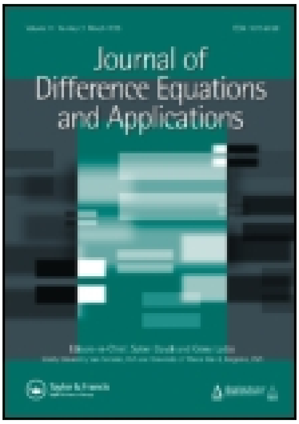

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Quadratic decomposition of symmetric semi-classical polynomial sequences of even class: an example from the case s = 2

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Quadratic decomposition of symmetric semi-classical polynomial sequences of even class: an example from the case $s = 2$

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In the present work, we deal with the quadratic decomposition of symmetric semi-classical polynomial sequences of even class. An example from class 2 is settled. We give an integral representation of the considered symmetric form.

Keywords: orthogonal polynomials; symmetric sequences; semi-classical sequences; quadratic decomposition

AMS(MOS) Subject Classification: 33C45; 42C05

1. Introduction

Recently, some authors have studied symmetric semi-classical polynomials of class 2 through the framework of quadratic decomposition [2,13]. But in those papers, there are no general results on symmetric semi-classical polynomials of class $s = 2$, the case $s = 1$ was carried out in Ref. [1]. It is the aim of this paper to give some general properties of symmetric semi-classical polynomials of even class. The case of odd class will be settled in a forthcoming paper.

2. Preliminary results

Let $\mathcal{P}$ be the vector space of polynomials with coefficients in $\mathbb{C}$ and $\mathcal{P}'$ its dual. We denote by $\langle v,f\rangle$ the effect of $v \in \mathcal{P}'$ on $f \in \mathcal{P}$. For $n \geq 0$, $(v)_n = \langle v, x^n\rangle$ are the moments of v.

In particular, a linear form is called symmetric if all of its moments of odd order are 0 [3]. We define in the space $\mathcal{P}'$ the derivative v' of the form v by $\langle v',f\rangle := -\langle v,f'\rangle$, the left multiplication by a polynomial hv by $\langle hv,f\rangle := \langle v, hf\rangle$, the shifted form $h_d v$ by $\langle h_d v,f\rangle := \langle v, h_d f\rangle = \langle v,f(\mathrm{d}x)\rangle$ and the inverse multiplication by a polynomial of degree 1 $(x-c)^{-1}v$, through [4]

$$\left\langle (x-c)^{-1}v,f\right\rangle := \langle v, \theta_c f\rangle \ \text{ with } \ \left(\theta_c f\right)(x) := \frac{f(x)-f(c)}{x-c}, \quad f \in \mathcal{P}.$$

We also denote $(f(\xi))(x) = f(x)$ for the dummy variable ξ.

Let us introduce the operator $\sigma : \mathcal{P} \to \mathcal{P}$ defined by $\sigma(f)(x) := f(x^2)$. By duality, the linear form $\sigma(v)$ is defined by $\langle \sigma(v),f\rangle = \langle v, \sigma(f)\rangle$. Thus, we have the following

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elementary properties of operator σ in the space $\mathcal{P}'$ that we will need in Section 3, see [1,12],

$$f(x)\sigma(v) = \sigma\big(f(x^2)v\big), \quad \sigma(v') = 2(\sigma(xv))', \quad v \in \mathcal{P}', \; f \in \mathcal{P}. \tag{2.1}$$

Let us recall that a form w is called regular if we can associate with it a monic polynomial sequence (MPS) $\{W_n\}_{n\geq 0}$ with $\deg W_n = n$ such that $\langle w, W_m W_n\rangle = r_n \delta_{n,m}$, $r_n \neq 0,\; n,m \geq 0$. Sequence $\{W_n\}_{n\geq 0}$ is said to be a monic orthogonal polynomial sequence (MOPS) with respect to the linear form w, which fulfils a second-order recurrence relation

$$\begin{aligned} &W_0(x) = 1, \quad W_1(x) = x - \beta_0, \\ &W_{n+2}(x) = (x - \beta_{n+1})W_{n+1}(x) - \gamma_{n+1} W_n(x), \quad \gamma_{n+1} \neq 0, \; n \geq 0. \end{aligned} \tag{2.2}$$

A MPS $\{W_n\}_{n\geq 0}$ is called symmetric if $W_n(-x) = (-1)^n W_n(x)$, $n \geq 0$, see [3,10]. Chihara [3] showed that if $\{W_n\}_{n\geq 0}$ is orthogonal with respect to the form w, then the following assertions are equivalent:

(a) $\{W_n\}_{n\geq 0}$ is symmetric,
(b) in the corresponding recurrence formula (2.2), $\beta_n = 0$, $\forall n \geq 0$ and
(c) w is symmetric.

