frac{\alpha}{\alpha_r} + k - \frac{1}{2}}}
\]

Furthermore, projecting on $\Lambda_r \Phi$ and using the fact that $\mathcal{L}_r (\Lambda_r \Phi) = -\Phi$, we have

\[
-(V_{i,0}, \Lambda_r \Phi)_{L^2_{\rho_r}} = (F_{i,0}, \Lambda_r \Phi)_{L^2_{\rho_r}} + d_i \left\| \Lambda \Phi \right\|_{L^2_{\rho_r}}^2
\]
and we choose $d_i$ to enforce
\[ (V_{i,0}, \Lambda_r \Phi)_{L^2_{r,v}} = 0. \] (2.34)

- Also, since $\partial^2_r G(r,0) = -\Lambda_r \Phi(r)$, we have for $j = 1$
\[ [L_r + 1] V_{i,1}(r) = F_{i,1}(r) - \frac{c_i}{2} \Lambda_r \Phi - \frac{d_i}{2} (\Lambda_r + 2) \Lambda_r \Phi. \]

We choose $c_i$ to enforce
\[ (F_{i,1}, \Lambda_r \Phi)_{L^2_{r,v}} - \frac{c_i}{2} \| \Phi \|^2_{L^2_{r,v}} - \frac{d_i}{2} (\Lambda_r + 2) \Lambda_r \Phi = 0. \] (2.35)

Thus, we may apply Lemma 2.9, and hence the exists a unique $V_{i,1}$ such that
\[ (V_{i,1}, \Lambda_r \Phi)_{L^2_{r,v}} = 0, \] (2.36)

and which in view of the above estimate for $F_{i,j}$ satisfies
\[ |\partial^k_r V_{i,1}(r)| \lesssim_n \frac{1}{\langle r \rangle^{\frac{2}{p-1} + k - \frac{1}{n}}}. \]

Note that (2.17) follows from (2.34) and (2.36).

- Finally, for $j \geq 2$, we may apply Lemma 2.9, and hence the exists a unique $V_{i,j}$ which in view of the above estimate for $F_{i,j}$ satisfies
\[ |\partial^k_r V_{i,j}(r)| \lesssim_n \frac{1}{\langle r \rangle^{\frac{2}{p-1} + k - \frac{1}{n}}}. \]

**step 3** Proof of the error estimate. We are now in position to prove the error estimate (2.19). As all terms of the type $b^i Z^{2j}$ for $1 \leq i \leq n$ and $0 \leq j \leq n$ in (2.31) vanish due to the choice of $V_{i,j}$, and in view of the estimates for $\Phi$, $G$, $H_{a,j}$, as well as the estimates of step 2 above for $V_{i,j}$, we infer
\[ \forall Z \in \Omega_5, \quad |\partial^k_r \partial^j_r \Psi_{b}^{(1)}| \lesssim_n \frac{b^{n+1} + b |Z|^{2n+2-k}}{\langle r \rangle^{\frac{2}{p-1} + \frac{1}{n} + j}} \quad \forall j, k. \]

Also, we have for $Z \in \Omega_5$ and $0 \leq j + k \leq 2$
\[ \left| \partial^j_r \partial^k_r \left( (G + V_b)^p - G^p - pG^{p-1} V_b \right) \right| \lesssim_n \frac{1}{\langle r \rangle^{\frac{2}{p-1} + \frac{1}{n} + j}}, \]

where we used the fact that $j + k \leq p$, since $p > 5$ and $j + k \leq 2$, which ensures that the above expression does not contain negative powers of $G + V_b$. In view of the support of $1 - \mu_b$, we deduce for $Z \in \Omega_5$ and $0 \leq j + k \leq 2$
\[ \left| \partial^j_r \partial^k_r \left( (1 - \mu_b) \left( (G + V_b)^p - G^p - pG^{p-1} V_b \right) \right) \right| \lesssim_n \frac{b^{n+1}}{\langle r \rangle^{\frac{2}{p-1} + \frac{1}{n} + j}} \]

where we used the fact that $n \geq p$ in the last inequality. The other terms of $\Psi_{b}^{(2)}$ defined in (2.32) are estimated using (2.26) (2.27) (2.28) (2.29) which leads to
\[ \forall Z \in \Omega_5, \quad |\partial^j_r \partial^k_r \Psi_{b}^{(2)}| \lesssim_n \frac{b^{n+1} + b |Z|^{2n+2-k}}{\langle r \rangle^{\frac{2}{p-1} + \frac{1}{n} + j}}, \quad 0 \leq j + k \leq 2. \]

In view of the decomposition (2.30) for $\Psi_{b}$, we immediately infer from the estimates for $\Psi_{b}^{(1)}$ and $\Psi_{b}^{(2)}$ the error estimate (2.19) for $\Psi_{b}$.
**step 4** Computation of $F_{1,0}$ and $F_{1,1}$. In view of the definition of $F_{i,j}$ in (2.33), we have

$$F_{1,0} + F_{1,1}Z^2 = \partial_2^2 G(r, Z) + p\Phi^{p-1}V_{1,0}Z^2H_{r-1,1}(r) + O(Z^4).$$

We compute the Taylor expansion

$$\partial_2^2 G = -\Lambda_r\Phi(r) + \frac{3}{2}Z^2(2\Lambda_r\Phi + \Lambda_r^2\Phi)(r) + O(Z^4), \tag{2.37}$$

which yields

$$F_{1,0} = -\Lambda_r\Phi(r), \quad F_{1,1} = \frac{3}{2}(2\Lambda_r\Phi + \Lambda_r^2\Phi)(r) + p\Phi^{p-1}V_{1,0}H_{r-1,1}(r). \tag{2.38}$$

**Proof of (2.37).** Recall that we have

$$G(r, Z) = \frac{1}{\mu(Z)^{\frac{2}{p-1}}} \Phi\left(\frac{r}{\mu(Z)}\right), \quad \mu(Z) = \sqrt{1 + Z^2}.$$

Then

$$\partial_2 G = -\frac{\mu'}{\mu} \frac{1}{\mu^{\frac{2}{p-1}}} \Lambda_r\Phi\left(\frac{r}{\mu(Z)}\right). \tag{2.39}$$

We further compute:

$$\partial_2^2 G = \frac{1}{(1 + Z^2)^{\frac{2}{p-1}} \mu^{\frac{2}{p-1}}} \left[(Z^2 - 1)\Lambda_r\Phi + Z^2\Lambda_r^2\Phi\right]\left(\frac{r}{\mu}\right). \tag{2.40}$$

We now Taylor expand at $Z = 0$ and obtain in particular using the uniform bound on $\Lambda_r^i\Phi(r), \ i = 1, 2, 3$:

$$\frac{1}{\mu^{\frac{2}{p-1}}} \Lambda_r\Phi\left(\frac{r}{\mu}\right) = \Lambda_r\Phi(r) - \frac{Z^2}{2}\Lambda_r^2\Phi(r) + O(Z^4) \tag{2.41}$$

which yields the Taylor expansion at the origin:

$$\partial_2^2 G = -\Lambda_r\Phi(r) + \frac{3}{2}Z^2(2\Lambda_r\Phi + \Lambda_r^2\Phi)(r) + O(Z^4).$$

This concludes the proof of (2.37).

**step 5** Computation of $V_{1,0}$, $d_1$ and $c_1$. From (2.33) for $j = 0$, we have

$$\mathcal{L}_r V_{1,0}(r) = F_{1,0}(r) + d_1 \Lambda_r\Phi$$

which together with (2.38) yields

$$\mathcal{L}_r V_{1,0}(r) = (d_1 - 1)\Lambda_r\Phi.$$

Since we choose $d_1$ to enforce the orthogonality (2.34), we immediately deduce

$$V_{1,0} = 0, \quad d_1 = 1,$$

which proves in particular (2.20).

Next, recall from (2.35) that we choose $c_1$ to enforce

$$(F_{1,1}, \Lambda_r\Phi)_{L^2_{\nu r}} - \frac{c_1}{2} \|\Lambda_r\Phi\|^2_{L^2_{\nu r}} - \frac{d_1}{2} \left((\Lambda_r + 2)\Lambda_r\Phi, \Lambda_r\Phi\right)_{L^2_{\nu r}} = 0$$
which together with (2.38) and the computation of \( V_{1,0} \) and \( d_1 \) above yields

\[
c_1 = 4 + \frac{2(\Lambda^2 \Phi, \Lambda \Phi)_{L^2_{2r}}}{\|\Lambda \Phi\|_{L^2_{2r}}^2} = 2(2 - s_c) + \frac{\|r\Lambda \Phi\|_{L^2_{2r}}^2}{2\|\Lambda \Phi\|_{L^2_{2r}}^2}.
\]

where we used in the last inequality the following computation

\[
(\Lambda f, f)_{L^2_{2r}} = -s_c\|f\|_{L^2_{2r}}^2 + \frac{1}{4}\|rf\|_{L^2_{2r}}^2, \quad s_c = \frac{3}{2} - \frac{2}{p - 1}.
\]

This finishes the proof of (2.21) and hence of Lemma 2.7. \( \square \)

**Lemma 2.10** (High order localized approximate solution). Let \( n \in \mathbb{N}^* \) such that \( n \geq p \). For \( 0 < \delta < \delta(n) \ll 1 \) and \( 0 < b < b(n) \ll 1 \) small enough, let \((V_b, B(b), M(b))\) be the approximate solution given by Lemma 2.7. Let an even cut off function

\[
\chi_\delta(z) = \chi \left( \frac{Z}{\delta} \right), \quad \chi(\sigma) = \begin{cases} 1 & \text{for } |\sigma| \leq 1, \\ 0 & \text{for } |\sigma| \geq 2 \end{cases}
\]

and let

\[
\tilde{\Phi}_b = \Phi_b + \tilde{v}_b \text{ where } \tilde{v}_b = \chi_\delta v_b \text{ and } v_b(z) = V_b(Z).
\]

Then, \( \tilde{\Phi}_b \) satisfies

\[
-bB(b)\partial_b \tilde{\Phi}_b + \left( \frac{1}{2} - M(b) \right) \Lambda Y \tilde{\Phi}_b - \Delta \tilde{\Phi}_b + \tilde{\Phi}_b^p = \tilde{\Psi}_b
\]

where

\[
\tilde{\Psi}_b = b(\chi_\delta - 1)\partial_Z^2 G + B(b)(\chi_\delta - 1)Z\partial_Z G + \tilde{\Psi}_b^{(0)}
\]

and where \( \tilde{\Psi}_b^{(0)} \) is estimated by

\[
|\partial_j^p \partial_Z^k \tilde{\Psi}_b^{(0)}| \lesssim \delta^{n+1} + b|Z|^{2n+2-k} \frac{C}{(r)^{p-1} - 1} 1_{|Z| \leq \delta}, \quad 0 \leq j + k \leq 2.
\]

Furthermore, \( \tilde{\Phi}_b \) satisfies also

\[
|\tilde{\Phi}_b|_{b=0} = \Phi, \quad \frac{\partial \tilde{\Phi}_b}{\partial b}_{|b=0} = -\frac{1}{2}(P_2 + 2R_1)(z)\Lambda_r \Phi.
\]

**Proof.** Since \( v_b(z) = V_b(Z) \), and in view of the equation (2.18) satisfied by \( V_b \), we infer

\[
\mathcal{L}_Y v_b = \partial^2_\delta \Phi_b + b B(b)(\partial_b \Phi_b + \partial_b v_b) + M(b)(\Lambda Y \Phi_b + \Lambda Y v_b) + F(v_b) + \Psi_b
\]

with \( \Psi_b \) satisfying (2.19). Since \( \tilde{v}_b = \chi_\delta v_b \), we infer

\[
\mathcal{L}_Y \tilde{v}_b = \chi_\delta \partial^2_\delta \Phi_b + b B(b)(\chi_\delta \partial_b \Phi_b + \partial_b \tilde{v}_b) + M(b)(\Lambda Y \Phi_b + \Lambda Y \tilde{v}_b) + F(\tilde{v}_b) + \tilde{\Psi}_b^{(0)}
\]

with

\[
\tilde{\Psi}_b^{(0)} = \chi \left( \frac{Z}{\delta} \right) \Psi_b + \left[ \frac{1}{2} Z \chi' - \frac{b}{\delta^2} \chi'' - \frac{B(b)}{\delta} \chi' - M(b)Z \chi \right] \left( \frac{Z}{\delta} \right) V_b
\]

\[
- \frac{2b}{\delta} \chi' \left( \frac{Z}{\delta} \right) \partial_Z V_b + \left( 1 - \chi \left( \frac{Z}{\delta} \right) \right) G^p + \chi \left( \frac{Z}{\delta} \right) (G + V_b)^p - \left( G + \chi \left( \frac{Z}{\delta} \right) V_b \right)^p.
\]
where we have used the equation (2.3) for equality. Plugging (2.47), we infer

\[ \Phi_b \sim \Phi_b \sim \Phi_b, \]

which is (2.46). This concludes the proof of the lemma.

\[ \square \]

which together with (2.15), (2.20) yields

\[ \partial \phi^b(z) = V_b(z), \quad \partial_b \phi^b(z) = \partial_b V_b(z) + \frac{1}{2b} Z \partial Z V_b(z) \]

which is (2.44). Finally, we prove (2.46). We compute from (2.2):

\[ \Phi_b|_{b=0} = \Phi \]

and

\[ \frac{\partial \Phi_b}{\partial b} = -\frac{\partial b \mu_b}{\mu_b} \frac{1}{\mu_b} \Lambda_r \Phi \left( \frac{r}{\mu_b} \right) = -\frac{z^2}{2 \mu_b^2} \Lambda_r \Phi \left( \frac{r}{\mu_b} \right), \quad \mu_b = \sqrt{1 + b z^2}, \]

and hence

\[ \frac{\partial \Phi_b}{\partial b} |_{b=0} = -\frac{z^2}{2} \Lambda_r \Phi = -\frac{1}{2} (P_2 + 2 P_0)(z) \Lambda_r \Phi \]

where we used from (1.12):

\[ P_2(z) = z^2 - 2, \quad P_0(z) = 1. \]

Moreover, we have

\[ v_b(z) = V_b(z), \quad \partial_b v_b(z) = \partial_b V_b(z) + \frac{1}{2b} Z \partial Z V_b(z) \]

which together with (2.15), (2.20) yields

\[ (\partial_b \phi^b)|_{b=0} = (\partial_b \phi^b)|_{b=0} = 0. \]

Hence, we infer

\[ (\phi^b)|_{b=0} = \Phi, \quad \frac{\partial \phi^b}{\partial b} |_{b=0} = -\frac{1}{2} (P_2 + 2 P_0)(z) \Lambda_r \Phi \]

which is (2.46). This concludes the proof of the lemma. \[ \square \]

\[ ^1 \text{Recall that } Z = \sqrt{b z}. \]
3. The bootstrap argument

3.1. Setting of the bootstrap. We set up in this section the bootstrap analysis of the flow for a suitable set of finite energy initial data. The solution will be decomposed in a suitable geometrical way using by now standards arguments, see [15, 18].

