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THE BOUNDED L^2 CURVATURE CONJECTURE [after S. Klainerman, I. Rodnianski and J. Szeftel]

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1. INTRODUCTION

General relativity is one of the pillars of modern theoretical physics. Mathematically, it consists in the study of Lorentzian manifolds, typically in dimension $3+1$, satisfying the so-called *Einstein equations*, a system of geometric partial differential equations for the components of the Lorentzian metric of the manifold. In the absence of any matter source [(1)], they take the form

$$\mathrm{Ric}(g) = 0, \tag{1}$$

where $\mathrm{Ric}(g)$ denotes the Ricci tensor [(2)] of a Lorentzian metric g.

As was understood already by Einstein himself, equations (1) are of wave type so that, being in particular evolution equations, the natural problem associated to them is the Cauchy problem. The initial data, formulated geometrically, consists in a triplet

1. They are then typically called the Einstein *vacuum* equations.
2. For the reader unfamiliar with some of notions in geometry needed to read this text, certain basic definitions of Lorentzian and Riemannian geometry are given at the beginning of Section 2.

(Σ, h, k) where Σ is a manifold of dimension [3] 3, h is a Riemannian metric on Σ, k is a symmetric 2-tensor and the following system of constraint equations holds

$$R^{(h)} - |k|^2 + (tr_h k)^2 = 0, \tag{2}$$

$$\mathrm{div}_h\, k - \nabla^{(h)}\, tr_h k = 0, \tag{3}$$

where $R^{(h)}$ is the scalar curvature of h, $\nabla^{(h)}$ is the Levi-Civita connection associated to h, $\mathrm{div_h}\, k$ is the divergence of k, and $tr_h k := k_{ab}h^{ab}$ denotes the trace of the 2-tensor k.

A solution to the initial value problem associated to (Σ, h, k), which we shall call a development of the corresponding data, is then a Lorentzian manifold $(\mathcal{M}, g)$ of dimension $3+1$ satisfying the Einstein equations (1) and such that there exists an embedding of Σ into $\mathcal{M}$ with (h, k) coinciding with the first and second fundamental forms of the embedding.

The bounded L^2 curvature conjecture, originally proposed by Klainerman in 1999 [16], roughly states that the initial value problem for the Einstein equations should be well-posed in the class of data (Σ, h, k) such that the Ricci curvature tensor of h and the first derivatives of k are in L^2_{loc}. Since the curvature tensor depends on the derivatives of the metric up to second order and since k encodes the data for its time derivative, this means that we should be able to control the solutions assuming only L^2 bounds on no more than two derivatives of the initial metric. This conjecture was recently proven by S. Klainerman, I. Rodnianski and J. Szeftel in the series of works

- *S. Klainerman, I. Rodnianski, J. Szeftel, The bounded L^2 curvature conjecture. arXiv:1204.1767, 91 pp.* (This is the main part of the series in which the proof is completed based on the results of the papers below.)
- *J. Szeftel, Parametrix for wave equations on a rough background I: regularity of the phase at initial time. arXiv:1204.1768, 145 pp.*
- *J. Szeftel, Parametrix for wave equations on a rough background II: control of the parametrix at initial time. arXiv:1204.1769, 84 pp.*
- *J. Szeftel, Parametrix for wave equations on a rough background III: space-time regularity of the phase. arXiv:1204.1770, 276 pp.*
- *J. Szeftel, Parametrix for wave equations on a rough background IV: Control of the error term. arXiv:1204.1771, 284 pp.*
- *J. Szeftel, Sharp Strichartz estimates for the wave equation on a rough background. arXiv:1301.0112, 30 pp.*

The proof of the bounded L^2 curvature conjecture can be seen as the culminant point of a long sequence of works concerning the study of well-posedness for semi-linear and quasilinear wave equations applied to General Relativity and other geometric wave equations. This question has a long history, starting with the pioneer work of Choquet-Bruhat establishing existence and local uniqueness of solutions to (1) in the smooth

3. The equations can naturally be posed in higher or lower dimensions, but in this text, we shall consider only the $3+1$ dimensional case.

case [4]. While we shall not try in this text to give an exhaustive treatment of the history of geometric wave equations with low regularity assumptions, the remaining of this introduction aims at providing enough information concerning its developments so as to be able to highlight the main characteristics of the proof of the bounded L^2 curvature conjecture and in particular, explain why this result is the first of its kind. We start with some standard properties of the Einstein equations.

1.1. The Einstein equations as a system of quasilinear wave equations

The Einstein equations (1) being geometric, a choice of gauge, such as a choice of coordinates, is necessary so as to transform (1) into a system of partial differential equations amenable to various techniques from analysis. A popular choice for (1) is the *wave gauge* (also called harmonic or de Donder gauge). In the *wave gauge*, we consider a system of coordinates $(x^\mu)_{\mu=0,..,3}$ on a Lorentzian manifold $(\mathcal{M}, g)$ such that for each μ, x^μ is a solution to the linear wave equation on $(\mathcal{M}, g)$, i.e. $\Box_g x^\mu = 0$ where $\Box_g$ is the wave operator associated [5] to g

$$\Box_g := g^{\mu\nu} D_\mu D_\nu,$$

where D denotes the Levi-Civita connection associated to g. In wave gauge, the components of $\mathrm{Ric}(g)$ simplify, so that (1) reduces to

$$\Box_g\, g_{\mu\nu} = \mathcal{Q}_{\mu\nu}(\partial g, \partial g), \tag{4}$$

where $\mathcal{Q}_{\mu\nu}(\partial g, \partial g)$ denotes a quadratic form in the first derivatives of g with coefficients depending only on the components of g.

Since the principal symbol of $\Box_g$, $g^{\mu\nu}\xi_\mu\xi_\nu$, is hyperbolic and depends on the solution itself, we see that the above system is a system of quasilinear wave equations.

1.2. The scale invariance of the equations

Note that the equations (4) are invariant under scaling: if g is a solution, then for any $\lambda > 0$, so is $(x^\mu) \to g(\lambda x^\mu)$. The 3-dimensional Sobolev space which is left invariant under this transformation is $\dot{H}^{s_c}(\mathbb{R}^3)$ with $s_c = 3/2$ being the critical exponent. Scaling symmetries play an important role in the study of non-linear wave equations. Roughly speaking, for H^s regularity with $s > s_c$, one can shrink the size of the data while making the time of existence larger, while for $s < s_c$, one typically expects ill-posedness.

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4. In fact, the starting point of this history could arguably be placed around the 30's with the work of Darmois [11] on the Cauchy problem for the Einstein equations with analytic data and that of Stellmacher [38] on the uniqueness of solutions for smooth data. See also the work of Lichnerowicz [28].

5. In the whole text, the Einstein summation convention will be used. For instance, $g^{\mu\nu} D_\nu D_\mu$ stands for $\sum_{\mu,\nu=0,..,3} g^{\mu\nu} D_\mu D_\nu$.

1.3. The energy estimate and the classical method

The fundamental estimate lying at the core of any well-posedness theory for (1) (or any other system of quasilinear wave equations with spatial dimension greater than 1) is the energy estimate [6].

For (4), the energy estimate gives (here t is a local time coordinate, $x = x^1, x^2, x^3$ denotes spatial coordinates and $A \lesssim B$ stands for $A \leq CB$ where $C > 0$ is some universal constant), for any $t \geq t_0$

$$||\partial g(t)||_{L^2_x} \lesssim ||\partial g(t_0)||_{L^2_x} \exp\left(C \int_{t_0}^{t} ||\partial g(s)||_{L^\infty_x} ds\right). \tag{5}$$

This estimate closes provided one has a uniform bound on

$$\int_{t_0}^{t} ||\partial g(s)||_{L^\infty_x} ds. \tag{6}$$

The classical method consists in estimating ∂g pointwise using the Sobolev embedding $H^s(\mathbb{R}^n) \hookrightarrow L^\infty(\mathbb{R}^n)$, for $s > n/2$. Hence, in dimension 3, the classical method for proving well-posedness for quasilinear wave equations such as (4) requires $s > 5/2$. It leads to the following theorem

Theorem 1.1 (Classical local existence [12, 14]). — *Let (Σ, h, k) be an initial data set for the Einstein vacuum equations* (1). *Assume that Σ can be covered by a locally finite system of coordinate charts, related to each other by C^1 diffeomorphisms, such that $(h, k) \in H^s_{loc}(\Sigma) \times H^{s-1}_{loc}(\Sigma)$ with $s > \frac{5}{2}$. Then there exists (up to diffeomorphism) a unique* [7] *maximal globally hyperbolic development of the data for which the image of Σ by the embedding is a Cauchy hypersurface* [8].

There are several motivations to improve the regularity requirements in the above theorem. Local well-posedness for rougher data typically implies better control of the solutions. For instance, it can be seen as a way of detecting singularities under the form of a breakdown criterion [9]. When the time of existence of solutions is controlled by their norms in some functional spaces corresponding to a conserved quantity, it typically can be upgraded to a global existence result. Recalling the definition in Section 1.2 of

6. There are of course, several other ingredients necessary to prove well-posedness for (1) using the reduced equations (4), such as the propagation of constraints, higher order energy estimates obtained after commutation and estimates on the difference of solutions. We shall not give the details of this in this text.

7. The original proof in [12, 14] actually requires one more derivative for the uniqueness. The fact that uniqueness holds at the same level of regularity as the existence has been obtained in [33].

8. That is any inextendible causal curve intersects this hypersurface. Lorentzian manifolds having a Cauchy hypersurface are called *globally hyperbolic* spacetimes. This condition is necessary for the global uniqueness property.

9. See for instance [25, 50, 35, 32] for several studies of breakdown criteria in the context of the Einstein equations.

the critical exponent s_c corresponding to the scale invariance of the equations, note that a local in time existence result at the critical level would automatically guarantee that smooth solutions arising from data with small critical norms are global in time. This would be similar to the ϵ-regularity results for elliptic and parabolic pdes and their applications to the uniform control of solutions. For quasilinear wave equations, the only cases where critical well-posedness have been obtained are in $1+1$ dimensions or for spherically symmetric solutions of higher dimensional problems. In these cases, the L^2 norms are dropped in favor of critical bounded variation (BV) type norms. Glimm's existence result [13] can for instance be interpreted as a global well-posedness theory for data with small BV norms. An illustration of the importance of critical well-posedness in the context of general relativity is contained in Christodoulou's proof of the weak cosmic censorship for the spherically-symmetric Einstein-scalar field system [6] which makes use of his previous existence result of solutions in weigthed, critical, BV type spaces [5]. In more than one dimension, the BV norms do not however propagate and the only results of well-posedness at the critical level concern semi-linear equations. Some of the most famous recent results where developing a critical well-posedness theory led to the understanding of the global properties of the solutions for a geometric semi-linear system of wave equations are contained in the works [27, 46, 40, 39] on wave maps.

1.4. Strichartz estimates for semilinear and quasilinear wave equations

In view of (5), the main obstruction to close the energy estimate is present even for the simpler semilinear wave equation

$$\Box\psi = \mathcal{Q}(\partial\psi, \partial\psi), \tag{7}$$

where $\Box = -\partial_t^2 + \partial_x^2 + \partial_y^2 + \partial_z^2$ denotes the usual wave operator associated with the Minkowski metric. One of the tools that can be used to lower the regularity needed for well-posedness of equations such as (7) are the so-called *Strichartz estimates*: For any $\epsilon > 0$, sufficiently regular solutions ϕ of the wave equation $\Box\phi = 0$ satisfy

$$||\partial\phi||_{L^2_t L^\infty_x} \lesssim ||\phi_{|t=0}||_{H^{2+\epsilon}_x} + ||\partial_t\phi_{|t=0}||_{H^{1+\epsilon}_x}.$$

We see that the Strichartz estimates can control (6) by taking advantage of the time integral, contrary to the classical method. This leads to a well-posedness theory for (7) for initial data in $H^{2+\epsilon} \times H^{1+\epsilon}$, see [34].

Returning to (4), there are several strong obstructions to developing Strichartz estimates in this quasilinear setting. At the linear level, one needs to understand how to prove Strichartz estimates for wave equations such as

$$\Box_g\psi = 0$$

under low regularity assumptions on the metric g. The first breakthrough in this direction came with the results of Bahouri-Chemin [2, 1] and then Tataru in [48, 49], where they obtained Strichartz estimates with a small loss of derivatives. The best results for well-posedness for quasilinear wave equations using Strichartz type estimates were obtained in [22, 37]. Compared to the previous work mentioned, they make use of the important observation (first used in [19]) that the metric itself is a solution of a wave equation. This led to a well-posedness theory for initial data in $H^{2+\epsilon} \times H^{1+\epsilon}$, i.e. at the same regularity level than the optimal regularity for the semi-linear wave equation (7) when no assumptions are made on the structure of the quadratic form $\mathcal{Q}$.

1.5. Bilinear estimates and the null condition

Since without more information on the structure of the non-linearity in (7), it is known that this is the best that can be achieved (see [30, 29]), any improvement on the above result must exploit some specific cancellations in (7) valid only for some systems.

