Lectures on Geometric Kinetic Equations

Jacques Smulevici*

October 5, 2016

Contents

*Laboratoire de Mathématiques, Université Paris-Sud 11, bât. 425, 91405 Orsay, France.

1 Preliminary remarks

These lectures have been designed for the June 2016 summer school on "Geometric analysis of wave and fluids", organized at MIT, Boston. In particular, I expect the students to have follow the lectures of the first week and thus, to be familiar with the vector field method for the standard wave equation on Minkowski space and the basics of Lorentzian geometry.

It is clear by now that the vector field method provides a robust framework for the study of quasi-linear wave equations. One of the aims of these lectures is to show that it has in fact a much wider range of applications, as we will develop a vector field type method for kinetic transport equations. We will however pursue the analogies between wave and kinetic equations a bit further. For instance, we know that for non-linear wave equations in low dimensions, *structure* (think null-condition) plays a very important role and we can naturally wonder if there are analogues of these structural properties for non-linear systems of kinetic equations.

Apart from trying to understand kinetic equations using approaches initially developed for wave equations, I will also present some standard techniques from kinetic theory: we all need to learn the basics.

For simplicity, I will often fix the number of spatial dimensions below to 3, but the interested reader can easily derive generalizations to arbitrary numbers of dimensions.

These lectures contain many exercices as well as some open problems, the main difference being that the author of these notes is only certain to know how to solve the exercices. I strongly encourage the students to do at least part of the exercices.

Most of the results and new techniques (in particular the vector field method for Vlasov fields) presented below have been obtained through my joint work with Jérémie Joudioux and David Fajman. We refer thus to our work [FJS15] for details.

These lectures are still in preparation. In particular, with probability one, they contain typos/mistakes or inaccurate references. Feel free to email me for comments.

Signature conventions: $-+++$ for a Lorentzian metric.

2 Introduction: the basic equations

We are interested in the description of a large ensemble of particles, which for simplicity, we assume all have same rest mass m. The equations and the definitions of the objects of study depend on the physical situation we are considering. In the non-relativistic case, the equations are derived from Newtonian mechanics while in the general relativistic case, the equations come from General Relativity. In all cases, the main unknown will be a counting or *distribution* function f, depending on space-time location as well as velocities. Given a point x in space-time and a velocity v, $f(x,v)$ represents the density of particles at position x and velocity v, so that if $d\mu_{x,v}$ is a volume form in phase space, then $f(x,v)d\mu_{x,v}$ should represent the number of particles in that infinitesimal volume.

2.1 The relativistic setting

An easy example is the flat Minkowski space. If $(t,x) \in \mathbb{R}^4$, $v \in \mathbb{R}^3$, then $f(t,x,v)dx^4 dv^3$ gives the number of particles at space-time location (t,x) with 4-velocity[1] $(v^\alpha) := (v^0 := \sqrt{m^2+|v|^2}, v)$. In a general, time-oriented, Lorentzian manifold (M,g), f will be defined on a submanifold of the tangent-bundle, namely, the set of points $(x,v) \in TM$ such that $g_x(v,v) = -m^2$, with v future directed[2]. This submanifold is called the *mass shell* and will be denoted $\mathscr{P}$. We denote by $\pi : TM \to M$ the canonical projection. By a small abuse of notation, we will denote by the same letter the induced projection on $\mathscr{P}$.

In the absence of an external force field, the particles are free-falling and their trajectories are given by the geodesics of the Lorentzian manifold. In these lectures, we shall neglect all possible collisions between particles, which implies that the distribution function f should be *constant* along geodesics.

To write down the equation satisfied by f, let (x^α) be a coordinate system. Given any vector V in T_xM, we know that can decompose V as $V = V^\alpha \partial_{x^\alpha}$. This implies that given a coordinate system (x^α), there is a *natural*[3] coordinate system on TM, given by (x^α, v^α). The (v^α) are called *conjugate* coordinates to the (x^α). Let thus now consider a geodesic $\gamma : s \to \gamma(s)$. Recall that on the mass shell, we have at each point $(x,v) \in \mathscr{P}$, $g_x(v,v) = -m^2$, i.e.

$$g_{00} v^0 v^0 + 2 g_{0i} v^i v^0 + g_{ij} v^i v^j + m^2 = 0,$$

Assume that ∂_{x^0} is timelike (so that $g_{00} < 0$) and that the level set of x^0 are spacelike hypersurfaces (so that g_{ij} is positive definite), then

$$v^0 = \frac{g_{0i} v^i \pm \sqrt{\left(g_{0i} v^i\right)^2 - g_{00}\left(g_{ij} v^i v^j + m^2\right)}}{-g_{00}},$$

with only the + sign kept above if we want only future directed vectors and our manifold is time oriented by the vector field ∂_{x^0}.

Thus, in this setting, we can used (x^α, v^i) where $1 \le i \le 3$ as a coordinate system on the mass shell, the remaining v^0 being obtained by the above formula.

Assume now that $\gamma(s) = (\gamma^\alpha(s))$ in some coordinate system. Then, the geodesic equations reads

$$\frac{d^2\gamma^\alpha}{ds^2} + \Gamma^\alpha_{\beta\sigma} \frac{d\gamma^\beta}{ds} \frac{d\gamma^\sigma}{ds} = 0, \tag{1}$$

where the $\Gamma := \Gamma(g)$ are the Christoffel symbols of the Lorentzian metric. (Recall that

$$\Gamma^\alpha_{\beta\sigma} = \frac{1}{2} g^{\alpha\gamma}\left(g_{\gamma\beta,\sigma} + g_{\gamma\sigma,\beta} - g_{\sigma\beta,\gamma}\right).$$

[1] Actually, this is only the 4 velocity when $m = 1$, otherwise, it corresponds to the 4-momentum. In these lectures, I will, by a small abuse of language use the word velocity often instead of momentum.

[2] Note that a future directed timelike vector is never zero. On the other hand, for null vectors, we adopt here the convention that a future directed null vector is never 0.

[3] Here, by a small abuse of notation, we denote by the same symbols the x^α and their lifts to TM. In the remainder of this text, we will use freely these types of abuse of notation for simplicity.

Let f now be a distribution function which is constant along geodesics. Since f is defined on the mass shell, we can identify locally f with a function of (x^α, v^i).

Now, f being constant along geodesics imply that $f\left(\gamma^\alpha(s), \frac{d\gamma^i}{ds}(s)\right)$ should be constant in s. Differentiating, we obtain that f should solve

$$0 = \frac{d\gamma^\alpha}{ds}\frac{\partial f}{\partial x^\alpha} + \frac{d^2\gamma^i}{ds^2}\frac{\partial f}{\partial v^i},$$

which using the geodesic equation (1) can be rewritten as

$$0 = \frac{d\gamma^\alpha}{ds}\frac{\partial f}{\partial x^\alpha} - \Gamma^i_{\beta\sigma}\frac{d\gamma^\beta}{ds}\frac{d\gamma^\sigma}{ds}\frac{\partial f}{\partial p^i}.$$

Since this relation should be true for any future timelike (or null if $m = 0$) geodesic, we have obtained that f must solve

$$0 = v^\alpha \frac{\partial f}{\partial x^\alpha} - \Gamma^i_{\beta\sigma} v^\beta v^\sigma \frac{\partial f}{\partial v^i}, \tag{2}$$

at each point $(x, v) \in \mathscr{P}$. This is a linear transport equation, called the *Vlasov* equation, whose characteristics are, by construction, the future timelike or null geodesics of the space-time.

In the particular case of Minkowski space in Cartesian coordinates (so that the Christoffel symbols vanishes and $t = x^0$), the Vlasov equation becomes

$$v^\alpha \partial_{x^\alpha} f = 0,$$

where $v^0 = \sqrt{m^2 + |v|^2}$.

We will refers to this as the "free transport" (even though this might be physically incorrect since (2) is also describing free falling particles). Let us define the free transport operator as $T = v^\alpha \partial_{x^\alpha}$. Note that T actually depends on the value of m. In particular, there will be an important difference between the massless particles $m = 0$ and the massive ones $m > 0$.

Exercices:

1. Write the Vlasov equation in a general Lorentzian manifold for charged particles of charge q in presence of an external electromagnetic field described by a 2-form (Faraday tensor) F. (recall that the trajectory γ of a charged particle is given by
$$\nabla_{\dot\gamma}\dot\gamma^\alpha = qF^\alpha_\beta \dot\gamma^\beta.)$$
2. Assume that f is a (sufficiently regular) distribution function defined on all of TM. By restriction, f defines a regular distribution function on $\mathscr{P}$, denoted here g. Compute $\partial_{x^\alpha} g$ and $\partial_{v^i} g$ in terms of $\partial_{x^\alpha} f$ and $\partial_{v^\alpha} f$, where (x^α, v^α) denotes a standard coordinate system on TM as introduced above.

2.2 Remarks on the regularity of the distribution functions

We have seen that f can be expressed in coordinates (at least locally) as a function of (x^α, v^i), with x^0 to be thought of as a time function.

Recall also the definition of the mass-shell: $\mathscr{P}$ is the set of points $(x, v) \in TM$ such that $g_x(v, v) = -m^2$, where v is future-directed (and non-zero if $m = 0$).

The distribution functions that we consider in this lecture notes can all be constructed by solving an initial value problem. Assuming that the initial data is, say, smooth and compactly supported, then the resulting distribution function will be smooth and its restriction to a spatial section will have compact support.

As is classical, all that is actually needed below is a finite number of derivatives together with polynomial type decay for the initial data depending only on the dimension.

Note moreover that in physics, f represents a particle density, and is thus required to be positive. This will not play a role in what we do (and in fact is not preserved by differentiation) so I will ignore this from now on.

2.3 The non-relatistivic case

The classical equivalent of the above free transport operator is the operator $\partial_t + v^i\partial_{x^i}$. If the particles are moving in a force field F so that their trajectories is given by Newton's equation

$$m\ddot{x}^i = F^i$$

then we obtain the equation

$$v^\alpha \partial_{x^\alpha} f - F^i \partial_{v^i} f = 0,$$

with $v^0 = 1$.

3 The free transport equations

The equations that we want to consider first are

1. The classical (i.e. non-relativistic) free transport equation

$$\partial_t f + v^i \partial_{x^i} f = 0,$$

   where $f = f(t,x,v)$, $(t,x,v) \in \mathbb{R}_t \times \mathbb{R}^n_x \times \mathbb{R}^n_v$.

2. The relativistic transport equations

$$v^\alpha \partial_{x^\alpha} f = 0,$$

   where $v^0 = \sqrt{m^2 + |v|^2}$, $m \geq 0$ is a constant, $f = f(t,x,v)$, $(t,x,v) \in \mathbb{R}_t \times \mathbb{R}^n_x \times \mathbb{R}^n_v$ for $m > 0$ while $(t,x,v) \in \mathbb{R}_t \times \mathbb{R}^n_x \times \mathbb{R}^n_v \setminus \{0\}$ if $m = 0$.

We will write any of the above equation of $T(f) = 0$ where T is one of the free transport operators.

Let us note the following lemma.

Lemma 3.1. *Let f solve $T(f) = 0$. Let χ be a smooth function $\chi : \mathbb{R} \to \mathbb{R}$, then $T(\chi(f)) = 0$.*
Moreover, we have that $T(|f|) = 0$ in the sense of distribution.

For the proof in the case of the absolute value, recall that if f is $W^{1,1}$ then $|f|$ is in $W^{1,1}$ and that $\partial|f| = \frac{f}{|f|}\partial f$ in the sense of distribution.

3.1 Conservation laws

Lemma 3.2. *Let f solves $T(f)=0$. Then, for all t, and any $1 \leq p < +\infty$, we have*

$$||f(t,x,v)||_{L^p_{x,v}} = ||f(0,x,v)||_{L^p_{x,v}}.$$

In fact, for any sufficiently regular function $\chi : \mathbb{R} \to \mathbb{R}$,

$$||\chi(f(t,x,v))||_{L^p_{x,v}} = ||\chi(f(0,x,v))||_{L^p_{x,v}}.$$

(These last conserved quantities are called the Casimir invariants*).*

3.2 The method of characteristics

The transport equations are first order, and can thus be solved by the method of characteristics. It follows that if f solves $T(f)=0$, then

$$f(t,x,v) = f(0, x - \frac{v}{v^0} t, v),$$

where $v^0 = 1$ in the non-relativistic case and $v^0 = \sqrt{m^2 + |v|^2}$ otherwise.

Exercice: How would you generalize this to the Vlasov equation on a general Lorentzian manifold ?

Lemma 3.3. *In view of the above formula, we can take $p = +\infty$ in Lemma 3.2.*

3.3 Decay estimates for transport equations: heuristics

Let us recall that f solves a transport equation. f is thus constant along the characteristics, and there can be therefore no global decay of f, for instance the quantity $||f(t,.,.)||_{L^\infty_{x,v}}$ does not decay in t. In fact, it is a conserved quantity in view of the previous lemma.

The decaying quantities will be v integrals of f. For instance, in the classical case, we can consider the quantity

$$\rho(t,x) := \int_{v \in \mathbb{R}^3} f(t,x,v) dv. \tag{3}$$

(replace f by $|f|$ in the above formula if f is not positive.)

Assume first that f has initially compact support in x, say within a ball of radius R. In view of the representation formula,

$$f(t,x,v) = f(0, x - vt, v)$$

we see that we need $|x - vt| \leq R$ for a velocity v to contribute to the integral. In other words, the integral, for $t > 0$, is computed on a ball centered at $\frac{x}{t}$ and of radius $\frac{R}{t}$. Thus, we have a bound

$$||\rho(t,x)|| \lesssim ||f(0,.,.)||_{L^\infty_{x,v}} \frac{R^3}{t^3}$$

and we see that $\rho(t,x)$ decays, the decay being due to the v support of $f(t,x,.)$ shrinking.

Remark 3.1.

- *The above decay estimates is a bit crude, however, it has been used in many situations.*
- *The v integrals such as $\rho(t,x)$ are often called* velocity averages *in the litterature.*

3.4 Decay estimates from the method of characteristics

A variation of the above argument will first help us to remove the compact support assumptions and then apply it to the relativistic case.

We start again with the non-relativistic case.

Let f thus solves

$$T(f) = 0, \quad T = \partial_t + v^i \partial_{x^i}.$$

Now, we have

$$\begin{aligned}
\rho(t,x) &= \int_{v\in\mathbb{R}^3} f(t,x,v)dv \\
&= \int_{v\in\mathbb{R}^3} f(0,x-vt,v)dv \\
&\leq \int_{v\in\mathbb{R}^3} \sup_{w\in\mathbb{R}^3} f(0,x-vt,w)dv
\end{aligned}$$

For $t > 0$, we can apply the change of variable $y = x - vt$ to get

$$\rho(t,x) \leq \frac{1}{t^3} \int_{y\in\mathbb{R}^3} \sup_{w\in\mathbb{R}^3} f(0,y,w)dv.$$

We have thus obtained the decay estimates

$$||\rho(t,x)||_{L^\infty} \leq \frac{1}{t^3} ||f(0,.,.)||_{L^1_x L^\infty_v}.$$

Consider now the relativistic case with $m > 0$. We have again

$$\begin{aligned}
\rho(t,x) &= \int_{v\in\mathbb{R}^3} f(t,x,v)dv && (4) \\
&= \int_{v\in\mathbb{R}^3} f(0,x-\frac{v}{v^0}t,v)dv && (5) \\
&\leq \int_{v\in\mathbb{R}^3} \sup_{w\in\mathbb{R}^3} f(0,x-\frac{v}{v^0}t,w)dv && (6)
\end{aligned}$$

We now want to apply the change of variable $y = x - \frac{v}{v^0}t$. Note however that as $v \to +\infty$, $\left|\frac{v}{v^0}\right| \to 1$. This implies that for large v, we are trying to map a $3d$ space, to a sphere, i.e. the change of variable degenerates for large v. The way this would be reflected in our estimates is that if you compute the Jacobian of the change of variable, then this Jacobian is not uniformly bounded in v. For simplicity, we therefore assume that f has compact v support. Let us define

$$V = \sup\left\{|v|,\, v\in\mathbb{R}^3 : \exists x\in\mathbb{R}^3, f(0,x,v)\neq 0\right\}$$

(where the sup should be replaced by an essential supremum if f is, say, not continuous). Then, we can bound the Jacobian by a constant depending only on V.

Thus, we obtain a decay estimate of the form

$$\rho(t,x) \leq C(V)\frac{1}{t^3} \int_{y\in\mathbb{R}^3} \sup_{w\in\mathbb{R}^3} f(0,y,w)dv,$$

where $C(V) > 0$ depends only V and $C(V) \to +\infty$ as $V \to +\infty$.

Exercice:

1. Compute explicity the Jacobian in the above change of variable and give an estimate for $C(V)$.

2. in the massless case, show, using the method of characteristics as above, that the following decay estimate holds

$$\begin{aligned}\int_{v\in\mathbb{R}^3\setminus\{0\}} f(t,x,v)dv^3 &\lesssim \frac{1}{t^2}V^3\int_{v\in\mathbb{S}^2_t}\sup_{w\in\mathbb{R}^3} f(0,x-v,w)dv,\\ &\lesssim \frac{1}{t^2}V^3\sup_x\sup_{r\geq 0}\int_{v\in\mathbb{S}^2_r}\sup_{w\in\mathbb{R}^3} f(0,x-v,w)dv,\end{aligned}$$

where $\mathbb{S}_r$ denotes the round sphere of radius r centered at 0, and V is again an upper bound on the size of the v support of the initial distribution.

The above decay estimates are very easy to prove. However,

1. one needs a good control of the characteristics for this method to work. For instance, in the Lorentzian setting, one would need to control the exponential map, as well as derivative of the exponential map for this to work.