Geometrical decomposition of the flow. We start by showing the existence of the suitable decomposition.

**Lemma 3.1** (Geometrical decomposition). There exists \( \hat{b} > 0 \) and \( \kappa > 0 \) small enough such if

\[
0 < \hat{b} \leq b \text{ and } \|w\|_{L^\infty} \leq \kappa,
\]

and

\[
u = \tilde{\Phi}_b + w,
\]

then \( u \) has a unique decomposition

\[
u = \frac{1}{\mu^{\frac{p-1}{2}}} \left( \tilde{\Phi}_b + \sum_{j=-\ell_0}^{-2} \sum_{M=0}^{M(j)} a_{j,M} \phi_{j,2M} + \varepsilon \left( \frac{x}{\mu} \right) \right),
\]

where \( \varepsilon \) satisfies the orthogonality conditions

\[
(\varepsilon, \phi_{j,2M})_{L^2_{\rho_Y}} = 0, \quad -\ell_0 \leq j \leq -1, \quad 0 \leq M \leq M(j),
\]

and with

\[
|\mu - 1| + |b - \hat{b}| + \sum_{j=-\ell_0}^{-2} \sum_{M=0}^{M(j)} |a_{j,M}| \lesssim \|w\|_{L^\infty}. \tag{3.1}
\]

Furthermore, for \( K \) such that

\[
K \geq 1 + \max_{-\ell_0 \leq j \leq -1} M(j), \tag{3.2}
\]

and \( q \) such that

\[
q > 1 \quad \text{and} \quad \frac{q + 1}{p - 1} > 2, \tag{3.3}
\]

we have

\[
\|\varepsilon\|_{H^2_{\rho_Y}} + \|\nabla \varepsilon\|_{L^{2q+2}_{\rho_Y}} + \left( \int \frac{\varepsilon^2}{1 + 2K \rho_r} dY \right)^{\frac{1}{2}} + \left( \int \frac{\|\nabla \varepsilon\|_{2q+2}^2}{1 + 2K \rho_r} dY \right)^{\frac{1}{2q+2}} + \|v\|_{W^{1,2q+2}} \lesssim \hat{b}^{-\frac{2}{q}} (\|w\|_{H^2} + \|w\|_{W^{1,2q+2}}) \tag{3.4}
\]

where

\[
v = \sum_{j=-\ell_0}^{-2} \sum_{M=0}^{M(j)} a_{j,M} \phi_{j,2M} + \varepsilon.
\]

*Proof.* It is a classical consequence of the implicit function theorem.

**step 1** Existence of the decomposition of \( U \) and proof of (3.1). We introduce the smooth maps

\[
F(w, \mu, b, (a_{j,M})_{-\ell_0 \leq j \leq -2, 0 \leq M \leq M(j)}) = \mu^{\frac{2}{p-1}} \left( \tilde{\Phi}_b + w \right) (\mu x) - \tilde{\Phi}_b - \sum_{j=-\ell_0}^{-2} \sum_{M=0}^{M(j)} a_{j,M} \phi_{j,2M}
\]
and

\[ G = \left( (F, \phi_{j,M})_{L^2_{\rho_y}}, -\ell_0 \leq j \leq -1, 0 \leq M \leq M(j) \right). \]

We immediately check that \( G(0, 1, b, \ldots, 0) = 0 \). Also, from (2.46), (2.5) and Lemma 2.5, we have

\[
(L \Phi_{a,M}, \phi_{j,2M})_{L^2_{\rho_y}} = 0, \quad -\ell_0 \leq j \leq -2, \quad 0 \leq M \leq M(j),
\]

and hence, we deduce that

\[
\frac{\partial G}{\partial (\mu, b, (a_{j,M})_{-\ell_0 \leq j \leq -2, 0 \leq M \leq M(j)})}\bigg|_{(0, 1, 0, \ldots, 0)} = \begin{pmatrix} A & 0 \\ 0 & I \end{pmatrix}
\]

where \( I \) is the \( N \) by \( N \) identity matrix with the integer \( N \) is given by

\[ N = \sum_{j=-\ell_0}^{-1} (1 + M(j)) \]

and where \( A \) is the following 2 by 2 matrix

\[
\begin{pmatrix}
(\frac{\partial \Phi_b}{\partial b}\big|_{b=0}, \phi_{j,-1,0})_{L^2_{\rho_y}} & (L \Phi, \phi_{-1,0})_{L^2_{\rho_y}} \\
(\frac{\partial \Phi_b}{\partial b}\big|_{b=0}, \phi_{j,-1,2})_{L^2_{\rho_y}} & (L \Phi, \phi_{-1,2})_{L^2_{\rho_y}}
\end{pmatrix}.
\]

Since we have

\[
|A| = \frac{1}{\|P_0\|_{L^2_{\rho_y}} \|P_2\|_{L^2_{\rho_y}}} \left| \frac{1}{2} (P_2 + 2P_0) (z) \Lambda \Phi, P_0 \Lambda \Phi \bigg|_{L^2_{\rho_y}} \right| (L \Phi, P_0 \Lambda \Phi)_{L^2_{\rho_y}} \neq 0,
\]

we deduce that

\[
\frac{\partial G}{\partial (\mu, b, (a_{j,M})_{-\ell_0 \leq j \leq -2, 0 \leq M \leq M(j)})}\bigg|_{(0, 1, b, 0, \ldots, 0)}
\]

is invertible. Since \( 0 < b \leq \hat{b} \ll 1 \), we infer by continuity and the fact that the set of invertible matrices is open that

\[
\frac{\partial G}{\partial (\mu, b, (a_{j,M})_{-\ell_0 \leq j \leq -2, 0 \leq M \leq M(j)})}\bigg|_{(0, \hat{b}, 0, 0, \ldots, 0)}
\]

is invertible. In view of the implicit function theorem, for \( \kappa > 0 \) small enough, for any

\[
\|w\|_{L^\infty} \leq \kappa
\]

there exists \( (\mu, b, (a_{j,M})_{-\ell_0 \leq j \leq -2, 0 \leq M \leq M(j)}) \) and

\[
\varepsilon = F\left(w, \mu, b, (a_{j,M})_{-\ell_0 \leq j \leq -2, 0 \leq M \leq M(j)} \right)
\]

such that

\[
u = \tilde{\Phi}_b + \frac{1}{\mu^{p-1}} \left( \Phi_b + \sum_{j=-\ell_0}^{-2} \sum_{M=0}^{M(j)} a_{j,M} \phi_{j,2M} + \varepsilon \right) \left( \frac{x}{\mu} \right),
\]

\[
(\varepsilon, \phi_{j,2M})_{L^2_{\rho_y}} = 0, \quad -\ell_0 \leq j \leq -1, \quad 0 \leq M \leq M(j),
\]
and the estimate (3.1) holds true for the parameters, i.e.

$$|\mu - 1| + |b - \tilde{b}| + \sum_{j=-L_0}^{-2} \sum_{M=0}^{M(j)} |a_{j,M}| \lesssim \|w\|_{L^\infty}.$$ 

**Step 2** Proof of (3.4). Recall that we have defined $\varepsilon$ as

$$\varepsilon = \mu^{\frac{2}{p-1}} (\tilde{\Phi}_b + w)(\mu x) - \sum_{j=-L_0}^{-2} \sum_{M=0}^{M(j)} a_{j,M} \phi_{j,2M}.$$ 

We infer

$$\varepsilon = \tilde{\varepsilon} + \mu^{\frac{2}{p-1}} w(\mu Y) - \sum_{j=-L_0}^{-2} \sum_{M=0}^{M(j)} a_{j,M} \phi_{j,2M}.$$ 

where we have introduced the notation

$$\tilde{\varepsilon} = (\mu - 1) \int_0^1 \left( 1 + \sigma(\mu - 1) \right)^{\frac{2}{p-1}} \Lambda_Y \tilde{\Phi}_b ((1 + \sigma(\mu - 1)) Y) d\sigma$$

$$+ (b - \tilde{b}) \int_0^1 \partial_b \tilde{\Phi}_{b+\sigma(b-\tilde{b})}(Y) d\sigma.$$ 

We estimate

$$\|\varepsilon\|_{H^2_{\rho_Y}} + \|\nabla \varepsilon\|_{L^{2q+2}_{\rho_Y}} + \left( \int \frac{\varepsilon^2}{1 + z^2 K} \rho_Y dY \right)^{\frac{1}{2}} + \left( \int \frac{|\nabla \varepsilon|^{2q+2}}{1 + z^2 K} \rho_Y dY \right)^{\frac{1}{2q+2}} + \|v\|_{W^{1,2q+2}}$$

$$\lesssim \|\tilde{\varepsilon}\|_{H^2_{\rho_Y}} + \|\nabla \varepsilon\|_{L^{2q+2}_{\rho_Y}} + \left( \int \frac{\varepsilon^2}{1 + z^2 K} \rho_Y dY \right)^{\frac{1}{2}} + \left( \int \frac{|\nabla \varepsilon|^{2q+2}}{1 + z^2 K} \rho_Y dY \right)^{\frac{1}{2q+2}} + \|\tilde{\varepsilon}\|_{W^{1,2q+2}}$$

$$+ \|w\|_{H^2} + \|w\|_{W^{1,2q+2}} + \sum_{j=-L_0}^{-2} \sum_{M=0}^{M(j)} |a_{j,M}|,$$

where we used the fact that for $-L_0 \leq j \leq -2$ and $0 \leq M \leq M(j)$, we have

$$\left( \int \frac{\phi_{j,2M}^2}{1 + z^2 K} \rho_Y dY \right)^{\frac{1}{2}} + \left( \int \frac{|\nabla \phi_{j,2M}|^{2q+2}}{1 + z^2 K} \rho_Y dY \right)^{\frac{1}{2q+2}} \lesssim \left( \int \frac{P_{j,2M}^2}{1 + z^2 K} dz \right)^{\frac{1}{2}} + \left( \int \frac{(P_{j,2M}^2)^{2q+2}}{1 + z^2 K} dz \right)^{\frac{1}{2q+2}}$$

in view of the choice

$$K \geq 1 + \max_{-L_0 \leq j \leq -1} M(j).$$

Together with the estimate for $a_{j,M}$ derived in step 1, we infer

$$\|\varepsilon\|_{H^2_{\rho_Y}} + \|\nabla \varepsilon\|_{L^{2q+2}_{\rho_Y}} + \left( \int \frac{\varepsilon^2}{1 + z^2 K} \rho_Y dY \right)^{\frac{1}{2}} + \left( \int \frac{|\nabla \varepsilon|^{2q+2}}{1 + z^2 K} \rho_Y dY \right)^{\frac{1}{2q+2}} + \|v\|_{W^{1,2q+2}}$$

$$\lesssim \|\tilde{\varepsilon}\|_{H^2_{\rho_Y}} + \|\nabla \varepsilon\|_{L^{2q+2}_{\rho_Y}} + \left( \int \frac{\varepsilon^2}{1 + z^2 K} \rho_Y dY \right)^{\frac{1}{2}} + \left( \int \frac{|\nabla \varepsilon|^{2q+2}}{1 + z^2 K} \rho_Y dY \right)^{\frac{1}{2q+2}} + \|\tilde{\varepsilon}\|_{W^{1,2q+2}}$$

$$+ \|w\|_{H^2} + \|w\|_{W^{1,2q+2}} + \sum_{j=-L_0}^{-2} \sum_{M=0}^{M(j)} |a_{j,M}|,$$

where we used the fact that $q > 1$ and the Sobolev embedding in $\mathbb{R}^4$ in the last inequality.
We still need to estimate $\tilde{\varepsilon}$. We have
\[
\Lambda_Y \Phi_\nu(Y) = \frac{1 - Z^2}{1 + Z^2} \frac{1}{\mu^{p+1}} \Lambda_\nu \Phi \left( \frac{r}{\mu} \right),
\]
which together with the decay of $\Phi$, the fact that $\tilde{\Phi}_b = \Phi_b + \tilde{\nu}_b$ and the estimates for $\tilde{\nu}_b$ yields
\[
|\partial_r \partial_{\tilde{Z}} \Lambda_Y \tilde{\Phi}_b(Y)| \lesssim \frac{1}{(\langle r \rangle + |Z|)^{\frac{2}{p+1} - \frac{4}{n}}}, \quad |\partial_r \partial_{\tilde{Z}} \tilde{\Phi}_b(Y)| \lesssim \frac{1}{b(\langle r \rangle + |Z|)^{\frac{2}{p+1} - \frac{4}{n}}}
\]
In view of the definition of $\tilde{\varepsilon}$, we infer
\[
\|\tilde{\varepsilon}\|_{H^2_{\rho_Y}} + \|\nabla \tilde{\varepsilon}\|_{L^{2q+2}_{\rho_Y}} + \left( \int \frac{\varepsilon^2}{1 + z^{2K}} \rho_r dY \right)^{\frac{1}{2}} + \left( \int \frac{|\nabla \tilde{\varepsilon}|^{2q+2}}{1 + z^{2K}} \rho_r dY \right)^{\frac{1}{2q+2}} + \|\tilde{\varepsilon}\|_{W^{1,2q+2}} \lesssim b^{-\frac{1}{4}q + 1} \|\varepsilon\|_{L^{\infty}}.
\]
Together with the estimate for the parameters $b$ and $\mu$, we infer
\[
\|\hat{\varepsilon}\|_{H^2_{\rho_Y}} + \|\nabla \hat{\varepsilon}\|_{L^{2q+2}_{\rho_Y}} + \left( \int \frac{\varepsilon^2}{1 + z^{2K}} \rho_r dY \right)^{\frac{1}{2}} + \left( \int \frac{|\nabla \hat{\varepsilon}|^{2q+2}}{1 + z^{2K}} \rho_r dY \right)^{\frac{1}{2q+2}} + \|\hat{\varepsilon}\|_{W^{1,2q+2}} \lesssim b^{-\frac{1}{4}q} \|\varepsilon\|_{L^{\infty}}.
\]
Coming back to $\varepsilon$, we deduce
\[
\|\varepsilon\|_{H^2_{\rho_Y}} + \|\nabla \varepsilon\|_{L^{2q+2}_{\rho_Y}} + \left( \int \frac{\varepsilon^2}{1 + z^{2K}} \rho_r dY \right)^{\frac{1}{2}} + \left( \int \frac{|\nabla \varepsilon|^2}{1 + z^{2K}} \rho_r dY \right)^{\frac{1}{2}} + \|\varepsilon\|_{W^{1,2q+2}} \lesssim b^{-\frac{1}{4}q} \|\varepsilon\|_{H^2} + \|\varepsilon\|_{W^{1,2q+2}},
\]
which is (3.4). This concludes the proof of the lemma.