Furthermore, the MPSs $\{P_n\}_{n\geq 0}$ and $\{R_n\}_{n\geq 0}$, given by the following quadratic decomposition

$$W_{2n}(x) = P_n(x^2), \quad W_{2n+1}(x) = xR_n(x^2), \quad n \geq 0, \tag{2.3}$$

are orthogonal, where $u = \sigma(w)$ and $v = \gamma_1^{-1} x\sigma(w)$ are their corresponding regular forms, respectively. For more details for the quadratic decomposition of MOPS, see [8].

3. Quadratic decomposition of symmetric semi-classical polynomial sequences of even class

Let us recall that a regular form w is called semi-classical if there exist two polynomials Φ monic and Ψ, with $\deg \Psi \geq 1$, such that w satisfies the following differential equation

$$(\Phi w)' + \Psi w = 0. \tag{3.1}$$

The pair (Φ, Ψ) is not unique. Equation (3.1) can be simplified if and only if there exists a root c of Φ such that $\Phi'(c) + \Psi(c) = 0$ and $\langle w, \theta_c\Psi + \theta_c^2\Phi\rangle = 0$. Then w fulfils the equation

$$((\theta_c\Phi)w)' + \big\{\theta_c\Psi + \theta_c^2\Phi\big\}w = 0.$$

We call the class of w, the minimum value of the integer $\max(\deg\Phi - 2, \deg\Psi - 1)$ for all pairs satisfying (3.1). The class of semi-classical form w is $s = \max(\deg\Phi - 2, \deg\Psi - 1)$, if and only if the following condition is satisfied [10]

$$\prod_{c\in\mathcal{Z}_\Phi} \big\{|\Phi'(c) + \Psi(c)| + \big|\big\langle w, \theta_c^2\Phi + \theta_c\Psi\big\rangle\big|\big\} \neq 0, \tag{3.2}$$

where $\mathcal{Z}_\Phi$ is the set of 0s of Φ.

When the form w is of class s, the corresponding MOPS is also called semi-classical of class s.

An important result, given in Ref. [1], characterizes the elements of the linear differential equation satisfied by any symmetric semi-classical form. We have the following proposition.

PROPOSITION 3.1. *Let w be a symmetric semi-classical form satisfying equation* (3.1).

If w is of even class, then Φ is even and Ψ is odd. If w is of odd class, then Φ is odd and Ψ is even.

In this section, we are concerned with the quadratic decomposition of the symmetric semi-classical (MOPS) $\{W_n\}_{n\geq 0}$ of class $2s$ with respect to the normalized form w ($(w)_0 = 1$) satisfying equation (3.1).

From Proposition 3.1, polynomials Φ is even and Ψ is odd. Let ϕ and ψ be two polynomials such that $\Phi(x) = \phi(x^2)$ and $\Psi(x) = x\psi(x^2)$.

3.1 Class and differential functional equation of the form u

Multiplying equation (3.1) by x, we obtain

$$(x\phi(x^2)w)' + \big(x^2\psi(x^2) - \phi(x^2)\big)w = 0. \tag{3.3}$$

Applying operator σ in (3.3) and from its properties given in (2.1), the form $u = \sigma(w)$ satisfies the following differential equation

$$\begin{aligned} &(\Phi_u u)' + \Psi_u u = 0, \\ &\Phi_u(x) = x\phi(x); \quad \Psi_u(x) = \frac{1}{2}\big(x\psi(x) - \phi(x)\big). \end{aligned} \tag{3.4}$$

THEOREM 3.2. *Let w be a symmetric semi-classical form of class 2s satisfying* (3.1). *Then the linear form $u = \sigma(w)$ is also semi-classical of class s satisfying* (3.4).

For the proof, we use the following result.