2. without extra work, we need a compact support assumption in v for them to work and we do not obtain directly spatial decay, or improved decay outside or near the light-cones in the massless or massive cases.

3. These decay estimates are not functional inequalities (that is to say they only apply to solutions of the equations, compare with the Klainerman-Sobolev inequality for instance).

The vector field approach that we shall now present will remediate to all of this. The estimates that we have obtained all required some L^∞ bounds in v of the data. We will replace these by requiring that some (weighted) v derivatives are integrable. By Sobolev, the latter is of course a stronger requirement to the former. In other words, the aim of the vector field approach is not to obtain sharper estimates as far as regularity is concerned. The main advantage will be that in non-linear applications, we can mostly forget the characteristics of the system and treat it in a way very similar to say, a system of non-linear waves. In particular, there will be no need to control the v support of the solution and in fact, no need for any boundedness in v of the initial support.

4 The vector field approach to decay of velocity averages

First, let us recall that there are two approaches to the vector field method for the wave equation: the original approach of Klainerman, which is based mostly on commutators, and the approach following Morawetz, where the key piece is to prove an *integrated* decay estimate for a local or weighted energy.

We shall consider mostly the vector field method following the approach of Klainerman, but let us mention immediately that there also exists integrated decay type estimates for velocity averages.

Let us now recall what are the key ingredients to Klainerman's vector field method for the wave equation

1. A coercive conservation law: in the case of the wave equation this is simply the energy estimate.
2. A set of commuting vector fields: for the wave equation, they are given by the Killing vector fields, and the scaling vector fields (thus parts of the conformal Killing fields). The set of commuting vector fields is therefore tied to the symmetries of the equation.
3. Weighted identities between the algebra of the commuting vector fields: for instance the usual derivative ∂_t can be rewritten as

$$\partial_t = \frac{tS - x^i \Omega_{0i}}{t^2 - |x|^2},$$

where S denotes the scaling vector field and the Ω_{0i} are the Lorentz boosts.
4. The Klainerman-Sobolev inequality, which is derived from the usual Sobolev inequalities and a careful analysis of the weights, some of them coming from the vector fields idendities, some of them from the volume forms.
5. Finally, using the conservation laws, one can relate the norms appearing on the right-hand side of the Klainerman-Sobolev inequality to the initial norm of the data.

To obtain a vector field method for kinetic equations, we shall therefore obtain an analogue for each of the above ingredients.

The conservation laws that we will use is simply the conservation of the $L^1_{x,v}$ norms[4] of f.

The next ingredient is the construction of the commuting vector fields. We will do it geometrically for the relativistic transport equations.

4.1 Complete lift

The commuting vector fields will be given by the *complete lifts* of the Killing vector fields.

Let M be a smooth manifold and X be a vector field on M. Let ϕ_s denote the flow of X. For each s (for simplicity, we assume here that X is complete, but the constructions below are local), ϕ_s is a map

$$\phi_s : M \to M$$

and by definition, since ϕ_s is the flow of X, we have $\phi_s(x) = \gamma(s)$ with γ the unique curve starting at x and which solves

$$\dot{\gamma} = X \circ \gamma.$$

Given $x \in M$, recall that the differential of ϕ_s is a linear map:

$$d\phi_s(x) : T_x M \to T_{\phi_s(x)} M.$$

[4]Other L^p norms of f could be used, and would lead to sharp decay estimates of velocity averages of f^p. The L^1 norm is needed if we want to obtain sharp estimates for velocity averages of f only.

Recall that by definition, given a vector in $v \in T_x M$, if we take a curve in M α with $\alpha(0) = x$, $\dot\alpha(0) = v$, then $t \to \phi_s \circ \alpha(t)$ is a curve and $d\phi_s(x)(v) = \frac{d(\phi_s \circ \alpha)}{dt}(t = 0)$. Moreover, in coordinates

$$[d\phi_s(x)(v)]^\alpha = \frac{\partial x^\alpha \circ \phi_s}{\partial x^\beta} v^\beta.$$

If now $v \in T_x M$, we can consider the point $(\phi_s(x), d\phi_s(x)(v)) \in TM$. When s moves, this defines a curve in TM

$$s \to (\phi_s(x), d\phi_s(x)(v)) \in TM.$$

Its tangent vector at $s = 0$ is a vector in the tangent space of TM at the point (x, v) i.e. a vector in $T_{(x,v)}TM$, denoted here $\tilde{X}_{(x,v)}$.

Lemma 4.1. *The map: $(x, v) \to \tilde{X}_{(x,v)}$ is smooth and thus defines a vector field on TTM called the complete lift of the vector field X. In local canonical coordinates,*

$$\tilde{X} = X^\alpha \partial_{x^\alpha} + v^\gamma \frac{\partial X^\alpha}{\partial x^\gamma} \partial_{v^\alpha}.$$

Proof. Exercice. □

Remark 4.1. *The map $X \to \tilde{X}$ depends only the differentiable structure of M, and is thus independent of any metric structure on M.*

Why do we care about complete lifts? Let X be a rotation in Minkowski space, for instance $X = y\partial_x - x\partial_y$. Then, a quick computation shows that $[T, X] = v^y \partial_x - v^x \partial_y$, for T the free transport operator. Thus, the rotations do not commute in general with our equation. On the other hand, the complete lifts of the rotations will commute, as explained in the following lemma.

Lemma 4.2 (Commutation of the transport operator with the complete lifts)**.** *Let X be a vector field on $\mathbb{R}_t \times \mathbb{R}^n_x$ and $\tilde{X}$ its complete lift. Let $T = v^\alpha \partial_{x^\alpha}$ be the relativistic transport operator (massless or massive), then*

$$[T, \tilde{X}] = v^\beta v^\sigma \frac{\partial X^\alpha}{\partial x^\beta x^\sigma} \partial_{v^\alpha}.$$

In particular, if X is any of the Killing fields of the Minkowski space then $[T, \tilde{X}] = 0$.

Things are more complicated for the conformal Killing fields because it will depends on the value of the mass. However, we have in all cases

Lemma 4.3. *Let $S = x^\alpha \partial_{x^\alpha}$ be the scaling vector field. Then*

$$[T, S] = T,$$

for T the relativistic massless or massive transport operator.

Note that here, we are not using its complete lift (but its vertical lift cf later in the lectures).

Exercice:

1. Let (M, g) be a Lorenztian manifold and $T = v^\alpha \partial_{x^\alpha} - v^\alpha v^\beta \Gamma^\sigma_{\alpha\beta} \partial_{v^\sigma}$ its geodesics vector field. Prove that for any vector field X on M, we have

$$[T, \tilde{X}] = v^\alpha v^\beta \left[\nabla_\alpha \nabla_\beta X^\sigma - R^\sigma{}_{\beta\alpha\gamma} X^\gamma \right] \partial_{v^\sigma} f.$$

2. The equation $\nabla_\alpha \nabla_\beta X^\sigma - R^\sigma{}_{\beta\alpha\gamma} X^\gamma = 0$ is called the *Jacobi* equation. Prove that any Killing fields of g solve the Jacobi equation.

4.2 The complete lifts of the Killing vector fields of the Minkowski space

Now we have a good receipe to construct commuting vector fields, let us write them down.

The complete lift of the Killing fields of the Minkowski space are given by

- translations: $\widehat{\partial_{x^\alpha}} = \partial_{x^\alpha}$.
- Rotations (including Lorentz boost): with $\Omega_{\alpha\beta} = x_\alpha \partial_{x^\beta} - x_\beta \partial_{x^\alpha}$, we get $\widehat{\Omega_{\alpha\beta}} = x_\alpha \partial_{x^\beta} - x_\beta \partial_{x^\alpha} + v_\alpha \partial_{v^\beta} - v_\beta \partial_{v^\alpha}$.

As for the usual Killing fields of Minkowski space, the commutator of two complete lifts is linear combination of complete lifts with constant coefficient coefficients.

4.3 Restrictions of the complete lifts

Now, recall that the distribution functions we want to consider are actually defined only on the mass shell, not on the whole tangent bundle. Fortunately, complete lifts of Killing fields are always tangent to the mass shell, and hence their restrictions to the mass shell are honest differential operators.

Lemma 4.4. *Let (M, g) be a Lorentzian manifold, and $\mathscr{P}$ its mass shell. Let X be a Killing field and $\widehat{X}$ its complete lift. Then, $\widehat{X}$ is tangent to $\mathscr{P}$ and thus, for any (sufficiently regular) distribution function f defined on $\mathscr{P}$, $\widehat{X}(f)$ is well defined.*

Proof. Let X be Killing. Without loss of generality, choose a coordinate system on M such that X coincide with one of the partial derivative: $X = \partial_{x^i}$. Then, $\widehat{X} = \partial_{x^i}$ while a 1-form normal to the mass shell is given by $\nu = g_{\alpha\beta,\gamma} v^\alpha v^\beta dx^\gamma + 2 g_{\alpha\beta} v^\beta dv^\alpha$. Since $g_{\alpha\beta}$ must be independent of x^i in the (x^α) coordinate system, it follows that $< \nu, \widehat{X} >= 0$, i.e. $\widehat{X}$ is tangent to the mass shell. □

Note that the restrictions of the x^α translations and the usual (not hyperbolic) rotations are given by the same expressions in the (x^α, v^i) coordinate system, while for the hyperbolic rotations, one obtain

$$\widehat{\Omega_{0i}} = t \partial_{x^i} + x^i \partial_t + v^0 \partial_{v^i},$$

where $v^0 = \sqrt{m^2 + |v|}$.

Let $\mathbb{P}$ be the set of all Killing fields (the Poincaré algebra).

We will denote by

$$\widehat{\mathbb{P}} \equiv \{ \widehat{Z} \mid Z \in \mathbb{P} \},$$

the set of all complete lifts and by

$$\mathbb{K} = \widehat{\mathbb{P}} \cup \{S\},$$

where $S = x^\alpha \partial_{x^\alpha}$ is the scaling vector field (not its complete lift).

4.4 Velocity averages and the complete lift of the Killing fields

We have found our commuting vector fields, but we still need to understand how to exploit them. Recall that what decays are not the distribution functions themselves, but their velocity averages. Thus, we would like to estimate quantities of the form

$$Z\left(\int_v f dv\right)$$

for Z a vector field. However, if we integrate in v first, then $\int_v f dv$ are quantities defined on M not TM (or rather $\mathscr{P}$), so where how are we going to make use or our complete lift ? The answer is contained is the following lemma:

Vector fields and the operator of averaging in v essentially commute in the following sense.

Lemma 4.5. *Let f be a regular distribution function for the massless case. Then,*

- *for any translation ∂_{x^α}, we have*

$$\partial_{x^\alpha}\left[\rho(f)\right] = \rho\left(\partial_{x^\alpha}(f)\right) = \rho\left(\widehat{\partial_{x^\alpha}}(f)\right).$$

- *for any rotation Ω_{ij}, $1 \leq i, j, \leq n$, we have*

$$\Omega_{ij}\left[\rho(f)\right] = \rho\left(\widehat{\Omega_{ij}}(f)\right),$$

  where $\widehat{\Omega_{ij}}$ is the complete lift of the vector field Ω_{ij}.

- *for any Lorentz boost Ω_{0i}, $1 \leq i \leq n$, we have*

$$\Omega_{0i}\left[\rho(f)\right] = \rho\left(\widehat{\Omega_{0i}}(f)\right) + 2\rho\left(\frac{v^i}{|v|} f\right).$$

- *for the scaling vector field S, we have*

$$S\left[\rho(f)\right] = \rho\left(\widehat{S}(f)\right) + (n+1)\rho(f).$$

- *finally, all the above equalities hold (almost everywhere) with f replaced by $|f|$, for instance*

$$S\left[\rho(|f|)\right] = \rho\left(\widehat{S}(|f|)\right) + (n+1)\rho(|f|).$$

Proof. Let us consider, for instance, a Lorentz boost $\Omega_{0i} = t\partial_{x^i} + x^i \partial_t$, then

$$\Omega_{0i}\left[\rho(f)\right] \quad = \quad \int_v \left(t\partial_{x^i} + x^i \partial_t\right)(f)|v| dv. \tag{7}$$

On the other hand, note that

$$\begin{aligned}\int_v \left(t\partial_{x^i} + x^i\partial_t\right)(f)|v|dv &= \int_v \left(t\partial_{x^i} + x^i\partial_t + |v|\partial_{v^i}\right)(f)|v|dv - \int_v |v|^2\partial_{v^i}(f)dv \\ &= \int_v \widehat{\Omega_{0i}}(f)|v|dv + 2\int_v \frac{v^i}{|v|}(f)|v|dv \\ &= \rho\left(\widehat{\Omega_{0i}}(f)\right) + 2\rho\left(\frac{v^i}{|v|}f\right),\end{aligned}$$

using an integration by parts in v^i. The other cases can all be treated similarly, the translations being trivial since $\widehat{\partial_{x^\alpha}} = \partial_{x^\alpha}$. That f can be replaced by $|f|$ follows from the standard property of differentiation of the absolute value[5]. □

In the massive case, we have the following lemma, whose proof is left to the reader since it is very similar to the above.

Lemma 4.6. *Let f be a regular distribution function for the massive case. Then,*

- *for any translation ∂_{x^α}, we have*

$$\partial_{x^\alpha}\left[\rho(f)\right] = \rho\left(\partial_{x^\alpha}(f)\right) = \rho\left(\widehat{\partial_{x^\alpha}}(f)\right).$$

- *for any rotation Ω_{ij}, $1 \le i, j, \le n$, we have*

$$\Omega_{ij}\left[\rho(f)\right] = \rho\left(\widehat{\Omega_{ij}}(f)\right),$$

  where $\widehat{\Omega_{ij}}$ is the complete lift of the vector field Ω_{ij}.

- *for any Lorentz boost Ω_{0i}, $1 \le i \le n$, we have*

$$\Omega_{0i}\left[\rho(f)\right] = \rho\left(\widehat{\Omega_{0i}}(f)\right) + 2\rho_m\left(\frac{v^i}{v^0}f\right).$$

- *finally, all the above equalities holds with f replaced by $|f|$.*

Remark 4.2. *Although we do not have for all commutation vector fields $Z\rho = \rho\widehat{Z}$, we do have that $|Z\rho(|f|)| \lesssim \rho\left(|\widehat{Z}(f)|\right) + \rho(|f|)$ and this is all we shall need from the above. Note also that if we were looking at other moments, then similar formulae would hold with different coefficients. For instance, we have $\Omega_{0i}\int_v f d\mu_m = \int_v \widehat{\Omega}_{0i} f d\mu_m$ for sufficiently regular f.*

Exercice:

Let (M, g) be Lorentzian manifold, X a Killing field of the metric g and $T_{\mu\nu}[f]$ an energy momentum tensor of some regular Vlasov field f:

$$T_{\mu\nu}[f] = -\int_v f v_\mu v_\nu \sqrt{g}\frac{dv^3}{v_0}.$$

Prove that $\mathscr{L}_X T_{\mu\nu}[f] = T_{\mu\nu}[\widehat{X}(f)]$. (note also that above, we did not always use the Lie derivative...)

[5] Recall that $f \in W^{1,1}$ implies that $|f| \in W^{1,1}$ with $\partial|f| = \frac{f}{|f|}\partial f$ almost everywhere. See for instance [LL97], Chap 6.17.

4.5 Vector field identities

The following classical vector field identities will be used later.

Lemma 4.7. *The following identities hold:*

$$\begin{aligned}(t^2-r^2)\partial_t &= tS - x^i\Omega_{0i},\\ (t^2-r^2)\partial_i &= -x^j\Omega_{ij} + t\Omega_{0i} - x^iS,\\ (t^2-r^2)\partial_r &= t\frac{x^i}{r}\Omega_{0i} - rS.\end{aligned}$$

Furthermore,

$$\partial_s \equiv \frac{1}{2}(\partial_t+\partial_r) = \frac{S+\omega^i\Omega_{0i}}{2(t+r)}, \quad \overline{\partial}_i \equiv \partial_i - \omega_i\partial_r = \frac{\omega^j\Omega_{ij}}{r} = \frac{-\omega_i\omega^i\Omega_{0j}+\Omega_{0i}}{t}. \tag{8}$$

4.6 Decay estimates for velocity averages of massless distribution via the vector field method

We are now ready to prove decay estimates for velocity averages using the vector field method, at least for massless fields. For massive fields, we will need a little bit of extra knowledge, as we will use the so-called *hyperboloidal foliation.*

Theorem 4.1 (Klainerman-Sobolev inequalities for velocity averages of massless distribution functions)**.** *Let f be a regular distribution function for the massless case defined on $[0,T]\times\mathbb{R}^n_x\times\left(\mathbb{R}^n_v\setminus\{0\}\right)$ for some $T>0$. Then, for all $(t,x)\in[0,T]\times\mathbb{R}^n_x$,*

$$\rho(|f|)(t,x) \lesssim \frac{1}{(1+|t-|x||)(1+|t+|x||)^{n-1}}\|f\|_{\mathbb{K},n}(t), \tag{9}$$

where

$$\|f\|_{\mathbb{K},k}(t) \equiv \sum_{|\alpha|\le k}\sum_{\widehat{Z}^\alpha\in\widehat{\mathbb{K}}^{|\alpha|}}\int_{\Sigma_t}\rho\left(\left|\widehat{Z}^\alpha f\right|\right)(t,x)dx.$$

Proof. The proof is very similar to the proof of the usual Klainerman-Sobolev inequality for the wave equation, apart from a technical problem due to the presence of the absolute value. In the usual approach, one typically starts by applying Sobolev's Lemma. Here, the absolute value present us from doing, so we will reprove Sobolev's Lemma as we prove our decay estimate.