**Description of the initial data.** We now pick an initial data close to $\tilde{\Phi}_b$ up to scaling, where $\tilde{\Phi}_b$ has been constructed in Lemma 2.10, and assume in the coordinate of the above geometrical decomposition
\[
u_0 = \frac{1}{\lambda_0^{p+1}} \left( \tilde{\Phi}_b + \nu_0 \right) \left( \frac{x}{\lambda_0} \right),
\]
with
\[
\nu_0 = \psi_0 + \varepsilon_0, \quad \psi_0 = \sum_{j=-\ell_0}^{-2} \sum_{M=0}^{M(j)} (a_{j,M})_0 \phi_{j,2M}(Y)
\]
and $\varepsilon_0$ satisfies the following orthogonality conditions
\[
(\varepsilon_0, \phi_{j,2M})_{L^2_{\rho_Y}} = 0, \quad -\ell_0 \leq j \leq -1, \quad 0 \leq M \leq M(j).
\]


Let $K > 0$ be a large enough universal constant such that in particular (3.2) holds true, and define
\[ \nu_K(z) = \frac{1}{1 + z^{2K}} \] (3.8)

Let a large enough integer $q$ such that in particular (3.3) holds true, and pick $n \geq n(K)$ large enough and $s_0 > s_0(n,K)$ large enough. Pick parameters $\lambda_0, b_0, (a_{j,M})_0$ and a profile $\varepsilon_0$ which satisfy the initial bounds:

- rescaled solution:
  \[ \lambda_0 = e^{-\frac{s_0}{2}}; \] (3.9)

- control of the $b$ parameter:
  \[ b_0 = \frac{1}{c_1s_0} \] (3.10)

where the constant $c_1 > 0$ is given by (2.21);

- initial control of the unstable modes:
  \[ \sum_{j=-\ell_0}^{-2} \sum_{M=0}^{M(j)} |a_{j,M}(0)|^2 \leq \frac{1}{s_0^4}; \] (3.11)

- initial control of the exponentially localized norm:
  \[ \|\varepsilon_0\|_{H^2_{\nu'}} + \|\nabla\varepsilon_0\|_{L^2_{\nu'}} < \frac{1}{s_0^2}; \] (3.12)

- control of polynomially localized norms:
  \[ \int \nu_K\varepsilon_0^2 \rho_Y dY \leq \frac{1}{s_0^{2K}}, \quad \int \nu_K|\nabla\varepsilon_0|^{2q+2} \rho_Y dY \leq \frac{1}{s_0^{2q+2K}}; \] (3.13)

- initial control of the global $W^{1,2q+2}$ norm:
  \[ \|\varepsilon_0\|_{W^{1,2q+2}} < \frac{1}{s_0}. \] (3.14)

**Remark 3.2.** Note that the above properties of the initial data $u_0$ can be obtained by applying Lemma 3.1 to an initial data of the form
\[ u_0 = \tilde{\Phi}_b + w_0 \] (3.15)
where $\tilde{\Phi}_b$ has been constructed in Lemma 2.10 and where
\[ 0 < b_0 \ll 1 \quad \text{and} \quad \|w_0\|_{W^{1,2q+2}} + \|w_0\|_{H^2} \leq \frac{b_0^{2n}}{b_0}. \] (3.16)

Indeed, the decomposition (3.5) (3.6) (3.7) immediately follows from Lemma 3.1. Then, we may choose $s_0$ as
\[ s_0 = \frac{1}{c_1b_0} \]
so that (3.10) holds true. In view of our assumptions on $w_0$, this yields in particular
\[ \|w_0\|_{W^{1,2q+2}} + \|w_0\|_{H^2} \leq \frac{1}{s_0^{2n}}, \]
and the estimates (3.11) (3.12) (3.13) (3.14) immediately follow from the bounds (3.1) (3.4). Finally, we may always renormalize the initial data to enforce (3.9).
Renormalized flow. From a standard continuity in time argument, as long as the solution remains close to \( \Phi \) up to scaling in \( L^2_{\rho Y} \), we may introduce the time dependent geometrical decomposition

\[
    u(t, x) = \frac{1}{\lambda(t)^{p-1}} U(s, Y), \quad Y = \frac{s}{\lambda(t)}
\]

with

\[
    U = \tilde{\Phi}_b(t) + v, \quad v = \psi + \varepsilon
\]

and

\[
    \psi = -2 \sum_{j=-\ell_0}^{M(j)} \sum_{M=0}^j a_{j,M}(t) \phi_{j,2M}(Y)
\]

We claim the following bootstrap proposition.

\[
    \left( \varepsilon(t), \phi_{j,2M} \right)_{L^2_{\rho Y}} = 0, \quad -\ell_0 \leq j \leq -1, \quad 0 \leq M \leq M(j).
\]

The above decomposition is continuously differentiable with respect to time from standard parabolic regularizing effects. Consider the renormalized time

\[
    s(t) = \int_0^t \frac{d\tau}{\lambda^2(\tau)} + s_0,
\]

then from (3.17):

\[
    \partial_s U - \frac{\lambda_s}{\lambda} \Delta U = \Delta U + U^p
\]

which together with (2.15), (2.3) yields the \( v \) equation:

\[
    (b_s + bB(b)) \partial_s \tilde{\Phi}_b - \left( \frac{\lambda_s}{\lambda} + \frac{1}{2} - M(b) \right) \Lambda \tilde{\Phi}_b + \partial_s v + \mathcal{L} v
\]

\[
    = \tilde{\Psi}_b + \left( \frac{\lambda_s}{\lambda} + \frac{1}{2} \right) \Lambda v + F(v)
\]

where

\[
    F(v) = F_1 + F_2, \quad F_1 = p \left( \tilde{\Phi}_b^{p-1} - \Phi^{p-1} \right) v, \quad F_2 = \left( \tilde{\Phi}_b + v \right)^p - \hat{\Phi}_b^p - p \hat{\Phi}_b^{p-1} v.
\]

We may equivalently develop \( v = \psi + \varepsilon \) and obtain the \( \varepsilon \) equation:

\[
    \partial_s \varepsilon + \mathcal{L} \varepsilon = \tilde{\Psi}_b - \text{Mod} + L(\varepsilon) + F(v)
\]

where Mod encodes the modulation equations

\[
    \text{Mod} = -2 \sum_{j=-\ell_0}^{M(j)} \sum_{M=0}^j [(a_{j,M})_s + (\lambda_j + M)a_{j,M}] \phi_{j,2M} - \left( \frac{\lambda_s}{\lambda} + \frac{1}{2} \right) \Lambda \psi
\]

\[
    - \left( \frac{\lambda_s}{\lambda} + \frac{1}{2} - M(b) \right) \Lambda \tilde{\Phi}_b + (b_s + bB(b)) \partial_s \tilde{\Phi}_b
\]

and we defined the linear error

\[
    L(\varepsilon) = \left( \frac{\lambda_s}{\lambda} + \frac{1}{2} \right) \Lambda \varepsilon.
\]

We claim the following bootstrap proposition.

**Proposition 3.3 (Bootstrap).** Given \( q \) large enough satisfying in particular (3.3), \( K \geq K(q) \) large enough satisfying in particular (3.2), \( n \geq n(K, q) \) large enough and \( s_0(n, K, q) \) large enough, then for all \( \lambda_0, b_0, \varepsilon_0 \) satisfying (3.9), (3.10), (3.12), (3.13), (3.14) and the orthogonality conditions (3.7), there exist \( (a_{j,M}(0))_{-\ell_0 \leq j \leq -2, 0 \leq M \leq M(j)} \) satisfying (3.11) such that the solution starting from \( u_0 \) given by (3.5), decomposed according to (3.17) satisfies for all \( s \geq s_0 \):
• control of the scaling:

\[ 0 < \lambda(s) < e^{-\frac{4}{s}}; \]  

(3.26)

• control of the b parameter:

\[ \frac{1}{10c_1s} < b(s) < \frac{10}{c_1s}; \]  

(3.27)

• control of the unstable modes:

\[ \sum_{j=-\ell_0}^{-2} \sum_{M=0}^{M(j)} |a_{j,M}(s)|^2 \leq \frac{1}{s^n}; \]  

(3.28)

• control of the exponentially localized norm:

\[ \|\varepsilon(s)\|_{H^2} < \frac{1}{s^{\frac{1}{2}}}; \]  

(3.29)

and

\[ \|\nabla\varepsilon\|_{L^{2+2}} < \frac{1}{s^{\frac{1}{2}}}; \]  

(3.30)

• control of polynomially localized norms:

\[ \int \nu_K|\varepsilon(s)|^2 \rho_r dY \leq \frac{1}{s^{K+1}}, \quad \int \nu_K|\nabla\varepsilon(s)|^{2q+2} \rho_r dY \leq \frac{1}{s^{2q+K+1}}; \]  

(3.31)

• control of the global \( W^{1,2q+2} \) norm:

\[ \|v(s)\|_{W^{1,2q+2}} < \frac{1}{s^{\delta_q}} \]  

(3.32)

for some small enough \( \delta_q > 0 \).

Proposition 3.3 is the heart of the analysis, and the corresponding solutions are easily shown to satisfy the conclusions of Theorem 1.1. The strategy of the proof follows [14, 19]: we prove Proposition 3.3 by contradiction using a topological argument à la Brouwer: given \( \lambda_0, b_0, \varepsilon_0 \) satisfying (3.9), (3.10), (3.12), (3.13), (3.14), (3.7), we assume that for all \( (a_{j,M}(0))_{-\ell_0 \leq j \leq -2, 0 \leq M \leq M(j)} \) satisfying (3.11), the exit time

\[ s^* = \sup \{ \ s \geq s_0 \ \text{such that} \ (3.26), (3.27), (3.28), (3.29), (3.31), (3.32) \ \text{holds on} \ [s_0, s) \} \]  

(3.33)

is finite

\[ s^* < +\infty \]  

(3.34)

and look for a contradiction for \( s_0 \geq s_0(n, K, q) \) large enough. From now on, we therefore study the flow on \([s_0, s^*)\) where (3.26), (3.27), (3.28), (3.29), (3.31), (3.32) hold. Using a bootstrap method we show that the bounds (3.26), (3.27), (3.29), (3.31), (3.31), (3.32) can be improved, implying that at time \( s^* \) necessarily the unstable modes have grown and (3.28) reaches its boundary. Since 0 is a linear repulsive equilibrium for these modes, this will contradict Brouwer fixed point theorem.
3.2. Modulation equations. We now compute the modulation equations which
describe the time evolution of the parameters. They are computed in the self-similar
zone, and involve the ρ weighted norm.

Lemma 3.4 (Modulation equations). There holds the modulation equations:

\[ \left| \frac{\lambda}{\lambda} + \frac{1}{2} - M(b) \right| + |b_s + bB(b)| + \sum_{j=-\ell_0}^{M} \sum_{M=0}^{M} |(a_{j,M})_s + (\lambda_j + M)a_{j,M}| \]
\[ \lesssim b^{n+1} + b \left( \|\varepsilon\|_{L^2_{\rho_Y}} + \sum_{j=-\ell_0}^{M} \sum_{M=0}^{M} |a_{j,M}| \right). \] (3.35)

Proof. This lemma follows from the choice of orthogonality conditions (3.19) and
the explicit properties of the refined reconnecting profile \( \tilde{\Phi}_b \). The control of the
nonlinear term relies in an essential way on (3.32) which from Sobolev implies for \( q \)
large enough the \( L^\infty \) smallness

\[ \|v\|_{L^\infty} \lesssim \|v\|_{W^{1,2q+2}} \lesssim \frac{1}{sq} \ll 1. \] (3.36)

We take the \( L^2_{\rho_Y} \) scalar product of (3.23) with \( \phi_{j,2M} \) and compute from (3.19):

\[ (\text{Mod}, \phi_{j,2M})_{L^2_{\rho_Y}} = (\tilde{\Psi}_b, \phi_{j,2M})_{L^2_{\rho_Y}} + (L(\varepsilon) + F(v), \phi_{j,2M})_{L^2_{\rho_Y}}. \]

The error term in controlled from (2.44) (2.45) thanks to the space localization of
the \( \rho_Y dY \) measure:

\[ \| (\tilde{\Psi}_b, \phi_{j,2M})_{L^2_{\rho_Y}} \| \lesssim b^{n+1}. \]

The linear term is estimated by integration by parts

\[ \| (L(\varepsilon), \phi_{j,2M})_{L^2_{\rho_Y}} \| \lesssim \left| \frac{\lambda}{\lambda} + \frac{1}{2} \right| \|\varepsilon\|_{L^2_{\rho_Y}}. \]

For the nonlinear term, we recall (3.22). We estimate:

\[ |\partial_b \Phi_b| = \left| -\frac{z^2}{2(1 + bz^2)} \frac{1}{\mu_b^{p+1}} \Lambda \Phi \left( \frac{r}{\mu_b} \right) \right| \lesssim |z|^2, \quad |\partial_b \tilde{b}| \lesssim 1 \]

which using \( \|\tilde{\Phi}_b\|_{L^\infty} \lesssim 1 \) implies the pointwise bound

\[ \|\tilde{\Phi}_b^{p-1} - \Phi^{p-1}\| \lesssim \int_0^b \left| \tilde{\Phi}_b^{p-2} \partial_b \tilde{\Phi}_b \right| db \lesssim b(1 + |z|^2) \] (3.37)

and hence

\[ |(F_1(v), \phi_{j,2M})_{L^2_{\rho_Y}}| \lesssim b \left| (v, (1 + |z|^2)\phi_{j,2M})_{L^2_{\rho_Y}} \right| \lesssim b \left( \|\varepsilon\|_{L^2_{\rho_Y}} + \sum |a_{j,M}| \right). \]

For the remaining nonlinear term, we use the rough \( L^\infty \) bound \( \|v\|_{L^\infty} + \|\tilde{\Phi}_b\|_{L^\infty} \leq 1 \)
and the confining measure:

\[ |(F_2(v), \phi_{j,2M})_{L^2_{\rho_Y}}| \lesssim \int (|v|^2 + |v|^p)|\phi_{j,2M}|\rho_Y dY \lesssim \int |v|^2|\phi_{j,2M}|\rho_Y dY \lesssim b \|v\|_{L^2_{\rho_Y}} \lesssim b \|\varepsilon\|_{L^2_{\rho_Y}} + b \left( \sum |a_{j,M}| \right) \]

where we used the fact that \( v = \psi + \varepsilon \) and the rough bound

\[ \|v\|_{L^2_{\rho_Y}} \leq \|\varepsilon\|_{L^2_{\rho_Y}} + \sum |a_{j,M}| \leq b \] (3.38)
which follows from (3.27) (3.28) (3.29). We therefore have obtained the following identity:

\[
\left| (\text{Mod}, \phi_{j,2M})_{L^2_{\rho_Y}} \right| \lesssim b^{n+1} + \left( \left| \frac{\lambda_s}{\lambda} + \frac{1}{2} - M(b) \right| + b \right) \| \varepsilon \|_{L^2_{\rho_Y}} + b \sum |a_{j,M}|. \quad (3.39)
\]

We now compute the lhs of (3.39) for the various values of \(j\).