In [17], Klainerman and Machedon developed a class of estimates, more precisely *bilinear* estimates, for non-linearities satisfying the so-called *null condition.* This type of non-linearity is known to appear in many interesting systems of semi-linear equations arising from physics, such as the wave map, Yang-Mills or Maxwell-Klein-Gordon equations. In the case of the Yang-Mills equations, this led to a well-posedness result for $s = s_c + 1/2$, where $s_c = 1/2$ is the critical exponent in that case [10]. The importance of the null condition, first identified by Klainerman, was initially remarked in several works concerning small data global existence for quasilinear wave equations in $3d$ with quadratic non-linearities [15, 4]. Under this condition, the most dangerous non-linear terms can be rewritten using linear combinations of the *null forms*

$$Q_0(\phi, \psi) = \partial_t \phi \, \partial_t \psi - \nabla \psi \cdot \nabla \phi, \tag{8}$$

$$Q_{ij}(\phi, \psi) = \partial_i \phi \, \partial_j \psi - \partial_i \psi \, \partial_j \phi, \quad 1 \leq i, j \leq 3. \tag{9}$$

There are however several obstructions to developing this program for the Einstein equations, the first one being that the Einstein equations in wave gauge (4) do not satisfy the null condition [11]!

1.6. Strategy for a proof of the bounded L^2 curvature conjecture

In view of the above discussion, if one thinks of improving upon the previous low regularity results following the bilinear estimates approach, there are two fundamental steps to first achieve

- A exhibit a new formulation of the Einstein equations in which some version of the null condition is satisfied.
- B provide an appropriate framework for deriving bilinear estimates for the null forms appearing the previous step.

10. Note that this result has been improved to $s > 3/4$ for small data in the so-called temporal gauge [47].

11. See [3]. The equations (4) satisfy only a weak form of the null condition, which is sufficient to proving small data global existence but provides no improvement for low regularity issues [31].

Since however all known proofs of bilinear estimates in more than 1 spatial dimension are based on an explicit representation of the solutions, also called parametrix, in order to achieve step 2, it seems also necessary to

C construct a parametrix Φ_F for solutions to the inhomogeneous linear scalar wave equation $\square_g \phi = F$, derive appropriate bounds for Φ_F and the corresponding error term [12] $E = F - \square_g \Phi_F$ and exploit them to derive the desired bilinear estimates.

As it turns out, the proof of several bilinear estimates of Step B reduces to the proof of L^4 Strichartz estimates for (a frequency localized version of) the parametrix of step C. Thus, the final ingredient is

D Prove sharp L^4 Strichartz estimates for a frequency localized version of the parametrix of step C.

Finally, let us note that

– all the above steps need to be implemented using only hypothetical L^2 bounds for the curvature tensor in order to be consistent with the conjectured result,

– the proof of any of the above steps typically relies on the results of the other steps, so that several continuity arguments are needed to close all the estimates.

1.7. Statement of the main results

The main result obtained in [26, 41, 42, 43, 44, 45] can be summarized as follows [13]

THEOREM 1.2. — *Let $(\mathcal{M}, g)$ be the maximal globally hyperbolic development of some asymptotically flat initial data for the Einstein vacuum equations* (1). *Assume that $(\mathcal{M}, g)$ admits a maximal foliation by the level hypersurfaces Σ_t of some time function t and that Σ_0 coincides with some initial slice with induced metric h and second fundamental form k, such that* $\mathrm{Ric}(h) \in L^2(\Sigma_0)$, $\nabla k \in L^2(\Sigma_0)$. *Finally, assume that $r_{\mathrm{vol}}(\Sigma_0, 1) > 0$, where $r_{\mathrm{vol}}(\Sigma_0, 1)$ is the volume radius on scale less than* 1. *Then,*

1. *L^2 regularity. There exists a time*
$$T = T\left(||\mathrm{Ric}(h)||_{L^2(\Sigma_0)}, ||\nabla k||_{L^2(\Sigma_0)}, r_{\mathrm{vol}}(\Sigma_0, 1)\right) > 0$$
and a constant
$$C = C\left(||\mathrm{Ric}(h)||_{L^2(\Sigma_0)}, ||\nabla k||_{L^2(\Sigma_0)}, r_{\mathrm{vol}}(\Sigma_0, 1)\right) > 0$$
such that the following estimates hold
$$\begin{aligned} ||\mathrm{Riem}(g)||_{L^\infty[0,T]L^2(\Sigma_t)} &\leq C, \\ ||\nabla k||_{L^\infty[0,T]L^2(\Sigma_t)} &\leq C, \\ \inf_{0\leq t\leq T} r_{\mathrm{vol}}(\Sigma_t, 1) &> 1/C. \end{aligned}$$

12. Here contrary to the flat case when g is the Minkowski metric and $\square_g$ the usual d'Alembertian, the parametrix is only approximate, leading to the presence of the error term $E = F - \square_g \Phi_F$.

13. All relevant notions of geometry such as the definition of the maximal foliation and volume radius are recalled in Section 2.

2. *Higher regularity. Assuming higher regularity on the data, we also have, for any* $m > 0$ *such that, for all all multi-index* $|i| \leq m$, $||\nabla^{(i)}\mathrm{Ric}(h)|| + ||\nabla^{(i)}k||_{L^2(\Sigma_0)} < +\infty$,

$$\sum_{|\alpha|\leq m} ||D^\alpha \mathrm{Riem}(g)||_{L^\infty[0,T]L^2(\Sigma_t)} \leq C_m \left(\sum_{|i|\leq m} ||\nabla^{(i)}\mathrm{Ric}(h)||_{L^2(\Sigma_0)} + ||\nabla^{(i)}\nabla k||_{L^2(\Sigma_0)} \right),$$

where C_m *depends only on the previous* C *and* m.

3. *In particular, the time of existence of a classical solution depends only on* $||\mathrm{Ric}(h)||_{L^2(\Sigma_0)}$, $||\nabla k||_{L^2(\Sigma_0)}$ *and* $r_{\mathrm{vol}}(\Sigma_0, 1)$.

Remark 1.1. — *Since the curvature is at the level of 2 derivatives of g, the* L^2 *norm of the Riemann tensor is* 1/2 *derivative above a scaling invariant norm. Nonetheless, the above result still contains some criticality, in that the curvature tensor in* L^2 *is the minimum regularity allowing for the control of the radius of injectivity of null hypersurfaces. Now, in more than one space dimension, all known derivation of bilinear estimates are based on the construction of parametrices, whose control depends crucially on that of null hypersurfaces. Thus, all presently known techniques break down below the regularity of Theorem 1.2 and any result improving upon the above, if it exists, would most likely require a completely different approach.*

Remark 1.2. — *The above result has a clear physical interpretation, contrary to the previous well-posedness result for* $g \in H^{2+\epsilon}$. *Indeed, the flux of gravitational energy radiated through a hypersurface can be measured by the* L^2 *norm of the curvature tensor.*

Remark 1.3. — *The above result provides the first low regularity estimates for quasilinear wave equations for which the whole structure of the equations matters, not just the principal part, as was the case for instance in the* $H^{2+\epsilon}$ *previous results of* [22, 37]. *In our opinion, this is a major conceptual improvement upon previous works.*

Using finite speed of propagation arguments [14], the scale invariance of the equations and the Cheeger-Gromov convergence theory for Riemannian manifolds, the proof of Theorem 1.2 can be reduced to the following small data version (see [26] Section 2.3 for details)

Theorem 1.3 (Small data version). — *Let* $(\mathcal{M}, g)$ *be the maximal globally hyperbolic development of some asymptotically flat initial data for the Einstein vacuum equations* (1). *Assume that* $(\mathcal{M}, g)$ *admits a maximal foliation by the level hypersurfaces* Σ_t *of*

14. This must be used with some care due to the presence of the constraint equations. In the present set of works, a solution to this problem is presented using the Corvino-Schoen gluing techniques [9, 10].

some time function t and that Σ_0 coincides with some initial slice with induced metric h and second fundamental form k, such that

$$\begin{aligned} ||\mathrm{Ric}(h)||_{L^2(\Sigma_0)} &\leq \epsilon, \\ ||\nabla k||_{L^2(\Sigma_0)} &\leq \epsilon, \\ r_{\mathrm{vol}}(\Sigma_0, 1) &\geq 1/2. \end{aligned}$$

Then,

1. L^2 *regularity. There exists an $\epsilon_0 > 0$ such that if $0 < \epsilon < \epsilon_0$, the solution exists up to time $t = 1$ and*

$$\begin{aligned} ||\mathrm{Riem}(g)||_{L^\infty[0,1]L^2(\Sigma_t)} &\lesssim \epsilon, \\ ||\nabla k||_{L^\infty[0,1]L^2(\Sigma_t)} &\lesssim \epsilon, \\ \inf_{0\leq t\leq 1} r_{\mathrm{vol}}(\Sigma_t, 1) &> 1/4. \end{aligned}$$

2. *Higher regularity. Assuming higher regularity on the data, we also have, for any $m > 0$ such that, for all all multi-index $|i| \leq m$, $||\nabla^{(i)}\mathrm{Ric}(h)|| + ||\nabla^{(i)}k||_{L^2(\Sigma_0)} < +\infty$,*

$$\sum_{|\alpha|\leq m} ||D^\alpha \mathrm{Riem}(g)||_{L^\infty[0,1]L^2(\Sigma_t)} \lesssim \left(\sum_{|i|\leq m} ||\nabla^{(i)}\mathrm{Ric}(h)||_{L^2(\Sigma_0)} + ||\nabla^{(i)}\nabla k||_{L^2(\Sigma_0)} \right).$$

1.8. Acknowledgements

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1.9. Structure of the text

In the next section, we recall some basic definitions of Lorentzian and Riemanian geometry. We also present the maximal foliation and the notations that we will use in this text. In Section 3, we present the quasi-linear Yang-Mills formulation of the equations. In Section 4, the general proof of Theorem 1.3 is given, assuming the validity of the bilinear estimates. In the following section, the parametrix needed for the bilinear estimates is presented and the proof of the bilinear estimates sketched, under several assumptions concerning the control of this parametrix and the validity of sharp L^4 Strichartz estimates. Section 6 is devoted to estimating the parametrix error while Section 7 presents the control of the null foliations tied to the phase functions of the parametrix. In Section 8, we explain how the phase functions are initialized so as to control the parametrix initially. Finally, the last section presents the sharp L^4 Stricharz estimates needed for the proof of some of the bilinear estimates.

2. PRELIMINARIES: THE GEOMETRIC SETTING

In this section, we recall certain basic definitions of Lorentzian geometry and present the maximal time foliation that will be used throughout this text.

2.1. Basic definitions

First a few definitions.

Definition 2.1. — *A* $(3+1)$ *Lorentzian manifold is a* 4 *dimensional manifold* $\mathcal{M}$ *endowed with a Lorentzian metric* g*, i.e. a non-degenerate symmetric covariant* 2*-tensor of signature* $(-+++)$.

The basic Lorentzian manifold, called the Minkowski or flat space, is $\mathbb{R}^4$ endowed with the Lorentzian metric given in Cartesian coordinates by $diag(-1,1,1,1)$.

As in Riemannian geometry, given a Lorentzian metric, there exists a unique (torsion-free) compatible connection D, called the Levi-Civita connection. Compatibility means as usual that $Dg = 0$. Recall that if $(x^\mu)_{\mu=0,..,3}$ denotes a local system of coordinates on $\mathcal{M}$, the covariant derivatives D_α are then defined as $D_\alpha := D_{\partial_\alpha}$. Using D, one can define as usual the Riemman curvature tensor R. Its components can be computed as follows. For any vector field $X = X^\alpha \partial_\alpha$, we have

$$D_\alpha D_\beta X^\gamma - D_\beta D_\alpha X^\gamma = R^\gamma{}_{\rho\alpha\beta} X^\rho.$$

Thus, R essentially measures the non-commutativity of the covariant derivatives D_α. Recall also the definition of the Ricci tensor

Definition 2.2. — *The Ricci tensor* $\mathrm{Ric}(g)$ *of any Lorentzian metric* g *is the symmetric* 2 *tensor whose components in any system of local coordinates are*

$$\mathrm{Ric}(g)_{\alpha\beta} = R^\rho{}_{\alpha\rho\beta}.$$

Thus, the Ricci tensor $\mathrm{Ric}(g)$ is a partial trace of the Riemann tensor [(15)].

2.2. First and second fundamental forms

Recall also that given any hypersurface Σ, the metric g induces on Σ a symmetric 2 tensor h, called the first fundamental form or induced metric. Σ is then called null or characteristic (respectively spacelike and timelike) if the signature of h is $(0++)$ (respectively $(+++)$ and $(-++)$). Given a non-characteristic hypersurface, there exists, up to a sign, a unique unit normal N vector field defined on Σ. The second fundamental form of Σ is then defined as

$$k(X,Y) = -g(X, D_Y N),$$

15. Another way to try to understand the Ricci tensor is to remember that when $\mathrm{Ric}(g) = 0$ then the Riemann tensor is invariant under conformal transformation. Hence, the Ricci tensor "measures" deviation from conformal invariance of the curvature.

where X and Y are vector fields tangent to Σ. It turns out that k is a symmetric 2-tensor.