Let $(t,x)\in[0,T]\times\mathbb{R}^n_x$ and assume first that $|x|\notin[t/2,3/2t]$ and $t+|x|\ge 1$. Let ψ be defined as

$$\psi : y \to \rho\left(|f|(t,x+(t+|x|)y)\right),$$

where $y=(y_1,y_2,..,y_n)$. Note that

$$\partial_{y_i}\psi(y) = \partial_{y^i}\left[\rho\left(|f|(t,x+(t+|x|)y\right)\right] = (t+|x|)\partial_{x^i}\left(\rho\left[|f|\right]\right)(t,x+(t+|x|)y).$$

Assume now that $|y|\le 1/4$. Using the fact that we are away from the light-cone and the condition on $|y|$, it follows that

$$1/C \le \frac{|t+|x||}{|t-|x+(t+|x|)y|} \le C,$$

for some $C>0$. It then follows from the vector field identities of Lemma 4.7 that

$$|\partial_{y^i}\rho(|f|(t,x+(t+|x|)y)| \lesssim \sum_{Z\in\mathbb{K}}\left|Z\left(\rho\left[|f|\right]\right)\right|(t,x+(t+|x|)y).$$

From Lemma 4.5, we then obtain that

$$\begin{aligned}\left|\partial_{y^i}\rho\left[|f|(t,x+(t+|x|)y)\right]\right| &\lesssim \sum_{|\alpha|\le 1,\widehat{Z}^\alpha\in\widehat{\mathbb{K}}^{|\alpha|}}\left|\rho\left[\widehat{Z}(|f|)\right]\right|(t,x+(t+|x|)y)+\rho(|f|)(t,x+(t+|x|)y),\\ &\lesssim \sum_{|\alpha|\le 1,\widehat{Z}^\alpha\in\widehat{\mathbb{K}}^{|\alpha|}}\left|\rho\left[\widehat{Z}^\alpha(|f|)\right]\right|(t,x+(t+|x|)y),\\ &\lesssim \sum_{|\alpha|\le 1,\widehat{Z}^\alpha\in\widehat{\mathbb{K}}^{|\alpha|}}\rho\left[\left|\widehat{Z}^\alpha(|f|)\right|\right](t,x+(t+|x|)y),\\ &\lesssim \sum_{|\alpha|\le 1,\widehat{Z}^\alpha\in\widehat{\mathbb{K}}^{|\alpha|}}\rho\left[\left|\widehat{Z}^\alpha(f)\right|\right](t,x+(t+|x|)y),\end{aligned}$$

where we have used in the last line that for any vector field $\widehat{Z}$, $\left|\widehat{Z}(|f|)\right|=\left|\widehat{Z}(f)\right|$ (almost everywhere and provided f is sufficiently regular), which essentially follows from the fact that $\partial|f|=\frac{f}{|f|}\partial f$ almost everywhere if $f\in W^{1,1}$. Let now $\delta=\frac{1}{16n}$, so that if $|y_i|\le\delta^{1/2}$ for all $1\le i\le n$, we then have $|y|\le 1/4$. Applying now a 1 dimensional Sobolev inequality in the variable y_1, we have

$$\begin{aligned}|\psi(0)|=\rho\left[|f|\right](t,x) &\lesssim \int_{|y_1|\le\delta^{1/2}}\left(\left|\partial_{y_1}\psi(y_1,0,..,0)\right|+\left|\psi(y_1,0,..,0)\right|\right)dy_1,\\ &\lesssim \int_{|y_1|\le\delta^{1/2}}\left(\sum_{|\alpha|\le 1,\widehat{Z}^\alpha\in\widehat{\mathbb{K}}^{|\alpha|}}\rho\left[\left|\widehat{Z}^\alpha(f)\right|\right]\left(t,x+(t+|x|)(y_1,0,..,0)\right)\right)dy_1.\end{aligned}$$

We can now apply a 1 dimensional Sobolev inequality in the variable y_2 and repeat the previous argument, with $|Z^\alpha(f)|$ replacing $|f|$, to obtain

$$|\psi(0)|\lesssim\int_{|y_1|\le\delta^{1/2}}\int_{|y_2|\le\delta^{1/2}}\left(\sum_{|\alpha|\le 2,\widehat{Z}^\alpha\in\widehat{\mathbb{K}}^{|\alpha|}}\rho_0\left[\left|\widehat{Z}^\alpha(f)\right|\right]\left(t,x+(t+|x|)(y_1,y_2,..,0)\right)\right)dy_1dy_2.$$

Repeating the argument up to exhaustion of all variables, we obtain that

$$\begin{aligned}&\rho_0\left[|f|\right](t,x)\\ &\lesssim\int_{|y_1|\le\delta^{1/2}}\int_{|y_2|\le\delta^{1/2}}..\int_{|y_n|\le\delta^{1/2}}\left(\sum_{|\alpha|\le n,\widehat{Z}^\alpha\in\widehat{\mathbb{K}}^{|\alpha|}}\rho\left[\left|\widehat{Z}^\alpha(f)\right|\right]\left(t,x+(t+|x|)(y_1,y_2,..,y_n)\right)\right)dy_1dy_2..dy_n.\end{aligned}$$

Applying the change of variable $z=(t+|x|)y$ gives us a $(t+|x|)^n$ factor which completes the proof of the inequality in this particular case. The case where $(t+|x|)\le 1$ follows from simpler considerations and is therefore left to the reader.

Let us thus turn to the case where $x\in[t/2,3/2t]$ and $(t+|x|)\ge 1$. Note that it then follows that $t>2/5$ and $|x|>1/3$. Let us introduce spherical coordinates $(r,\omega)\in[0,+\infty)\times\mathbb{S}^{n-1}$, such that $x=r\omega$ and denote by q the optical function $q\equiv r-t$. Let $v(t,q,\omega)\equiv\rho_0(f)(t,(t+q)\omega)$.

Note that $\partial_q v=\partial_r\rho$, $q\partial_q v=(r-t)\partial_r$ and that there exist constants C_{ij} such that

$$\partial_\omega v=\partial_\omega\left(\rho_0(f)(t,(q+t)\omega)\right)=\sum_{i<j}C_{ij}\Omega_{ij}\rho_0(f),$$

where the Ω_{ij} are the rotation vector fields.

Let $q_0=|x|-t$. We need to prove that

$$t^{n-1}(1+|q_0|)|v(t,q_0,\omega)|\lesssim\|f\|_{\mathbb{K},n}(t).$$

Using a one dimensional Sobolev inequality, we have for any $\omega \in \mathbb{S}^{n-1}$.

$$|v(t,q_0,\omega)| \lesssim \int_{|q|<t/4} \sum_{|\alpha|\le 1} \left| \left(\partial_q^\alpha v\right)(t,q+q_0,\eta) \right| dq.$$

Note now that

$$\left(\partial_q v\right)(t,q+q_0,\omega) = \left(\partial_r \rho(f)\right)(t,q+q_0,\omega) = \rho(\partial_r(|f|))$$

and thus

$$|\partial_q v(t,q+q_0,\omega)| \quad \lesssim \quad \rho(|\partial_r f|)(t,q+q_0,\omega),$$

where we have used again the properties of the derivatives of the absolute value. Let now $(\omega_1,\omega_2,..,\omega_{n-1})$ be a local coordinate patch in a neighbourhood of the point $\omega \in \mathbb{S}^{n-1}$. Using again a $1-d$ inequality, we have

$$\begin{aligned} \left|\rho_0(|\partial_r^\alpha f|)(t,q+q_0,\omega)\right| \quad &\lesssim \quad \int_{\omega_1} \left|\partial_{\omega_1}\rho(|\partial_r^\alpha f|)(t,q+q_0,\omega+(\omega_1,0,..,0))\right| d\omega_1 \\ &\quad + \int_{\omega_1} \left|\rho(|\partial_r^\alpha f|)(t,q+q_0,\omega+(\omega_1,0,..,0))\right| d\omega_1. \end{aligned}$$

Since ∂_{ω_1} can be rewritten in terms of the rotation vector fields, it follows from Lemma 4.5 that

$$\left|\partial_{\omega_1}\rho_0(|\partial_r^\alpha f|)\right| \lesssim \sum_{|\beta|\le 1} \rho\left(\left|\widehat{Z}^\beta \partial_r^\alpha f\right|\right).$$

Repeating until exhaustion of the number of variables on $\mathbb{S}^{n-1}$ and using that $\partial_r = \frac{x^k}{|x|}\partial_{x^k}$ and the commutation properties between $\widehat{Z}^\alpha$ and ∂_r, we obtain that

$$|\rho(|f|)(t,q+q_0,\omega)| \lesssim \int_{|q|\le t/4} \int_{\eta\in\mathbb{S}^{n-1}} \sum_{|\alpha|\le n} \rho\left(\left|\widehat{Z}^\alpha(f)\right|\right)(t,q+q_0,\eta)\, dq\, d\sigma_{\mathbb{S}^{n-1}}.$$

Now since in the domain of integration, $r = t+q+q_0 = q+|x| \sim t$, we have

$$\begin{aligned} t^{n-1}|\rho(|f|)(t,q+q_0,\omega)| \quad &\lesssim \quad \sum_{|\alpha|\le n} \int_{|q|\le t/4} \int_{\eta\in\mathbb{S}^{n-1}} \rho\left(\left|\widehat{Z}^\alpha(f)\right|\right)(t,q+q_0,\eta) r^{n-1} dq\, d\sigma_{\mathbb{S}^{n-1}}, \\ &\lesssim \quad \sum_{|\alpha|\le n} \int_{t/4\le r\le 7t/4} \int_{\eta\in\mathbb{S}^{n-1}} \rho\left(\left|\widehat{Z}^\alpha(f)\right|\right)(t,r,\eta) r^{n-1} dr\, d\sigma_{\mathbb{S}^{n-1}} \\ &\lesssim \quad \sum_{|\alpha|\le n} \int_{t/4\le|y|\le 7t/4} \rho\left(\left|\widehat{Z}^\alpha(f)\right|\right)(t,y)\, dy, \end{aligned}$$

which concludes the proof when $|q_0| \le 1$.

Assume now that $|q_0| > 1$. Let $\chi \in C_0^\infty(-1/2,1/2)$ be a smooth cut-off function such that $\chi(0) = 1$ and define $V_{q_0}(t,q,\omega) \equiv \chi((q-q_0)/q_0)v(t,q,\omega)$. To get the extra factor of $|q_0|$, we apply the method used above replacing the function v by the function $(s,\eta) \to V_{q_0}(t,q_0+q_0 s,\eta)$ and applying first a $1-d$ Sobolev inequality in s on $|s| < 1/2$. The extra power of q_0 appearing are then absorbed since $|q_0+q_0 s| \sim |q_0|$ in the region of integration and since $(r-t)\partial_r$ can be expressed as a linear combination of commutation vector fields from Lemma 4.7 (with coefficients homogeneous of degree 0). The rest of the proof is similar to the one just given when $|q_0| \le 1$ and therefore omitted. □

Since the norm on the right-hand side is conserved for solutions of the homogeneous massless transport equations, we obtain in particular,

Theorem 4.2 (Decay estimates for velocity averages of massless distribution functions [FJS15])**.** *Let f be a regular distribution function for the massless case, a solution to $T(f) = 0$ on $\mathbb{R}_t \times \mathbb{R}^n_x \times \left(\mathbb{R}^n_v \setminus \{0\}\right)$. Then, for all $(t, x) \in \mathbb{R}_t \times \mathbb{R}^n_x$,*

$$\rho(|f|)(t,x) \lesssim \frac{1}{(1+|t-|x||)(1+|t+|x||)^{n-1}} \|f\|_{\mathbb{K},n}(0). \tag{10}$$

4.7 Decay estimates for velocity averages of massive distribution via the vector field method

The analogy between the wave equation and massless particles is replaced in that case by the analogy between the Klein Gordon equation and massive particles, where the Klein Gordon equation (with mass $m = 1$) is

$$(\Box - 1)\psi = 0, \quad \psi := \psi(t,x).$$

Now, how is the vector field method working for Klein Gordon fields ? We could try to repeat the same estimates as the one obtained for the wave equation but there is one caveat: the scaling vector field does not commute with $\Box - 1$. Without the scaling vector field, we can no longer use the vector field idendities of Lemma 4.7.

Instead, the traditional techniques (see [Kla93]) use

1. The hyperboloidal foliation.
2. Vector field identities on each hyperboloidal slice.
3. the extra control of the L^2 norm of ψ in the energy estimates.

Let us explain each ingredient, starting with the foliation.

4.7.1 The hyperboloidal foliation

Let us fix global Cartesian coordinates (t, x^i), $1 \le i \le n$ on $\mathbb{R}^{n+1}$ and denote by Σ_t the hypersurface of constant t. The hypersurfaces Σ_t, $t \in \mathbb{R}$ then give a complete foliation of $\mathbb{R}^{n+1}$.

The hyperboloidal foliation is defined as follows. For any $\rho > 0$, define H_ρ by

$$H_\rho = \left\{(t,x) \,\middle|\, t \ge |x| \text{ and } t^2 - |x|^2 = \rho^2\right\}.$$

For any $\rho > 0$, H_ρ is thus only one sheet of a two sheeted hyperboloid[6]

Note that

$$\bigcup_{\rho \ge 1} H_\rho = \left\{(t,x) \in \mathbb{R}^{n+1} \,\middle|\, t \ge (1+|x|^2)^{1/2}\right\}.$$

The above subset of $\mathbb{R}^{n+1}$ will be referred to as *the future of the unit hyperboloid.* On this set, we will use as an alternative to the Cartesian coordinates (t, x) the following two other sets of coordinates.

[6] The hyperboloidal foliation was originally introduced in [Kla85] in the context of the non-linear Klein-Gordon equation. For more recent applications, see [WW15] and [LM15] which concern the stability of the Minkowski space for the Einstein-Klein-Gordon system.

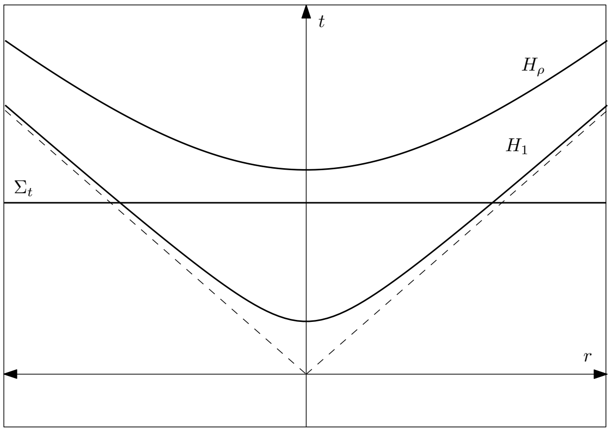

Figure 1: The H_ρ foliations in the (t, r) plane, $\rho > 1$

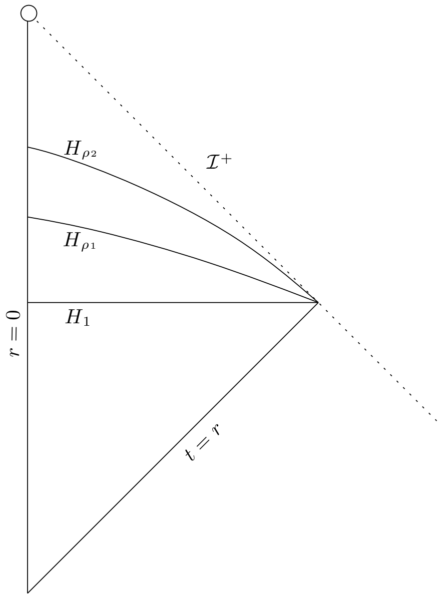

Figure 2: The H_ρ foliations in a Penrose diagram of Minkowski space, $\rho_2 > \rho_1 > 1$

Spherical coordinates

We first consider spherical coordinates (r,ω) on $\mathbb{R}^n_x$, where ω denotes spherical coordinates on the $n-1$ dimensional spheres and $r = |x|$. (ρ, r, ω) then defines a coordinate system on the future of the unit hyperboloid. These new coordinates are defined globally on the future of the unit hyperboloid apart from the usual degeneration of spherical coordinates and at $r = 0$.

Pseudo-Cartesian coordinates

These are the coordinates $(y^0, y^j) \equiv (\rho, x^j)$. These new coordinates are also defined globally on the future of the unit hyperboloid.

For any function defined on (some part of) the future of the unit hyperboloid, we will move freely between these three sets of coordinates.

4.7.2 Geometry of the hyperboloids

The Minkowski metric η is given in (ρ, r, ω) coordinates by

$$\eta = -\frac{\rho^2}{t^2}\left(d\rho^2 - dr^2\right) - \frac{2\rho r}{t^2} d\rho dr + r^2 \sigma_{\mathbb{S}^{n-1}},$$

where $\sigma_{\mathbb{S}^{n-1}}$ is the standard round metric on the $n-1$ dimensional unit sphere, so that for instance

$$\sigma_{\mathbb{S}^2} = \sin\theta^2 d\theta^2 + d\phi^2,$$

in standard (θ, ϕ) spherical coordinates for the 2-sphere. The 4 dimensional volume form is thus given by

$$\frac{\rho}{t} r^{n-1} d\rho dr d\sigma_{\mathbb{S}^{n-1}},$$

where $d\sigma_{\mathbb{S}^{n-1}}$ is the standard volume form of the $n-1$ dimensional unit sphere.