**\(a_{j,M}\) terms, \(j \leq -2\).** First observe from (2.15), (2.20) the bounds

\[
\| \nabla_y^k \Lambda \tilde{v}_b \|_{L^2} + \| \nabla_y^k \partial_y \tilde{v}_b \|_{L^2_{\rho_Y}} \lesssim b, \quad k = 0, 1, 2,
\]

which together with the computations

\[
\nabla \Phi_b = \frac{1 - b z^2}{1 + b z^2} \frac{1}{\mu_b} \Lambda \Phi \left( \frac{r}{\mu_b} \right), \quad \partial_y \Phi_b = \frac{z^2}{2 \mu_b} \frac{1}{\mu_b} \Lambda \Phi \left( \frac{r}{\mu_b} \right)
\]

yields

\[
\| \nabla_y^k (\Lambda \tilde{v}_b - \Lambda \Phi) \|_{L^2_{\rho_Y}} + \| \nabla_y^k \left( \partial_y \tilde{v}_b + \frac{1}{2} z^2 \Lambda \Phi \right) \|_{L^2_{\rho_Y}} \lesssim b, \quad k = 0, 1, 2. \quad (3.41)
\]

Hence, we have in particular

\[
(\partial_y \tilde{v}_b, \phi_{j,2M})_{L^2_{\rho_Y}} = O(b), \quad (\Lambda \tilde{v}_b, \phi_{j,2M})_{L^2_{\rho_Y}} = O(b).
\]

We conclude from (3.24) using the orthonormality of eigenfunctions, separation of variables and the rough bound (3.38):

\[
(\text{Mod}, \phi_{j,2M})_{L^2_{\rho_Y}} = [(a_{j,M})_* + (\lambda_j + M) a_{j,M}] \| \phi_{j,2M} \|_{L^2_{\rho_Y}}^2 + O \left( b \left( \left| \frac{\lambda_s}{\lambda} + \frac{1}{2} - M(b) \right| + |a_{j,M}| + |b s + b B(b)| \right) \right). \quad (3.42)
\]

**Scaling terms.** We compute from (3.41):

\[
(\Lambda \tilde{v}_b, \Lambda \Phi)_{L^2_{\rho_Y}} = \| \Lambda \Phi \|_{L^2_{\rho_Y}}^2 + O(b)
\]

and hence:

\[
(\text{Mod}, \Lambda \Phi)_{L^2_{\rho_Y}} = - \left( \left| \frac{\lambda_s}{\lambda} + \frac{1}{2} - M(b) \right| \right) \| \Lambda \Phi \|_{L^2_{\rho_Y}}^2 + O(b) \quad (3.43)
\]

**\(b\) equation.** We compute from (3.41):

\[
(\Lambda \tilde{v}_b, (z^2 - 2) \Lambda \Phi)_{L^2_{\rho_Y}} = O(b)
\]

\[
(\partial_y \tilde{v}_b, (z^2 - 2) \Lambda \Phi)_{L^2_{\rho_Y}} = - \frac{1}{2} \| (z^2 - 2) \Lambda \Phi \|_{L^2_{\rho_Y}}^2 + O(b)
\]

from which using the orthogonality of eigenfunctions:

\[
(\text{Mod}, (z^2 - 2) \Lambda \Phi)_{L^2_{\rho_Y}} = - \frac{1}{2} \| (z^2 - 2) \Lambda \Phi \|_{L^2_{\rho_Y}}^2 (1 + O(b))(b s + b B(b)) + O \left( b \left( \left| \frac{\lambda_s}{\lambda} + \frac{1}{2} - M(b) \right| + \sum |a_{j,M}| \right) \right). \quad (3.44)
\]

**Conclusion.** Injecting (3.42), (3.43), (3.44) into (3.39) yields (3.35). \(\square\)
3.3. Inner $H^2$ bounds with exponential localization. We now turn to the control of the flow in exponentially weighted norms which is an elementary consequence of the spectral gap estimate (2.9), the dissipative structure of the flow, the $L^\infty$ bound (3.36) to control the non linear term and the explicit form of the refined reconnecting $\tilde{\Phi}_b$ profiles which generate the leading order error term.

**Lemma 3.5** (Lyapunov control of exponentially weighed norms). There holds the pointwise differential bounds:

\[
\frac{d}{ds} \varepsilon_\rho^2 \varepsilon + c \varepsilon^2_{H^1_\rho} \lesssim b^{2n+2} + (\|v\|_{L^\infty}^2 + b^2) \sum |a_{j,M}|^2, \quad (3.45)
\]

\[
\frac{d}{ds} \|\nabla \varepsilon\|_{H^1_\rho}^2 \lesssim b^{2n+2} + \|a_{j,M}\|^{2q+2} + b^{(2q+2)(n+1)}, \quad (3.46)
\]

\[
\frac{d}{ds} \|L^\rho \varepsilon\|_{H^1_\rho}^2 + c \|L^\rho \varepsilon\|_{H^1_\rho}^2 \lesssim b^{2n+2} + (b + \|v\|_{L^\infty}) \|\varepsilon\|_{H^1_\rho}^2 \quad (3.47)
\]

for some universal constant $c > 0$.

**Proof.** **Step 1** $L^2$ exponential bound. We compute from (3.23):

\[
\frac{1}{2} \frac{d}{ds} \|\varepsilon\|_{L^2_\rho}^2 = (\varepsilon, \partial_s \varepsilon)_{L^2_\rho} = -(\mathcal{L} \varepsilon, \varepsilon)_{L^2_\rho} + (\tilde{\Phi}_b + L(\varepsilon) - \text{Mod} + F(v), \varepsilon)_{L^2_\rho} \quad (3.48)
\]

and estimate all terms in the above identity.

We start with the nonlinear term (3.22). Recall the variance bound\(^2\)

\[
\|Yu\|_{L^2_\rho} \lesssim \|u\|_{H^1_\rho} \quad (3.49)
\]

which together with the pointwise bound (3.37) ensures

\[
|((\Phi^{-1} - \Phi^{-1})_\rho, \varepsilon)_{L^2_\rho} \lesssim b((1 + |z|^2) \varepsilon, \varepsilon)_{L^2_\rho} \lesssim b\|\varepsilon\|_{H^1_\rho}^2 .
\]

We now estimate using the rough $L^\infty$ bound $\|v\|_{L^\infty} \ll 1$:

\[
|F_2(v, \varepsilon)|_\rho \lesssim \int |\varepsilon| \rho Y dY \lesssim \|v\| \|\rho Y\| \lesssim \delta \|\varepsilon\|_{L^2_\rho}^2 + C_s \|v\|_{L^\infty}^2 \|\rho Y\| \lesssim \|v\|_{L^2_\rho}^2 + \|v\|_{L^\infty}^2 \sum |a_{j,M}|^2 .
\]

To estimate the $L$ term, we use the rough bound from (3.35):

\[
|((\lambda \varepsilon - \frac{1}{2}) + |b_0 + b B(b)| + |(a_{j,M})_s - (\lambda j + M)a_{j,M}| \lesssim b \quad (3.50)
\]

which implies using (2.42), (3.49):

\[
|((\lambda, L(\varepsilon))_{L^2_\rho} \|v\|_{L^2_\rho} \lesssim b((1 + |Y|) \varepsilon)_{L^2_\rho} \lesssim b\|\varepsilon\|_{H^1_\rho}^2 . \quad (3.51)
\]

The leading order term $\tilde{\Phi}_b$ term is estimated in brute force from (2.44) (2.45) using the exponential localization of the measure:

\[
\|\varepsilon, \tilde{\Phi}_b\|_{L^2_\rho} \lesssim b^{n+1} \|\varepsilon\|_{L^2_\rho} .
\]

\(^2\)see for example [6], Appendix A.
To control the modulation parameters, we use (3.41), (3.19), (3.35) to estimate:
\[
|\langle \varepsilon, \text{Mod} \rangle| \lesssim b \left[ \sum |a_{j,M}| + b^{n+1} + b\|\varepsilon\|_{L^2_\nu} \right] \| (1 + |Y|) \varepsilon \|_{L^2_\nu} \lesssim \delta \|\varepsilon\|^2_{H^1_\nu} + c_3 b^{2n+4} + c_6 b^2 \sum |a_{j,M}|^2.
\]
Injecting the collection of above bounds into (3.48) and using the spectral gap estimate (2.9) with the choice of orthogonality conditions (3.19) yields (3.45).

**step 2** \(W^{1,2q+2}\) exponential bound. Let \(q\) be a large enough integer. Let
\[
\varepsilon_i = \partial_i \varepsilon, \; i = 1, 2, 3, 4,
\]
then from (3.23):
\[
\partial_s \varepsilon_i + (\mathcal{L} + 1) \varepsilon_i = \partial_i \left[ \tilde{\Psi}_b - \text{Mod} + L(\varepsilon) + F(v) \right] + p(p-1)\Phi^{-2} \partial_i \Phi \varepsilon. \tag{3.52}
\]
We then compute:
\[
\frac{1}{2q+2} \int dY \int_0^{\varepsilon_i^{2q+1}} \partial_s \varepsilon_i = \int \varepsilon_i^{2q+1} \partial_s \varepsilon_i
\]
\[
= - \left( (\mathcal{L} + 1) \varepsilon_i, \varepsilon_i^{2q+1} \right)_{L^2_\nu} + \left( \varepsilon_i^{2q+1}, \partial_i \left[ \tilde{\Psi}_b - \text{Mod} + L(\varepsilon) + F(v) \right] \right)_{L^2_\nu}
\]
and estimate all terms in the above identity.

We integrate by parts to compute:
\[
\left( \left( \Delta - \frac{1}{2} Y \cdot \nabla \right) \varepsilon_i, \varepsilon_i^{2q+1} \right)_{L^2_\nu} = \int \frac{1}{\rho_Y} \nabla \cdot (\rho_Y \nabla \varepsilon_i) \varepsilon_i^{2q+1} \rho_Y dY
\]
\[
= -(2q + 1) \int \varepsilon_i^{2q} |\nabla Y \varepsilon_i|^2 \rho_Y dY = -(2q + 1) \int |\nabla Y (\varepsilon_i^{q+1})|^2 \rho_Y dY.
\]
We apply the spectral gap estimate (2.9) to \(\varepsilon_i^{q+1}\) and conclude that there exists \(c > 0\), and for all \(A > 0\) large enough, there exists \(C_A\) such that
\[
\int |\nabla Y (\varepsilon_i^{q+1})|^2 \rho_Y dY \geq c \int |\nabla (\varepsilon_i^{q+1})|^2 \rho_Y dY + A \int (\varepsilon_i^{q+1})^2 \rho_Y dY
\]
\[
- C_A \sum_{j,M: j \leq j(A), M(A)} (\varepsilon_i^{q+1}, \phi_{j,2M})^2_{L^2_\nu},
\]
where \(j, M \leq j(A), M(A)\) are the indices corresponding to all eigenvalues \(\mu_{j,2M}\) of \(\mathcal{L}_\nu\) that satisfy \(\mu_{j,2M} \leq A\). Hence choosing \(A\) large enough compared to \(\|\Phi\|_{L^\infty}\), we infer
\[
- \left( (\mathcal{L} + 1) \varepsilon_i, \varepsilon_i^{2q+1} \right)_{L^2_\nu} \leq -c \int |\nabla (\varepsilon_i^{q+1})|^2 \rho_Y dY - \frac{A}{2} \int (\varepsilon_i^{q+1})^2 \rho_Y dY
\]
\[
+ C_A \sum_{j,M: j \leq j(A), M(A)} (\varepsilon_i^{q+1}, \phi_{j,2M})^2_{L^2_\nu}.
\]
We now estimate using Hölder and the polynomial growth of eigenmodes $|\phi_{j,2M}| \lesssim |Y|^{c(M)}$:

$$
\left(\varepsilon_i^{q+1}, \phi_{j,2M}\right)_{L^2_{\rho \gamma}}^2 \lesssim \left( \int |\varepsilon_i|^{2q} |\phi_{j,2M}|^2 \rho Y dY \right) \left( \int |\varepsilon_i|^2 \rho Y dY \right)
$$

$$
\lesssim \left( \int |\varepsilon_i|^{2q+2} \rho Y dY \right)^{\frac{2q}{2q+2}} \left( \int |\varepsilon_i|^2 \rho Y dY \right)
$$

$$
\leq \delta \int |\varepsilon_i|^{2q+2} \rho Y dY + c_{\delta} \left( \int |\varepsilon_i|^2 \rho Y dY \right)^{q+1}.
$$

and hence, for $\delta$ small enough compared to $C_A$, $j(A)$ and $M(A)$, we infer

$$
- \left( (L + 1)\varepsilon_i, \varepsilon_i^{2q+1}\right)_{L^2_{\rho \gamma}} \leq -c \int |\nabla (\varepsilon_i^{q+1})|^2 \rho Y dY - \frac{A}{4} \int (\varepsilon_i^{q+1})^2 \rho Y dY + C_A \|\varepsilon\|_{H^2_{\rho Y}}^{2q+2}.
$$

(3.53)

The leading order error term is controlled from (2.45):

$$
\left| \varepsilon_i^{2q+1} \partial_i \tilde{\Psi}_b \right|_{L^2_{\rho \gamma}} \lesssim \int |\varepsilon_i|^{2q+2} \rho Y dY + \int |\partial_i \tilde{\Psi}_b|^{2q+2} \rho Y dY \lesssim \int |\varepsilon_i|^{2q+2} \rho Y dY + b^{(2q+2)(n+1)}.
$$

We integrate by parts and use (A.1) to estimate:

$$
|\varepsilon_i^{2q+2}, \partial_i \Lambda \varepsilon|_{L^2_{\rho \gamma}} \lesssim \int (1 + |Y|^2) \varepsilon_i^{2q+2} \rho Y dY \lesssim \int \left[ \varepsilon_i^{2q+2} + |\nabla Y (\varepsilon_i^{q+1})|^2 \right] \rho Y dY
$$

and hence from (3.50):

$$
|\varepsilon_i^{2q+1}, L(\varepsilon)|_{L^2_{\rho \gamma}} \lesssim b \int \left[ \varepsilon_i^{2q+2} + |\nabla Y (\varepsilon_i^{q+1})|^2 \right] \rho Y dY.
$$