2.3. Maximal foliations

Assume that $(\mathcal{M}, g)$ can be foliated by the level hypersurfaces Σ_t of a time function t. Let T denote the unit normal to Σ_t and (h, k) denote its first and second fundamental forms respectively. The foliation is said to be *maximal* if $tr_h k := k_{ab}h^{ab} = 0$. For a maximal foliation, the constraint equations (2) on any Σ_t reduce to

$$R^{(h)} = |k|^2 \tag{10}$$

$$\mathrm{div}_h\, k = 0. \tag{11}$$

Let us also recall that if n is the lapse of the foliation, i.e $n^{-2} := -g(Dt, Dt)$, then, on each Σ_t, n satisfies the elliptic equation

$$\Delta_h n = n|k|^2,$$

where Δ_h is the induced Laplace-Beltrami operator on Σ_t.

We will consider only asymptotically flat Lorentzian manifolds, i.e. Lorentzian metric such that g approaches the Minkowski metric at spatial infinity. In particular, n will approach 1 at spatial infinity.

Recall that the curvature tensor is constructed by applying 2 covariant derivatives and commuting, while the definition of k only involves the first derivatives of the metric. We will call *structure equations* the equations linking the curvature tensor with any geometrical object (such as k, the Ricci coefficients or the Christoffel symbols) whose definition depends only on the first derivatives of the metric. For instance, the structure equations of the maximal foliation are given by

$$\begin{aligned} D_T k_{ab} + n^{-1} D_a D_b n + k_{ac} k_b^c &= R_{aTbT}, \\ D_a k_{bc} - D_b k_{ac} &= R_{cTab}, \\ \mathrm{Ric}(h)_{ab} - k_{ac} k_b^c &= R_{aTbT}. \end{aligned}$$

2.4. Indices and notations

$(\mathcal{M}, g)$ will always denote a $3+1$ dimensional Lorentzian manifold. The Levi-Civita connection associated to g will be denoted by D, its Riemann tensor by $\mathrm{Riem}(g)$ or sometimes in components just as $R_{\mu\nu\sigma\rho}$. The greek indices $\mu, \nu, \sigma, \rho, ..$ will be indices related to a coordinate system on $\mathcal{M}$.

If (Σ_t) is a maximal foliation, we will denote by $h = h(t)$ and $k = k(t)$ the induced metric and second fundamental form on a Σ_t hypersurface, $\nabla^{(h)}$ or ∇, the Levi-Civita connection associated to h and $\mathrm{Ric}(h)$ its Ricci tensor. The indices a, b, c, d will refer to a decomposition of tensors using an arbitrary frame on a Σ_t, while the indices i, j, k will be used to refer to a particular orthonormal frame (e_1, e_2, e_3) to be constructed below. On $\mathcal{M}$, we will often use a frame (e_0, e_1, e_2, e_3) where e_0 is a vector field normal to Σ_t while e_1, e_2, e_3 are tangent to Σ_t. ∂ will denote any derivative of a scalar quantity in any of the directions e_1, e_2, e_3, while $\boldsymbol{\partial}$ will denote any derivative of a scalar quantity relative in any of the directions e_0, e_1, e_2, e_3.

We shall use below several functions depending on a parameter $\omega \in \mathbb{S}^2$. If u is a function defined on $\mathcal{M} \times \mathbb{S}^2$ (or $\Sigma \times \mathbb{S}^2$), we will denote by $\partial_\omega u$, a scalar function defined on $\mathcal{M} \times \mathbb{S}^2$ (respectively $\Sigma \times \mathbb{S}^2$) obtained by applying a vector field tangent to $\mathbb{S}^2$ of size $\lesssim 1$ with respect to the usual metric of $\mathbb{S}^2$ to u. For instance, if standards spherical coordinates (θ, ϕ) are used on $\mathbb{S}^2$, then $\partial_\omega u$ will be used to denote any of $\partial_\theta u$, $\partial_\phi u$.

Remark 2.1. — *Our notation is similar to that used in* [26, 41, 42, 43, 44, 45] *apart from the fact that they use bold letters, such as* $\mathbf{g}$*, to denote tensors relative to* $\mathcal{M}$*, and normal letters for tensor related to* Σ_t*. Thus, they use* $\mathbf{g}$ *for the Lorentzian metric and* g *for the induced metric (while we use* g *for the Lorentzian metric and* h *for the induced metric).*

2.5. Volume radius

On top of assumptions on the curvature and the second fundamental form, Theorems 1.2 and 1.3 also need an assumption on the *volume radius* whose definition is recalled here.

Definition 2.3. — *Let* (Σ, h) *denote a Riemannian manifold and let* $B_r(p)$ *denote the geodesic ball of center* $p \in \Sigma$ *and radius* $r > 0$*. The volume radius at* p *and at scales less than* r*, denoted by* $r_{\mathrm{vol}}(p, r)$*, is defined as*

$$r_{\mathrm{vol}}(p, r) = \inf_{0 < r' \leq r} \frac{|B_{r'}(p)|}{(r')^3},$$

where $|B_{r'}(p)|$ *denotes the Riemannian volume of* $B_{r'}(p)$*.*

2.6. Basic notions for null hypersurfaces

Null hypersurfaces of a Lorentzian manifold are by definition hypersurfaces for which the induced metric is degenerate. Equivalently, a hypersurface is null if its normal direction is tangent to the hypersurface itself.

In this text, null hypersurfaces will appear as level sets of a scalar function u, (sometimes called an optical function) which will be solution to the eikonal equation $g(Du, Du) = 0$. A typical such function is therefore a function defined on $\mathcal{M}$. However, we will in fact need to construct u depending on a frequency parameter $\omega \in \mathbb{S}^2$. Thus, u will in fact be a function defined on $\mathcal{M} \times \mathbb{S}^2$,

$$\begin{array}{rcl} \mathcal{M} \times \mathbb{S}^2 & \to & \mathbb{R} \\ u : (p, \omega) & \to & u(p, \omega). \end{array}$$

The reason behind the extra ω parameter is that we want to construct a phase function ϕ defined on the cotangent bundle $T\mathcal{M}^*$ or, more precisely on the submanifold of $T\mathcal{M}^*$, composed of points of (p, ξ) such that $g(\xi, \xi) = 0$. Since this last equation is homogeneous (cones are conical !), we can factorize out $|\xi|$ and therefore construct our phase as a function on $\mathcal{M} \times \mathbb{S}^2$ [16]. The phase function ϕ will also be homogeneous of degree 1 in $|\xi|$, so that $\phi(p,\xi) = |\xi|\, u(x,\omega)$, with $\omega \in \mathbb{S}^2$. We shall also refer to u as the phase function.

Often, we will derive formulae or estimates which are transparent in ω (for instance uniform estimates in ω). In this case, we will by a small abuse of notations, drop the ω dependence. We will also often use the symbols ${}^{\omega}u$ to denote the function $p \in \mathcal{M} \to u(p,\omega)$.

For instance, ω being fixed, the level sets of $p \in \mathcal{M} \to u(p,\omega)$ will be denoted as $\mathcal{H}_{{}^{\omega}u}$ or sometimes simply $\mathcal{H}_u$. Note that with the function u to be constructed below, the level sets $\mathcal{H}_u$ will be diffeomorphic to $\mathbb{R}^3$ and the intersection $\mathcal{H}_u \cap \Sigma_t$ will be diffeomorphic to 2-planes.

More generally, we will denote by $\mathcal{H}$ a general null hypersurface. Similarly to a spacelike hypersurface, if L is a normal to a null hypersurface $\mathcal{H}$, we can define a null second fundamental form (unique up to a normalization for L) by

$$\chi(X,Y) := g(X, D_Y L),$$

for X, Y $\mathcal{H}$-tangent vector fields. This defines a symmetric tensor on $\mathcal{H}$. Particularly important in this text will be its trace [17] $tr\chi$. Indeed, if u is a solution to the eikonal equation and L is the normal to $\mathcal{H}_u$ such that $g(T,L) = -1$ then we have the formula

$$\Box_g u = T(u)\, tr\chi, \tag{12}$$

where $tr\chi$ denote the trace of the null second fundamental form associated to L.

The following technical definition will be needed later.

Definition 2.4 (Weakly regular null hypersurface). — *Let $\mathcal{H}$ be a null hypersurface with future null normal L verifying $g(L,T) = -1$. Let also $N = L - T$. Let $\nabla\!\!\!\!/$ be the induced connection on $\mathcal{H} \cap \Sigma_t$. $\mathcal{H}$ is said to be* weakly regular *if*

$$||DL||_{L^3(\mathcal{H})} + ||DN||_{L^3(\mathcal{H})} \lesssim 1,$$

and the following Sobolev inequality holds, for any scalar function f defined on $\mathcal{H}$ such that the right-hand side is finite,

$$||f||_{L^6(\mathcal{H})} \lesssim ||\nabla\!\!\!\!/ f||_{L^2(\mathcal{H})} + ||L(f)||_{L^2(\mathcal{H})} + ||f||_{L^2(\mathcal{H})}.$$

16. More precisely, for any any $p \in \mathcal{M}$, the characteristic submanifold of $T_p\mathcal{M}^*$ composed of non-zero covectors ξ such that $g(\xi,\xi) = 0$ is diffeomorphic to two disconnected copies of $\mathbb{R} \times \mathbb{S}^2$.

17. Recall that if e_A, e_B are $\mathcal{H}$-tangent vector fields such that (e_A, e_B) form an orthonormal basis of the orthogonal complement of L in $T\mathcal{H}$, then $tr\chi := \chi(e_A,e_A) + \chi(e_B,e_B)$.

3. THE EINSTEIN EQUATIONS AS A QUASILINEAR YANG-MILLS SYSTEM

The Yang-Mills formulation of the Einstein equations relies on Cartan theory of moving frames. Instead of decomposing tensors thanks to coordinate induced vector fields, we start first by fixing a moving orthonormal frame, i.e. a set of vector fields e_α, $\alpha = 0, .., 3$ such that

$$g(e_\alpha, e_\beta) = diag(-1, 1, 1, 1).$$

Let $(x^\mu)_{\mu=0,..,3}$ be a system of local coordinates. We can define a 1-form with values in the set of anti-symmetric matrices as

$$A_\mu dx^\mu = g(D_\mu e_\beta, e_\alpha) dx^\mu.$$

We will identify the set of anti-symmetric matrices with the Lie algebra of the restricted Lorentz group $SO(3,1)$. Hence, we can think of the one-form A as defining a connection for the Lie group $SO(3,1)$.

Using the definitions of the Riemann tensor and the connection 1-form A, we have

$$R(e_\alpha, e_\beta, \partial_\mu, \partial_\nu) = D_\mu(A_\nu)_{\alpha\beta} - D_\nu(A_\mu)_{\alpha\beta} + (A_\nu)_\alpha^\gamma (A_\mu)_{\gamma\beta} - (A_\mu)_\alpha^\gamma (A_\nu)_{\gamma\beta}. \tag{13}$$

Recall that A is a 1-form with values in Lie algebra $so(3,1)$. We endow $so(3,1)$ with the natural Lie bracket arising from matrix multiplication

$$([A_\mu, A_\nu])_{\alpha\beta} = (A_\mu)_\alpha^\gamma (A_\nu)_{\gamma\beta} - (A_\nu)_\alpha^\gamma (A_\mu)_{\gamma\beta}.$$

With this notation, equation (13) becomes

$$R(e_\alpha, e_\beta, \partial_\mu, \partial_\nu) = D_\mu(A_\nu)_{\alpha\beta} - D_\nu(A_\mu)_{\alpha\beta} - ([A_\mu, A_\nu])_{\alpha\beta}.$$

Now the expression on the right-hand side is exactly the expression for the curvature of the connection A:

$$(F_{\mu\nu})_{\alpha\beta} := D_\mu(A_\nu)_{\alpha\beta} - D_\nu(A_\mu)_{\alpha\beta} - ([A_\mu, A_\nu])_{\alpha\beta}.$$

The usual covariant derivative of the Riemann tensor can be rewritten using the F curvature tensor

$$D_\sigma R_{\alpha\beta\mu\nu} = \left({}^{(A)}D_\sigma F_{\mu\nu}\right)_{\alpha\beta} := (D_\sigma F_{\mu\nu})_{\alpha\beta} + ([A_\sigma, F_{\mu\nu}])_{\alpha\beta},$$

where

$${}^{(A)}D_\sigma U := D_\sigma U + [A_\sigma, U]$$

is the covariant derivative associated to the connection 1-form A for $so(3,1)$ valued tensors U.

Note that we are using two sets of indices, the *internal* indices $\alpha, \beta, ..$, which refer to the orthonormal frame (e_α), and the *external* indices $\mu, \nu, ..$ which refer to an arbitrary frame, such as a coordinate induced basis. In most of the computations below, the internal indices will be irrelevant, and we shall just drop them.

The curvature tensor F satisfies the so-called Bianchi idendities (which are of course equivalent to the Bianchi idendities for R)

$$ {}^{(A)}D_\sigma F_{\mu\nu} +^{(A)} D_\mu F_{\nu\sigma} +^{(A)} D_\nu F_{\sigma\mu} = 0. $$

Moreover, it follows from the Einstein equations $\mathrm{Ric}(g) = 0$ that the Riemann tensor is divergence free, which in view of the above leads to

$$ (14) \qquad {}^{(A)}D^\mu F_{\mu\nu} = 0. $$

Recall also that the Einstein equations implies that the divergence operator commutes with the covariant derivative D. Using the definition of F in terms of A and (14), we obtain

$$ (15) \qquad \square_g A_\nu - D_\nu \left(D^\mu A_\mu\right) = J_\nu, $$

where J_ν is the 1-form

$$ J_\nu := D^\mu \left([A_\mu, A_\nu]\right) - [A^\mu, F_{\mu\nu}], $$

which satisfies $D^\mu J_\mu$.