The Minkowski metric induces on each of the H_ρ a Riemannian metric given by

$$ds^2_{H_\rho} = \frac{\rho^2}{t^2} dr^2 + r^2 \sigma_{\mathbb{S}^{n-1}}.$$

A normal differential form to H_ρ is given by $tdt - rdr$ while $t\partial_t + r\partial_r$ is a normal vector field. Since

$$\eta\left(t\partial_t + r\partial_r, t\partial_t + r\partial_r\right) = -\rho^2,$$

the future unit normal vector field to H_ρ is given by the vector field

$$\nu_\rho \equiv \frac{1}{\rho}\left(t\partial_t + r\partial_r\right). \tag{11}$$

Finally, the induced volume form on H_ρ, denoted $d\mu_{H_\rho}$, is given by

$$d\mu_{H_\rho} = \frac{\rho}{t} r^{n-1} dr d\sigma_{\mathbb{S}^{n-1}}.$$

4.7.3 The particle vector field and the stress energy tensor of Vlasov fields

For the massless case, we used the conservation in t of $||f(t,.,.)||_{L^1_{x,v}}$. For the massive case, we want to use the hyperboloidal foliation, so we are looking for a conservation of some norm of f of the form $\int_{H_\rho}\int_v f$ where H_ρ is the hypersurface of constant ρ. Unfortunately, a naive attempt such as

$$\int_{H_\rho}\int_v f dv d\mu_\rho$$

does not lead to a conserved quantity in ρ.

We could try to look for a conservation law by hand, but we can also just remember that the Vlasov equation, like the wave equation, has an energy-momentum tensor.

Let us first define a volume form in v

$$d\mu_m \equiv \frac{dv^1 \wedge \ldots \wedge dv^n}{v^0} = \frac{dv}{\sqrt{m^2 + |v|^2}}, \tag{12}$$

where as usual $m = 0$ in the massless case.

Remark 4.3. *In the massless case, the volume form* $\frac{dv}{|v|}$ *is singular near* $v = 0$*. In the remainder of this article, we will however study mostly energy densities, which introduce an additional factor of* $|v|^2$ *in the relevant integrals and thus remove this singular behaviour near* $v = 0$*. Note also that, in dimension* 2 *or greater,* $f \in L^\infty_{loc}$ *is sufficient for* $\frac{f}{|v|} \in L^1_{loc}$*.*

We now define the particle vector field in the case of massive particles as

$$N^\mu_m \equiv \int_{\mathbb{R}^n} f v^\mu d\mu_m,$$

and in the case of massless particles as

$$N^\mu_0 \equiv \int_{\mathbb{R}^n\setminus\{0\}} f v^\mu d\mu_0,$$

as well as the energy momentum tensors

$$T^{\mu\nu}_m \equiv \int_{\mathbb{R}^n} f v^\mu v^\nu d\mu_m,$$

and

$$T^{\mu\nu}_0 \equiv \int_{\mathbb{R}^n\setminus\{0\}} f v^\mu v^\nu d\mu_0,$$

where $d\mu_m$ and $d\mu_0$ are the volume forms defined in (12). More generally, we can define the higher moments

$$M^{\alpha_1\dots\alpha_p}_m \equiv \int_{\mathbb{R}^n} f v^{\alpha_1}\dots v^{\alpha_p} d\mu_m,$$

and similarly for the massless system.

The interest in any of the above quantities is that if f is a solution to the associated massless or massive transport equations, then these quantities are divergence free. Indeed, we have

$$\partial_\mu T^{\mu\nu}_0 = \int_{\mathbb{R}^n\setminus\{0\}} \mathbf{T}_0(f) v^\nu d\mu_0, \quad (13)$$

$$\partial_\mu T^{\mu\nu}_m = \int_{\mathbb{R}^n} \mathbf{T}_m(f) v^\nu d\mu_m. \quad (14)$$

We will be interested in particular in the energy densities

$$\rho_0(f) \equiv T_0(\partial_t,\partial_t) = \int_{\mathbb{R}^n\setminus\{0\}} f|v|dv, \quad (15)$$

for the massless case, while for the massive case we define

$$\rho_{\mathrm{m}}(f) \equiv T_m(\partial_t,\partial_t) = \int_{\mathbb{R}^n} f v^0 dv. \quad (16)$$

In the following, we will denote by $\rho(f)$ any of the quantities $\rho_{\mathrm{m}}(f)$ or $\rho_0(f)$ depending on whether we are looking at the massive or the massless relativistic operator.

In the massive case, we will also make use of the following energy density

$$\chi_m(f) \equiv T_m(\partial_t, \nu_\rho), \quad (17)$$

where ν_ρ is the future unit normal to H_ρ introduced in Section 4.7.2. We compute

$$\begin{aligned} \chi_{\mathrm{m}}(f) &= \int_{v\in\mathbb{R}^n} f v_0 \left(\frac{t}{\rho} v_0 + \frac{r}{\rho} v^r\right) d\mu_m, \\ &= \int_{v\in\mathbb{R}^n} f \left(\frac{t}{\rho} v^0 - \frac{x^i}{\rho} v_i\right) dv. \end{aligned}$$

The following lemma will be used later.

Lemma 4.8 (Coercivity of the energy density normal to the hyperboloids)**.** *Assuming that $t \geq r$, we have*

$$\chi_m(f) \geq \frac{t}{2\rho} \int_{v\in\mathbb{R}^n} f \left[\left(1-\frac{r}{t}\right)\left((v^0)^2 + v_r^2\right) + r^2\sigma_{AB} v^A v^B + m^2\right] \frac{dv}{v^0}. \tag{18}$$

Proof. Using that

$$(v^0)^2 = v_r^2 + r^2\sigma_{AB} v^A v^B + m^2$$

where σ_{AB} denotes the components of the metric $\sigma_{\mathbb{S}^n}$ and v^A, v^B are the angular velocities, we have

$$(v^0)^2 = \frac{(v^0)^2}{2} + \frac{1}{2}\left(v_r^2 + r^2\sigma_{AB} v^A v^B + m^2\right)$$

and thus

$$v^0\left(\frac{t}{\rho} v^0 - \frac{x^i}{\rho} v_i\right) = \frac{t}{2\rho}\left((v^0)^2 + v_r^2 + r^2\sigma_{AB} v^A v^B + m^2 - 2\frac{x^i}{t} v_i v^0\right).$$

The lemma now follows from

$$(v^0)^2 + v_r^2 - 2\frac{x^i}{t} v_i v^0 \geq \left(1-\frac{r}{t}\right)\left((v^0)^2 + v_r^2\right),$$

assuming $t \geq r$. □

Remark 4.4.

- *Since $(v^0)^2 \geq v_r^2$, we will use* (18) *in the form*

$$\chi_m(f) \geq \frac{t}{2\rho} \int_{v\in\mathbb{R}^n} f \left[\left(1-\frac{r}{t}\right)(v^0)^2 + r^2\sigma_{AB} v^A v^B + m^2\right] d\mu_m.$$

- *We also remark that*

$$\chi_m(|f|) \geq \frac{m^2}{2} \int_v |f| \frac{dv}{v^0} = \frac{m^2}{2} \rho_m\left(\frac{|f|}{(v^0)^2}\right),$$

  since $\frac{t}{2\rho} \geq \frac{1}{2}$, and, furthermore,

$$\chi_m(|f|) \geq \frac{t-r}{2\rho} \rho_m(|f|) = \frac{\rho}{2(t+r)} \rho_m(|f|).$$

- *Finally, independently of Lemma 4.8, since by the Cauchy-Schwarz inequality for Lorentzian metrics, as the vectors ν_ρ, and v are both timelike future directed,*

$$\left|\frac{tv^0 - x^i v_i}{\rho}\right| = |\langle v, \nu_\rho\rangle| \geq |v||\nu_\rho| = m, \text{ where } |v| = |g(v,v)|^{\frac{1}{2}},$$

  we get immediately

$$\int_v |f| dv \leq \int_v \frac{1}{m}\left|\frac{tv^0 - x^i v_i}{\rho}\right| |f| dv = \frac{1}{m}\chi_m(|f|).$$

As in the massless case, let us define the following norms for massive fields

Definition 4.1. *Let f be a regular distribution function for the massive case defined on $\bigcup_{1\le\rho\le P} H_\rho \times \mathbb{R}^n_v$. For $k\in\mathbb{N}$, we define, for all $\rho\in[1,P]$,*

$$\|f\|_{\mathbb{P},k}(\rho) \equiv \sum_{|\alpha|\le k}\sum_{\widehat{Z}^\alpha\in\widehat{\mathbb{P}}^{|\alpha|}} \int_{H_\rho} \chi_m(|\widehat{Z}^\alpha f|)\,d\mu_{H_\rho}. \tag{19}$$

4.8 Klainerman-Sobolev inequalities and decay estimates: massive case

In the massive case $m>0$, we will prove

Theorem 4.3 (Klainerman-Sobolev inequalities for velocity averages of massive distribution functions[FJS15])**.** *Let f be a regular distribution function for the massive case defined on $\bigcup_{1\le\rho<P} H_\rho \times \mathbb{R}^n_v$ for some $P\in[1,+\infty]$. Then, for all $(t,x)\in\bigcup_{1\le\rho<P} H_\rho$,*

$$\int_{v\in\mathbb{R}^n} |f|(t,x,v)\frac{dv}{v^0} \lesssim \frac{1}{(1+t)^n}\|f\|_{\mathbb{P},n}(\rho(t,x)), \tag{20}$$

where $\rho(t,x) = (t^2-|x|^2)^{1/2}$.

Proof. Recall from Remark 4.4, that

$$\chi_m(|f|)(t,x) \ge m^2\frac{t}{2\rho}\int_v f\,d\mu_m = m^2\frac{t}{2\rho}\int_{v\in\mathbb{R}^n}|f|(t,x,v)\frac{dv}{v^0} \tag{21}$$

and thus, note that

$$\int_{H_\rho}\chi_m(|f|)(t,x)\,d\mu_{H_\rho} \ge \int_{H_\rho} m^2\frac{t}{2\rho}\int_{v\in\mathbb{R}^n}|f|(t,x,v)\frac{dv}{v^0}\,d\mu_{H_\rho}. \tag{22}$$

Let (t,x) be fixed in $\bigcup_{1\le\rho\le P} H_\rho$ and define the function ψ in the (y^α) system of coordinates (see end of Section 4.7.1) as follows

$$\psi(y^0,y^j) \equiv \int_{v\in\mathbb{R}^n}|f|(y^0,x^j+ty^j)\,d\mu_m.$$

Similarly to the proof of the massless case, we apply first a one $1-d$ Sobolev inequality in the variable y^1

$$\int_{v\in\mathbb{R}^n}|f|(y^0,x^j)\,d\mu_m = |\psi(y^0,0)| \lesssim \int_{|y^1|\le 1/(8n)^{1/2}}\left[\left|\partial_{y^1}\psi\right|(y^0,y^1,0,..,0)+|\psi|(y^0,y^1,0,..,0)\right]dy^1.$$

Now $\partial_{y^1}\psi = \frac{t}{t(y^0,x^1+ty^1,x^2,..,x^n)}\Omega_{01}$, where the t in the numerator is that of the point (t,x) while $t(y^0,x^j+ty^j)\equiv\left((y^0)^2+(x^j+ty^j)^2\right)^{1/2}$ is the time of the point defined in the y^α coordinates by (y^0,x^j+ty^j). Now if $|x|\le t/2$, then it follows from the conditions $|y^1|\le\frac{1}{(8n^{1/2})}\le\frac{1}{8}$ that $(y^0)^2\ge 3/4t^2$ and thus that $|\frac{t}{t(y^0,x^1+ty^1,x^2,..,x^n)}|\le C$ for some uniform $C>0$. On the other hand if $|x|\ge t/2$, then it follows from the conditions $|y^1|\le\frac{1}{(8n^{1/2})}\le\frac{1}{8}$ that $|x^j+ty^j|\ge 3/8t$, where $y^j=(y^1,0,..,0)$. Thus, we have for $|y^1|\le 1/(8n^{1/2})$

$$\left|\partial_{y^1}\psi\right|(y^0,y^1,0,..,0) \lesssim \left|\int_v \Omega_{01}|f|(y^0,x^1+ty^1,x^2,..,x^n,v)\,d\mu_m\right|.$$

The remainder of the proof is then similar to the massless case. We have

$$\left|\int_v \Omega_{01}(|f|)(y^0, x^1 + ty^1, x^2, .., x^n, v)\, d\mu_m\right| \lesssim \sum_{|\alpha|\leq 1} \int_v |\widehat{Z}^\alpha f|(y^0, x^1 + ty^1, x^2, .., x^n, v)\, d\mu_m.$$

Inserting in the Sobolev inequality and repeating up to exhaustion of all the variables (the fact that for all j, $|y^j| \leq \frac{1}{(8n^{1/2})}$, guarantees that $|y| = (\sum_{j=1}^n |y^j|^2)^{1/2} \leq 1/8$ so that we still have $\frac{t}{t(y^0, x^j + ty^j)} \sim 1$), we obtain

$$\int_v |f|(y^0, x^1, x^2, .., x^n, v)\, d\mu_m \quad \lesssim \quad \sum_{|\alpha|\leq n} \int_{|y|\leq 1/8} \int_v |\widehat{Z}^\alpha f|(y^0, x^j + ty^j, v)\, d\mu_m\, dy.$$

Recall that the volume form on each of the H_ρ is given in spherical coordinates by $\frac{\rho}{t} r^{n-1} dr d\sigma$, or in y^α coordinates by $\frac{y^0}{t} dy$. Thus, we have

$$\begin{aligned}
\int_v |f|(y^0, x^1, x^2, .., x^n, v)\, d\mu_m \quad &\lesssim \quad \sum_{|\alpha|\leq n} \int_{|y|\leq 1/8} \int_v |\widehat{Z}^\alpha f|(y^0, x^j + ty^j, v)\, d\mu_m \frac{t(y^0, x^j + ty^j)}{y^0} d\mu_{H_\rho}, \\
&\lesssim \quad \frac{t(y^0, x^j)}{y^0} \sum_{|\alpha|\leq n} \int_{|y|\leq 1/8} \int_v |\widehat{Z}^\alpha f|(y^0, x^j + ty^j, v)\, d\mu_m\, d\mu_{H_\rho},
\end{aligned}$$

where we have used again that $t(y^0, x^j + ty^j) \sim t(y^0, x^j)$ in the region of integration. Applying the change of coordinates $z^j = ty^j$ and noticing that the quantities on the right-hand side are controlled by the estimate (22) applied to $\widehat{Z}^\alpha(f)$ completes the proof. □

Since the norm on the right-hand side of (20) is conserved if f is a solution to the massive transport equation, we obtain, as a corollary, the following pointwise decay estimate.

Theorem 4.4 (Pointwise decay estimates for velocity averages of massive distribution functions [FJS15])**.** *Let f be a regular distribution function for the massive case satisfying the massive transport equation $\mathbf{T}_m(f) = 0$ on $\bigcup_{1\leq\rho<+\infty} H_\rho \times \mathbb{R}^n_v$. Then, for all $(t, x) \in \bigcup_{1\leq\rho<+\infty} H_\rho$,*

$$\int_{v\in\mathbb{R}^n} |f|(t, x, v) \frac{dv}{v^0} \lesssim \frac{1}{(1+t)^n} \|f\|_{\mathbb{P},n}(\rho = 1).$$

Finally, let us mention the following improved decay for derivatives.

Proposition 4.1 (Improved decay estimates for derivatives of velocity averages of massive distribution functions)**.** *Let f be a regular distribution function for the massive case satisfying the massive transport equation $\mathbf{T}_m(f) = 0$ on $\bigcup_{1\leq\rho<+\infty} H_\rho \times \mathbb{R}^n_v$. Then, for all $i \in \mathbb{N}$, for all multi-indices l and for all $(t, x) \in \bigcup_{1\leq\rho<+\infty} H_\rho$,*

$$\left| v_\rho^i \partial_y^l \int_{v\in\mathbb{R}^n} f(t, x, v) \frac{dv}{v^0} \right| \lesssim \frac{1}{(1+t)^{n+|l|} \rho^i} \|f\|_{\mathbb{P},n+i+l}(\rho = 1),$$

where $v_\rho = \frac{x^\alpha \partial_{x^\alpha}}{\rho}$ is the future unit normal to H_ρ and ∂_y^l is a combination of $|l|$ vector fields among the ∂_{y^k}, $1 \leq k \leq n$, which are tangent to the H_ρ.

Proof. We have $v_\rho = \frac{S}{\rho}$ with S the scaling vector field. On the other hand, recall that S essentially commutes with the massive transport operator, so that in particular $\mathbf{T}_m(S(f)) = 0$ if $\mathbf{T}_m(f) = 0$. Thus, $\int_v S(f)\frac{dv}{v^0} = S\left(\int_v f\frac{dv}{v^0}\right)$ satisfies the same decay estimates as $\int_v f\frac{dv}{v^0}$, which shows the improved decay for $v_\rho(\int_v f\frac{dv}{v^0})$. The higher order derivatives follow similarly. Indeed, using that $S(\rho) = \rho$, we have for instance $S^2(f) = \rho^2 v_\rho^2(f) + S(f)$. Applying the decay estimates for the velocity averages of $S^2(f)$ and $S(f)$ gives the correct improved decay for velocity averages of $v_\rho^2(f)$. Higher normal derivatives can be treated similarly. Finally, the improved decay for tangential derivatives of velocity averages is an easy consequence of the fact that $\partial_{y^k} = \frac{1}{t}\Omega_{0k}$. □

4.9 Distribution functions for massive particles with compact support in x

The decay estimates for massive fields require that the initial data be given on the initial hyperboloid H_1 instead of a more traditional $t = const$ hypersurface. We will explain here how we can go from the $t = 0$ hypersurface to H_1 provided the initial data on $t = 0$ has compact support in x. For simplicity, consider the homogeneous massive transport equation with initial data f_0 given at $t = 0$. Assume that the support of f_0 is contained in the ball of radius R. Without loss of generality, we may translate the problem in time, so that we now consider the problem with data at time $t = \sqrt{R^2+1}$.

$$\mathbf{T}_m(f) = 0, \tag{23}$$

$$f(t = \sqrt{R^2+1}) = f_0. \tag{24}$$

Now, by the finite speed of propagation, the solution to this problem vanishes outside of the cone

$$\mathscr{C}(R) \equiv \left\{(t,r,\omega)\ /\ t - r = \sqrt{R^2+1} - R,\ \omega \in \mathbb{S}^{n-1},\ t \geq \sqrt{R^2+1}\right\} \cup \left\{(t,r,\omega)\ /\ t + r = \sqrt{R^2+1} + R,\ \omega \in \mathbb{S}^{n-1},\ t \leq \sqrt{R^2+1}\right\}$$

depicted below. Thus, the trace of f on H_1 is compactly supported and as a consequence, the norm appearing on the right-hand side of the decay estimate is finite. Recall also that the decay estimates hold for $t \geq \sqrt{1+|x|^2}$. On the other hand, the region $t < \sqrt{1+|x|^2}$ would coresponds here to the exterior of $\mathscr{C}(R)$. Thus, for compactly supported initial data given on some $t = const$ hypersurfaces, we can apply our decay estimates and obtain a $1/t^n$ decay uniformly in x.