Also, we have

$$
\left| \varepsilon_i^{2q+1}, p(p - 1)\Phi^{p-2} \partial_i \Phi \varepsilon \right|_{L^2_{\rho \gamma}} \lesssim \int |\varepsilon_i|^{2q+1} |\varepsilon| \rho Y dY \lesssim \int |\varepsilon_i|^{2q+2} \rho Y dY.
$$

We now turn to the control of the nonlinear term. We first estimate using $\|\tilde{\Phi}_b\|_{L^\infty} + \|\nabla \tilde{\Phi}_b\|_{L^\infty} \lesssim 1$, Hölder and the polynomial growth of $\psi$:

$$
|\partial_i F_1(v), \varepsilon_i^{2q+1}|_{L^2_{\rho \gamma}} \lesssim \int |\varepsilon_i|^{2q+1} (|v| + |\nabla v|) \rho Y dY
$$

$$
\lesssim \int |\varepsilon_i|^{2q+2} \rho Y dY + \int (|v| + |\nabla v|)^{2q+2} \rho Y dY
$$

$$
\lesssim \int |\varepsilon_i|^{2q+2} \rho Y dY + \sum |a_{j,M}|^{2q+2}.
$$

We now compute

$$
\nabla F_2(v) = p \nabla Y v \left[ (\tilde{\Phi}_b + v)^{p-1} - \tilde{\Phi}_b^{p-1} \right]
$$

$$
+ p \nabla Y \tilde{\Phi}_b \left[ (\tilde{\Phi}_b + v)^{p-1} - \tilde{\Phi}_b^{p-1} - (p - 1) \tilde{\Phi}_b^{p-2} v \right]
$$

and estimate by homogeneity with the $L^\infty$ bound (3.36):

$$
|F_2(v)| \lesssim |v|^2, \quad |\nabla Y F_2(v)| \lesssim |\nabla Y v||v| + |v|^2 \lesssim |\nabla Y v| + |v|
$$

(3.54)

and hence the same bound as above:

$$
|\partial_i F_1(v), \varepsilon_i^{2q+1}|_{L^2_{\rho \gamma}} \lesssim \int |\varepsilon_i|^{2q+1} (|v| + |\nabla v|) \rho Y dY \lesssim \int |\varepsilon_i|^{2q+2} \rho Y dY + \sum |a_{j,M}|^{2q+2}.
$$
The collection of above bounds for $i = 1, 2, 3, 4$ yields (3.46) provided the constant $A$ in (3.53) has been chosen large enough.

**Step 3** $\dot{H}^2$ exponential bound. Let
\[
\varepsilon(2) = L_Y \varepsilon,
\]
then $\varepsilon(2)$ satisfies the orthogonality conditions (3.19) and the equation from (3.23):
\[
\partial_s \varepsilon(2) + L_Y \varepsilon(2) = L_Y \left[ \tilde{\Psi}_b - \text{Mod} + L(\varepsilon) + F(v) \right]
\]  
(3.55)
and hence
\[
\frac{1}{2} \frac{d}{ds} \|\varepsilon(2)\|_{L^2_{t,\rho}}^2 = \left( -L_Y \varepsilon(2) + L_Y \left[ \tilde{\Psi}_b - \text{Mod} + L(\varepsilon) + F(v) \right], \varepsilon(2) \right)_{L^2_{t,\rho}}.
\]  
(3.56)
The main forcing term is estimated in brute force using (2.44) (2.45):
\[
(L_Y \tilde{\Psi}_b, \varepsilon(2))_{L^2_{t,\rho}} \lesssim b^{n+1} \|\varepsilon(2)\|_{L^2} \leq c_b b^{2n+2} + \delta \|\varepsilon(2)\|_{H^1_{t,\rho}}^2.
\]  
(3.55)
The Mod terms are controlled using (3.41), (3.19), (3.35) which yield:
\[
(L_Y \text{Mod}, \varepsilon(2))_{L^2_{t,\rho}} \lesssim b \sum |a_{j,M}| + b^{n+1} + b \|\varepsilon\|_{L^2_{t,\rho}} \|((1 + Y)\varepsilon(2))_{L^2_{t,\rho}}
\lesssim \delta \|\varepsilon(2)\|_{H^1_{t,\rho}}^2 + c_b b^{2n+4} + c_b b^2 \sum |a_{j,M}|^2.
\]

For the $L(\varepsilon)$ term, we use the commutator relation
\[
[\Delta Y, \Lambda Y] = 2\Delta Y
\]  
(3.57)
to compute
\[
[L_Y, \Lambda Y] = [-\Delta Y + \Lambda Y - p\Phi^{p-1}, \Lambda Y] = -2\Delta Y + p(p-1)\Phi^{p-2} r \partial_r \Phi
\]  
\[
= 2(L_Y - \Lambda Y + p\Phi^{p-1}) + p(p-1)\Phi^{p-2} r \partial_r \Phi
\]  
from which using (2.42) (3.51), (3.49) and $\|\Phi\|_{L^\infty} + \|\Lambda \Phi\|_{L^\infty} \lesssim 1$:
\[
\|\varepsilon(2), L_Y \Lambda Y \varepsilon\|_{L^2_{t,\rho}} \lesssim \|\varepsilon(2)\|_{H^1_{t,\rho}} + \|\varepsilon(2)\|_{H^1_{t,\rho}}^2 \lesssim \|\varepsilon(2)\|_{H^1_{t,\rho}}^2 + \|\varepsilon\|_{H^1_{t,\rho}}^2
\]  
(3.56)
and hence from (3.50):
\[
|\varepsilon(2), L_Y L(\varepsilon))_{L^2_{t,\rho}} \| \lesssim b \left( \|\varepsilon(2)\|_{H^1_{t,\rho}}^2 + \|\varepsilon\|_{H^1_{t,\rho}}^2 \right).
\]  
(3.55)
It remains to estimate the nonlinear term. We first integrate by parts since $L_Y$ is self-adjoint for $(\cdot, \cdot)_{L^2_{t,\rho}}$:
\[
|\varepsilon(2), L_Y F, \varepsilon(2))_{L^2_{t,\rho}} | = \left| \langle \nabla F, \nabla \varepsilon(2) \rangle + \left( \frac{2}{p-1} F - p\Phi^{p-1} F, \varepsilon(2) \right) \right|_{L^2_{t,\rho}}.
\]
We recall the decomposition (3.22). For the first term, we need to deal with the fact that the difference $\tilde{\Phi}_b - \Phi$ is not $L^\infty$ small for $|Z| \gtrsim 1$. We first estimate pointwise using (2.20)
\[
|\partial_\theta \Phi_b| = \left| \frac{1}{2b} Z \partial_\theta G \right| = -\frac{1}{b} \frac{Z^2}{\mu \mu^{p+1}} \Lambda \left( \frac{r}{\mu(Z)} \right) \leq \frac{Z^2}{b} = z^2, \quad \|\partial_\theta \tilde{\Phi}_b\| \lesssim b
\]  
(3.58)
and similarly for higher derivatives, and hence the pointwise bound
\[
|\partial_\theta \tilde{\Phi}_b| + |\nabla \partial_\theta \tilde{\Phi}_b| \lesssim 1 + z^2.
\]  
(3.58)
This implies
\[
|\nabla^k (\tilde{\Phi}_b^{p-1} - \Phi^{p-1})| = \left| (p - 1) \int_0^b \nabla_Y \left( \Phi_b^{p-2} \partial_b \tilde{\Phi}_b \right) db \right| \lesssim b(1 + z^2), \quad k = 0, 1. \quad (3.59)
\]

We first estimate:
\[
\left| \left( \frac{2}{p - 1} - p \Phi^{p-1} \right) F_1, \varepsilon \right|_{L^2_{\rho Y}} \lesssim b \int (1 + |Y|^2) |\varepsilon(2)|^2 \rho_Y dY \lesssim b \|\varepsilon(2)\|_{H^1_{\rho Y}}^2.
\]

Next:
\[
\begin{align*}
&|((\tilde{\Phi}_b^{p-1} - \Phi^{p-1}) \nabla_Y v, \nabla_Y \varepsilon(2))_{L^2_{\rho Y}}| \\
&\quad \lesssim \delta \|\nabla_Y \varepsilon(2)\|_{L^2_{\rho Y}}^2 + c_\delta b^2 \sum |a_{j,M}|^2 + c_\delta \int |\tilde{\Phi}_b^{p-1} - \Phi^{p-1}|^2 |\nabla_Y \varepsilon|^2
\end{align*}
\]
and
\[
\begin{align*}
&|\int (\nabla (\tilde{\Phi}_b^{p-1} - \Phi^{p-1})^2 \varepsilon^2 + |\tilde{\Phi}_b^{p-1} - \Phi^{p-1}|^2 |\nabla_Y \varepsilon|^2) \rho_Y dY | \\
&\quad \lesssim \frac{\varepsilon^2 + |\nabla_Y \varepsilon|^2}{1 + |z|^{2K}} \rho_Y dY + \delta \|\nabla_Y \varepsilon(2)\|_{L^2_{\rho Y}}^2 + c_\delta b^2 \sum |a_{j,M}|^2
\end{align*}
\]
and hence the control of the first nonlinear term:
\[
|\nabla F_1, \nabla \varepsilon(2))_{L^2_{\rho Y}} | \lesssim \delta \|\nabla \varepsilon(2)\|_{L^2_{\rho Y}}^2 + c_\delta b^2 \sum |a_{j,M}|^2 + e^{-\frac{2}{\sqrt{\delta}}} \int \frac{\varepsilon^2 + |\nabla_Y \varepsilon|^2}{1 + |z|^{2K}} \rho_Y dY.
\]

For the second nonlinear term, we compute explicitly
\[
\begin{align*}
\nabla F_2(v) &= p \nabla_Y v \left[ (\tilde{\Phi}_b + v)^{p-1} - \tilde{\Phi}_b^{p-1} \right] \\
&\quad + p \nabla_Y \tilde{\Phi}_b \left[ (\tilde{\Phi}_b + v)^{p-1} - (p - 1) \tilde{\Phi}_b^{p-2} v \right].
\end{align*}
\]
We estimate by homogeneity with the $L^\infty$ bound (3.36):
\[
|F_2(v)| \lesssim |v|^2, \quad |\nabla_Y F_2(v)| \lesssim |\nabla_Y v||v| + |v|^2 \quad (3.60)
\]
and hence the bound using (3.36) again:
\[
\begin{align*}
&|\nabla F_2(v), \nabla \varepsilon(2))_{L^2_{\rho Y}} | + \left| \left( \frac{2}{p - 1} - p \Phi_b^{p-1} F_2(v) \right) , \varepsilon \right|_{L^2_{\rho Y}} \\
&\quad \lesssim \int |v| |\nabla_Y v| + |v|^2 | |\nabla \varepsilon(2)||\rho_Y dY + \int |\varepsilon(2)||v|^2 \rho_Y dY \\
&\quad \leq \delta \|\varepsilon(2)\|_{H^1_{\rho Y}}^2 + C_\delta \left[ \int |v|^4 |\nabla_Y v|^2 \rho_Y dY + \int |v|^4 \rho_Y dY \right] \leq \delta \|\varepsilon(2)\|_{H^1_{\rho Y}}^2 + C_\delta \|v||_{H^1_{\rho Y}}^2 \|v||_{H^1_{\rho Y}}^2 \\
&\quad \leq \delta \|\varepsilon(2)\|_{H^1_{\rho Y}}^2 + C_\delta \|v||_{L^\infty}^2 \left[ \|v||_{H^1_{\rho Y}}^2 + \sum |a_{j,M}|^2 \right].
\end{align*}
\]
The collection of above bounds together with the spectral gap estimate (2.9) and 
the orthogonality conditions (3.19) injected into (3.56) yields (3.61).

3.4. Inner $W^{1,2q+2}$ bounds with polynomial localization in $z$. The bounds of 
Lemma 3.5 rely in an essential way on the spectral gap estimate (2.9) which demands 
a Gaussian like localization measure. Once these bounds are known, they can be 
turned into polynomially weighted bounds provided the weight is strong enough, 
and the approximate solution of Lemma 2.7 has been developed to a sufficiently high 
order.

**Lemma 3.6** (Lyapunov control of polynomially weighted norms). Let $K \geq K(q)$ a 
large enough constant and recall (3.8):

$$\nu_K(z) = \frac{1}{1 + z^{2K}}.$$ 

Then there holds the pointwise differential bounds:

$$\frac{d}{ds} \| \varepsilon \sqrt{\nu_K} \|^2_{L^2_{p_Y}} + \frac{K}{8} \| \varepsilon \sqrt{\nu_K} \|^2_{L^2_{p_Y}} + \| \sqrt{\nu_K} \nabla \varepsilon \|^2_{L^2_{p_Y}}$$

$$\lesssim \| \varepsilon \|^2_{L^2_{p_Y}} + bK^{q+\frac{1}{2}} + (\| v \|^2_{L^\infty} + b^2) \sum |a_{j,M}|^2,$$

(3.61)

$$\frac{d}{ds} \left( \int |\nabla \varepsilon|^{2q+2} \nu_K \rho_Y dY \right) + \frac{K}{16q + 16} \int |\nabla \varepsilon|^{2q+2} \nu_K \rho_Y dY$$

$$\lesssim \| \varepsilon \|^2_{L^2_{p_Y}} + \int |\nabla \varepsilon|^{2q+2} \rho_Y dY + b^{2q+K+\frac{3}{2}} + \sum |a_{j,M}|^{2q+2}.$$ (3.62)

**Remark 3.7.** We more precisely need $K \gtrsim \| \Phi \|_{L^\infty}^{p-1}$ in order to absorb the potential 
terms in the energy estimates below. Also the constants in the rhs of (3.61), (3.62) 
do not depend on $K$.

**Proof of Lemma 3.6.** This follows from a brute force energy identity using the weight 
$\frac{1}{1 + |z|^{2K}}$ to overcome the bounded potential $\Phi^{p-1}$.