3.1. The choice of frames

Recall that $\mathcal{M}$ is assumed to be foliated by the level sets (Σ_t) of a time function t. We will consider an orthonormal frame (e_0, e_1, e_2, e_3) such that $e_0 = T$ is the future unit normal to the Σ_t foliation and e_i, $1 \leq i \leq 3$, is an orthonormal frame tangent to the Σ_t. Using that k is traceless due to the maximal foliation condition and the definition of A, we can express some components of A in terms of k

$$ \begin{aligned} (A_i)_{0j} = (A_j)_{0i} &= -k_{ij}, \\ (A_0)_{0i} &= -n^{-1}\nabla_i n. \end{aligned} $$

With the divergence free property of k, we obtain

$$ \nabla^i (A_i)_{0j} = k_i^m (A_i)_{mj}. $$

Note that we still have some freedom left in the choice of the e_i's. Two such frames are related by O(3) matrices: $\tilde{e}_i = O_i^j e_j$. The connection 1-form for the new frame can be rewritten in terms of the old. Schematically,

$$ \tilde{A}_m = O A_m O^{-1} + (\partial_m O) O^{-1}. $$

This freedom will be used to construct a frame such that the Coulomb type gauge condition $\nabla^l (A_l) = A^2$ is satisfied, where A^2 denotes terms which are quadratic in the components of A.

3.2. Main equations for A

The main equations for the connection 1-form in Coulomb gauge can be summarized as follows

Proposition 3.1. — *Let $\boldsymbol{A} = (A_0, A)$ denote the components of the connection 1-form for an orthonormal frame $(e_0, e_i), 1 \leq i \leq 3$, such that e_0 coincides with the future unit normal to the maximal foliation and the Coulomb gauge condition $\mathrm{div}A = A^2$ holds. Then, the Einstein equations reduce to*

$$\Delta A_0 = \boldsymbol{A}\boldsymbol{\partial}A + \boldsymbol{A}\partial(A_0) + \boldsymbol{A}^3, \tag{16}$$

$$\Box_g A_i + \partial_i(\partial_0 A_0) = A^j \partial_j A_i + A^j \partial_i A_j + A_0 \partial\boldsymbol{A} + A\boldsymbol{\partial}(A_0) + \boldsymbol{A}^3, \tag{17}$$

where $\boldsymbol{A}$ denotes any components of the connection 1-form, while A_i or A denotes only the e_i components.

The existence of a frame such that the Coulomb gauge condition is satisfied is garanteed by the following Uhlenbeck type Lemma (see Lemma 4.2 in [26]),

Lemma 3.2. — *Let (Σ, h) be a 3 dimensional asymptotically Euclidean manifold. Let $\tilde{A}$ denote a 1-form connection corresponding to an orthonormal frame and assume that the following bounds are satisfied*

$$||\tilde{A}||_{L^2(\Sigma)} + ||\nabla\tilde{A}||_{L^2(\Sigma)} + ||\mathrm{Ric}(h)||_{L^2(\Sigma)} \leq \delta, \quad r_{\mathrm{vol}}(\Sigma, 1) \geq 1/4,$$

where $r_{\mathrm{vol}}(\Sigma, 1)$ is the volume radius on scale less than 1. Then, there exists a $\delta_0 > 0$ such that if $0 < \delta \leq \delta_0$, there exists a connection 1-form A on Σ, such that

- *A, $\nabla A \in L^2(\Sigma)$, and $\nabla\nabla A \in L^2(\Sigma)$ provided that $\nabla\nabla\tilde{A} \in L^2(\Sigma)$,*
- *A satisfies the Coulomb gauge condition*

$$\mathrm{div}A = A^2,$$

- *the following bounds are satisfied*

$$||A||_{L^2(\Sigma)} + ||\nabla A||_{L^2(\Sigma)} \leq \delta.$$

4. SKETCH OF THE PROOF OF THE BOUNDED L^2 CURVATURE CONJECTURE

The proof of Theorem 1.3 is based on a bootstrap argument. We will first describe the main bootstrap assumptions and briefly sketch the proof of their improvements. Two constants $M \geq 1$ and $\epsilon > 0$ will be used below. M will be chosen sufficiently large later depending only on universal constants. ϵ will be chosen sufficiently small so that in particular, $M\epsilon$ will be as small as wanted.

4.1. The Bootstrap assumptions and their improvements

We will assume that the following bootstrap assumptions hold on an interval $[0, T^*]$, for some $0 < T^* \leq 1$.

– Bootstrap curvature assumptions.

$$||\text{Riem}(g)||_{L_t^\infty L^2(\Sigma_t)} \leq M\epsilon$$

and

$$||\text{Riem}(g).L||_{L_t^\infty L^2(\mathcal{H})} \leq M\epsilon,$$

where $\text{Riem}(g).L$ denotes any component of $\text{Riem}(g)$ with at least one index contracted with L and where $\mathcal{H}$ denotes an arbitrary weakly regular null hypersurface.

– Bootstrap assumptions for the connection $\mathbf{A} = (A_0, A)$.

$$||A||_{L_t^\infty L^2(\Sigma_t)} + ||\boldsymbol{\partial} A||_{L_t^\infty L^2(\Sigma_t)} + ||A||_{L_t^2 L^7(\Sigma_t)} + ||\partial B||_{L_t^2 L^7(\Sigma_t)} \leq M\epsilon,$$

where $B = (-\Delta)^{-1}\text{curl}\, A$, and

$$||A_0||_{L_t^\infty L^2(\Sigma_t)} + ||\boldsymbol{\partial} A_0||_{L_t^\infty (L^2(\Sigma_t)\cap L^3(\Sigma_t))} + ||A_0||_{L_t^2 L^\infty(\Sigma_t)} + ||\partial \boldsymbol{\partial} A_0||_{L_t^\infty L^{3/2}(\Sigma_t)} \leq M\epsilon,$$

– Bilinear assumptions I.

$$||A^j \partial_j A||_{L^2(\mathcal{M})} + ||R_{\ \cdot\ \cdot\ j0} \partial^j B||_{L^2(\mathcal{M})} \lesssim M^3 \epsilon^2,$$

– Bilinear assumptions II.

$$||(-\Delta)^{-1/2} \left(Q_{ij}(A, A)\right)||_{L^2(\mathcal{M})} \lesssim M^3 \epsilon^2,$$

where Q_{ij} is the null form $Q_{ij}(\phi, \psi) = \partial_i \phi \partial_j \psi - \partial_j \phi \partial_i \psi$, and

$$||(-\Delta)^{-1/2} \left(\partial(A^l)\partial_l A\right)||_{L^2(\mathcal{M})} \lesssim M^3 \epsilon^2,$$

– Trilinear assumptions.
Let Q be the Bell-Robinson tensor

$$Q_{\alpha\beta\gamma\delta} = R_{\alpha\ \gamma}^{\ \lambda\ \sigma} R_{\beta\lambda\delta\sigma} + ({}^\star R)_{\alpha\ \gamma}^{\ \lambda\ \sigma} ({}^\star R)_{\beta\lambda\delta\sigma}\,,$$

where ${}^\star R$ is the Hodge dual of R. Then,

$$\left| \int_M Q_{ij\gamma\sigma} k^{ij} T^\gamma T^\delta \right| \lesssim M^4 \epsilon^3.$$

As a consequence of the small data assumptions and Lemma 3.2, it follows that the bootstrap assumptions are valid at least for a small $T^* > 0$. The following proposition states that the above bootstrap assumptions can be improved.

Proposition 4.1 (Improved bootstrap assumptions). — *There exists an* $\epsilon_0 > 0$ *such that if* $0 < \epsilon < \epsilon_0$ *and the above bootstrap assumptions hold on* $[0, T^\star]$*, then we have*

– *Improved curvature and connection estimates*

$$\begin{aligned}
||\mathrm{Riem}(g)||_{L_t^\infty L^2(\Sigma_t)} &\lesssim \epsilon + M^2\epsilon^{3/2} + M^3\epsilon^2,\\
||\mathrm{Riem}(g).L||_{L^2(\mathcal{H})} &\lesssim \epsilon + M^2\epsilon^2 + M^3\epsilon^{3/2},\\
||A||_{L_t^\infty L^2(\Sigma_t)} + ||\boldsymbol{\partial} A||_{L_t^\infty L^2(\Sigma_t)} &\lesssim \epsilon + M^2\epsilon^{3/2} + M^3\epsilon^2,\\
||A_0||_{L_t^2 L^\infty(\Sigma_t)} + ||\boldsymbol{\partial} A_0||_{L_t^\infty L^3(\Sigma_t)} + ||\partial\boldsymbol{\partial} A_0||_{L_t^\infty L^{3/2}(\Sigma_t)} &\lesssim \epsilon + M^2\epsilon^{3/2} + M^3\epsilon^2,
\end{aligned}$$

– *Improved bilinear and trilinear estimates*

$$\begin{aligned}
||A^j\partial_j A||_{L^2(\mathcal{M})} + ||A^j\partial_j(\boldsymbol{\partial} B)||_{L^2(\mathcal{M})} + +||R\ {}_{\cdot\ \cdot\ j0}\partial^j B||_{L^2(\mathcal{M})} &\lesssim M^2\epsilon^2,\\
||(-\Delta)^{-1/2}\left(Q_{ij}(A,A)\right)||_{L^2(\mathcal{M})} &\lesssim M^2\epsilon^2,\\
||(-\Delta)^{-1/2}\left(\partial(A^l)\partial_l A\right)||_{L^2(\mathcal{M})} &\lesssim M^2\epsilon^2,\\
\left|\int_M Q_{ij\gamma\sigma}k^{ij}T^\gamma T^\delta\right| &\lesssim M^3\epsilon^3.
\end{aligned}$$

– *Improved (non-sharp) Strichartz estimates*

$$||A||_{L_t^2 L_x^7(\Sigma_t)} + ||\partial B||_{L_t^2 L_x^7(\Sigma_t)} \lesssim M\epsilon.$$

In conjunction with an argument of propagation of high regularity and estimates controlling the volume radius, the above propositions enable to extend the time of existence of the solutions up to $t = 1$ and to obtain that all the (improved) bootstrap assumptions are true on $[0, 1]$, which, in particular, establishes Theorem 1.3.

4.2. Main ideas for the proof of the improved bootstrap assumptions

We refer to [26] for the detailed proof of Proposition 4.1 and present here only the principal ideas.

The core of the argument is to control A. For this, it is clear that one needs to eliminate the term $\partial_i(\partial_0 A_0)$ from the A_i equations (17). Moreover, we also need to exhibit the crucial null structure. To this end, one could try to project the equations onto divergence free vector fields, but the resulting commutator is hard to work with. Instead, the variable

$$B := (-\Delta)^{-1}\mathrm{curl}\, A.$$

will be used. Since then $A = \mathrm{curl}\,(B) + l.o.t$, it is sufficient to estimate B in order to improve the bootstrap assumptions.

This is done in several steps

– Step 1: derive a wave equation for B with estimates for $\Box_g B$.
– Step 2: derive energy estimates for solutions to the wave equation $\Box_g\phi = F$.
– Step 3: derive curvature estimates, i.e. L^2 bounds on $|\mathrm{Riem}(g)|$ on Σ_t and on weak null hypersurfaces.
– Step 4: improve the bilinear estimates,

where in all the above steps, the original bootstrap assumptions can be used. Now, Step 2 and Step 3 are in some sort classical. Step 3 was for instance at the core of the monumental work of Christodoulou-Klainerman establishing the stability of the Minkowksi space (see [8]). Bilinear estimates, even in flat space, are based on an explicit representation of the solutions, i.e a parametrix, and the proof of Step 4 is no exception. Assuming the existence of such a parametrix, a preliminary version of step 4 had been obtained previously by Klainerman-Rodnianski in [20]. In the next section, we will present some basic notions concerning the parametrix constructed in [41, 42, 43, 44], so that we can eventually present a sketch of the proof of the bilinear estimates on the basis of this parametrix.

5. THE PARAMETRIX AND THE BILINEAR ESTIMATES

Let $u_\pm$ be two families of scalar functions defined on $\mathcal{M}\times\mathbb{S}^2$, as introduced in Section 2.6, solutions of the eikonal equation

$$g(Du, Du) = 0, \tag{18}$$

where Du denotes the spacetime gradient of a scalar function u, $Du := -g^{\alpha\beta}\partial_\beta u$.

The notation ${}^\omega u_\pm(t,x) = u_\pm(t,x,\omega)$, will be used, (t,x) being coordinates on $\mathcal{M}$. The equation (18) is a transport equation and thus, requires a prescription for the initial data. The initial data ${}^\omega u_\pm(0,x)$ will be chosen carefully and this choice, with its consequences is the subject of [41, 42], presented briefly in Section 8 of this text.