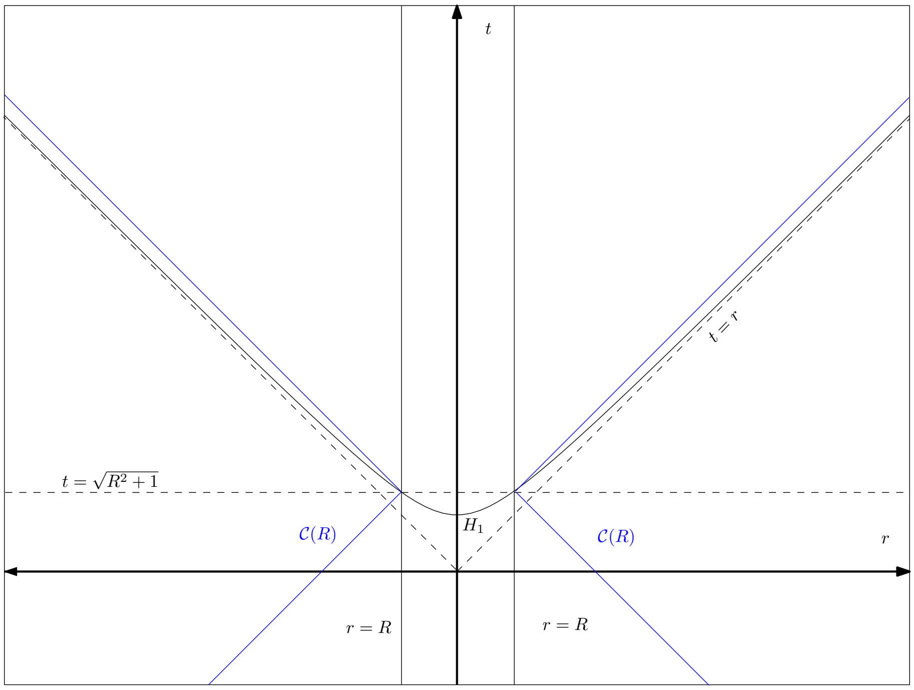

Figure 3: The trace of a distribution function with compact support on H_1.

4.10 The classical case

Finally, let us show how the same type of techniques can be used to derive decay estilmates for the classical transport equation

$$\partial_t f + v^i \partial_{x^i} f = 0.$$

Exercice:

1. Show that the following set of vector fields commute with the classical transport operator
   - The translations ∂_{x^α}.
   - The complete lift of the usual rotations $\Omega^x_{ij} + \Omega^v_{ij}$.
   - Vector fields associated with the Galilean invariance[7] $t\partial_{x^i} + \partial_{v^i}$.
   - Spatial scaling $x^i \partial_{x^i} + v^i \partial_{v^i}$
   - Space-time scaling $x^\alpha \partial_{x^\alpha}$.
2. Prove the following decay estimates

$$\int_v |f|(t,x,v)dv \lesssim \frac{1}{1+|x|+|t|} \sum_{|\alpha|\leq n} ||\widehat{Z}^\alpha f(0,x,v)||_{L^1_{x,v}},$$

   where the $\widehat{Z}^\alpha$ are compositions of $|\alpha|$ commuting vector fields.

[7] Another point of view is to consider them (formally) as degenerate versions (as $c \to +\infty$) of Lorentz boosts.

5 Yet another proof of decay for velocity averages

5.1 Weights preserved by the flow

Recall that in a general Lorentzian manifold with metric g, if γ is a geodesic with tangent vector $\dot{\gamma}$ and K denotes a Killing field, then $g(\dot{\gamma}, K)$ is preserved along γ.

Exercice: prove indeed that is K is Killing and $s \to \gamma(s)$ is a geodesic, then

$$\frac{d}{ds} g(K, \dot{\gamma}) = 0.$$

In this section, we explain how to transpose and exploit this fact for the free transport operators.

We define the sets of weights

$$\Bbbk_m \equiv \{v^\alpha x^\beta - x^\alpha v^\beta, v^\alpha\}, \quad m > 0 \tag{25}$$

and

$$\Bbbk_0 \equiv \{x^\alpha v_\alpha, v^\alpha x^\beta - x^\alpha v^\beta, v^\alpha\}. \tag{26}$$

The following lemma can be easily checked.

Lemma 5.1. *Let T_m be the relativistic transport operator of mass m. For all $\mathfrak{z} \in \Bbbk_m$, $[\mathbf{T}_m, \mathfrak{z}] = 0$.*

The weights in $\Bbbk_m$ or $\Bbbk_0$ also have good commutation properties with the vector fields in $\widehat{\mathbb{P}}$ and $\widehat{\mathbb{K}}$.

Lemma 5.2. *For any $\mathfrak{z} \in \Bbbk_m$ and any $\widehat{Z} \in \widehat{\mathbb{P}}$,*

$$[\widehat{Z}, \mathfrak{z}] = \sum_{\mathfrak{z}' \in \Bbbk_m} c_{\mathfrak{z}'} \mathfrak{z}'$$

where the $c_{\mathfrak{z}'}$ are constant coefficients.

Similarly for any $\mathfrak{z} \in \Bbbk_0$ and any $\widehat{Z} \in \widehat{\mathbb{K}}$,

$$[\widehat{Z}, \mathfrak{z}] = \sum_{\mathfrak{z}' \in \Bbbk_0} d_{\mathfrak{z}'} \mathfrak{z}',$$

for some constant coefficients $d_{\mathfrak{z}'}$.

Proof. This follows from straightforward computations. □

In the classical case, the weights that are preserved by the flow are the one associated with the Galilean symmetries.

Lemma 5.3. *Let $\mathfrak{z} = v^i, x^i - v^i t, x^i v^j - x^j v^i$, then $\left(\partial_t + v^i \partial_{x^i}\right) \mathfrak{z} = 0$.*

Again, these $\mathfrak{z}$ weights have good commutation properties with the commuting vector fields.

5.2 L^p decay of velocity averages

These extra weights can be used to prove decay estimates of velocity averages.

We start with the easy classical case:

Proposition 5.1. *Let f be a regular distribution function. Then, for all $t > 0$,*

$$\left\|\int_v f(t,.,v)dv\right\|_{L^2_x} \lesssim \frac{1}{t^3/2}||f < x - vt >^{n/2+}||_{L^2_{x,v}}.$$

In particular, if f solves the free transport equation $\partial_t + v^i\partial_{x^i} f = 0$, then

$$||\int_v f(t,.,v)dv||_{L^2_x} \lesssim \frac{1}{t^3/2}||f(0,.,.) < x >^{n/2+}||_{L^2_{x,v}}.$$

(Here $n/2+$ is any number of the form $n/2+\epsilon$ with $\epsilon > 0$ and the constant in the inequalities a priori degenerate as $\epsilon \to 0$. Moreover, $< . >$ is the Japanese bracket, meaning $< x >= \sqrt{1+|x|^2}$.)

Proof. We have

$$\begin{aligned} \int_v f dv &= \int_v f \frac{< x - vt >^{n^+/2}}{< x - vt >^{n^+/2}} dv, && (27)\\ &\leq ||f < x - vt >^{n^+/2}||_{L^2_v}||< x - vt >^{-n^+/2}||_{L^2_v}, && (28) \end{aligned}$$

using the Cauchy-Schwarz inequality. This leads to

$$||\int_v f dv||_{L^2_x} \leq ||f < x - vt >^{n^+/2}||_{L^2_{x,v}}||< x - vt >^{-n^+/2}||_{L^\infty_x L^2_v}.$$

Let us thus compute

$$||< x - vt >^{-n/2^+}||^2_{L^2_v} = \int_{v\in\mathbb{R}^n} < x - vt >^{-n^+} dv.$$

Aplying again the change of variable $w = x - vt$ we get the results. □

Exercice:

- Write decay estimates for L^p norms of $\int_v f dv$ using the same type of arguments.
- We consider the relativistic massive transport operator. Again, we would like to use the Cauchy-Schwarz inequality (or Hölder) as follows

$$\begin{aligned} \int_v f dv &= \int_v f < v^0x - vt >^{n^+/2}< v^0x - vt >^{-n^+/2} dv, && (29)\\ &\leq ||f < v^0x - vt >^{n^+/2}||_{L^2}||< v^0x - vt >^{-n^+/2}||_{L^2_v}. && (30) \end{aligned}$$

  The difficulty comes from the second term on the right-hand side, as it does not have uniform L^∞ decay. A possible solution is to add extra v weights:

$$\int_v f dv \leq ||f < v^0 >^q< v^0x - vt >^{n^+/2}||_{L^2}||< v^0 >^{-q}< v^0x - vt >^{-n^+/2}||_{L^2_v}.$$

  Prove an L^2_x decay estimate for q sufficiently large. (Start with $n = 1$ for simplicity).
- Write an L^p analogue of these estimates.

5.3 Interlude: a vector field approach to decay of solutions to Schrödiner equations

Consider the $1d$ Schrödinger equation[8],

$$(i\partial_t + \partial_{xx})(u) = 0.$$

This equation admits several conservations, but in particular, you can check easily *mass conservation*

$$||u(t)||_{L^2_x} = ||u(0)||_{L^2_x}.$$

Using, say, the fundamental solution, it is easy to see that $|u(t)|^2 \lesssim 1/t^d$. Is there a vector field way to capture this ?

Again, we need to find the commuting vector fields, and they typically need to contains weights in t or x to get decay.

After trial and error, one can derive that (formally) if u solves the Schrödinger equation, so does

$$it\partial_x u + \frac{x}{2}u.$$

Unfortunately, because of the second term, the operator $u \to it\partial_x u + \frac{x}{2}u$ is not a pure differential operator, i.e. it does not correspond to a vector field.

The idea is to rewrite it by adding an *phase* function, as follows

$$it\partial_x u + \frac{x}{2}u = e^{\frac{ix^2}{4t}} it\partial_x \left(e^{-\frac{ix^2}{4t}} u\right).$$

Exercice: use the above formula to prove decay estimates for solutions to the Schródinger equation.

(Small) Open problem: The Airy equation reads

$$\partial_t + \partial_x^3.$$

It is a typical linear dispersive pde, that arises naturally in some model of fluid dynamics (for instance, it is the linear part of the so called Korteweg-de-Vries (KdV) equation). Using a Fourier decomposition of the solutions, one can easily see that solutions to the Airy equation will decay like $t^{-1/3}$.

Does there exists a vector field approach to this decay, perhaps using a similar *phase* conjugation ?

6 Null condition and null decomposition of tensors for Vlasov fields

In the case of wave equations, we know that due to the slow decay $(1/t)$ of waves in $3d$, we need *structural* conditions on the non-linearities for global solutions to exist[9]. The standard structures are the *null forms*

$$\begin{aligned} Q_0(\phi,\psi) &= g(d\phi, d\psi), \\ Q_{ij}(\phi,\psi) &= (\partial_{x^i}\psi\partial_{x^j}\phi - \partial_{x^j}\psi\partial_{x^i}) \end{aligned}$$

[8] I would like to thanks Sun Jin Oh for explaining me this example.

[9] See the seminal work of F. John [Joh79].

Where is the improvement coming ?

Improvement for decay:

In terms of decay, recall that the tangential derivative to the light cone, such as $\partial_t + \partial_r$, can be rewritten as

$$L = \sum_{|\alpha|=1} \frac{1}{t} c_\alpha Z^\alpha.$$

Thus, they generate a $1/t$ extra decay compared to a random vector field.

So the idea is that in the null forms Q we want to check that each product always contains a tangential derivative.

This leads to formulae such as

$$Q_0(\phi, \psi) = \frac{1}{t} \sum_{|\alpha|,|\beta|=1} c_{\alpha\beta} Z^\alpha(\phi) Z^\beta(\psi)$$

and the extra $1/t$ is the reason for the gain.

Improvement for regularity:

Imagine that we want to prove an energy estimate for an equation such as

$$\Box\psi = \partial\psi.\partial\psi.$$

After multiplying the equation by $\partial_t\psi$ and running the usual arguments for the energy estimate, we need to control the error term

$$\int_{t,x} |\partial_t\psi.\partial\psi.\partial\psi|(s,x)ds \lesssim \int_t |\partial_t\psi|_{L^2_x}(s) \times |\partial\psi.\partial\psi|_{L^2_x}(s)ds.$$

The first factor on the right hand side can be absorbed using a Gronwall type argument but we still need a bound on the second. This can be done using the Sobolev inequality, but implies that we need to control ψ in $H^{n/2+1+\epsilon}$.

However, for null forms, we can do much better:

Theorem 6.1 (Klainerman-Machedon estimates (in dimension 3) [KM93])**.** *For Q a null form, if*

$$\begin{aligned} \Box\psi &= Q(\psi,\phi), \\ \Box\phi &= Q(\psi,\phi) \end{aligned}$$

then, locally in time

$$||Q(\partial\psi.\partial\phi)||_{L^2_{t,x}} \lesssim E(\partial\psi) + E(\psi) + E(\phi) + E(\partial\phi).$$

where E is the usual energy for the wave equation.

So there is a gain of $1/2+\epsilon$. (without the null forms, we can actually also improve to just $H^{2+\epsilon}$ regularity, using the so-called Strichartz estimates, see for instance [PS93]).

Moreover, we are also familiar with the idea of decomposing tensors into tensor components using a *null frame*. A typical null frame in Minkowski space is given by

$\bar{L} := \partial_t - \partial_r, L := \partial_t + \partial_r, e_1, e_2$, where e_1, e_2 are a local orthonormal frame of vector fields tangent to the 2-spheres of constant t and r.

If we consider say a Maxwell 2-form F, one can prove (see [CK90]), using the standard vector field method (including the Morawetz vector field as a multiplier)

$$|F_{AL}| \lesssim (1+t+|x|)^{-5/2}, \quad |F_{A\bar{L}}| \lesssim (1+t+|x|)^{-1}(1+t-|x|)^{-3/2}, \ldots$$

and these types of estimates can easily be saturated.

We will present here some analogue of these estimates in the case of massless Vlasov fields. (Note that the dynamics of massless fields are also relevant to the massive fields, in the sense that many special properties of massive fields degenerate as either $|v| \to +\infty$ or $m \to 0$.)

One of the main motivations comes from the non-linear problems.

Let us illustrate this with the so-called *Einstein-Vlasov system*[10]

$$Ric(g) - \frac{1}{2} g R(g) = T$$

where $Ric(g)$ and $R(g)$ denotes respectively the Ricci tensor and Ricci scalar of some Lorentzian metric g and where $T = T[f]$ is the energy momentum tensor of a Vlasov field f which solves the Vlasov equation

$$v^\alpha \partial_{x^\alpha} f - v^\gamma v^\beta \Gamma^i_{\gamma\beta}(g) \partial_{v^i} f = 0,$$

where $\Gamma^i_{\gamma\beta}(g)$ are the Christoffel symbols of the metric g.

We recall that, schematically, the first equation takes the form (in so called wave coordinate)

$$\Box_g g_{\mu\nu} = Q_{\mu\nu}(\partial g, \partial g) + T_{\mu\nu}[f],$$

where $\Box_g$ is the wave operator associated with the metric g and Q is some bilinear form.

Consider massless particles. In 3d, the decay that we have obtained so far gives would imply that for g close to the Minkowskis metric, $|T_{\mu\nu}[f]|$ decays like $1/t$. This will lead to logarithmic growth in the energy estimates.

This type of logarithmic obstacle for small data global existence is classic for non-linear waves with quadratic non-linearities in low dimension and without any special structure on the equation, the best we can hope is an almost global existence (say existence of solutions up to time $\exp(1/\epsilon)$ where ϵ measure the "size" of the solution initiallly).

Thus, we also need a form of the null condition here for the terms coming from $T[f]$.

Moreover, we will also need a form of the null condition in the Vlasov equation.

We will give below several illustrations of null conditions for Vlasov fields, either at the linear level or for model problems .

[10] See [Rin13] for a thorough presentation of this system.

6.1 Null decomposition of the energy-momentum tensor and decay

Consider again the energy-momentum tensor of massless Vlasov fields in the flat Minkowski space.

$$T_{\mu\nu}[f] = \int_{v\in\mathbb{R}^3} f v_\mu v_\nu \frac{dv}{v^0}.$$

Let $L = \partial_t + \partial_r$. If we want to consider couple systems of wave and particles, we need to understand why T_{LL} should decay better than say $T_{\underline{L}\underline{L}}$.

One way is, as above, to exploit the weights propagated by the flow.