**Step 1** $L^2$ weighted bound. From (3.23):

$$\frac{1}{2} \frac{d}{ds} \| \varepsilon \sqrt{\nu_K} \|^2_{L^2_{p_Y}} = (\partial_s \varepsilon, \nu_K \varepsilon)_{L^2_{p_Y}}$$

$$= -(\mathcal{L}_Y \varepsilon, \nu_K \varepsilon)_{L^2_{p_Y}} + (\widetilde{\Psi}_b + L(\varepsilon) - \text{Mod} + F(v), \nu_K \varepsilon)_{L^2_{p_Y}}.$$ 

We integrate by parts to compute:

$$\int \left( -\Delta_Y \varepsilon + \frac{1}{2} Y \cdot \nabla \varepsilon \right) \nu_K \varepsilon \rho_Y dY$$

$$= \int \left[ -\frac{1}{\rho^2} \partial_r (r^2 \rho \partial_r \varepsilon) - \partial_z \varepsilon + \frac{1}{2} z \partial_z \varepsilon \right] \nu_K(z) r^2 \rho_Y dr dz$$

$$= \int |\nabla Y \varepsilon|^2 \nu_K \rho_Y dY - \int \varepsilon^2 \left( \frac{(z \nu_K')'}{4} + \frac{\nu'_K}{2} \right) \rho_Y dY$$

and hence

$$-(\mathcal{L}_Y \varepsilon, \frac{\varepsilon}{1 + z^{2K}})_{L^2_{p_Y}} = - \int |\nabla Y \varepsilon|^2 \nu_K \rho_Y dY$$

$$+ \int \varepsilon^2 \left( -\frac{2}{p - 1} + p \Phi^{p-1} \right) \nu_K + \frac{(z \nu_K')'}{4} \rho_Y dY.$$ (3.63)
We now observe that for \(|z| \geq z(K)|
\[
\frac{(zvK)'}{4} + \frac{v''}{2}K \leq -\frac{K}{4|z|^2K} \tag{3.64}
\]
and hence for \(K \gtrsim 1 + \|\Phi\|_{L^\infty}\):
\[
-(\mathcal{L}Y\varepsilon, \nuK\varepsilon)_{L^2_{v\nu}} \leq -\int |\nabla Y| \varepsilon^2 \nuK dY - \frac{K}{8} \|\varepsilon \sqrt{\nuK}\|_{L^2_{v\nu}}^2 + C_K \|\varepsilon\|_{L^2_{v\nu}}, \tag{3.65}
\]
where the last term controls the region \(|z| \leq z(K)|
. The leading order term is estimated from (2.44) (2.45):
\[
|(\varepsilon, \nuK \tilde{\Phi}_b)_{L^2_{v\nu}}| \leq \delta \|\varepsilon \sqrt{\nuK}\|_{L^2_{v\nu}}^2 + c_\delta \int_{|z| \leq 25} \frac{1}{1 + |z|^{2K}} \left[ b^{2\nu+2} + b^2 \right] |Z|^{4n+4} \ dz
\]
\[+ c_\delta b^2 \int_{|z| \geq 4} \frac{dZ}{1 + |z|^{2K}}. \]
We estimate after changing variables \(Z = z\sqrt{b}\):
\[
\int_{|Z| \leq 25} \frac{1}{1 + |z|^{2K}} \left[ b^{2\nu+2} + b^2 \right] |Z|^{4n+4} \rho_r \ dz \lesssim b^{2\nu+2} + \int_{|Z| \leq 25} \frac{b^2}{\sqrt{b}} \frac{1}{|Z|^{2K}} |Z|^{4n+4} \ dZ
\]\[+ b^2 \int_{|Z| \geq 4} \frac{1}{\sqrt{b}} \frac{1}{|Z|^{2K}} |Z|^{4n+4} \ dZ \lesssim b^{K+\frac{1}{2}} \]
provided \(n \geq n(K)|\) has been chosen large enough in Lemma 2.7. We next integrate by parts like for the proof of (2.42) to compute:
\[
|(\nuK\varepsilon, \Lambda Y\varepsilon)_{L^2_{v\nu}}| \lesssim \int \nuK (1 + r^2) \varepsilon^2 \rho_r r^2 \ dr \ dz \lesssim \int \nuK (|\nabla \varepsilon|^2 + \varepsilon^2) \rho_r r^2 \ dr \ dz \tag{3.66}
\]
where we used (A.1) in the last step, and hence from (3.50):
\[
|(L\varepsilon, \nuK\varepsilon) \lesssim \|\|\nabla \varepsilon \sqrt{\nuK}\|_{L^2_{v\nu}}^2 + \|\varepsilon \sqrt{\nuK}\|_{L^2_{v\nu}}^2\). \tag{3.67}
\]
To estimate the modulation equation terms, we first observe from (2.7) that
\[
\|\sqrt{\nuK} \phi_{j,2M}\|_{L^2_{v\nu}} + \|\Lambda \tilde{\Phi}_b \sqrt{\nuK}\|_{L^2_{v\nu}} + \|\partial_t \tilde{\Phi}_b \sqrt{\nuK}\|_{L^2_{v\nu}} \lesssim 1, \quad -\ell_0 \leq j \leq -1, \quad 0 \leq M \leq M(j) \tag{3.67}
\]
provided \(K|\) satisfies (3.2) and hence from (3.35):
\[
\|\text{Mod} \sqrt{\nuK}\|_{L^2_{v\nu}} \lesssim \sum |(a_{j,M})_s - (\lambda_j + M)a_{j,M}| + \left| \frac{\lambda_s}{\lambda} + 1 \right| \sum |a_{j,M}| + \left| \frac{\lambda_s}{\lambda} + 1 \right| - M(b)| + |b_s + bB(b)| \]
\[
\lesssim b^{n+1} + b \left( \|\varepsilon\|_{L^2_{v\nu}} + \sum |a_{j,M}| \right) \tag{3.68}
\]
which yields the bound:
\[
|(\text{Mod}, \nuK\varepsilon)_{L^2_{v\nu}}| \lesssim b^{2n+1} + b\|\varepsilon\|_{L^2_{v\nu}}^2 + b^2 \sum |a_{j,M}| + \|\varepsilon \sqrt{\nuK}\|_{L^2_{v\nu}}^2. \]

The small linear term is estimated in brute force using \(\|\tilde{\Phi}_b\|_{L^\infty} + \|\Phi\|_{L^\infty} \lesssim 1:
\[
|(F_1(\nu), \nuK\varepsilon)_{L^2_{v\nu}}| \lesssim \int (|\varepsilon| + |\psi|) \nuK \varepsilon \varepsilon^{-\frac{r^2}{2}} r^2 \ dz \leq \frac{K}{20} \|\varepsilon \sqrt{\nuK}\|_{L^2_{v\nu}}^2 + \sum |a_{j,M}|^2
\]
The leading order term is estimated from (2.44) (2.45):
\[-\int v_K|\varepsilon|^2\rho_r r^2 dr dz \leq \frac{K}{20} \int v_K|\varepsilon|^2\rho_r r^2 dr dz + \int v_K|\varepsilon|^4\rho_r r^2 dr dz\]
\[-\int v_K|\varepsilon|^2\rho_r r^2 dr dz + \|v\|_{L^\infty}^2 \int v_K|\varepsilon|^2\rho_r r^2 dr dz \leq \frac{K}{20} \int v_K|\varepsilon|^2\rho_r r^2 dr dz\]
where we used (3.67) in the last step, and (3.61) follows.

**step 2** \(W^{1,2q+2}\) weighted bound. Let \(\varepsilon_i = \partial_i \varepsilon\), for \(i = 1, 2, 3, 4\). We compute from (3.52):
\[
\frac{1}{2q + 2} \int v_K \varepsilon_i^{2q+2} \rho_r dr dY = \int v_K \varepsilon_i^{2q+1} \partial_i \varepsilon_i = -\left( (L + 1) \varepsilon_i, \nu \varepsilon_i^{2q+1} \right)_{L^2_{pr}} + \left( \varepsilon_i^{2q+1}, \nu \partial_i \left[ \bar{\Psi}_b - \text{Mod} + L(\varepsilon) + F(v) \right] \right)_{L^2_{pr}} + \left( \varepsilon_i^{2q+1}, \nu \partial_i \left( p - 1 \right) \Phi^{p-2} \Phi \right)_{L^2_{pr}}.
\]

We integrate by parts to compute:
\[
\int \left( -\Delta Y \varepsilon_i + \frac{1}{2} Y \cdot \nabla Y \varepsilon_i \right) v_K \varepsilon_i^{2q+1} \rho_r dY = \int \left[ -\frac{1}{2} \frac{\partial}{\partial r^2} (r^2 \rho_r \partial_r \varepsilon_i) - \frac{1}{2} \frac{\partial}{\partial r^2} (\varepsilon_i^{2q+1}) \right] v_K(z) \varepsilon_i^{2q+1} \rho_r r^2 dr dz = (2q + 1) \int \varepsilon_i^{2q+1} |\partial_i \varepsilon_i|^2 v_K \rho_r r^2 dr dz + (2q + 1) \int (\varepsilon_i^{2q+1})^2 v_K \rho_r r^2 dr dz - \frac{1}{2q + 2} \int \varepsilon_i^{2q+2} \left( \frac{z v_K}{2} + v''_K \right) \rho_r r^2 dr dz \geq c \int |\nabla Y(\varepsilon_i^{q+1})|^2 v_K \rho_r dY + \frac{K}{8q + 8} \int \varepsilon_i^{2q+2} v_K \rho_r dY - CK \int \varepsilon_i^{2q+2} \rho_Y dY
\]
where we used (3.64) in the last step, and hence
\[-\left( (L + 1) \varepsilon_i, \nu \varepsilon_i^{2q+1} \right)_{L^2_{pr}} \leq -c \int |\nabla Y(\varepsilon_i^{q+1})|^2 v_K \rho_r dY + \frac{K}{16q + 16} \int \varepsilon_i^{2q+2} v_K \rho_r dY\]
+ \(CK \int \varepsilon_i^{2q+2} \rho_Y dY\).

The leading order term is estimated from (2.44) (2.45):
\[
\left( \varepsilon_i^{2q+1}, \nu \partial_i \bar{\Psi}_b \right)_{L^2_{pr}} \leq \int v_K \left( |\varepsilon_i^{2q+2} + |\partial_i \bar{\Psi}_b|^{2q+2} \right) \rho_r dY
\leq \int v_K |\varepsilon_i^{2q+2} |\rho_r dY + \int_{|z| \leq 2d} \frac{1}{1 + |z|^{2K}} \left[ b^{2n+2} + b|z|^{4n+4} \right]^{2q+2} dz + b^{2q+2} \int_{|z| \geq d} \frac{1}{1 + |z|^{2K}} dz \leq \int v_K |\varepsilon_i^{2q+2} |\rho_r dY + b^{2q+2} d^{2K+2}.
\]
We next integrate by parts and use (A.1) to estimate:

\[ |(\nu K \varepsilon_i^{2q+1}, \partial_t \Lambda Y \varepsilon_i)_{L^2_{\mu}}| \lesssim \int \nu K (1+r^2) \varepsilon_i^{2q+2} \rho_r dY \lesssim \int \nu K \left[ \varepsilon_i^{2q+2} + |\nabla Y(\varepsilon_i^{q+1})|^2 \right] \rho_r dY \]

and hence from (3.50):

\[ |(\nu K \varepsilon_i^{2q+1}, \partial_t L(\varepsilon))_{L^2_{\mu}}| \lesssim b \int \nu K \left[ \varepsilon_i^{2q+2} + |\nabla Y(\varepsilon_i^{q+1})|^2 \right] \rho_r dY. \]

The modulation equation terms are estimated in brute force for \( K \geq K(q) \) large enough from (3.35):

\[ |(\nu K \varepsilon_i^{2q+1}, \partial_t \text{Mod})_{L^2_{\mu}}| \lesssim \int \nu K \varepsilon_i^{2q+2} \rho_r dY + \int |\partial_t \text{Mod}|^{2q+2} \nu K \rho_r dY \]

\[ \lesssim \int \nu K \varepsilon_i^{2q+2} \rho_r dY + b^{2q+2} \left[ \|\varepsilon\|^{2q+2}_{L^2_{\rho}} + \sum |a_{j,M}|^{2q+2} \right] + b^{2q+K+\frac{3}{2}}. \]

Also, we have

\[ \left( \varepsilon_i^{2q+1}, \nu K p(p-1) \Phi^{p-2} \partial_t \Phi \right)_{L^2_{\mu}} \lesssim \int |\varepsilon_i|^{2q+1} |\varepsilon| \nu K \rho_r dY \lesssim \int |\varepsilon_i|^{2q+2} \nu K \rho_r dY. \]

For the nonlinear term, we estimate in brute force from (3.54):

\[ |(\nu K \varepsilon_i^{2q+1}, \partial_t F(v))_{L^2_{\mu}}| \lesssim \int (|v| + |\nabla Y v|) \nu K |\varepsilon_i|^{2q+1} \rho_r dY \lesssim \int |\varepsilon_i|^{2q+2} \nu K \rho_r dY + \sum |a_{j,M}|^{2q+2}. \]

The collection of above bounds concludes the proof of (3.62). \( \square \)

3.5. Outer global \( W^{1,2q+2} \) bound. We recall

\[ v = \varepsilon + \psi \]

and now aim at propagating an unweighted global \( W^{1,2q+2} \) decay estimate for \( v \). We rewrite (3.21) as

\[ \partial_s v - \Delta_Y v - \frac{\lambda_s}{\lambda} \Lambda_Y v = h, \quad (3.69) \]

with

\[ h = \Phi_b + \left( \frac{\lambda_s}{\lambda} + \frac{1}{2} - M(b) \right) \Lambda \Phi_b - (b_s + b B(b)) \partial_t \Phi_b + \tilde{F}(v), \quad \tilde{F} = (\Phi_b + v)^p - \Phi_b^p. \]

Lemma 3.8 (Global \( W^{1,2q+2} \) bound). There holds the Lyapunov type monotonicity formula

\[ \frac{d}{ds} \left\{ \int |v|^{2q+2} dY \right\} + c \int |v|^{2q+2} dY \lesssim b^{2q+\frac{1}{2}} + \frac{1}{b^K} \int |v|^{2q+2} dY, \quad (3.70) \]

\[ \frac{d}{ds} \left\{ \int |\nabla_Y v|^{2q+2} dY \right\} + c \int |\nabla_Y v|^{2q+2} dY \lesssim b^{2q+\frac{1}{2}} + \frac{1}{b^K} \int |\nabla_Y v|^{2q+2} + |v|^{2q+2} dY, \quad (3.71) \]

for some universal constant \( c(q) > 0 \).