ω being fixed, let $\mathcal{H}_{{}^\omega u_\pm}$ denote the level sets of ${}^\omega u_\pm$. $\mathcal{H}_{{}^\omega u_\pm}$ are then weakly regular null hypersurfaces and their normals are tangent to themselves and unique up to a choice of normalization. We will denote by ${}^\omega L_\pm$ the normals to $\mathcal{H}_{{}^\omega u_\pm}$ normalized by $g({}^\omega L_\pm, T) = -1$. Finally, we defined ${}^\omega N_\pm$ by

$${}^\omega N_\pm :={}^\omega L_\pm - T.$$

By construction, ${}^\omega N_\pm$ are unit vector fields tangent to Σ_t and normal to the 2-planes $\mathcal{H}_{{}^\omega u_\pm}\cap\Sigma_t$.

For any pair of functions $f_\pm:\mathbb{R}^3\to\mathbb{R}$, we define the scalar function on $\mathcal{M}$

$$\psi[f_+,f_-] = \int_{\mathbb{S}^2}\int_{\lambda\in\mathbb{R}_+} e^{i\lambda {}^\omega u_+} f_+(\lambda\omega)\lambda^2 d\lambda d\omega + \int_{\mathbb{S}^2}\int_{\lambda\in\mathbb{R}_+} e^{i\lambda {}^\omega u_-} f_-(\lambda\omega)\lambda^2 d\lambda d\omega. \tag{19}$$

In [42] and [44], under appropriate assumptions on the phase functions $u_\pm$, it is proven that given ϕ_0 and ϕ_1 two scalar functions on Σ_0, then there exists a unique pair of functions (f_+,f_-) such that

$$\psi[f_+,f_-]_{|\Sigma_0} = \phi_0,\quad \partial_0\psi[f_+,f_-]_{|\Sigma_0} = \phi_1. \tag{20}$$

Moreover, the following estimates hold

$$||\lambda f_+||_{L^2(\mathbb{R}^3)} + ||\lambda f_-||_{L^2(\mathbb{R}^3)} \lesssim ||\nabla\phi_0||_{L^2(\Sigma_0)} + ||\phi_1||_{L^2(\Sigma_0)}, \tag{21}$$

$$||\lambda^2 f_+||_{L^2(\mathbb{R}^3)} + ||\lambda^2 f_-||_{L^2(\mathbb{R}^3)} \lesssim ||\nabla^2\phi_0||_{L^2(\Sigma_0)} + ||\nabla\phi_1||_{L^2(\Sigma_0)}, \tag{22}$$

as well as the estimates on the error $\Box_g\psi[f_+, f_-]$

$$\begin{aligned} ||\Box_g\psi[f_+, f_-]||_{L^2(\mathcal{M})} &\lesssim M\epsilon\left(||\nabla\phi_0||_{L^2(\Sigma_0)} + ||\phi_1||_{L^2(\Sigma_0)}\right), \\ ||\partial\Box_g\psi[f_+, f_-]||_{L^2(\mathcal{M})} &\lesssim M\epsilon\left(||\nabla^2\phi_0||_{L^2(\Sigma_0)} + ||\nabla\phi_1||_{L^2(\Sigma_0)}\right). \end{aligned}$$

These results are briefly presented in Sections 6 and 8 of this text.

Given any pair of functions ϕ_0 and ϕ_1, we can then define $\Psi[\phi_0, \phi_1]$ by $\Psi[\phi_0, \phi_1] := \psi[f_0, f_1]$, where (f_0, f_1) are as in the above theorem. Using an iteration scheme based on the Duhamel formula, one can then represent any exact solution to the wave equation as a sum of approximate solutions constructed inductively. We refer to [26] for the details.

Remark 5.1. — *Note that the parametrix* (19) *is the sum of two half-waves, one for u_+ and one for u_-, corresponding to the fact that the characteristic manifold is the sum of two disconnected components. In all the estimates below, we will drop the $\pm$ and estimate only one of the half-wave since the proofs are identical.*

5.1. Improving the bilinear estimates

In this section, we shall give a sketch of the proof of the bilinear estimates, focusing only on the typical example (see Proposition 4.1).

$$||A^j\partial_j A||_{L^2(\mathcal{M})} \lesssim M^2\epsilon^2.$$

Note that $A^j\partial_j A$ does not look a priori like the typical null forms (8) and (9). However, recall that the estimates on the connection are not obtained directly for A but for B with $B = (-\Delta)^{-1}\mathrm{curl}\, A$. In fact, one has the following lemma

Lemma 5.1 (Writing A in terms of B). — *One has*

$$A = \mathrm{curl}\,(B) + E$$

with E satisfying

$$||\partial E||_{L^\infty_t L^3(\Sigma_t)} + ||\partial^2 E||_{L^\infty_t L^{3/2}\Sigma_t)} + ||E||_{L^2_t L^\infty(\Sigma_t)} \leq M^2\epsilon^2.$$

Now recalling that $\mathrm{curl}\,(B)_j = \epsilon_{jmn}\partial_m(B_n)$, where ϵ_{jmn} is the totally anti-symmetric symbol, we see that

$$A^j\partial_j A = \epsilon_{jmn}\partial_m(B_n)\partial_j(A) + ..$$

where the .. corresponds to some error terms coming from E in the above lemma which are easier to estimate. In $\epsilon_{jmn}\partial_m(B_n)\partial_j(A)$, we naturally recognise a null form of type (9).

To exploit this structure, it is important that B itself satisfies a wave equation and that we control both the error term $\Box_g B$ and the initial data.

LEMMA 5.2 (Estimates for B). — *B satifies $\square_g B = F$ with*

$$||\partial F||_{L^2(\mathcal{M})} \lesssim M^2\epsilon^2,$$
$$||\boldsymbol{\partial} B(0)||_{L^2(\Sigma_0)} + ||\partial^2 B(0)||_{L^2(\Sigma_0)} + ||\partial\partial_0 B(0)||_{L^2(\Sigma_0)} \lesssim M\epsilon.$$

The above estimates allows us to use the parametrix introduced above to represent B. Replacing B by $\int_{\mathbb{S}^2}\int_{\lambda\in\mathbb{R}^+} e^{i\lambda^\omega u} f(\lambda\omega)\lambda^2 d\lambda d\omega$ in $\epsilon_{jmn}\partial_m(B_n)\partial_j(A)$ leads to the quantity

$$\int_{\mathbb{S}^2}\int_{\lambda\in\mathbb{R}^+} \epsilon_{jm}\left(e^{i^\omega \lambda u}\right)_m \partial_j(A) f(\lambda\omega)\lambda^2 d\lambda d\omega.$$

Now since $\left(e^{i\lambda^\omega u}\right)_m = i\lambda\partial_m u e^{i\lambda^\omega u}$ and since the gradient of ${}^\omega u$ on Σ_t is given by $\nabla^\omega u = {}^\omega b^{-1\omega} N$, with ${}^\omega N$ a unit normal to $\Sigma_t \cap \mathcal{H}_{{}^\omega u}$ (so that ${}^\omega b = |\nabla^\omega u|^{-1}$), the bilinear estimate controlling $A^j(\partial_j A)$ reduces to an $L^2(\mathcal{M})$ estimate on

$$\int_{\mathbb{S}^2}\int_{\lambda\in\mathbb{R}^+} e^{i\lambda^\omega u}\, {}^\omega b^{-1}\epsilon_{jm}\, {}^\omega N_m \partial_j(A) f(\lambda\omega)\lambda^2 d\lambda d\omega. \tag{23}$$

Using the anti-symmetry of ϵ_{jm} and the definition of ${}^\omega N$, note that the quantity $\epsilon_{jm}N_m\partial_j(A)$ satisfies

$$||\epsilon_{jm}N_m\partial_j(A)||_{L^\infty_u L^2(\mathcal{H}_u)} \lesssim ||\nabla\!\!\!\!/ A||_{L^\infty_u L^2(\mathcal{H}_u)},$$

and the last term can be bounded via energy estimates (for B) by $M\epsilon$. To estimate (23), one can then proceed as follows. First,

$$\left|\left|\int_{\mathbb{S}^2}\int_{\lambda\in\mathbb{R}^+} e^{i\lambda u}\, {}^\omega b^{-1}\epsilon_{jm}N_m\partial_j(A) f(\lambda\omega)\lambda^2 d\lambda d\omega\right|\right|_{L^2(\mathcal{M})} \lesssim$$
$$\int_{\mathbb{S}^2}\left|\left|\int_{\lambda\in\mathbb{R}^+} e^{i\lambda u}\, {}^\omega b^{-1}\epsilon_{jm}N_m\partial_j(A) f(\lambda\omega)\lambda^2 d\lambda\right|\right|_{L^2(\mathcal{M})} d\omega.$$

To estimate the $L^2(\mathcal{M})$ norm below the ω integral, we consider on $\mathcal{M}$ a coordinate system of the form $(t, {}^\omega u, x')$, where x' denotes coordinates on $\Sigma_t \cap \mathcal{H}_{{}^\omega u}$. Thus, for each ω, we have a different coordinate sytem.

Now,

$$\left|\left|\int_{\lambda\in\mathbb{R}^+} e^{i\lambda u}\epsilon_{jm}N_m\partial_j(A) f(\lambda\omega)\lambda^2 d\lambda\right|\right|_{L^2(\mathcal{M})} \leq$$
$$||{}^\omega b^{-1}||_{L^\infty(\mathcal{M})}\, ||\epsilon_{jm}N_m\partial_j(A)||_{L^\infty_{{}^\omega u} L^2(\mathcal{H}_{{}^\omega u})} \left|\left|\int_{\lambda\in\mathbb{R}^+} e^{i\lambda^\omega u} f(\lambda\omega)\lambda^3 d\lambda\right|\right|_{L^2_{{}^\omega u} L^\infty(\mathcal{H}_{{}^\omega u})}.$$

Observe that for the last term on the right-hand side, the $L^\infty(\mathcal{H}_{{}^\omega u})$ norm is in fact not needed, since the quantity is independent of the point (t, x') in $\mathcal{H}_{{}^\omega u}$. Moreover, by Plancherel, the last term is nothing else than $||f\lambda^3||_{L^2_\lambda}$. Since it follows from the construction of ${}^\omega u$ (see [43]) that $\sup_{\omega\in\mathbb{S}^2} ||{}^\omega b^{-1}||_{L^\infty(M)} \lesssim 1$, we obtain

$$\left|\left|\int_{\mathbb{S}^2}\int_{\lambda\in\mathbb{R}^+} e^{i\lambda u\,\omega} b^{-1}\epsilon_{jm}{}^{\omega}N_m\partial_j(A)f(\lambda\omega)\lambda^2 d\lambda d\omega\right|\right|_{L^2(\mathcal{M})} \lesssim$$
$$\sup_{\omega\in\mathbb{S}^2}||\epsilon_{jm}{}^{\omega}N_m\partial_j(A)||_{L^\infty_{\omega_u}L^2(\mathcal{H}_{\omega_u})}\int_{\mathbb{S}^2}||f\lambda^3||_{L^2_\lambda}d\omega.$$

Since we already know that $\sup_{\omega\in\mathbb{S}^2}||\epsilon_{jm}{}^{\omega}N_m\partial_j(A)||_{L^\infty L^2(\mathcal{H}_{\omega_u})} \lesssim M\epsilon$ and since $\int_{\mathbb{S}^2}||f\lambda^3||_{L^2_\lambda}d\omega = ||\lambda^2 f||_{L^3(\mathbb{R})} \lesssim M\epsilon$, using the estimates on B, one obtains the improved estimates

$$||A^j\partial_j(A)||_{L^2(\mathcal{M})} \lesssim M\epsilon^2.$$

Let us also mention that the improved bootstrap estimates concerning the second set of bilinear assumptions for the null forms $(-\Delta)^{-1/2}Q_{ij}$ can be reduced to the proof of an $L^4(\mathcal{M})$ Strichartz estimate. This Strichartz estimate is obtained in [45] and presented in this text in Section 9. The same Strichartz estimate also provides the improved $L^2_tL^7_x$ Strichartz estimates for A and B of Proposition 4.1.

6. CONTROL OF THE PARAMETRIX ERROR

Recall the parametrix introduced in the previous section

$$\int_{\mathbb{S}^2}\int_{\lambda\in\mathbb{R}_+} e^{i\lambda^\omega u} f(\lambda\omega)\lambda^2 d\lambda d\omega.$$

Applying the wave operator $\Box_g$ to the above expression using that ${}^\omega u$ is a solution to the eikonal equation gives us the error

$$Ef(t,x) := i\int_{\mathbb{S}^2}\int_{\lambda\in\mathbb{R}_+} e^{i\lambda^\omega u}\Box_g^\omega u f(\lambda\omega)\lambda^3 d\lambda d\omega.$$

The goal of this section is to outline the proof of the following theorem

Theorem 6.1. — *Let u be a phase function on $\mathcal{M}\times\mathbb{S}^2$ satisfying the regularity assumptions obtained in* [43]. *Then, we have the estimate*

$$||Ef||_{L^2(\mathcal{M}} \lesssim \epsilon||\lambda f||_{L^2(\mathbb{R}^3)}.$$

In view of formula (12),

$$\Box_g^\omega u =^\omega b^{-1\,\omega} tr\chi.$$

The results of [43] contain in particular the following regularity for b and $tr\chi$

$$||tr\chi||_{L^\infty} + ||\nabla\!\!\!/ tr\chi||_{L^2_{x'}L^\infty_t} + ||b-1||_{L^\infty} + ||\nabla b||_{L^\infty_u L^2(\mathcal{H}_u)} \lesssim \epsilon, \tag{24}$$

where x' denotes coordinates on $\mathcal{H}_u\cap\Sigma_t$ and where we have dropped all ω subscripts since all the above estimates are uniform in ω.