Indeed, in the same sense that one can write vector identities such as

$$(t+r)L = \sum_{|\alpha|=1} c_\alpha Z^\alpha$$

where c_α are homogeneous of degree zero, we have the weights identities

Lemma 6.1. *Let $v_\alpha = m_\alpha$ be the mulplier associated to any of the translation vector fields. Let $\omega^i = x^i/|x|$ and let $\partial_r = \omega^i\partial_i$ and $m_r = \omega^i m_i$, $i = 1,..,3$. Finally let m_S be the multiplier associated with the scaling vector field. Then, we have*

$$v_t = \frac{t m_S - x^i \omega_{0i}}{t^2-r^2} \tag{31}$$

$$v_r = \frac{t\omega^i\omega_{0i} - r m_s}{t^2-r^2} \tag{32}$$

$$v_i = \frac{-x^j\omega_{ij} + t\omega_{0i} - x_i m_S}{t^2-r^2} = -\frac{x_i m_s}{t^2-r^2} + \frac{x_i x^j \omega_{0j}}{t(t^2-r^2)} + \frac{\omega_{0i}}{t}. \tag{33}$$

In particular,

$$v_L = \frac{1}{2}(v_t + v_r) = \frac{m_s + \omega^i\omega_{0i}}{2(t+r)}, \tag{34}$$

$$\overline{v}_i := v_i - \omega_i m_r = \frac{\omega^j\omega_{ij}}{r} = -\frac{\omega_i\omega^j\omega_{0j} + \omega_{0i}}{t}. \tag{35}$$

Here $\overline{v}_i$ is the projection to the cone of the v_i vector. If v is a covector, we shall we shall denote by $\overline{v}$ the tangential[11] part of v (i.e. $<\underline{L}, \overline{v}>= 0$, with $\underline{L} = \partial_t - \partial_r$). Thus, $|\overline{v}| = |v_L| + |\overline{v_i}|$.

We then have the estimates

Lemma 6.2. *Let $q = r - t$. Then,*

$$(1 + t + |q|)|\overline{v}| + (1 + |q|)|v| \lesssim \sum_I |m^I(v)|.$$

where the sum is over all possible multipliers corresponding to translation, angular momentum and scaling.

[11] Note than we have $v_L = -\frac{1}{2}v^q, v_q = -\frac{1}{2}v^L$, thus, the tangential part of the velocity (index down) corresponds to the transvertial part of the momentum, index up.

Proof. The proof is very similar to the one of the standard vector field method. Note first that if $|t|+|r| \leq 1$, then the estimate holds since the usual translation are included in the sum. The idendity for $|\overline{v}|$ following directly from the previous lemma. □

The conclusion is that we get improved decay for $T_{LL}[f]$ but the norm on the right-hand side will have extra weights corresponding to the $\mathfrak{z}$ weights.

6.2 A null form for a wave/particle interaction

Consider the partially non-linear problem

$$\begin{aligned} T(f) &= v^0 Q(\partial\phi, f), \\ \Box\phi &= 0, \end{aligned}$$

where T is the massless relativistic transport operator.

Here $Q(\partial\phi, f)$ is a linear combination of terms of the form $\partial\phi.f$ with constant (or homogeneous) coefficients and the extra v^0 is here only because of homogeneity.

Exercice:

1. Write a commutator formula for the above equation after N commutations.
2. Write the energy estimates.
3. Show that solutions to the above problem have the same decay as in the linear case in dimension $n \geq 4$.

In dimension 3, since $\partial\phi$ decays a priori no faster than $1/t$, standard energy estimates for the above transport equation would lead to a log growth.

As in the standard application of the theory of null forms, we want to understand whether there might exist special non-linearities, for which this log growth is absent.

We will consider the following special non-linearity:

$$Q(\partial\phi, f) = \frac{1}{v^0} T(\phi) f,$$

where $T(\phi) = v^\alpha \partial_{x^\alpha}(\phi)$.

To understand why this non-linearity is better, we first show that, like Q_0, this special non-linearity can be eliminated using a simple transformation.

(recall that if $\Box\phi = Q_0(\phi,\phi)$ and ϕ is small then, setting $\phi = \ln(1+v)$, then v solves the homogeneous wave equation. This is the so-called "Niremberg example".)

Indeed, with the above choice of Q, the equation becomes

$$T(f) = T(\phi) f,$$

which can be rewritten as

$$T(f e^{-\phi}) = 0$$

and using $g = f e^{-\phi}$ as our unknown, we can apply all the linear estimates to g and then go back to our original variable.

In the actual non-linear physical systems, such as the Einstein-Vlasov, Vlasov-Norström or Vlasov-Maxwell, the non-linearities typically have a different structure, closer to

$$T(f) + Q(\partial_x \phi, \partial_v f) = 0$$

In the (massless) Vlasov-Norström case, one has for instance

$$T(f) + T(\phi) v^i \partial_{v^i} f = 0. \tag{36}$$

The extra v derivatives acting on f is a true annoyance, as ∂_{v^i} does not belongs to our algebra of commuting fields. Note however that $v^i \partial_{v^i}$ does commute with T, in the sense that $[T, v^i \partial_{v^i}] = -T$.

So we will enhance our algebra of commuting fields to include $v^i \partial_{v^i}$.

However, with (36), we can no longer eliminate the non-linearity by our trick. So we need to estimate products of the form $T(\phi) v^i \partial_{v^i} f$. We therefore need an estimate for products of the form $T(\phi) g$, where g solves a transport equation.

Why is this product better ? For wave equations, we need to see some sort of derivatives tangential to the light-cone for a product to be better. We therefore force the apparition of $\partial_t + \partial_r$ as follows

$$\begin{aligned} T(\phi) g &= v^\alpha \partial_{x^\alpha} \phi . g, \\ &= v^0 \left(\partial_t + \frac{v^i}{v^0} \partial_{x^i} \right) (\phi) g, \\ &= v^0 \left(\partial_t + \partial_r - \partial_r + \frac{v^i}{v^0} \partial_{x^i} \right) (\phi) g, \\ &= v^0 (\partial_t + \partial_r) (\phi) g \\ &\quad + v^0 \left(-\partial_r + \frac{v^i}{v^0} \partial_{x^i} \right) (\phi) g. \end{aligned}$$

Now the first term on the RHS is good, (it has extra decay) so we focus on the second.

$$\begin{aligned} v^0 \left(-\partial_r + \frac{v^i}{v^0} \partial_{x^i} \right) (\phi) g &= v^0 \left(-\frac{x^i}{|x|} \partial_{x^i} + \frac{v^i}{v^0} \partial_{x^i} \right) (\phi) g \\ &= v^0 \left(-\frac{x^i}{|x|} + \frac{v^i}{v^0} \right) \partial_{x^i} \phi. \end{aligned}$$

Now, away from the light-cone, we know that we can get extra decay and therefore close the estimates. Near the light-cone on the other hand $|x| \sim t$, so let us pretend that we can rewrite the above factor as

$$\frac{1}{t} \left(-x^i + t \frac{v^i}{v^0} \right) = \frac{1}{t} \mathfrak{z}^i$$

where $\mathfrak{z}^i$ are part of the weights we can propagated along the characteristic flow. Thus, we can incorporate these weights in the energy estimates for g.

6.2.1 Applications to the massless Vlasov-Nordström system

We consider in this section, the so-called Vlasov-Norström system. The Norström theory is a scalar theory of gravity that still contains gravitational wave. Formally, one can obtain it from the usual Einstein equations by considering only conformal deformation of Minkowski space and by removing all non-linear wave interactions.

We will denote by $\mathbf{T}_\phi$ the transport operator defined by

$$\mathbf{T}_\phi \equiv v^\alpha \frac{\partial}{\partial x^\alpha} - v^\alpha \nabla_\alpha \phi \cdot v^i \frac{\partial}{\partial v^i},$$

i.e.

$$\mathbf{T}_\phi = \mathbf{T}_0 - \mathbf{T}_0(\phi) \cdot v^i \partial_{v^i}.$$

The massless Vlasov-Nordström system can then be rewritten as

$$\Box \phi = 0, \tag{37}$$

$$\mathbf{T}_\phi(f) = 0, \tag{38}$$

which we complement by the initial conditions

$$\phi(t=0) = \phi_0, \quad \partial_t \phi(t=0) = \phi_1, \tag{39}$$

$$f(t=0) = f_0, \tag{40}$$

where (ϕ_0, ϕ_1) are sufficiently regular functions defined on $\mathbb{R}^n_x$ and f_0 is a sufficiently regular function defined on $\mathbb{R}^n_x \times \left(\mathbb{R}^n_v \setminus \{0\}\right)$.

By sufficiently regular, we mean that all the computations below make sense. We will eventually require that $\mathscr{E}_N[\phi_0, \phi_1] < +\infty$ where $\mathscr{E}_N$ is the energy norm defined by (44) and similarly, we will also require below that $||f_0||_{\mathbb{K},N} < +\infty$ (with some additional weights in the case of dimension 3). Provided N is large enough (depending only on n), these two regularity requirements are then enough so that all the computations below are justified. In the remaining, we will therefore omit any further mention of regularity issues.

6.2.2 Commutation formula for $\mathbf{T}_\phi$

Recall the algebra of commutation fields $\widehat{\mathbb{K}}$. Let $\widehat{\mathbb{K}}_0 = \widehat{\mathbb{K}} \cup \{v^i \partial_{v^i}\}$.

Lemma 6.3. *For any $\widehat{Z} \in \widehat{\mathbb{K}}_0$,*

$$\begin{aligned} [\mathbf{T}_\phi, \widehat{Z}] &= c_Z T + \left[-c_Z T(\phi) + T(Z(\phi))\right] v^i \partial_{v^i}, \\ &= c_Z \mathbf{T}_\phi + T(Z(\phi)) v^i \partial_{v^i}, \end{aligned}$$

where $c_Z = 0$ if $\widehat{Z} \in \widehat{\mathbb{K}}$ and $c_Z = 1$ if $\widehat{Z} = S$, and where Z is the non-lifted field corresponding to $\widehat{Z}$ if $\widehat{Z} \in \widehat{\mathbb{K}}$ and $Z = S$ if $\widehat{Z} = S$.

Proof. Note first that for all $\widehat{Z} \in \widehat{\mathbb{K}}$, we have $[\widehat{Z}, v^i \partial_{v^i}] = 0$. We then compute

$$\begin{aligned} \left[\mathbf{T}_\phi, \widehat{Z}\right] &= [T, \widehat{Z}] - \left[T(\phi) v^i \partial_{v^i}, \widehat{Z}\right] \\ &= [T, \widehat{Z}] + \widehat{Z}\left(T(\phi)\right) v^i \partial_{v^i} + T(\phi) \left[\widehat{Z}, v^i \partial_{v^i}\right] \\ &= \left[T, \widehat{Z}\right] + \left(\left[\widehat{Z}, T\right] \phi + \mathbf{T}_0(\widehat{Z}\phi)\right) v^i \partial_{v^i}. \end{aligned}$$

To conclude the proof of the lemma, replace all the instances of $\left[T, \widehat{Z}\right]$ by $c_Z T$ according to Lemma 4.2. □

Iterating the above, one obtains

Lemma 6.4. *Let f be a solution to* (38)*. For any multi-index α, we have the commutator estimate*

$$\left|[\mathbf{T}_\phi, \widehat{Z}^\alpha] f\right| \leq C \sum_{\substack{|\beta|\leq|\alpha|,|\gamma|\leq|\alpha|,\\ |\beta|+|\gamma|\leq|\alpha|+1}} |T(Z^\gamma(\phi))| \cdot |\widehat{Z}^\beta f|, \tag{41}$$

where the $Z^\gamma \in \mathbb{K}^{|\gamma|}$ and the $\widehat{Z}^\beta \in \widehat{\mathbb{K}}_0^{|\beta|}$ and $C > 0$ is some constant depending only on $|\alpha|$.

6.2.3 Approximate conservation law

Similar to the conservation of the L^1 norm for the free transport operator, we have

Lemma 6.5. *Let h be a regular distribution function. Let g be a regular solution to $\mathbf{T}_\phi(g) = v^0 h$, with $v^0 = |v|$, defined on $[0,T]\times\mathbb{R}^n_x\times\left(\mathbb{R}^n_v\setminus\{0\}\right)$ for some $T > 0$. Then, for all $t \in [0,T]$,*

$$\begin{aligned}&\int_{\Sigma_t} \rho_0(|g|)(t,x)dx \\ &\leq \int_{\Sigma_0} \rho_0(|g|)(0,x)dx + \int_0^t \int_{\Sigma_s} \rho_0(|h|)(s,x)dxds + (n+1)\int_0^t \int_{\Sigma_s}\int_{v\in\mathbb{R}^n_v\setminus\{0\}} |T(\phi) f| dv dx ds.\end{aligned} \tag{42}$$

Proof. As before, this follows, after regularization of the absolute value, from integration by parts or an application of Stoke's theorem. The term $T(\phi)v^i\partial_{v^i}|f|$, which appears in the computation, gives rise after integration by parts in v to the last term in (42) since $\partial_{v^i}\left(v^i T(\phi)\right) = (n+1)T(\phi)$. □

6.2.4 Massless case in dimension $n \geq 4$

In this section, we first consider the case $n \geq 4$, the $3d$ case requiring the use of the null condition as explained above.

If ϕ is a solution to the wave equation, let us consider the energy at time $t = 0$

$$\mathcal{E}_N[\phi](t=0) \equiv \sum_{|\alpha|\leq N, |\alpha|\in\mathbb{K}^{|\alpha|}} \left|\left|Z^\alpha(\partial\phi)(t=0)\right|\right|^2_{L^2(\mathbb{R}^n_x)}. \tag{43}$$

Now if $\phi(t=0) = \phi_0$ and $\partial_t\phi(t=0) = \phi_1$, for pairs of sufficiently regular functions (ϕ_0,ϕ_1) defined on $\mathbb{R}^n_x$, then the above quantity can be computed purely in terms of ϕ_0, ϕ_1, so we define[12]

$$\mathcal{E}_N[\phi_0,\phi_1] \equiv \mathcal{E}_N[\phi](t=0). \tag{44}$$

Similarly, if f is a solution to (38) arising from initial data f_0 at $t = 0$, then we define

$$E_N[f](t=0) \equiv ||f||_{\mathbb{K},N}(t=0) \left(= \sum_{|\alpha|\leq N, \widehat{Z}^\alpha\in\widehat{\mathbb{K}}_0^\alpha} \left|\left|\rho_0\left(\widehat{Z}^\alpha(f)(t=0)\right)\right|\right|_{L^1(\mathbb{R}^n_x)}\right) \tag{45}$$

[12] The alternative to the approach we use here is to assume that (ϕ_0,ϕ_1) are regular initial data with decay fast enough in x, for instance by assuming compact support, so that the resulting $\mathcal{E}_N[\phi(t=0)]$ is finite. What we want to emphasize here is that the quantity $\mathcal{E}_N[\phi(t=0)]$ can in fact be computed purely in terms of the initial data (using the equation to rewrite second and higher time derivatives of ϕ in terms of spatial derivatives), and that this is all that is needed in terms of decay in x.

and we remark that this quantity can be computed purely in terms of f_0 so we will set

$$E_N[f_0] \equiv E_N[f](t=0).$$

We will prove

Theorem 6.2. *[FJS15]Let $n \geq 4$ and let $N \geq \frac{3n}{2}+1$. Let (ϕ_0, ϕ_1, f_0) be an initial data set for the massless Vlasov-Nordström system such that $\mathcal{E}_N[\phi_0,\phi_1] + E_N[f_0] < +\infty$. Then, the unique solution (f,ϕ) to* (37)-(38) *satisfying the initial conditions* (39)-(40) *verifies the estimates*

1. *Global bounds: for all $t \geq 0$,*

$$E_N[f](t) \leq e^{C\mathcal{E}_N^{1/2}[\phi_0,\phi_1]} E_N[f_0],$$

where $C > 0$ is a constant depending only on N, n.

2. *Pointwise estimates for velocity averages: for all $(t,x) \in [0,+\infty) \times \mathbb{R}^n_x$ and all multi-indices α satisfying $|\alpha| \leq N-n$,*

$$\rho_0(|\widehat{Z}^\alpha f|)(t,x) \lesssim \frac{e^{C\mathcal{E}_N^{1/2}[\phi_0,\phi_1]} E_N[f_0]}{(1+|t-|x||)(1+|t+|x||)^{n-1}}.$$

Proof. Let $N, n, \phi_0, \phi_1, f_0$ be as in the statement of the theorem. From the conservation of energy and the commutation properties of the Z^α with the wave operator, we have, for all t,

$$\mathcal{E}_N[\phi](t) = \mathcal{E}_N[\phi_0,\phi_1] \equiv \mathcal{E}_N.$$

Applying the standard decay estimates obtained via the vector field method to ϕ, we have for all multi-indices α satisfying $|\alpha| \leq N-(n+2)/2$ and for all $(t,x) \in \mathbb{R}_t \times \mathbb{R}^n_x$

$$|\partial Z^\alpha \phi(t,x)|^2 \lesssim \frac{\mathcal{E}_N[\phi](t)}{(1+|t-|x||)(1+|t+|x||)^{n-1}}. \tag{46}$$

Note that it follows from a standard existence theory for regular data that for all t, $E_N[f(t)] < +\infty$.