Proof of Lemma 3.8. step 1 \( L^{2q+2} \) bound. We compute from (3.69):

\[ \frac{1}{2q+2} \frac{d}{ds} \int v^{2q+2} dY = \int v^{2q+1} \partial_s v = \int v^{2q+1} \left[ \Delta_Y v + \frac{\lambda_s}{\lambda} \Lambda_Y v + h \right] dY. \]
The linear term is computed by integration by parts:

\[
\int v^{2q + 1} \left( \Delta_Y v + \frac{\lambda_s}{\lambda} \Lambda v \right) dY 
= -(2q + 1) \int v^{2q} |\nabla_Y v|^2 dY + \frac{\lambda_s}{\lambda} \left( \frac{2}{p - 1} - \frac{4}{2q + 2} \right) \int v^{2q + 2} dY.
\] (3.72)

Observe using (3.50) that

\[
\frac{\lambda_s}{\lambda} \left[ \frac{2}{p - 1} - \frac{4}{2q + 2} \right] \int v^{2q + 2} dY 
= \left( \frac{1}{2} + O(b) \right) \left( \frac{2}{p - 1} - \frac{4}{2q + 2} \right) \int v^{2q + 2} dY 
\leq -c \int v^{2q + 2} dY
\]

where \( c > 0 \) for \( b \) small enough and \( q \) large enough. Next by Hölder:

\[
\int v^{2q + 1} h dY \leq \delta v^{2q + 2} dY + c_5 \int h^{2q + 2} dY
\]

and we now estimate the \( h \) terms. First from (2.44) (2.45):

\[
\int |\tilde{\Psi}_b|^2 dY \lesssim \int_{|Z| \leq \delta} \frac{b^{n+1} + b |Z|^{2n+2}}{\langle r \rangle^{\frac{p}{2} - \frac{1}{q}}} dY 
+ b^{2q+2} \int_{|Z| \geq \delta} (|\partial_Z^2 G| + |Z \partial_Z G|)^{2q+2} dY 
\lesssim b^{2q+2 - \frac{1}{q}}
\]

for \( q \) large enough, where we used in the last inequality the fact that in view of (2.23) (2.39) (2.40), we have

\[
|\partial_Z^2 G| + |Z \partial_Z G| \lesssim \frac{1}{(1 + r^2 + |Z|^2)^{\frac{1}{p}}}
\]

In order to treat the modulation equation terms, we compute

\[
|\Lambda_Y \tilde{\Phi}_b| = \left| \frac{1}{\mu^{\frac{2}{p-1}}} \left( 1 - \frac{Z^2}{1 + Z^2} \right) \Lambda \Phi \left( \frac{r}{\mu} \right) \right| \approx \frac{1}{\mu^{\frac{2}{p-1}} + \langle r \rangle^{\frac{2}{p-1}}} \approx \frac{1}{(1 + r^2 + b z^2)^{\frac{1}{p}}}
\]

and hence for \( q \) large enough using (2.16):

\[
\int |\Lambda_Y \tilde{\Phi}_b|^{2q+2} \lesssim \frac{1}{\sqrt{b}}
\]

Similarly:

\[
|\partial_b \tilde{\Phi}_b| \lesssim \frac{1}{2b} |Z \partial_Z G| \lesssim \frac{1}{b} \frac{Z^2}{1 + Z^2} \frac{1}{\mu^{\frac{2}{p-1}}} \Lambda \Phi \left( \frac{r}{\mu} \right) \lesssim \frac{1}{b} \frac{1}{(1 + r^2 + b z^2)^{\frac{1}{p}}}
\]

and hence

\[
\int |\partial_b \tilde{\Phi}_b|^{2q+2} dY \lesssim \frac{1}{\sqrt{bb^{2q+2}}}
\]

We conclude using (3.35):

\[
\int \left| \left( \frac{\lambda_s}{\lambda} + \frac{1}{2} - M(b) \right) \Lambda \tilde{\Phi}_b - (b_s + b B(b)) \partial_b \tilde{\Phi}_b \right|^{2q+2} dY 
\lesssim \frac{1}{\sqrt{b} \left( b^n + \| \varepsilon \|_{L^p_{2r}} + \sum |a_{j,M}| \right)^{2q+2}} \lesssim \frac{b^n}{b}
\]

where we used in the last inequality the bounds (3.27) (3.28) (3.29).
We now turn to the nonlinear term whose control relies on the polynomially weighted bounds of Lemma 3.6. Indeed, we estimate by homogeneity

$$|\tilde{F}(v)| \lesssim \tilde{\Phi}_b |v| + \|v\|_{L^\infty} |v|$$

from which:

$$\int |\tilde{F}(v)|^{2q+2} dY \lesssim \int |\tilde{\Phi}_b|^{2q+2} |v|^{2q+2} + \|v\|^{2q+2}_{L^\infty} \int |v|^{2q+2} dY.$$ 

The second term is treated thanks to $\|v\|_{L^\infty} \ll 1$, and we split the first term using

$$\|\partial^k \tilde{\Phi}_b\|_{L^\infty(|Z| \geq A)} + \|\partial^k \tilde{\Phi}_b\|_{L^\infty(|r| \geq A)} < \delta \ll 1, \quad k = 0, 1$$

for $A$ large enough, and hence

$$\int |\tilde{\Phi}_b|^{2q+2} |v|^{2q+2} \lesssim \int_{|Z| \geq A} |\tilde{\Phi}_b|^{2q+2} |v|^{2q+2} + \int_{|r| \geq A} |\tilde{\Phi}_b|^{2q+2} |v|^{2q+2} \lesssim \delta \int |v|^{2q+2} dY + \frac{C(A)}{b^k} \int \frac{|v|^{2q+2}}{1 + |z|^{2k}} dY.$$ 

The collection of above bounds concludes the proof of (3.70).

**step 2** $W^{1,2q+2}$ bound. Let $v_i = \partial_i v$ for $i = 1, 2, 3, 4$. Then from (3.69):

$$\partial_s v_i - \Delta_Y v_i - \frac{\lambda_s}{\lambda} (v_i + \Lambda_Y v_i) = \partial_i h,$$

and hence:

$$\frac{1}{2q+2} \int v_i^{2q+2} dY = \int v_i^{2q+1} \partial_s v_i = \int v_i^{2q+1} \left[\Delta_Y v_i + \frac{\lambda_s}{\lambda} (v_i + \Lambda_Y v_i) + \partial_i h\right] dY.$$ 

The linear term is computed from (3.72):

$$\int v_i^{2q+1} \left[\Delta_Y v_i + \frac{\lambda_s}{\lambda} (v_i + \Lambda_Y v_i)\right] dY$$

$$= - (2q+1) \int v_i^{2q} |\nabla_Y v_i|^2 dY + \frac{\lambda_s}{\lambda} \left(1 + \frac{2}{p-1} - \frac{4}{2q+2}\right) \int v_i^{2q+2} dY$$

$$= - \left(\frac{1}{2} + O(b)\right) \left(1 + \frac{2}{p-1} - \frac{4}{2q+2}\right) \int v_i^{2q+2} dY$$

$$\leq -c \int v_i^{2q+2} dY.$$ 

Next, we have by Hölder:

$$\int v_i^{2q+1} \partial_i h dY \leq \delta v_i^{2q+2} dY + c_\delta \int (\partial_i h)^{2q+2} dY$$

and we now estimate the $\partial_i h$ terms. From (2.45):

$$\int |\partial_i \tilde{\Phi}_b|^{2q+2} \lesssim \int_{|Z| \leq 2\delta} \left|\frac{b^{n+1} + b|Z|^{2n+2-1}}{(r)^{\frac{2}{p-1}} - \frac{1}{n}}\right|^{2q+2} dY$$

$$+ b^{2q+2} \int_{|Z| \geq 2\delta} (|\partial_i \partial^2 Z G| + |\partial_i (Z \partial Z G)|)^{2q+2} dY$$

$$\lesssim b^{2q+2 - \frac{1}{2}}$$
for $q$ large enough. For the modulation equation terms, we estimate in brute force as above
\[ \int |\partial_t \Lambda Y \tilde{\Phi}_b|^{2q+2} dY \lesssim \frac{1}{\sqrt{b}} \int |\partial_t \partial_b \tilde{\Phi}_b|^{2q+2} dY \lesssim \frac{1}{\sqrt{b}} \]
from which using (3.35):
\[ \int \left| \left( \frac{\lambda_s}{\lambda} + \frac{1}{2} - M(b) \right) \partial_t \Lambda Y \tilde{\Phi}_b - (b_s + b B(b)) \partial_t \partial_b \tilde{\Phi}_b \right|^{2q+2} dY \lesssim b^\frac{1}{2} \]
We now estimate the nonlinear term by homogeneity:
\[ |\partial_i \hat{F}(v)| \lesssim |\partial_i \tilde{\Phi}_b||v| + |\partial_i v| \left[ \|v\|_{L^\infty}^{p-1} + |\tilde{\Phi}_b|^{p-1} \right] \]
from which for $A$ large enough using (3.73):
\[ \int |\partial_i \hat{F}(v)|^{2q+2} dY \lesssim \delta \int (|v|^{2q+2} + |v|_{L^\infty}^{2q+2}) dY + \int |z| \leq A, r \leq A \left| \frac{C(A)}{b^R} + 1 + |z|^{2K} \right| \rho_r dY. \]
The collection of above bounds concludes the proof of (3.71). \qed

3.6. Conclusion. We are now in position to conclude the proof of Proposition 3.3 which then easily implies Theorem 1.1.

Proof of Proposition 3.3. We recall that we are arguing by contradiction assuming (3.34). We first show that the bounds (3.26), (3.27), (3.29), (3.30), (3.31), (3.32) can be improved on $[s_0, s^*]$, and then, the existence of the data $a_{j,M}(0)$ follows from a classical topological argument à la Brouwer.

step 1 Improved control of the geometrical parameters. We estimate from (3.50):
\[ \frac{1}{2} (1 - \delta) < -\frac{\lambda_s}{\lambda} < \frac{1}{2} (1 + \delta), \quad 0 < \delta \ll 1 \]
which implies
\[ (\lambda(s) e^{\frac{s}{2}})' < 0, \quad (\lambda(s) e^{2s})' > 0 \]
and hence using (3.9)
\[ 0 < \lambda(s) < \lambda(s_0) e^{\frac{2s_0}{2}} e^{-\frac{s}{2}} < \frac{1}{2} e^{-\frac{s}{2}}, \]
and (3.26) is improved. For the $b$ parameter, we estimate from (3.35), (3.27), (3.28), (3.29) and (2.21):
\[ |b_s + c_1 b^2| \lesssim \frac{1}{s^3}, \quad c_1 > 0. \]
Hence
\[ \left| \frac{d}{ds} \left( -\frac{1}{b} \right) + c_1 \right| \lesssim \frac{1}{s^3 b^2} \lesssim \frac{1}{s} \]
which time integration using (3.10) yields:
\[ \left| -\frac{1}{b(s)} + c_1 s \right| \leq \sqrt{s} \quad \text{and thus} \quad \frac{1}{2c_1 s} < b(s) < \frac{2}{c_1 s} \]
which improves (3.27).
**step 2** Improved Sobolev bounds. We now systematically integrate in time the monotonicity formulas of Lemmas 3.5, 3.6, 3.8 to improve the bounds (3.29)-(3.32).

**Exponential norms.** We rewrite (3.45) using (3.27), (3.28), (3.29), (3.36) and obtain:

\[
\frac{d}{ds} \| \epsilon \|_{L^2_{r_Y}}^2 + c \| \epsilon \|_{L^2_{r_Y}}^2 \lesssim \frac{1}{s^{n+\delta_q}}
\]

which time integration with (3.12) yields:

\[
\| \epsilon(s) \|_{L^2_{r_Y}}^2 \leq \| \epsilon(0) \|_{L^2_{r_Y}}^2 e^{-c(s-s_0)} + s e^{-c(s-s_0)} \lesssim \frac{1}{s^{n+\delta_q}}
\]

(3.74)

where we used \((s/s_0)^{2n} e^{-c(s-s_0)} \leq 1\) for \(s \geq s_0\) since \(s \mapsto (s/s_0)^{2n} e^{-c(s-s_0)}\) is non increasing on \([s_0, +\infty)\) for \(s_0(n)\) large enough. We similarly rewrite (3.47):

\[
\frac{d}{ds} \| L_Y \epsilon \|_{L^2_{r_Y}}^2 + c \| L_Y \epsilon \|_{L^2_{r_Y}}^2 \lesssim \frac{1}{s^{n+\delta_q}}
\]

which similarly yields with (3.12):

\[
\| L_Y \epsilon(s) \|_{L^2_{r_Y}}^2 \lesssim \frac{1}{s^{n+\delta_q}}.
\]

(3.75)

We now recall

\[
(L_Y \epsilon, \epsilon)_\rho = \| \nabla \epsilon \|_{L^2_{r_Y}}^2 + \int \left( \frac{2}{p-1} - p\Phi^{p-1} \right) |\epsilon|^2 \rho_Y dY
\]

which together with (3.74), (3.75) implies:

\[
\| \nabla \epsilon \|_{L^2_{r_Y}}^2 \leq (L_Y \epsilon, \epsilon)_{L^2_{r_Y}} + C \| \epsilon \|_{L^2_{r_Y}}^2 \lesssim \frac{1}{s^{n+\delta_q}}.
\]

This implies from (A.2):

\[
\| \Delta \epsilon \|_{L^2_{r_Y}}^2 \lesssim (L_Y \epsilon, \epsilon)_{L^2_{r_Y}} + \| \epsilon \|_{H^1_{r_Y}}^2 \lesssim \frac{1}{s^{n+\delta_q}}
\]

which together with (3.74), (3.75) yields

\[
\| \epsilon \|_{H^2_{r_Y}}^2 \lesssim \frac{1}{s^{n+\delta_q}} < \frac{1}{2s^n}
\]

(3.76)

which improves (3.29). Similarly from (3.46), (3.27), (3.28), (3.29):

\[
\frac{d}{ds} \| \nabla \epsilon \|_{L^2_{r_Y}}^{2q+2} + c \| \nabla \epsilon \|_{L^2_{r_Y}}^{2q+2} \lesssim \frac{1}{s^{(q+1)(n+1)}}
\]

which time integration using (3.12) ensures:

\[
\| \nabla \epsilon \|_{L^2_{r_Y}}^{2q+2} \lesssim \frac{1}{s^{(q+1)(n+1)}} \leq \frac{1}{2s^n(q+1)}
\]

and (3.30) is improved.

**Polynomial norms.** We rewrite (3.61) using (3.27), (3.28), (3.29), (3.32) as:

\[
\frac{d}{ds} \| \sqrt{K} \|_{L^2_{r_Y}}^2 + \frac{K}{s} \| \sqrt{K} \|_{L^2_{r_Y}}^2 \lesssim \frac{1}{s^{K+\frac{1}{2}}}
\]
for $n \geq n(K)$ large enough, which time integration using (3.13) yields:

$$\|\varepsilon(s)\|_{L^2_{t\nu}}^2 \lesssim \frac{1}{s^{K+\frac{1}{2}}} < \frac{1}{2s^{K+1}}.$$ 

Similarly from (3.62):

$$\frac{d}{ds} \left( \int |\nabla \varepsilon|^{2q+2} \nu K \rho r dY \right) + \frac{K}{16q+16} \int |\nabla \varepsilon|^{2q+2} \nu K \rho r dY \lesssim \frac{1}{s^{K+\frac{1}{2}}}$$

which together with (3.13) ensures:

$$\int |\nabla \varepsilon(s)|^{2q+2} \nu K \rho r dY \lesssim \frac{1}{s^{K+\frac{1}{2}} + 1} < \frac{1}{2s^{K+2q+1}}.$$ 

This yields an improvement of (3.31).