One can view Theorem 6.1 as an L^2 bound for a Fourier integral operator with phase u and symbol $b^{-1}tr\chi$ satisfying only weak regularity, as we just recalled above.

6.1. The basic computation

Note that

$$\begin{aligned}
||Ef|| &\leq \int_{\mathbb{S}^2}\left|\left|b^{-1}tr\chi\int_{\lambda\in\mathbb{R}_+}e^{i\lambda u}f(\lambda\omega)\lambda^2 d\lambda\right|\right|_{L^2(\mathcal{M})}d\omega\\
&\leq \int_{\mathbb{S}^2}||b^{-1}tr\chi||_{L^\infty_u L^2(\mathcal{H}_u)}\left|\left|\int_{\lambda\in\mathbb{R}_+}e^{i\lambda u}f(\lambda\omega)\lambda^2 d\lambda\right|\right|_{L^2_u}d\omega\\
&\leq \epsilon||\lambda^2 f||_{L^2(\mathbb{R}^3)},
\end{aligned}$$

using Plancherel in λ, Cauchy-Schwarz in ω and the estimates on b and $tr\chi$ from (24). In the above computation, we failed to take advantage of any higher regularity we have on b and $tr\chi$ and the result is that we miss the conclusion of Theorem 6.1 by a power of λ. Theorem 6.1 would instead follow if we could exchange a power of λ for a derivative of $b^{-1}tr\chi$. The proof of Theorem 6.1 consists in three steps which roughly enable the trade mentioned above.

1. Using standard Littlewood-Paley decomposition for λ, decompose the error

$$Ef = \sum_{j\geq -1} E_j f$$

where $E_j f = E(\psi_j f)$, with $\psi_j(\lambda) = \psi(2^{-j})$ for $j \geq 0$ and $\psi_{-1}(\lambda) = \phi(\lambda)$, with ψ and ϕ being 2 smooth compactly supported functions with values in $[0,1]$ such that $1 = \sum_{j\geq 0}\psi_j + \phi$. This is a classical technique and the aim of this step is to prove

Proposition 6.2 (Almost orthogonality in frequency amplitude)
We have

$$||Ef||^2_{L^2(\mathcal{M})} \lesssim \sum_{j\geq -1}||E_j f||^2_{L^2(\mathcal{M})} + \epsilon^2||f||^2_{L^2(\mathbb{R}^3)}.$$

The first step enables to consider only $E_j f$ instead of Ef in Theorem 6.1. Repeating the basic computation would again lead to an extra power of λ, except that now $\lambda \sim 2^j$.

2. For each j, let $(\eta_j^\nu)_{\nu\in\Gamma}$ be a smooth partition of unity of $\mathbb{S}^2$:

$$1 = \sum_{\nu\in\Gamma}\eta_j^\nu(\omega), \quad \forall \omega\in\mathbb{S}^2,$$

such that Γ is a lattice on $\mathbb{S}^2$ of size 2^{-j} and η_j^ν has support of size 2^{-j}. We then decompose $E_j f$ as $E_j f = \sum_{\nu\in\Gamma}E_j^\nu f$, with $E_j^\nu f = E(\psi_j\eta_j^\nu f)$. The aim of the second step is to prove

Proposition 6.3 (Almost orthogonality in frequency angle)
We have

$$||E_j f||^2_{L^2(\mathcal{M})} \lesssim \sum_{\nu\in\Gamma} ||E_j^\nu f||^2_{L^2(\mathcal{M})} + \epsilon^2 ||\psi_j f||^2_{L^2(\mathbb{R}^3)}.$$

Step 2 allows us to consider only $E_j^\nu f$ instead of Ef or $E_j f$ in Theorem 6.1. Repeating the basic computation would lead to an extra $2^{j/2}$, i.e. there is a gain of $2^{j/2}$ compared to the previous computation, which comes from the fact that we are now exploiting some oscillations in ω.

3. The aim of the third step is to improve the estimate on $E_j^\nu f$ compared to the basic computation, so as to obtain

Proposition 6.4. — *The $E_j^\nu f$ satisfy*

$$||E_j^\nu f||_{L^2(\mathcal{M})} \lesssim \epsilon ||\eta_j^\nu \psi_j f||_{L^2(\mathbb{R}^3)}.$$

Propositions 6.2, 6.4 and 6.3 immediately lead to Theorem 6.1

$$\begin{aligned}
||Ef||^2_{L^2(\mathcal{M})} &\lesssim \sum_{j\geq -1} ||E_j f||^2_{L^2(\mathcal{M})} + \epsilon^2 ||f||^2_{L^2(\mathcal{M})},\\
&\lesssim \sum_{j\geq -1}\sum_{\nu\in\Gamma} ||E_j^\nu f||^2_{L^2(\mathcal{M})} + \epsilon^2 \sum_{j\geq -1} ||\psi_j f||^2_{L^2(\mathbb{R}^3)} + \epsilon^2 ||f||^2_{L^2(\mathbb{R}^3)},\\
&\lesssim \sum_{j\geq -1}\sum_{\nu\in\Gamma} \epsilon^2 ||\eta_j^\nu \psi_j f||^2_{L^2(\mathcal{M})} + \epsilon^2 \sum_{j\geq -1} ||\psi_j f||^2_{L^2(\mathbb{R}^3)} + \epsilon^2 ||f||^2_{L^2(\mathbb{R}^3)},\\
&\lesssim \epsilon^2 ||f||^2_{L^2(\mathbb{R}^3)}.
\end{aligned}$$

We refer to [44] for the detailed proof of Propositions 6.2, 6.4 and 6.3. The main ingredients are

- Geometric integration by parts: they are three different derivatives to consider corresponding to three different directions: the direction of the gradient of u, denoted L, the other two directions tangent to $\mathcal{H}_u$ denoted $\nabla\!\!\!\!/$ and the direction transveral to $\mathcal{H}_u$, denoted $\underline{L}$. The crucial observation is that the strongest control we have is typically on L derivatives, while we have less control on $\nabla\!\!\!\!/$ derivatives and even worse estimates for $\underline{L}$ derivatives. Thus, as much as possible, we want to perform integration by parts in L, then, when needed, one uses integration by parts in the other tangential directions and only as a last resort should one consider integration by parts in the $\underline{L}$ direction.
- To prove Proposition 6.4, one can exploit the regularity in ω of b and $tr\chi$ to freeze the ω dependence in these terms. This allows to reducing the proof of Proposition 6.4 to energy estimates for the wave equation.
- The hardest proof is that of Proposition 6.3. After using geometric integration by parts in tangential and L directions, the basic estimate leads to a log-loss, i.e. the

estimate barely fails. To overcome this, the strategy of [44] relies on a Littlewood-Paley decomposition of $tr\chi$

$$tr\chi = P_{\leq j}(tr\chi) + P_{>j}(tr\chi),$$

where $P_{\leq j}$ are geometric Littlewood-Paley (almost) projections as introduced in [23]. This allows to separate the high frequencies from the low ones so as to force, in the worse terms causing the log-loss obtained after geometric integration by parts, the tangential derivatives to fall only on the low frequencies.

7. CONTROL OF SPACETIME NULL FOLIATIONS

Recall that in order to prove Theorem 6.1, one needs precise control of the null foliation generated by the level sets of ${}^{\omega}u$ for each $\omega \in \mathbb{S}^2$ (see for instance the estimates (24)). Moreover, one also needs to investigate the regularity of these foliations with respect to ω.

These issues were settled in [43]. To present the results proven there, let us recall that $({}^{\omega}u)_{\omega\in\mathbb{S}^2}$ is a family of solutions of the eikonal equations $g(Du, Du) = 0$, indexed by $\omega \in \mathbb{S}^2$, with data on Σ_0 as constructed in [41]. In this section, we will often drop the ω index in the notation. In order to decompose and estimate geometric tensors, such as χ and the curvature tensor, we introduce the *null frame* (e_1, e_2, e_3, e_4) where $e_1 = T + N = bL'$, $e_2 = T - N$, and e_3, e_4 are arbitrary orthonormal vector fields tangent to the planes $P_{t,u} = \Sigma_t \cap \mathcal{H}_u$. Here $L' = -Du$, $b^{-1} = T(u)$ and N is a unit normal to the planes $P_{t,u} = \Sigma_t \cap \mathcal{H}_u$, tangent to Σ_t. The geometric information at the level of the first derivatives of the metric is contained in the following tensors

Definition 7.1 (Ricci coefficients). — *Given a null frame (e_1, e_2, e_3, e_4), the Ricci coefficients associated to it are the $P_{t,u}$ tangent tensors*

$$\chi_{AB} = \langle D_A e_4, e_B\rangle, \qquad \underline{\chi}_{AB} = \langle D_A e_3, e_B\rangle,$$
$$\zeta_A = \frac{1}{2}\langle D_3 e_4, e_A\rangle, \qquad \underline{\zeta}_A = \frac{1}{2}\langle D_4 e_3, e_A\rangle$$
$$\underline{\xi}_A = \frac{1}{2}\langle D_3 e_3, e_A\rangle.$$

The 2 tensors χ and $\underline{\chi}$ can be decomposed into their trace and traceless parts on the 2 surfaces $P_{t,u}$

$$tr\chi = g^{AB}\chi_{AB}, \qquad tr\underline{\chi} = g^{AB}\underline{\chi}_{AB},$$
$$\widehat{\chi} = \chi - \frac{1}{2}tr\chi g, \qquad \widehat{\underline{\chi}} = \underline{\chi} - \frac{1}{2}tr\underline{\chi} g.$$

The geometric information at the level of two derivatives of the metric is contained in the curvature tensor of g which can also be decomposed using the null frame as follows.

Definition 7.2. — *The null components of the curvature tensor R of the space-time metric g are defined as*

$$\alpha_{AB} = R(L, e_A, L, e_B), \qquad \beta_A = \frac{1}{2}R(e_A, L, \underline{L}, L),$$
$$\rho = \frac{1}{4}R(\underline{L}, L, \underline{L}, L), \qquad \sigma = \frac{1}{4}{}^\star R(\underline{L}, L, \underline{L}, L)$$
$$\underline{\beta}_A = \frac{1}{2}R(e_A, \underline{L}, \underline{L}, L), \qquad \underline{\alpha}_{AB} = R(\underline{L}, e_A, \underline{L}, e_B),$$

where ${}^\star R$ denotes the Hodge dual of the Riemann tensor.

Recall from Section 4.1, the bootstrap assumptions

$$||\mathrm{Riem}(g).L||_{L^\infty_u L^2(\mathcal{H}_u)} \leq M\epsilon,$$

where $\mathrm{Riem}(g).L$ denotes any component of $\mathrm{Riem}(g)$ with at least one index contracted with L. In view of the definition, this means all curvature components but the tensor $\underline{\alpha}$. Using the above null decomposition of curvature and connection, the main equations can be divided in two parts

1. The Null structure equations: these are equations relating the first derivatives of the Ricci coefficients to the curvature components.
2. The Bianchi idendities: for any Lorentzian manifold, the curvature tensor satisfies the identies
$$D_\gamma R_{\mu\nu\alpha\beta} + D_\nu R_{\gamma\mu\alpha\beta} + D_\mu R_{\nu\gamma\alpha\beta} = 0,$$
and a similar idendity holds for ${}^\star R$ if the Lorentzian manifold is a solution to the Einstein vacuum equations. These equations evaluated in a null frame give a set of equations at the level of the first derivatives of the null components of the curvature introduced above.

The decomposition of the connection and of the curvature tensor using orthonormal or null frames as well as the study of the solutions of the Einstein equations via the Bianchi equations for R and ${}^\star R$ has played a very important role in the study of global problems in relativity, starting with the monumental work of Christodoulou-Klainerman [8] on the stability of Minkowski space (see also [18] and [7]). In particular, two tools from [8] are useful here

– Commutation formulae: for instance, given U a tensor tangent to the $P_{t,u}$ surfaces, these formulae express the commutators $[\nabla\!\!\!/_A, \nabla\!\!\!/_3]U$, $[\nabla\!\!\!/_A, \nabla\!\!\!/_4]U$,$[\nabla\!\!\!/_4, \nabla\!\!\!/_3]U$, where $\nabla\!\!\!/$ denote the projection to $P_{t,u}$ of the covariant derivative, in terms of the curvature and Ricci coefficients.
– Decomposition of the main equations as a system of transport equations along null directions coupled to elliptic equations. For instance, the trace of χ satisfies the transport equation (called the Raychaudhuri equation)

$$L(tr\chi) = -\frac{1}{2}(tr\chi)^2 - |\widehat{\chi}|^2 - \bar{\delta}tr\chi, \tag{25}$$

where $\bar{\delta} = k_{NN} - n^{-1}N(n)$, with $n^{-2} = -g(Dt, Dt)$.