LEMMA 3.3. *Let c be a root of the polynomial ϕ and c_1 be a square root of c. We have*

$$\langle u, \theta_c^2\Phi_u + \theta_c\Psi_u\rangle = \frac{1}{2}\left\langle w, \theta_{c_1}^2\Phi + \theta_{c_1}\Psi\right\rangle, \tag{3.5}$$

$$\Phi_u'(c) + \Psi_u(c) = \frac{1}{2}c_1(\Phi'(c_1) + \Psi(c_1)). \tag{3.6}$$

Proof. We denote by $\phi_c = \theta_c\phi$. We have $\big(\theta_{c_1}\Phi\big)(x) = (x + c_1)\phi_c(x^2)$ and

$$\left(\theta_{c_1}^2\Phi\right)(x) = \phi_c(x^2) + 2c_1(x + c_1)\frac{\phi_c(x^2) - \phi_c\big(c_1^2\big)}{x^2 - c_1^2}.$$

This expression can be written in terms of operator σ, thus we have

$$\left(\theta_{c_1}^2\Phi\right)(x) = \big(\sigma(\phi_c)\big)(x) + 2(c_1x + c)\big(\sigma(\theta_c\phi_c)\big)(x). \tag{3.7}$$

Similarly, we verify that

$$\big(\theta_{c_1}\Psi\big)(x) = (\sigma(\psi))(x) + (c + c_1x)(\sigma(\theta_c\psi))(x). \tag{3.8}$$

Since w is symmetric and according to (3.7) and (3.8) we then obtain

$$\left\langle w, \theta_{c_1}^2 \Phi + \theta_{c_1} \Psi \right\rangle = \langle u, 2c(\theta_c \phi_c) + \phi_c + \psi + c(\theta_c \psi) \rangle. \tag{3.9}$$

On the other hand, we have

$$\left(\theta_c^2 \Phi_u\right)(x) = \left(\theta_c^2(\xi\phi(\xi))\right)(x) = \left(\theta_c(\phi + c\theta_c\phi)\right)(x) = \phi_c(x) + c(\theta_c\phi_c)(x),$$
$$(\theta_c \Psi_u)(x) = \frac{1}{2}\left(\theta_c(\xi\psi(\xi) - \phi(\xi))\right)(x) = \frac{1}{2}\left(\psi(x) + c(\theta_c\psi)(x) - \phi_c(x)\right).$$

Thus, we obtain

$$\left\langle u, \theta_c^2 \Phi_u + \theta_c \Psi_u \right\rangle = \frac{1}{2}\langle u, 2c(\theta_c \phi_c) + \phi_c + \psi + c(\theta_c \psi) \rangle. \tag{3.10}$$

Comparing (3.9) and (3.10), we find (3.5). Next, from (3.4), an easy calculation gives (3.6). □

Proof of Theorem 3.2. First, from (3.4), we have $\Phi_u'(0) + \Psi_u(0) = 2^{-1}\phi(0)$.

Now suppose $\Phi_u'(c) + \Psi_u(c) = 0$, then $c_1(\Phi'(c_1) + \Psi(c_1)) = 0$ from (3.6).

Since $\Phi'(0) + \Psi(0) = 0$, in any case $\Phi'(c_1) + \Psi(c_1) = 0$ which means $\langle w, \theta_{c_1}^2 \Phi + \theta_{c_1}\Psi \rangle \neq 0$ according to the assumption. Thus, from (3.5) of Lemma 3.3 $\langle u, \theta_c^2 \Phi_u + \theta_c \Psi_u \rangle \neq 0$, and so equation (3.4) is not simplified. Therefore, the class of u is $s_u = \max(\deg \Phi_u - 2, \deg \Psi_u - 1)$.

Furthermore, following the assumption we have $s = \max(\deg \phi - 1, \deg \psi)$.

When $\deg \psi \neq \deg \phi - 1$, we obtain $\deg \Psi_u - 1 = s$. On the contrary, $\deg \psi = \deg \phi - 1$ implies $\deg \Psi_u - 1 \leq s$. In any cases, it is easy to see that $s_u = s$. □

3.2 Class and differential functional equation of the form v

Let us now consider the form $v = \gamma_1^{-1} x u$. Thus, as was shown in Section 3.1, form v is also semi-classical. In fact, multiplying equation (3.4) by x, we get

$$(\Phi_v v)' + \Psi_v v = 0, \tag{3.11}$$

$$\Phi_v(x) = \Phi_u(x), \quad \Psi_v(x) = \Psi_u(x) - \phi(x), \tag{3.12}$$

where Φ_u and Ψ_u are the polynomials given in (3.4).