**Global norms.** We use the lossy bound

$$\int |v|^{2q+2} \nu K \rho r dY \lesssim \|v\|_{L^\infty} \int |v|^{2q+2} \nu K \rho r dY \lesssim \int |\varepsilon|^2 \rho r dY + \sum |a_{j,M}|^2$$

which injected into (3.70), (3.71) yields

$$\frac{d}{ds} \|v\|_{W^{1,2q+2}} \lesssim \frac{1}{s^{2q+\frac{1}{2}}} + \frac{1}{3K} \left[ \frac{1}{s^{K+1}} \right] \lesssim \frac{1}{s}. $$

Integrating in time using (3.14) ensures

$$\|v(s)\|_{W^{1,2q+2}} \lesssim \frac{1}{s}$$

which improves (3.32) provided $\delta_q$ has been chosen small enough.

**step 3** Brouwer fixed point argument. In view of the above improvements of (3.26), (3.27), (3.29), (3.30), (3.31), (3.32), we conclude from an elementary continuity argument that (3.34) implies the exit condition:

$$\sum_{j=-\ell_0}^{-2} \sum_{M=0}^{M(j)} (a_{j,M}(s^n))^2 = \frac{1}{(s^n)^n}. \quad (3.77)$$

On the other hand, we estimate from (3.35), (29):

$$\sum |(a_{j,M})_s + (\lambda_j + M)a_{j,M}| \lesssim \frac{1}{s^{\frac{1}{2}+1}}.$$ 

Also, from the non degeneracy (1.9), there exists $c > 0$ such that

$$\lambda_j + M \leq -c < 0, \quad -\ell_0 \leq j \leq -2, \quad 0 \leq M \leq M(j)$$

and hence

$$\frac{d}{ds} \sum s^n (a_{j,M}(s)) = s^n \sum a_{j,M} \left[ (a_{j,M})_s + \frac{n}{s} a_{j,M} \right]$$

$$= s^n \sum a_{j,M} \left[ (a_{j,M})_s + (\lambda_j + M)a_{j,M} + \frac{n}{s} a_{j,M} \right] - s^n \sum (\lambda_j + M)|a_{j,M}|^2$$

$$\geq c s^n \sum |a_{j,M}|^2 + O \left( \frac{s^n}{s^{n+1}} \right).$$
which implies from (3.77) the outgoing flux condition:

\[
\frac{d}{ds} \left\{ \sum_{j = -\ell_0}^{M(j)} \sum_{\mathcal{M} = 0}^{M(j)} s^n(a_{j,\mathcal{M}}(s))^2 \right\} \bigg|_{s = s^*} > \frac{c}{2} > 0.
\]

We conclude from standard argument that the map

\[
(a_{j,\mathcal{M}}(0)s^n_0 - \ell_0 \leq j \leq -2, 0 \leq \mathcal{M} \leq M(j) \rightarrow (a_{j,\mathcal{M}}(s_*)s^n_0 - \ell_0 \leq j \leq -2, 0 \leq \mathcal{M} \leq M(j))
\]

is continuous on the unit ball of \(\mathbb{R}^N\) where

\[
N = \sum_{j = -\ell_0}^{-2} (1 + M(j))
\]

and the identity on its boundary a contradiction to Brouwer’s Theorem. This concludes the proof of Proposition 3.3. \(\square\)

We are now in position to conclude the proof of Theorem 1.1.

**Proof of Theorem 1.1.** Let an initial data as in Proposition 3.3. The \(s\) time being global, the integration of the modulation equations (3.35) with the bounds (3.29) easily leads to the laws

\[
b(s) = \frac{1}{c_1 s} + O \left( \frac{1}{s^2} \right)
\]

and

\[
\frac{\lambda_s}{\lambda} = -\frac{1}{2} + M(b) + O \left( \frac{1}{s^2} \right) = -\frac{1}{2} + \frac{1}{s} + O \left( \frac{1}{s^2} \right)
\]

where we used the fact that \(d_1 = 1\) in the last equality in view of (2.21). Hence

\[
\lambda(s) = e^{-\frac{1}{2} + O(\log s)} = e^{-\frac{1}{2} s + O \left( \frac{\log s}{s} \right)}.
\]

This implies that the life time of the solution is finite

\[
T = \int_{s_0}^{+\infty} \lambda^2(s)ds < +\infty
\]

and the blow up is self similar

\[
T - t = \int_{s}^{+\infty} \lambda^2(s)ds = e^{-s} \left[ 1 + O \left( \frac{\log s}{s} \right) \right], \quad \lambda(t) = \sqrt{T - t(1 + o(1))}.
\]

The fact that the above construction defines a Lipschitz manifold of initial data in the \(W^{1,2q+2} \cap H^2\) topology is now classical, see [6], and the details are left to the reader. This concludes the proof of Theorem 1.1. \(\square\)

**Appendix A. Coercivity estimates**

**Lemma A.1** (Exponential Hardy). Let \(\nu(z) \geq 0\) and \(u(r, z)\) with cylindrical symmetry, then:

\[
\int \nu |u|^2 (1 + r^2)e^{-\frac{z^2}{\tau}} r^2 drdz \lesssim \int \nu (|u|^2 + |\nabla_Y u|^2) e^{-\frac{z^2}{\tau}} r^2 drdz. \tag{A.1}
\]

Moreover:

\[
\|\Delta_Y u\|^2_{L^2_rY} \lesssim \left\| -\Delta u + \frac{1}{2} Y \cdot \nabla u \right\|^2_{L^2_rY} + \|u\|^2_{H^1_rY}. \tag{A.2}
\]
Proof. step 1 Proof of (A.1). By density, we assume \( u \in D(\mathbb{R}^4) \). We use \( \partial_r \rho_r = -r \rho_r / 2 \) and integrate by parts to compute:

\[
\int_0^{+\infty} \frac{\varepsilon^2 r^2 e^{-\frac{\varepsilon^2}{r^2}}} r^2 dr = \int_0^{+\infty} 6\varepsilon^2 e^{-\frac{\varepsilon^2}{r^2}} dr + 4 \int_0^{+\infty} \partial_r \varepsilon \varepsilon e^{-\frac{\varepsilon^2}{r^2}} dr
\]

and hence from Hölder:

\[
\int_0^{+\infty} \varepsilon^2 r^2 e^{-\frac{\varepsilon^2}{r^2}} dr \lesssim \int_0^{+\infty} |\partial_r \varepsilon| r \varepsilon e^{-\frac{\varepsilon^2}{r^2}} dr + \int_0^{+\infty} \varepsilon^2 e^{-\frac{\varepsilon^2}{r^2}} dr
\]

\[
\leq c \delta \int_0^{+\infty} |\partial_r \varepsilon|^2 r^2 e^{-\frac{\varepsilon^2}{r^2}} dr + \delta \int_0^{+\infty} \varepsilon^2 r^2 e^{-\frac{\varepsilon^2}{r^2}} dr + \int_0^{+\infty} \varepsilon^2 e^{-\frac{\varepsilon^2}{r^2}} dr.
\]

Hence

\[
\int_0^{+\infty} \varepsilon^2 r^2 e^{-\frac{\varepsilon^2}{r^2}} dr \lesssim \int_0^{+\infty} [\varepsilon^2 + \varepsilon^2] r^2 e^{-\frac{\varepsilon^2}{r^2}} dr.
\]

We now multiply by \( \nu \) and integrate in \( z \), and (A.1) is proved.

step 2. Proof of (A.2). We compute:

\[
\left\| -\Delta u + \frac{1}{2} Y \cdot \nabla u \right\|_{L^2_{\rho Y}}^2 = \| \Delta u \|_{L^2_{\rho Y}}^2 + \frac{1}{4} \| Y \cdot \nabla u \|_{L^2_{\rho Y}}^2 - \int (\Delta u) Y \cdot \nabla u \rho_Y dY.
\]

To compute the crossed term, let \( u_{\lambda}(Y) = u(\lambda Y) \), then

\[
\int |\nabla u_{\lambda}(Y)|^2 \rho_Y dY = \frac{1}{\lambda^2} \int |\nabla u(Y)|^2 \rho_Y \left( \frac{Y}{\lambda} \right) dy
\]

and hence differentiating in \( \lambda \) and evaluating at \( \lambda = 1 \):

\[
2 \int \nabla u \cdot \nabla (Y \cdot \nabla u) \rho_Y dY = \frac{1}{\lambda^2} \int |\nabla u(Y)|^2 \rho_Y \left( \frac{Y}{\lambda} \right) dy
\]

i.e.

\[
2 \int Y \cdot \nabla u(Y) \partial u \rho_Y dY = \int |\nabla u|^2 (-2 \rho_Y - Y \cdot \nabla \rho_Y) dY
\]

which using \( \nabla \rho_Y = -\frac{1}{2} Y \rho_Y \) becomes:

\[
- \int (\Delta u) Y \cdot \nabla u \rho_Y dY = \frac{1}{4} \int |\nabla u|^2 |Y|^2 \rho_Y - \frac{1}{2} \int |Y \cdot \nabla u|^2 \rho_Y - \int \rho_Y |\nabla u|^2.
\]

Hence:

\[
\left\| -\Delta u + \frac{1}{2} Y \cdot \nabla u \right\|_{L^2_{\rho Y}}^2 = \| \Delta u \|_{L^2_{\rho Y}}^2 + \frac{1}{4} \int \rho_Y (|Y|^2 |\nabla u|^2 - |Y \cdot \nabla u|^2) - \int \rho_Y |\nabla u|^2
\]

\[
\geq \| \Delta u \|_{L^2_{\rho Y}}^2 - \| \nabla u \|_{L^2_{\rho Y}}^2
\]

which concludes the proof of (A.2). \( \square \)

Appendix B. Proof of Lemma 2.9

Recall that \( j \in \mathbb{N} \) and \( u_j(r) \) is the solution to

\[
(\mathcal{L}_r + j) u_j = f_j \text{ and } (u_1, \Lambda \Phi) = 0 \text{ if } j = 1,
\]

where \( f_j \) satisfies in the case \( j = 1 \)

\[
(f_1, \Lambda \Phi)_{L^2_{\rho r}} = 0.
\]
Recall also from Lemma 2.2 and (2.10) that $\mathcal{L}_r + j$ is a selfadjoint operator with domain $\mathcal{D}(\mathcal{L}_r) \subset L^2(r^2 \rho_r dr)$ and

$$\text{Ker}(\mathcal{L}_r + 1) = \langle \Lambda_r \Phi \rangle \text{ and Ker}(\mathcal{L}_r + j) = \{0\} \text{ for all } j \in \mathbb{N} \setminus \{1\}.$$

We immediately infer that we can solve uniquely

$$(\mathcal{L}_r + j)u_j = f_j \text{ and } (u_1, \Lambda_r \Phi) = 0 \text{ if } j = 1,$$

as long as

$$f_j \in L^2_{\rho_r} \text{ with } (f_1, \Lambda_r \Phi)_{L^2_{\rho_r}} = 0 \text{ in the case } j = 1,$$

and there holds the following trivial bound for $k \in \mathbb{N}$

$$\sum_{l=0}^{k+2} \| \partial^l_r u_j \|_{L^2_{\rho_r}} \lesssim_k \sum_{l=0}^k \| \partial^l_r f_j \|_{L^2_{\rho_r}} \lesssim_k \sum_{l=0}^k \| \langle r \rangle^{2-\eta} \partial^l_r f_j \|_{L^\infty}. \quad (B.1)$$

Next, we derive a pointwise bound for derivatives of $u_j$ in the region $r \geq 1$. There exists two independent solutions $\varphi_{1,j}$ and $\varphi_{2,j}$ of

$$(\mathcal{L}_r + j)\varphi = 0$$

smooth on $(0, +\infty)$ such that

$$\varphi_{1,j} \sim r^{2-1+2j-3} e^{-r^2 \frac{e^2}{\pi}}, \quad \varphi_{2,j} \sim \frac{1}{r^{2-1+2j}} \text{ as } r \to +\infty,$$

and their Wronskian

$$W := \varphi'_{1,j}\varphi_{2,j} - \varphi'_{2,j}\varphi_{1,j}$$

is given by

$$W = \frac{1}{r^{2} e^{\frac{e^2}{\pi}}}.$$

See for example Lemma 3.4 in [6] for a proof. Then, using variation of constants as well as the estimates (B.1) satisfied by $u_j$ implies that $u_j$ is given by

$$u_j(r) = \left( \int_1^{+\infty} f_j \varphi_{2,j}(r')^2 e^{-\frac{(r')^2}{4}} dr' \right) \varphi_{1,j}(r) + \left( a_j - \int_1^{r} f_j \varphi_{1,j}(r')^2 e^{-\frac{(r')^2}{4}} dr' \right) \varphi_{2,j}(r)$$

where the constant $a_j$ is given by

$$a_j = \frac{1}{\varphi'_{2,j}(1)} \left( u_j(1) - \left( \int_1^{+\infty} f_j \varphi_{2,j} r^2 e^{-\frac{r^2}{4}} dr \right) \varphi_{1,j}(1) \right).$$

This immediately yields the pointwise bound\(^3\)

$$\sum_{l=0}^k \| \langle r \rangle^{2-\eta} \partial^l_r u_j \|_{L^\infty(r \geq 1)} \lesssim_{k, \eta} |u_j(1)| + \sum_{l=0}^k \| \langle r \rangle^{2-\eta} \partial^l_r f_j \|_{L^\infty}.\$$

\(^3\)Note that we may take $\eta = 0$ for $j \geq 1$. Only the case $j = 0$ actually contains a log divergence and hence requires $\eta > 0$.\)
Finally, we derive a pointwise bound for $u$ in $r \leq 1$. By the Sobolev embedding in dimension 3 and (B.1), we have
\[
\sum_{l=0}^{k} \left\| \partial^{l}_{r} u_{j} \right\|_{L^{\infty}(r \leq 1)} \lesssim \sum_{l=0}^{k} \left\| \partial^{l}_{r} u_{j} \right\|_{H^{2}(r \leq 1)} \lesssim \sum_{l=0}^{k+2} \left\| \partial^{l}_{r} u_{j} \right\|_{L^{2}_{r}} \lesssim_{k} \sum_{l=0}^{k} \left\| \langle r \rangle^{\frac{2}{p-1} + l - \eta} \partial^{l} f_{j} \right\|_{L^{\infty}}.
\]
This concludes the proof of Lemma 2.9.

References


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