Typical elliptic equations take the form of Hodge systems giving the divergence and the curl of Ricci coefficients in terms of curvature coefficients. As an example, the second fundamental form satisfies

$$\begin{aligned} \operatorname{curl} k_{ij} &= {}^\star R_{\mu i \nu j} T^\mu T^\nu, \\ \operatorname{div} k &= 0. \end{aligned}$$

To state the main results proved in [43], let us introduce the following norms, for any $P_{t,u}$ tangent tensor F

$$\mathcal{N}_1(F) = ||F||_{L^2(\mathcal{H}_u)} + ||\nabla\!\!\!/ F||_{L^2(\mathcal{H}_u)} + ||\nabla\!\!\!/_L F||_{L^2(\mathcal{H}_u)},$$

as well as the trace norm

$$||F||_{L^\infty_{x'} L^2_t} = \sup_{x' \in \mathbb{R}^2} \left(\int_0^1 |F(t, x')|^2 dt \right).$$

Here (t, x') is a coordinate system on $\mathcal{H}$ where t is the usual time function and x' are constant along the null geodesic generated by L. Thus, the above norm corresponds to the supremum of the L^2 norm of the traces of F on the null geodesics generated by L over all such null geodesics. Finally, we will also need the norm

$$||F||_{L^2_{x'} L^\infty_t} = || \sup_{0 \leq t \leq 1} |F(t, x')| \, ||_{L^2(\mathbb{R}^2)}.$$

The following theorem is proven in [43].

Theorem 7.3. — *Let u be a solution to the eikonal equation $g(Du, Du) = 0$ with initial data as in* [41] *and assume that the curvature tensor of the spacetime satisfies the L^2 bounds*

$$||R||_{L^\infty_t L^2(\Sigma_t)} \leq \epsilon, \quad ||R.L||_{L^\infty_u L^2(\mathcal{H}_u)} \leq \epsilon.$$

Then,

1. *Null geodesics of L do not have conjugate points and distinct null geodesics do not intersect.*
2. *the following estimates are satisfied*
   – *Regularity of the lapse*

$$||n - 1||_{L^\infty} + ||\nabla n||_{L^\infty_t L^2_{x'}} + ||\nabla^2 n||_{L^\infty_t L^2_{x'}} + ||\nabla D_T n||_{L^\infty_t L^2_{x'}} \lesssim \epsilon,$$

   – *Regularity of the first fundamental form*

$$\mathcal{N}_1(k) + ||\nabla\!\!\!/_{\underline{L}} k_{AN}||_{L^\infty_u L^2(\mathcal{H}_u)} + ||\underline{L}(k_{NN})||_{L^\infty_u L^2(\mathcal{H}_u)} \lesssim \epsilon,$$

   – *Regularity of the null lapse*

$$||b - 1||_{L^\infty} + \mathcal{N}_1(b) + ||\nabla\!\!\!/^2 b||_{L^\infty_u L^2(\mathcal{H}_u)} + ||\nabla\!\!\!/ \nabla\!\!\!/_L b||_{L^\infty_u L^2(\mathcal{H}_u)} + ||\underline{L}(b)||_{L^2_{x'} L^\infty_t} \lesssim \epsilon,$$

– *Regularity of the null fundamental form*

$$\begin{aligned}||tr\chi||_{L^\infty} + ||\nabla\!\!\!/ tr\chi||_{L^2_{x'},L^\infty_t} + ||\underline{L}tr\chi||_{L^2_{x'},L^\infty_t} &\lesssim \epsilon,\\ ||\widehat{\chi}||_{L^2_{x'},L^\infty_t} + \mathcal{N}_1(\widehat{\chi}) + ||\nabla\!\!\!/_{\underline{L}}\widehat{\chi}||_{L^\infty_u L^2(\mathcal{H}_u)} &\lesssim \epsilon,\end{aligned}$$

– *Regularity of the twist form*

$$||\zeta||_{L^2_{x'},L^\infty_t} + \mathcal{N}_1(\zeta) \lesssim \epsilon.$$

In the above theorem, note that, as is usual for these types of wave equations, there are typically less control on transversal derivatives (i.e. in the direction of $\underline{L}$) than on derivatives tangential to the $\mathcal{H}_u$ (i.e. in the directions of L and $\nabla\!\!\!/$). Nonetheless, in [43], regularity estimates of $\underline{L}\underline{L}tr\chi$ and $\nabla\!\!\!/_{\underline{L}}(\zeta)$ (or rather on their Littlewood-Paley decompositions) are also obtained.

A second set of results obtained in [43] concerns the dependence of the foliation upon the ω parameter. We refer to [43] for the whole list of estimates obtained and only quote here the following excerpt.

THEOREM 7.4. — *Under the assumptions of the previous theorem, with N denoting a unit normal to the $P_{t,u}$ surfaces tangent to Σ_t, we have the estimates*

$$\begin{aligned}||\partial_\omega N||_{L^\infty} &\lesssim 1,\\ ||D\partial_\omega N||_{L^2_{x'},L^\infty_t} + ||\partial_\omega b||_{L^\infty} + ||\nabla\!\!\!/\partial_\omega b||_{L^2_{x'},L^\infty_t} + ||\partial_\omega\chi||_{L^2_{x'},L^\infty_t} + ||\partial_\omega\zeta||_{L^2_{x'},L^\infty_t} &\lesssim \epsilon.\end{aligned}$$

Furthermore, the tensor $\widehat{\chi}$ can be decomposed(18) *as*

$$\widehat{\chi} = \chi_1 + \chi_2$$

where χ_1 and χ_2 are two symmetric traceless 2 tensors tangent to $P_{t,u}$ satisfying

$$\sum_{i=1,2}\mathcal{N}_1(\chi_i) + ||\nabla\!\!\!/_{\underline{L}}\chi_i||_{L^\infty_u L^2(\mathcal{H}_u)} + ||\partial_\omega\chi_i||_{L^\infty_t L^2_{x'}} \lesssim \epsilon,$$

as well as, for any $2 \le p < +\infty$,

$$||\chi_1||_{L^p_t L^\infty_{x'}} + ||\partial_\omega\chi_2||_{L^p_t L^{4-}_{x'}} + ||\partial_\omega\chi_2||_{L^\infty_u L^{6-}(\mathcal{H}_u)} + ||\nabla\!\!\!/\partial_\omega\chi_2||_{L^\infty_u L^2(\mathcal{H}_u)} \lesssim \epsilon.$$

We refer to [43] for the proofs of the above theorem. Let us mention that the estimates of Theorem 7.3 concerning the regularity in (t,x) for χ and ζ had been obtained with respect to a geodesic foliation instead of a time foliation in [24, 21, 23]. The strategy in [43] is to use the estimates in geodesic foliation to derive the estimates in the time foliation. Two estimates can actually be directly translated from one foliation to the other, the estimates in L^∞ as well as the estimates in the trace norm. For the trace norm, recall that the (t,x') coordinate system is constructed by keeping the x' fixed along the geodesics generated by L. Thus, the only difference with the trace norm in the geodesic foliation comes from a change of parametrization of the geodesics, which is controled by the L^∞ bounds on n and b.

18. The point of this decomposition is that χ_1 has stronger regularity than $\widehat{\chi}$ in (t,x) while χ_2 has stronger regularity in ω.

8. THE PARAMETRIX AT INITIAL TIME

Recall that for given initial data (ϕ_0, ϕ_1) for the wave equation $\Box_g \psi = 0$, the parametrix aims at giving an approximate representation of ψ as an integral of type

$$\psi[f_-, f_+] = \int_{\mathbb{S}^2} \int_0^{+\infty} e^{i\lambda^\omega u_+} f_+(\lambda\omega)\lambda^2 d\lambda d\omega + \int_{\mathbb{S}^2} \int_0^{+\infty} e^{i\lambda^\omega u_-} f_-(\lambda\omega)\lambda^2 d\lambda d\omega.$$

where (f_-, f_+) are constructed from the data (ϕ_0, ϕ_1) and where u_-, u_+ are solutions to the eikonal equation. In the previous sections, we (briefly) explained how to estimate the error $\Box_g \psi[f_-, f_+]$, how to use the parametrix to derive bilinear estimates and how to propagate regularity of u_- and u_+. Therefore, we still need to

1. explain how to construct (f_-, f_+) from (ϕ_0, ϕ_1) and how to obtain the estimates (21). This will be done in Section 8.2.
2. explain how to construct initial data for the phases u_-, u_+. This will be done in Sections 8.1 and 8.3.

8.1. Construction of initial data for u_+ and u_-

Consider a scalar function u defined on $\Sigma \times \mathbb{S}^2$ where Σ denotes an initial spacelike hypersurface in $\mathcal{M}$ and assume that u has no critical point so that $N = \frac{\nabla u}{|\nabla u|}$ is well-defined. N is a unit normal to the surfaces P_u of constant u in Σ. As before, we can define a second fundamental form

$$\theta(e_A, e_B) = h(\nabla_{e_A} N, e_B),$$

where h denotes the induced metric on Σ and e_A, e_B are arbitrary vector fields tangent to P_u.

Recall that in the previous section, we obtained $tr\chi \in L^\infty(\mathcal{M})$, using transport equations along L. For this to hold, one needs the restriction of $tr\chi$ on the initial slice to lie in L^∞. Translated in terms of the data, using the maximal foliation condition $tr_g k = 0$, one needs

$$k_{NN} + tr\theta \in L^\infty(\Sigma),$$

or

$$-k_{NN} + tr\theta \in L^\infty(\Sigma).$$

In Section 8.3, we will present the results of [41], where the construction of a function u such that

$$-k_{NN} + tr\theta \in L^\infty_\omega L^\infty(\Sigma)$$

is obtained (the other one being completely analogous). The phases $u_\pm$ are then defined as

$$u_+(0, x, \omega) = u(x, \omega), \quad u_-(0, x, \omega) = -u(x, -\omega).$$

8.2. The construction of f_+ and f_-

Let

$$M_\pm f = \int_{\mathbb{S}^2}\int_{\lambda\in\mathbb{R}_+} e^{i\lambda u_\pm} f(\lambda\omega)\lambda^2 d\lambda d\omega$$

and

$$Q_\pm f = \int_{\mathbb{S}^2}\int_{\lambda\in\mathbb{R}_+} e^{i\lambda u_\pm}|\nabla u_\pm| f(\lambda\omega)\lambda^2 d\lambda d\omega.$$

Using that $T(u_\pm) = \mp|\nabla(u_\pm)|$, the conditions (20) can then be rewritten as

$$M_+ f_+ + M_- f_- = \phi_0, \tag{26}$$

$$Q_+(\lambda f_+) + Q_-(\lambda f_-) = i\phi_1. \tag{27}$$

Recalling the presentation of the parametrix at the beginning of Section 5, given data (ϕ_0, ϕ_1), we need to prove the existence and uniqueness [19] of (f_+, f_-) solving the above system satisfying the estimate

$$||\lambda f_+||_{L^2(\mathbb{R}^3)} + ||\lambda f_-||_{L^2(\mathbb{R}^3)} \lesssim ||\nabla\phi_0||_{L^2(\Sigma)} + ||\phi_1||_{L^2(\Sigma)}. \tag{28}$$

Note that in the flat case, one can take $(\Sigma, h, k) = (\mathbb{R}^3, \delta, 0)$, $u_\pm(t, x, \omega) = \mp t + x.\omega$, and $|\nabla u_\pm| = 1$. In particular, the operators $M_\pm$, $Q_\pm$ all coincide with the inverse Fourier transform $\mathcal{F}$ and $f_\pm$ are given by

$$f_\pm = \frac{1}{2}\left(\mathcal{F}\phi_0 \pm i\lambda^{-1}\mathcal{F}\phi_1\right).$$

The following theorems are proven in [42]

THEOREM 8.1. — *Let u be a phase function defined on $\Sigma\times\mathbb{S}^2$ as in* [41]. *Let b be a symbol defined on $\Sigma\times\mathbb{S}^2$ satisfying*

$$||b||_{L^\infty} + ||\nabla b||_{L^\infty_u L^2(P_u)} + ||\nabla\!\!\!/\nabla b||_{L^2(\Sigma)} \lesssim D, ||\partial_\omega b||_{L^2(\Sigma)} + ||\nabla\partial_\omega b||_{L^2(\Sigma)} \lesssim D,$$

as well as the decomposition $\nabla_N b = b_1^j + b_2^j$, where

$$||b_1^j||_{L^2(\Sigma)} \lesssim 2^{-j/2}D, \quad ||b_2^j||_{L^\infty_u L^2(P_u)} \lesssim D$$

and

$$||\nabla_N b_2^j||_{L^2(\Sigma)} + ||b_2^j||_{L^2_u L^\infty(P_u)} \lesssim D.$$

Let U denote the following Fourier integral operator

$$Uf(x) = \int_{\mathbb{S}^2}\int_{\mathbb{R}^+} e^{i\lambda u(x,\omega)} b(x,\omega) f(\lambda\omega)\lambda^2 d\lambda d\omega.$$

Then U is bounded on L^2 with operator norm $\lesssim D$.