Applying the Klainerman-Sobolev inequality (9), we obtain, for all multi-indices α satisfying $|\alpha| \leq N-n$ and for all $(t,x) \in \mathbb{R}_t \times \mathbb{R}^n_x$,

$$|\rho_0(\widehat{Z}^\alpha(f))(t,x)| \lesssim \frac{E_N[f](t)}{(1+|t-|x||)(1+|t+|x||)^{n-1}}.$$

From Lemma 6.5 and the commutator estimate (41), we have for all $t \geq 0$ and all multi-indices α,

$$\int_{\Sigma_t} \rho_0(|\widehat{Z}^\alpha f|)(t,x)dx \leq \int_{\Sigma_0} \rho_0(|\widehat{Z}^\alpha f|)(0,x)dx + \int_0^t \int_{\Sigma_s} \rho_0(|h^\alpha|)(s,x)dxds, \tag{47}$$

where[13]

[13] Note that the last term in the right-hand side of Lemma 6.5 is similar to the error terms arising from the commutator estimate of Lemma 6.4 and is therefore accounted for in the h^α error term in equation (47).

$$\begin{aligned} |h^\alpha| &\lesssim \frac{1}{v^0} \sum_{\substack{|\beta|\le|\alpha|,|\gamma|\le|\alpha|,\\ |\beta|+|\gamma|\le|\alpha|+1}} |T(Z^\gamma\phi)|\cdot|\widehat{Z}^\beta f| \\ &\lesssim \sum_{\substack{|\beta|\le|\alpha|,|\gamma|\le|\alpha|,\\ |\beta|+|\gamma|\le|\alpha|+1}} |\partial(Z^\gamma\phi)|\cdot|\widehat{Z}^\beta f|, \end{aligned}$$

so that

$$\rho_0(|h^\alpha|) \lesssim \sum_{\substack{|\beta|\le|\alpha|,|\gamma|\le|\alpha|,\\ |\beta|+|\gamma|\le|\alpha|+1}} |\partial(Z^\gamma\phi)|\rho_0\left(|\widehat{Z}^\beta f|\right),$$

since ϕ is independent of v. Integrating over x, we obtain, for all $s \in [0,t]$,

$$\int_{\Sigma_s} \rho_0(|h^\alpha|)(s,x)dx \lesssim \sum_{\substack{|\beta|\le|\alpha|,|\gamma|\le|\alpha|,\\ |\beta|+|\gamma|\le|\alpha|+1}} \int_{\Sigma_s} |\partial(Z^\gamma\phi)|\rho_0\left(|\widehat{Z}^\beta f|\right)(s,x)dx.$$

We now estimate each term in the above sum depending on the values of $|\gamma|$ and $|\beta|$.

If $|\beta| \le N-n$, we then apply the pointwise estimates on $\rho_0(\widehat{Z}^\beta(f))$ to obtain

$$\int_{\Sigma_s} |\partial(Z^\gamma\phi)|\rho_0\left(|\widehat{Z}^\beta f|\right)(s,x)dx \lesssim \int_{\Sigma_s} |\partial(Z^\gamma\phi)| \frac{E_N[f](s)}{(1+|s-|x||)(1+|s+|x||)^{n-1}} dx.$$

Applying the Cauchy-Schwarz inequality and using that

$$\left\|\frac{1}{(1+|s-|x||)(1+|s+|x||)^{n-1}}\right\|_{L^2_x} \lesssim \frac{1}{(1+s)^{(n-1)/2}}, \tag{48}$$

we obtain

$$\int_{\Sigma_s} |\partial(Z^\gamma\phi)|\rho_0\left(|\widehat{Z}^\beta f|\right)(s,x)dx \lesssim \mathcal{E}_N^{1/2}[\phi](s)\frac{E_N[f](s)}{(1+s)^{(n-1)/2}}. \tag{49}$$

If now $|\beta| > N-n$, then $|\gamma| \le |\alpha|+1-|\beta| \le N-(n+2)/2$ and thus, we also have (49), using this time the pointwise estimates on $\partial(Z^\gamma\phi)$ given by (46). Applying Grönwall's inequality finishes the proof of the theorem. □

6.2.5 Massless case in dimension $n = 3$

We now turn to the case of the dimension 3, where the slower pointwise decay of solutions to the wave equations leads to a slightly harder analysis. First, let us strengthen our norms for the Vlasov field.

For this, recall the algebra of weights $\mathbb{k}_0$ introduced in Section 5.1 and define a rescaled version κ_0 by

$$\kappa_0 \equiv (v^0)^{-1}\mathbb{k}_0 = \left\{\frac{\mathfrak{z}}{v^0} \,/\, \mathfrak{z} \in \mathbb{k}_0\right\},$$

where we recall that $v^0 = |v|$ in the massless case. If α is a multi-index, we will write $\left[\frac{\mathfrak{z}}{v^0}\right]^\alpha \in \kappa_0^{|\alpha|}$ to denote a product $|\alpha|$ elements of κ_0 and $\left[\frac{|\mathfrak{z}|}{v^0}\right]^\alpha$ in case we take the product of the absolute values of these elements.

Let us now define, for any regular distribution function f, the weighted norm

$$E_{N,q}[f] \equiv \sum_{\substack{|\alpha|\le N,\\ |\beta|\le q}} \sum_{\widehat{Z}^\alpha \in \widehat{\mathbb{K}}_0^{|\alpha|}} \int_{\Sigma_t} \rho_0\left(|\widehat{Z}^\alpha f| \left[\frac{|\mathfrak{z}|}{v^0}\right]^\beta\right)(x)dx \tag{50}$$

$$\left(= \sum_{\substack{|\alpha|\le N,\\ |\beta|\le q}} \sum_{\widehat{Z}^\alpha \in \widehat{\mathbb{K}}^{|\alpha|}} \int_{\Sigma_t} \int_{v\in\mathbb{R}^n\setminus\{0\}} \left(|\widehat{Z}^\alpha f|(x,v) \left[\frac{|\mathfrak{z}|}{v^0}\right]^\beta\right) v^0 dv dx\right),$$

where the weights $\frac{\mathfrak{z}}{v^0}$ lie in κ_0.

Theorem 6.3 (Asymptotic behaviour in dimension $n=3$[FJS15])**.** *Consider the dimension $n=3$. Let $N\ge 7$ and $q\ge 1$. Let (ϕ_0,ϕ_1,f_0) be an initial data set for the massless Vlasov-Nordström system such that $\mathcal{E}_N[\phi_0,\phi_1]+E_N[f_0]_{N,q}<+\infty$. Then, the unique solution (f,ϕ) to* (37)-(38) *satisfying the initial conditions* (39)-(40) *verifies the estimates*

1. *Global bounds with growth for the top order norms: for all $t\in\mathbb{R}_t$,*
$$E_{N,q}[f](t) \le (1+t)^{C\mathcal{E}_N^{1/2}[\phi_0,\phi_1]} E_{N,q}[f_0], \tag{51}$$
where $C>0$ is a constant depending only on N, n and q.
2. *Small data improvement for the low order norms: there exists an ε_0 (depending only on n,N,q) such that if $\mathcal{E}_N[\phi_0,\phi_1]\le\varepsilon_0$, then for all $t\in\mathbb{R}_t$,*
$$E_{N-(n+4)/2,q-1}[f](t) \le e^{C\mathcal{E}_N^{1/2}[\phi_0,\phi_1]} E_{N,q}[f_0]. \tag{52}$$
3. *Under the above smallness assumption, we also have the optimal pointwise estimates for velocity averages: for all $(t,x)\in\mathbb{R}_t\times\mathbb{R}^n_x$ and all multi-indices α satisfying $|\alpha|\le N-(3n+4)/2$ and all $|\beta|\le q-1$,*
$$\rho_0\left(\left|\widehat{Z}^\alpha(f)\left[\frac{\mathfrak{z}}{v^0}\right]^\beta\right|\right)(t,x) \lesssim \frac{e^{C\mathcal{E}_N^{1/2}[\phi_0,\phi_1]}E_{N,q}[f_0]}{(1+|t-|x||)(1+|t+|x||)^{n-1}}.$$

Proof. First, let us note that for all $\mathfrak{z}\in\mathbb{k}_0$, we have

$$v^i\partial_{v^i}\left(\frac{\mathfrak{z}}{v^0}f\right) = \frac{\mathfrak{z}}{v^0}v^i\partial_{v^i}f,$$

from which it follows that for all regular distribution functions g, $\left[\mathbf{T}_\phi, \frac{\mathfrak{z}}{v^0}\right]g=0$. Thus, we can upgrade Lemma 6.4 to

$$\left|\left[\mathbf{T}_\phi, \left[\frac{\mathfrak{z}}{v^0}\right]^\sigma \widehat{Z}^\alpha\right] f\right| \le C \sum_{\substack{|\beta|\le|\alpha|, |\gamma|\le|\alpha|,\\ |\beta|+|\gamma|\le|\alpha|+1}} |T(Z^\gamma\phi)|\cdot\left[\frac{|\mathfrak{z}|}{v^0}\right]^\sigma |\widehat{Z}^\beta f|, \tag{53}$$

where the $Z^\gamma\in\mathbb{K}_0^{|\gamma|}$, the $\widehat{Z}^\beta\in\widehat{\mathbb{K}}_0^{|\beta|}$, $\left[\frac{\mathfrak{z}}{v^0}\right]^\sigma\in\kappa_0^{|\sigma|}$ and $C>0$ is some constant depending only on $|\alpha|$. Applying arguments similar to those used in the $n\ge 4$ case yield

$$\begin{aligned} E_{N,q}[f](t) &\le E_{N,q}[f_0] + C\int_0^t \sum_{\substack{|\beta|\le|\alpha|, |\gamma|\le|\alpha|,\\ |\beta|+|\gamma|\le|\alpha|+1}} \sum_{|\sigma|\le q} \int_{\Sigma_s} |\partial(Z^\gamma\phi)|\rho_0\left(\left[\frac{|\mathfrak{z}|}{v^0}\right]^\sigma |\widehat{Z}^\beta f|\right)(s,x)dxds \\ &\le E_{N,q}[f_0] + C\int_0^t \mathcal{E}_N^{1/2}\frac{E_{N,q}[f](s)}{(1+s)}ds. \end{aligned} \tag{54}$$

Applying Grönwall inequality, we obtain (51).

Now assume that $\mathscr{E}_N \le \varepsilon_0$ with ε_0 small enough so that,

$$E_{N,q}[f](t) \le (1+t)^\delta E_{N,q}[f_0],$$

with $\delta = C\mathscr{E}_N^{1/2} < 1/2$.

The key to the improved estimates is the following decomposition of the transport operator T:

$$\begin{aligned} T &= v^0\partial_t + v^i\partial_{x^i} = v^0\left(\partial_t + \frac{x^i}{|x|}\partial_{x^i}\right) - v^0\frac{x^i}{|x|}\partial_{x^i} + v^i\partial_{x^i} \\ &= v^0\left(\partial_t + \frac{x^i}{|x|}\partial_{x^i}\right) + \frac{v^0x^i}{t|x|}(|x|-t)\partial_{x^i} + \frac{v^it - x^iv^0}{t}\partial_{x^i} \\ &= v^0\underbrace{\left(\partial_t + \frac{x^i}{|x|}\partial_{x^i}\right)}_{\text{outgoing derivatives}} - \frac{v^0}{t}\underbrace{\frac{x^i}{|x|}}_{\text{bounded}}\underbrace{\left(\frac{-x^j\Omega_{ij} + t\Omega_{0i} - x_iS}{t+r}\right)}_{\le C\left(|\Omega_{ij}|+|\Omega_{0i}|+|S|\right)} + v^0\frac{\mathfrak{z}}{v^0t}\partial_{x^i}, \end{aligned}$$

where the weight $\mathfrak{z}$ in the last term is $v^it - x^iv^0 \in \Bbbk_0$. Recall[14] now the following improved decay for outgoing derivatives of solutions to the wave equations: for all multi-indices α such that $|\alpha| \le N - (n+2)/2 - 1$,

$$\left|\left(\partial_t + \frac{x^i}{r}\partial_{x^i}\right)Z^\alpha(\phi)\right| \lesssim \frac{\mathscr{E}_N}{(1+t)^{3/2}}.$$

To estimate the second term, we need to obtain decay for $Z\phi$ as solution to the wave equation. This is done by integrating the decay of $\partial Z\phi$ coming from the Klainerman-Sobolev inequality along ingoing null rays. One then obtains:

$$\left|\left(\frac{-x^j\Omega_{ij} + t\Omega_{0i} - x_iS}{t+r}\right)Z^\alpha\phi\right| \lesssim \frac{\mathscr{E}_N}{(1+t)^{1/2}}.$$

As a consequence, it follows that for all multi-indices $|\alpha| \le N - (n+2)/2 - 1$,

$$|T(Z^\alpha\phi)| \lesssim \mathscr{E}_N v^0\left(\frac{1}{(1+t)^{3/2}} + \sum_{\mathfrak{z}\in\Bbbk_0}\frac{|\mathfrak{z}|}{v^0}\frac{1}{t(1+t)}\right).$$

Repeating the previous ingredients then gives (52). The pointwise estimates then follow from the Klainerman-Sobolev inequality (20). □

6.2.6 Improved regularity

Assume that ϕ solves the wave equations and f solves the free transport massless equation. Then,

$$\int_{\mathbb{R}^{3+1}}\int_v T(\phi)^2 f\frac{dv}{|v|}dx\,dt \lesssim \left(||\phi_{|t=0}||^2_{H^2} + ||\partial_t\phi_{|t=0}||^2_{H^1}\right)\int_{\mathbb{R}^3_x} f(0,x,v)|v|dx. \tag{55}$$

[14]This can be obtained from the usual Klainerman-Sobolev inequality and the formula for ∂_s in (8) by integration along the constant $t = |x|$ null lines.

7 Some standard systems of kinetic theory

7.1 The Vlasov-Poisson system

This is the system

$$\partial_t f + v^i \partial_{x^i} f \pm \nabla_x \phi . \nabla_v f = 0, \quad \Delta \phi = \rho(f), \tag{56}$$

where Δ is the Laplacian[15]. The variable x can be in $\mathbb{R}^n$, a bounded domain (in which case, extra boundary conditions must be specified) or some manifold. A typically studied case is that of $x \in T^n$ (corresponding to periodic boundary conditions).

There are many standard results on the VP system so here is an extremely non-exhaustive list:

- Global existence in dimension less than 3 (including) for arbitrarily large data. There are essentially two distinct approaches, the Pfaffelmoser approach [Pfa92] (see also [Sch91] for easier presentation following the same type of ideas), and the one via propagation of moments of Lions-Perthame [LP91] (see also [Pal14] for an recent improvement).
- Small data global existence and asymptotics in dimension 3 and larger. Recall that ρ decays like $1/t^n$. This leads to the decay rate for the force field $\nabla\phi$.

  $$|\nabla\phi| \lesssim \frac{1}{t^{n-1}}.$$

  Up to dimension 3, one can use these decay rates to prove global existence and derive some asymptotics of the solutions. Compared to a say, a standard quasilinear wave equations in dimension 3, one can study these equations with very little regularity on the initial data, so the original proof of [BD85] contained decay estimates for ρ but not for derivatives of ρ. To get sharp decay estimates for derivatives of ρ, you need to do more work, first results by [HRV11] and then [Smu15] using the vector field approach. Up to dimension 4, you can use the standard vector fields, but in dimension 3, you need a refined version, namely *modified vector fields*. Dimension 2 is an interesting (but hard) open problem.
  Open problem: consider the Vlasov Poisson system with two spatial dimensions. What are the asymptotics of the solutions for small initial data ?
- There are plenty of stationary solutions to the Vlasov Poisson system (in the attractive case) and a natural question is that of stability. Again, the scaling and non-linearities in these equations is such that, contrary to say the Einstein equations, you can prove results about orbital stability without understanding asymptotic stability. In fact, there are very few asymptotic stability results known for VP, even for quite simple stationnary state. The stronger orbital stability results are in [LMR12]. See also [Mou13] for a nice introduction to these questions and a presentation of the orbital results and methods of [LMR12].
- Landau damping. This is a classical phenomena in plasma theory. It was originally discover by Landau at the linearized level. A full proof in the non-linear

[15]Sometimes, extra boundary/decay/mean value conditions can be imposed on ρ to ensure unique solvability in the Poisson equation.

case was given by Villani-Mouhot [MV11] and there has been some slight improvements and simplified proofs since [BMM16]. Landau damping concerns the Vlasov-Poisson system for $x \in T^n$. The Poisson equation is slightly different to account for the periodicity:

$$\Delta\phi = \rho[f] - \int_{S^1} \rho[f].$$

Because of the periodic boundary conditions, the velocity averages do not need to decay anymore, instead they *homogenize*, that is to say they approach their mean value:

$$\left|\rho(f)(t,x) - \int_y \rho(f)(t,y)\right| \to 0$$

as t goes to ∞. How fast is the convergence ? This depends on the regularity. Let illustrate this for the free transport equation, using the typical method used first, and then using our vector field approach.

We thus consider (in 1d for simplicity)

$$(\partial_t + v\partial_x)(f) = 0,$$

for $f := f(t,x,v)$ with $(t,x,v) \in \mathbb{R} \times S^1 \times \mathbb{R}$.

The typical techniques are Fourier based. Let $\rho(f) = \int_v f dv$ and let ρ_k being its kth Fourier modes (in x):

$$\rho(f)(t) = \sum_{k\in\mathbb{Z}} \rho_k(t) e^{ikx}.$$

Let also $f_k(t,v)$ denotes the kth Fourier modes of f.

Applying the Fourier transform to this equation leads to

$$\partial_t f_k + ivkf_k = 0$$

which can be easily integrated as

$$f_k(t,v) = e^{-ivk} f_k(0,v).$$

Note that $\rho_k(f)(v) = \int_v f_k(t,v)dv$. One can then compute

$$\rho_k(f)(t) = \int_v f_k(t,v)dv = \int_v e^{-ivk} f_k(0, vdv = \tilde{f}_k(0,kt),$$

where $\tilde{f}_k(0,\eta)$ is the Fourier transform in v of $f_k(0,.)$.