19. The uniqueness actually follows immediately from the estimate (28).

The above theorem will be applied to the operators $M_\pm$ and $Q_\pm$. The results of [41] concerning the function u constructed in [41] and the foliation generated by u of Σ will ensure that the regularity conditions needed above are satisfied. Under extra conditions on u and on the constant D, one can actually show that the inverse bound also holds

$$||f||_{L^2(\mathbb{R}^3)} \lesssim ||Uf||_{L^2(\Sigma)}.$$

This leads to

Theorem 8.2. — *Let u be a phase function as constructed in* [41] *and satisfying assumptions* 1 *to* 6 *of Section 2.2 of* [41]. *Then, for any initial data (ϕ_0, ϕ_1) with $(\nabla\phi_0, \phi_1) \in L^2(\Sigma) \times L^2(\Sigma)$, there exists a unique solution (f_+, f_-) of the system* (26)-(27) *which moreover satisfies the estimate* (28).

We refer to [42] for the detailed proofs of Theorems 8.1 and 8.2. Central to the analysis are dyadic decompositions similar to the ones already presented in Section 6. Recall that 3 different dyadic decompositions were used in Section 6. First, one introduced the first and second dyadic decomposition in the frequency amplitude λ and in the frequency angles ω. This was however insufficient (see again the log-loss in Section 6) and a third decomposition was needed, this time in physical space for the symbol $tr\chi$, which led to the removal of the log-loss in conjunction with geometric integration by parts. For the proofs of Theorems 8.1 and 8.2, one again starts with dyadic decompositions in λ and ω. As in the previous case, log-losses appear and a third dyadic decomposition is introduced in order to remove them. However, on a spacelike slice, one cannot integrate by parts in the L direction, so the previous technique cannot be applied. Instead, one decomposes again in λ as follows. Given two angles $\nu, \nu' \in \mathbb{S}^2$ and $j \geq -1$, the frequency amplitude interval $[2^{j-1}, 2^{j+1}]$ is decomposed as

$$[2^{j-1}, 2^{j+1}] = \bigcup_{1 \leq k \leq |\nu-\nu'|^{-\alpha}} I_k,$$

where $\mathrm{diam}(I_k) \sim 2^j|\nu-\nu'|^\alpha$. This decomposition is used in conjunction with geometric integration by parts to remove the log-losses.

Another key element used in the proof of the above theorems is the existence of a global coordinate transformation tied to the phase function u. More precisely, for $\omega \in \mathbb{S}^2$, consider the following map

$$\begin{array}{rcl} \Sigma & \to & \mathbb{R} \times \left(T_\omega\mathbb{S}^2\right)^\star, \\ x & \to & \left(u(x,\omega), d_\omega u(x,\omega)\right), \end{array}$$

where $d_\omega u(x,\omega)$ denote the differential in ω of u at fixed x. Then, Σ is a C^1 diffeomorphism. Since $(T_\omega\mathbb{S}^2)^\star$ is diffeomorphic to $\mathbb{R}^2$, we can use any global coordinate system on $(T_\omega\mathbb{S}^2)^\star$ to obtain a C^1 diffeomorphism between Σ and $\mathbb{R}^3$, i.e. a global coordinate system on Σ.

8.3. Construction of the phases on the initial hypersurface

In Section 7, we presented several results concerning the propagation of regularity and control of the foliation generated by the level sets of a phase function u. For any of these results to hold, proper initial data for u must be constructed on the initial hypersurface, which we will denote by Σ in this section. This problem was solved in [41]. The aim of this section is to present the main results of [41] and to give some ideas concerning their proofs. We start by presenting the setting of [41].

Given a scalar function u defined on $\Sigma \times \mathbb{S}^2$, we shall denote as in Section 8.1, by θ the second fundamental form associated to a unit normal N of the surfaces P_u of constant u. Moreover, let a be the scalar defined by $a^{-1} = |\nabla u|$. Recall also that h denotes the induced Riemannian metric on Σ and k denotes the second fundamental form of Σ.

Using the definition of the Ricci tensor, one can derive the following equation for a

$$a^{-1}\not{\Delta}(a) = -\nabla_N tr\theta - |\theta|^2 + \mathrm{Ric}(h)_{NN}. \tag{29}$$

We wish to construct a phase function u defined on $\Sigma \times \mathbb{S}^2$ such that

– $u(x, \omega) \sim x.\omega$ as $|x| \to +\infty$.
– u has enough regularity in (x, ω) for the results of the previous sections to be applicable.
– In particular, $tr\theta - k_{NN} \in L^\infty$ and u has enough regularity in (x, ω) to prove Theorems 8.1 and 8.2.

To satisfy the first condition, the strategy of [41] is to modify the data (Σ, h, k) outside of some open set U and glue it to trivial data $(\mathbb{R}^3, \delta, 0)$. In the gluing area, the modified data is no longer a solution to the constraint equations (2) but it will be sufficient to satisfy the constraint only within U, since outside of U, one can exploit the smoothness of the data.

From the assumptions on the data, we have $R_{NN} \in L^2(\Sigma)$, so we may expect from (29) to control 2 derivatives of a in $L^2(\Sigma)$. Since a is at the level of the first derivatives of u, we thus expect naively to control only three derivatives of u in x in $L^2(\Sigma)$. The classical method to prove Theorems 8.1 and 8.2 is to use the $TT^\star$ argument coupled to several integration by parts in x. In dimention 3, this argument would need at least one more derivative of u than the above approach would give. Alternatively, one could try to use the $TT^\star$ argument coupled to several integration by parts in ω. Indeed, one can differentiate (29) using that R is independent of ω, to get an equation on $\partial_\omega a$ of the form

$$a^{-1}\not{\Delta}\partial_\omega a = 2\not{\nabla}\nabla_N a + ... \tag{30}$$

where the term on the right-hand side comes from the commutator $[\partial_\omega, \not{\nabla}]$. Thus, we need an estimate on $\nabla_N a$ to control $\partial_\omega a$. Now from (29), we expect to control only tangential derivatives of a, unless we can make the right-hand side play in our favor. Thus, we want to choose u (which amounts to fixing $tr\theta$) so that the term $\nabla_N tr\theta$ in the right-hand side of (29) would help us to control $\nabla_N a$ and so that $tr\theta - k_{NN} \in L^\infty(\Sigma)$. A first try would be to impose $tr\theta - k_{NN} = 0$ but together with (29), this does not provide control of $\nabla_N a$. A second guess would be to impose $tr\theta - k_{NN} = \nabla_N a$. Together with (29), this leads to an elliptic equation of the form $\nabla_N^2 a + a^{-1}\not\Delta a = \ldots.$ This allows to give the strong control $\nabla_N^2 a$ in $L^2(\Sigma)$ but implies that $\nabla_N a$ is in $H^1(\Sigma)$ only, and thus if $tr\theta - k_{NN} = \nabla_N a$, $tr\theta - k_{NN}$ does not a priori belong to L^∞. Instead, the condition [20]

$$tr\theta - k_{NN} = 1 - a$$

is imposed, which together with (29) leads to the parabolic equation

$$\nabla_N a - a^{-1}\not\Delta a = |\theta|^2 + \nabla_N k_{NN} + R_{NN}.$$

This then allows to control $\nabla_N a$ and thus, from (30), to gain control of $\partial_\omega a$. This leads to the following theorem (see [41])

THEOREM 8.3. — *Let (Σ, h, k) be an initial data set for the vacuum Einstein equations, such that $||\mathrm{Ric}(h)||_{L^2(\Sigma)} + ||\nabla k||_{L^2(\Sigma)} \leq \epsilon$. Then, there exists a phase function u defined on $\Sigma \times \mathbb{S}^2$ such that in x, we have the regularity*

$$\begin{aligned}
||a-1||_{L^\infty_u L^2(P_u)} + ||\nabla a||_{L^\infty_u L^2(P_u)} + ||a-1||_{L^\infty} + ||\not\nabla \nabla a||_{L^2(\Sigma)} &\lesssim \epsilon,\\
||tr\theta - k_{NN}||_{L^\infty(\Sigma)} + ||\nabla\theta||_{L^2(\Sigma)} &\lesssim \epsilon,\\
||\nabla_N a||_{L^\infty_u L^4(P_u)} + ||\nabla_N^2 a||_{L^2_u H^{-1/2}(P_u)} &\lesssim \epsilon,
\end{aligned}$$

while in ω, we have

$$\begin{aligned}
||\partial_\omega a||_{L^\infty(\Sigma)} + ||\not\nabla\partial_\omega a||_{L^\infty_u L^2(P_u)} + ||\not\nabla\partial_\omega a||_{L^2(\Sigma)} + ||\nabla_N\partial_\omega a||_{L^2_u H^{1/2}(P_u)} &\\
+ ||\nabla_N^2\partial_\omega a||_{L^2_u H^{-3/2}(P_u)} + ||\nabla\partial_\omega a||_{L^2(\Sigma)} &\lesssim \epsilon,\\
||\partial_\omega N||_{L^\infty(\Sigma)} &\lesssim 1,\\
||\partial_\omega^2 a||_{L^2_u H^{3/2}(P_u)} + ||\partial_\omega^2 a||_{L^\infty_u H^{1/2}(P_u)} + ||\nabla_N\partial_\omega^2 a||_{L^2_u H^{-1/2}(P_u)} + ||\nabla\partial_\omega^2\theta||_{L^2(\Sigma)} &\lesssim \epsilon,\\
||\partial_\omega^2 N||_{L^\infty(\Sigma)} + ||\partial_\omega^3 u||_{L^\infty_{loc}(\Sigma)} &\lesssim 1.
\end{aligned}$$

Moreover, in [41], further results are obtained, in particular estimates comparing u with a reference phase as well as estimates for the Littlewood-Paley decomposition of the normal derivative of a. Finally, using the above estimates for u, the existence and control of the global coordinate sytem mentioned at the end of Section 8.2 is also proven.

9. THE SHARP STRICHARTZ ESTIMATES

Recall that the proof of the second set of bilinear estimates in Section 4.1 can be reduced to a $L^4(\mathcal{M})$ Strichartz type estimate. More precisely, what is needed is a localized in frequency and localized in time version of Strichartz estimates. These Strichartz

20. Note that this equation is a sort of modified mean curvature flow.

estimates [21] have been obtained in [45]. There are several technical assumptions on the regularity of the phase, which we will ommit below. Let us simply mention that all these assumptions are compatible with the regularity of the phase functions presented in Section 7.

THEOREM 9.1. — *For $j \geq 0$, let ϕ_j denote the parametrix*

$$\phi_j = \int_{\mathbb{S}^2} \int_{\mathbb{R}_+} e^{i\lambda^\omega u} \psi(2^{-j}\lambda) f(\lambda\omega) \lambda^2 d\lambda d\omega,$$

where ψ is a smooth cut-off function supported in $[1/2, 2]$ and u is a phase function with regularity properties compatible with Section 7. Let (p, q) such that $p, q \geq 2$, $q < +\infty$ and $\frac{1}{p} + \frac{1}{q} \leq \frac{1}{2}$. Let $r = \frac{3}{2} - \frac{1}{p} - \frac{3}{q}$. Then, we have the Strichartz inequality [22]

$$||\phi_j||_{L^p[0,1] L^q(\Sigma_t)} \lesssim 2^{jr} ||\psi(2^{-j}\lambda) f||_{L^2(\mathbb{R}^3)}.$$

Via a classical TT^* argument, the above estimate can be reduced to the pointwise estimate

$$|K(t, x, s, y)| \lesssim \frac{1}{|t - s|}, \quad \forall (t, x, s, y) \in 2^j \mathcal{M}^2, \tag{31}$$

where K is the kernel

$$K(t, x, s, y) = \int_{\mathbb{S}^2} \int_{\mathbb{R}_+} e^{i\lambda 2^j \left(u(\frac{t}{2^j}, \frac{x}{2^j}, \omega) - u(\frac{s}{2^j}, \frac{y}{2^j}, \omega) \right)} \psi(\lambda)^2 \lambda^2 d\lambda d\omega.$$

Classicaly, one obtains these dispersive estimates by a stationnary phase argument. This relies on a Taylor expansion to second order of the phases, which would require $\partial_{t,x} \partial_\omega^2 u \in L^\infty$. However, the assumptions are such that we only have $\partial_{t,x} \partial_\omega u \in L^\infty$. To go over this issue, the strategy in [45], inspired by [36] and [37], is as follows.

First, integrating by parts in λ twice and using the compactness of the support of ψ, one obtains from the definition of K

$$|K(t, x, s, y)| \lesssim \int_{\mathbb{S}^2} \frac{1}{1 + 2^{2j} \left(u(\frac{t}{2^j}, \frac{x}{2^j}, \omega) - u(\frac{s}{2^j}, \frac{y}{2^j}, \omega) \right)^2} d\omega. \tag{32}$$

The aim is then to bound the difference of the phase functions from below to obtain (31) from (32). More precisely, the inequality

$$|u(t, x, \omega) - u(s, y, \omega)| \gtrsim |s - t| |\omega - \omega_0|^2,$$

for ω in the neighborhood of some $\omega_0 \in \mathbb{S}^2$ is obtained in [45]. The key point is that the above inequality can be obtained without the strong regularity assumption $\partial_{t,x} \partial_\omega^2 u \in L^\infty$.

21. Only $L_t^4 L_x^4$ Strichartz estimates are actually needed in the proof of the bounded L^2 curvature conjecture but since the proof of the Strichartz estimates for other exponents is not different, they are included in the statement of the theorem below.

22. Note that these estimates are the same than that of the flat space in dimension 3, which are known to be optimal.

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