Now assume that f_k is initially very regular in v. Then, its Fourier transform in v decays fast as $|\eta| \to +\infty$. Thus, as $t \to +\infty$, we have that $\rho_k(f)(t) \to 0$, unless $k = 0$.

It is then easy to see that $\rho(f)$ must converge (exercice: in which topology ?) to $\rho_0(f)$, which is exactly the mean value convergence mentioned above.

How would you capture this using the vector field method ?

Let $Z = t\partial_x + \partial_v$. As before, we can commute the free transport N times and thus control

$$\int_x \int_v |Z^N f|(t,x,v)dxdv$$

in terms of the initial data.

Now, recall the Wirtinger inequality. For a periodic function ψ in $W^{1,1}$,

$$\left|\psi(x) - \frac{1}{2\pi}\int_{S^1}\psi\right| \lesssim \int_{S^1} |\partial\psi|(y)dy.$$

Apply it to $\rho(f)$. This gives

$$|\rho(f) - \rho_0(f)| \lesssim \int_{S^1} |\partial_x \rho(f)|dx.$$

Rewrite $\partial_x\rho[f]$ as

$$\partial_x \rho[f] = \frac{1}{t}\rho[Z[f]].$$

Then, we get

$$|\rho(f) - \rho_0(f)| \lesssim \frac{1}{t}\int_{S^1} |\rho[Z[f]]|dx,$$

and we have obtain $1/t$ decay. To get further decay, note that we can reapply what we just did to $\rho[Z[f]]$, expect that now this quantity is mean free: $\int_x \rho[Z[f]]dx = 0$. Thus, we get that

$$|\rho[Z[f]]| \le \frac{1}{t}\int_{S^1} |\rho[Z^2[f]]|dx.$$

Iterating, we obtain that

$$|\rho(f) - \rho_0(f)| \lesssim \frac{1}{t^N}\int_{S^1} |\rho[Z^N[f]]|dx.$$

Using the conservation of the right-hand side in time, this implies that

$$|\rho(f) - \rho_0(f)| \lesssim \frac{1}{t^N}\int_{S^1} |\partial_v^N f(0,x,v)|dxdv.$$

Now, in physics, the distribution f could represent the number of charged particles. This distribution of charged particles then will generate an electromagnetic force E, which can be computed by solving the Poisson equation

$$\Delta\phi = \rho[f] - \int_{S^1}\rho[f], \quad E = \nabla\phi, \phi = \phi(t,x).$$

(here I have neglected all physical constants for simplicity)

To get uniqueness in the Poisson equation, we look for ϕ which are mean free : $\int_{S^1}\phi dx = 0$ (note that the physical quantity, the one that appears in the Vlasov equation, is actually E, not ϕ).

According to what we explained above, $\rho[f]$ goes to a constant, so the source term in the Poisson equation decays.

Exercice: Prove that the electric field E decays polynomially in t, with a rate of decay depending on the initial regularity of the data. (Harder: assuming the initial data to be real analytic, prove that the electric field decays exponentially in time.)

What happens is the the distribution of charges homogenize, and hence, there are no more "potential difference", and hence no electric field.

Homogeneous distributions: These are by definitions distributions functions which depends only on v. Any such distribution is actually a stationnary solutions to the Vlasov-Poisson system. Moreover the corresponding electric field vanishes. Landau damping is then the claim that if one perturbs some of these homogeneous solutions, the perturbed solutions should then converge quite rapidly towards a nearby homogeneous solutions. These special homogeneous solutions must verify a linear stability condition (typicaly called the *Penrose stability condition*). The work of Villani and Mouhot [MV11] can then been seen as a non-linear stability result for homogeneous linear stable distributions (but their analysis also contained improvement at the linear level).

The non-linear analysis is hard, so I will only here describe briefly the linear one.

Let us first write the linearized equation around a homogeneous solution $f_0(v)$. We obtain

$$\partial_t f + v\partial_x f + \nabla[\phi].\partial_v f_0 = 0.$$

We first Fourier in x:

$$\partial_t f_k + ivkf_k + ik\phi_k\partial_v f_0 = 0$$

We also Fourier in x the Poisson equation:

For $k \neq 0$,

$$-k^2\phi_k = \rho_k$$

which leads to

$$\partial_t f_k + ivkf_k - i\frac{k}{k^2}\rho_k\partial_v f_0 = 0$$

We solve this ode :

$$f_k(t) = e^{-ivkt} f_k(0,v) - \int_0^t e^{-ivk(t-s)}\frac{k}{k^2}\rho_k(s)\partial_v f_0(v)ds$$

and then integrate in v to obtain

$$\rho_k(t) = \tilde{f}_k(0,tk) + i\int_0^t \rho_k(s)\frac{k}{k^2}\tilde{f}_0(k(t-s))k(t-s)ds.$$

We are left with an equation for ρ_k of the form

$$a(t) = b(t) + \int_0^t a(t)W(t-s)ds.$$

This is a well known integral equation known at the Voltera equation. Typically, it is solved using the Laplace transform (a complex variant of the Fourier transform).

$$a^L(\lambda) = \int_0^{+\infty} e^{2\pi\lambda t} a(t)dt.$$

(well defined if a decay exponentially and λ small enough, otherwise use complex deformation)

Then, the above equation can be rewritten as

$$a^L = b^L + a^L W^L$$

which gives

$$a_L = \frac{b^L}{1 - W^L}.$$

provided that $1 - W^L \neq 0$.

The stability condition is then typically written as $|1 - W^L(\lambda)| > \kappa > 0$ (this should hold uniformly in λ in some complex strip).

Problem: this analysis allows to control ρ_k and proves decay of the electric field. However, compared with the linear case, it does not allow a control on norms such as $\int_{x,v} |Z^N(f)|dxdv$. Open problem: could we prove stronger stability bounds for the linear stability of homogeneous distributions ?

7.2 The Vlasov Norström system.

It is given by

$$\Box\phi = m^2 \int_v f \frac{dv}{\sqrt{m^2 + |v|^2}}, \tag{57}$$

$$\mathbf{T}_m(f) - \left(\mathbf{T}_m(\phi)v^i + m^2\nabla^i\phi\right)\frac{\partial f}{\partial v^i} = (n+1) f\, \mathbf{T}_m(\phi), \tag{58}$$

where $m = 0$ in the massless case and $m > 0$ in the massive case, $\Box \equiv -\partial_t^2 + \sum_{i=1}^n \partial_{x^i}^2$ is the standard wave operator of Minkowski space, ϕ is a scalar function of (t,x) and f is as before a function of (t,x,v^i) with $x \in \mathbb{R}^n$, $v \in \mathbb{R}^n$ if $m > 0$, $v \in \mathbb{R}^n \setminus \{0\}$ if $m = 0$. A good introduction to this system can be found in [Cal03]. See also the classical works [Cal06, Pal06].

Roughly speaking, the Vlasov-Nordström system can be derived from the Einstein-Vlasov system by considering only a special class of solutions (that of metrics conformal to the Minkowski metric) and by neglecting some of the non-linear interactions in the (Einstein part of) the equations. Since most of the simplifications concern difficulties which we already know how to handle (in the style of [CK93] or [LR10]) and since the method that we are using here is of the same type as the one used to study the Einstein vacuum equations, we believe it is a good model problem before addressing the full Einstein-Vlasov system via vector field methods.

Standard works on the VN system includes:

- Derivation of the system, basic properties and existence of steady states [Cal03].
- global existence for smooth, compactly initial data (compact in x, v) [Cal06] (for an older continuation criterion see [Pal06])
- weak solutions [CR04]
- Small data analysis [Fri04] (3d, defocusing and massive particles case only, no information on derivatives) and [FJS15] ($d \geq 4$, massive and massless case, no need for any sign condition, full sharp asymptotics)
- As in the Vlasov Poisson case, in dimension 3 and less, one cannot hope to close the estimates for the derivatives using just the basic vector fields.

7.3 The Vlasov-Maxwell system.

In the standard notation, this takes the form

$$\partial_t f + \frac{v^i}{v^0}.\partial_{x^i} f + (E + \frac{v}{v^0} \wedge B).\nabla_v f = 0, \tag{59}$$

$$\partial_t E + curl B = -\int_v f \frac{v dv}{v^0}, \quad \partial_t B + curl E = 0, \tag{60}$$

$$div E = \int_{v \in \mathbb{R}^3} f dv, \quad div B = 0 \tag{61}$$

In a more geometric form, we would write it as

$$v^\alpha \partial_{x^\alpha} + v^\beta F^a_\beta \partial_{v^a} f = 0, \tag{62}$$

$$\partial_{x^\alpha} F^{\alpha\beta} = j_\beta \tag{63}$$

where $F_{\alpha\beta}$ is a 2-form which is related to E and B as

$$F_{0i} = E_i, \quad B_i = \frac{-1}{2} \epsilon_{ijk} F^{jk}$$

and j_β is (proportional) to the particle current

$$j_\beta = \int_v v_\beta \frac{dv}{v^0}.$$

Exercice: prove formally the conservation of energy

$$\int_{x,v} f|v|^2 dv + 1/2 \int_x F_{\alpha\beta} F^{\alpha\beta} dx.$$

as well as the conservation of the number of particles $\int_{x,v} f \frac{dv}{v^0}$ and more generally of the Casimir invariants.

Standard works on the Vlasov-Maxwell equations include

- Small data for 3d case [GS87], without compactness assumption in v, but without sharp information on derivatives [Sch04].
- Conditional global existence in $3d$ (continuation criterion): [LS16, SAI10, Pal05, GS86, BGP03, Pal12, Pal14, Pal15, KS02]
- Lower dimensional global existence [GS97, GS98].

The global existence of regular solutions in dimension 3 has really become one of the main open problem for this system.

The basic continuation criterion takes the following form (see above for references)

Theorem 7.1. *Assume that f, F is a solution arising from smooth, compactly supported data that is regular on an interval of the form $[0, T)$ for some $T > 0$. Let $P(t)$ denotes*

$$P(t) = \sup\{|v|, v \in \mathbb{R}^3 : \exists x \in \mathbb{R}^3 \ / \ f(t, x, v) \neq 0\}.$$

Then, if P is uniformly bounded on $[0, T)$, there exists a (unique) smooth extension of f and F to a bigger interval $[0, T']$ with $T' > T$.

References

[BD85] C. Bardos and P. Degond. Global existence for the Vlasov-Poisson equation in 3 space variables with small initial data. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 2(2):101–118, 1985.

[BGP03] François Bouchut, François Golse, and Christophe Pallard. Classical solutions and the Glassey-Strauss theorem for the 3D Vlasov-Maxwell system. *Arch. Ration. Mech. Anal.*, 170(1):1–15, 2003.

[BMM16] Jacob Bedrossian, Nader Masmoudi, and Clément Mouhot. Landau Damping: Paraproducts and Gevrey Regularity. *Ann. PDE*, 2(1):2:4, 2016.

[Cal03] Simone Calogero. Spherically symmetric steady states of galactic dynamics in scalar gravity. *Classical Quantum Gravity*, 20(9):1729–1741, 2003.

[Cal06] Simone Calogero. Global classical solutions to the 3D Nordström-Vlasov system. *Comm. Math. Phys.*, 266(2):343–353, 2006.

[CK90] D. Christodoulou and S. Klainerman. Asymptotic properties of linear field equations in Minkowski space. *Comm. Pure Appl. Math.*, 43(2):137–199, 1990.

[CK93] D. Christodoulou and S. Klainerman. *The non-linear stability of the Minkowski space.* Princeton Mathematical Series, Princeton NJ, 1993.

[CR04] Simone Calogero and Gerhard Rein. Global weak solutions to the Nordström-Vlasov system. *J. Differential Equations*, 204(2):323–338, 2004.

[FJS15] D. Fajman, J. Joudioux, and J. Smulevici. A vector field method for relativistic transport equations with applications. *arXiv:1510.04939*, 2015.

[Fri04] Stefan Friedrich. Global Small Solutions of the Vlasov-Norstrom System. *arXiv:0407023*, 2004.

[GS86] Robert T. Glassey and Walter A. Strauss. Singularity formation in a collisionless plasma could occur only at high velocities. *Arch. Rational Mech. Anal.*, 92(1):59–90, 1986.

[GS87] Robert T. Glassey and Walter A. Strauss. Absence of shocks in an initially dilute collisionless plasma. *Comm. Math. Phys.*, 113(2):191–208, 1987.

[GS97] Robert Glassey and Jack Schaeffer. The "two and one-half-dimensional" relativistic Vlasov Maxwell system. *Comm. Math. Phys.*, 185(2):257–284, 1997.

[GS98] Robert T. Glassey and Jack Schaeffer. The relativistic Vlasov-Maxwell system in two space dimensions. I, II. *Arch. Rational Mech. Anal.*, 141(4):331–354, 355–374, 1998.

[HRV11] HyungJu Hwang, Alan D. Rendall, and Juan J.L. Velázquez. Optimal gradient estimates and asymptotic behaviour for the vlasov-poisson system with small initial data. *Archive for Rational Mechanics and Analysis*, 200(1):313–360, 2011.

[Joh79] Fritz John. Blow-up of solutions of nonlinear wave equations in three space dimensions. *Manuscripta Math.*, 28(1-3):235–268, 1979.

[Kla85] S. Klainerman. Global existence of small amplitude solutions to nonlinear Klein-Gordon equations in four space-time dimensions. *Comm. Pure Appl. Math.*, 38(5):631–641, 1985.

[Kla93] S. Klainerman. Remark on the asymptotic behavior of the Klein-Gordon equation in $\mathbf{R}^{n+1}$. *Comm. Pure Appl. Math.*, 46(2):137–144, 1993.

[KM93] S. Klainerman and M. Machedon. Space-time estimates for null forms and the local existence theorem. *Comm. Pure Appl. Math.*, 46(9):1221–1268, 1993.

[KS02] Sergiu Klainerman and Gigliola Staffilani. A new approach to study the Vlasov-Maxwell system. *Commun. Pure Appl. Anal.*, 1(1):103–125, 2002.

[LL97] Elliott H. Lieb and Michael Loss. *Analysis*, volume 14 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 1997.

[LM15] P. G. LeFloch and Y. Ma. The global nonlinear stability of Minkowski space for self-gravitating massive fields. *arXiv:1511.03324*, 2015.

[LMR12] Mohammed Lemou, Florian Méhats, and Pierre Raphaël. Orbital stability of spherical galactic models. *Invent. Math.*, 187(1):145–194, 2012.

[LP91] P.-L. Lions and B. Perthame. Propagation of moments and regularity for the 3-dimensional Vlasov-Poisson system. *Invent. Math.*, 105(2):415–430, 1991.

[LR10] Hans Lindblad and Igor Rodnianski. The global stability of Minkowski space-time in harmonic gauge. *Ann. of Math. (2)*, 171(3):1401–1477, 2010.

[LS16] Jonathan Luk and Robert M. Strain. Strichartz estimates and moment bounds for the relativistic Vlasov-Maxwell system. *Arch. Ration. Mech. Anal.*, 219(1):445–552, 2016.

[Mou13] Clément Mouhot. Stabilité orbitale pour le système de Vlasov-Poisson gravitationnel (d'après Lemou-Méhats-Raphaël, Guo, Lin, Rein et al.). *Astérisque*, (352):Exp. No. 1044, vii, 35–82, 2013. Séminaire Bourbaki. Vol. 2011/2012. Exposés 1043–1058.

[MV11] Clément Mouhot and Cédric Villani. On Landau damping. *Acta Math.*, 207(1):29–201, 2011.

[Pal05] Christophe Pallard. On the boundedness of the momentum support of solutions to the relativistic Vlasov-Maxwell system. *Indiana Univ. Math. J.*, 54(5):1395–1409, 2005.

[Pal06] C. Pallard. On global smooth solutions to the 3D Vlasov-Nordström system. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 23(1):85–96, 2006.

[Pal12] Christophe Pallard. Large velocities in a collisionless plasma. *J. Differential Equations*, 252(3):2864–2876, 2012.

[Pal14] Christophe Pallard. Space moments of the Vlasov-Poisson system: propagation and regularity. *SIAM J. Math. Anal.*, 46(3):1754–1770, 2014.

[Pal15] Christophe Pallard. A refined existence criterion for the relativistic Vlasov-Maxwell system. *Commun. Math. Sci.*, 13(2):347–354, 2015.

[Pfa92] K. Pfaffelmoser. Global classical solutions of the Vlasov-Poisson system in three dimensions for general initial data. *J. Differential Equations*, 95(2):281–303, 1992.

[PS93] Gustavo Ponce and Thomas C. Sideris. Local regularity of nonlinear wave equations in three space dimensions. *Comm. Partial Differential Equations*, 18(1-2):169–177, 1993.

[Rin13] Hans Ringström. *On the topology and future stability of the universe.* Oxford Mathematical Monographs. Oxford University Press, Oxford, 2013.

[SAI10] Reinel Sospedra-Alfonso and Reinhard Illner. Classical solvability of the relativistic Vlasov-Maxwell system with bounded spatial density. *Math. Methods Appl. Sci.*, 33(6):751–757, 2010.

[Sch91] Jack Schaeffer. Global existence of smooth solutions to the Vlasov-Poisson system in three dimensions. *Comm. Partial Differential Equations*, 16(8-9):1313–1335, 1991.

[Sch04] Jack Schaeffer. A small data theorem for collisionless plasma that includes high velocity particles. *Indiana Univ. Math. J.*, 53(1):1–34, 2004.

[Smu15] Jacques Smulevici. Small data solutions of the vlasov-poisson system and the vector field method. *arXiv:1504.02195*, 2015.

[WW15] J. Wang and Q. Wang. Global stability of Minkowski space for massive scalar fields. *talk given at the conference "GENERAL RELATIVITY - A Celebration of the 100th Anniversary" held at the Institut Henry-Poincaré*, Nov. 2015.