Theorem 3.4. *Let w be a linear form having the same meaning as in Theorem 3.2. Then the linear form $v = \gamma_1^{-1} x \sigma(w)$ is also semi-classical of class s_v satisfying*

$$(\Phi_v v)' + \Psi_v v = 0. \tag{3.13}$$

Moreover,

(i) *If $\phi(0) \neq 0$, then $s_v = s$, $\Phi_v(x) = x\phi(x)$ and $\Psi_v(x) = (1/2)\left(x\psi(x) - 3\phi(x)\right)$.*
(ii) *If $\phi(0) = 0$, then $s_v = s - 1$, $\Phi_v(x) = \phi(x)$ and $\Psi_v(x) = (1/2)\left(\psi(x) - (\theta_0\phi)(x)\right)$.*

For the proof, we need the following lemma.

LEMMA 3.5.

(i) *For all c root of the polynomial ϕ, we have the following*

$$\left\langle v, \theta_c^2\Phi_v + \theta_c\Psi_v\right\rangle = \frac{c}{\gamma_1}\left\langle u, \theta_c^2\Phi_u + \theta_c\Psi_u\right\rangle - \frac{1}{\gamma_1}\left(\Phi_u'(c) + \Psi_u(c)\right), \tag{3.14}$$

$$\Phi_v'(c) + \Psi_v(c) = \Phi_u'(c) + \Psi_u(c). \tag{3.15}$$

(ii) *The class of v depends only on the zero $x = 0$ of the polynomial ϕ.*

Proof. (i) According to (3.12), we have

$$\left\langle v, \theta_c^2\Phi_v + \theta_c\Psi_v\right\rangle = \frac{1}{\gamma_1}\left\langle u, x\left(\theta_c^2\Phi_u\right)(x) + x(\theta_c(\Psi_u - \phi))(x)\right\rangle. \tag{3.16}$$

Since $\Phi_u(x) = x\phi(x)$, we can write

$$(\theta_c\Phi_u)(x) = \left(\theta_c(\xi - c)\phi(\xi)\right)(x) + c(\theta_c\phi)(x) = \phi(x) + c(\theta_c\phi)(x).$$

So,

$$x\left(\theta_c^2\Phi_u\right)(x) = c\left(\theta_c^2\Phi_u\right)(x) + \phi(x) + c(\theta_c\phi)(x) - \Phi_u'(c). \tag{3.17}$$

Moreover, we have

$$x(\theta_c(\Psi_u - \phi))(x) = \Psi_u(x) - \phi(x) - \Psi_u(c) + c(\theta_c(\Psi_u - \phi))(x). \tag{3.18}$$

Adding (3.17) and (3.18), we obtain

$$x\left(\theta_c^2\Phi_u\right)(x) + x(\theta_c(\Psi_u - \phi))(x) = c\left(\theta_c^2\Phi_u + \theta_c\Psi_u\right)(x) - \left\{\Phi_u'(c) + \Psi_u(c)\right\} + \Psi_u(x).$$

Then (3.16) becomes

$$\left\langle v, \theta_c^2\Phi_v + \theta_c\Psi_v\right\rangle = \frac{c}{\gamma_1}\left\langle u, \theta_c^2\Phi_u + \theta_c\Psi_u\right\rangle - \frac{1}{\gamma_1}\left(\Phi_u'(c) + \Psi_u(c)\right) - \frac{1}{\gamma_1}\langle u, \Psi_u\rangle.$$

This yields (3.14), since $\langle u, \Psi_u\rangle = 0$, from (3.4). Next, it is easy to find (3.15) from (3.12).

(ii) Let c be a root of Φ_v, such that $c \neq 0$.

If $\Phi_v'(c) + \Psi_v(c) = 0$, it follows that $\langle v, \theta_c^2\Phi_v + \theta_c\Psi_v\rangle \neq 0$ from (3.14) and (3.15). Then, we cannot simplify equation (3.11) by $x - c$ for $c \neq 0$ and we get the desired result. □

Proof of Theorem 3.4. From (3.12), we have $\Phi_v'(0) + \Psi_v(0) = -(1/2)\ \phi(0)$.

(i) If $\phi(0) \neq 0$ and according to Lemma 3.5, we cannot simplify equation (3.11). Then the class of v is $s_v = \max(\deg\Phi_v - 2, \deg\Psi_v - 1)$.

Here if $\deg\phi \neq \deg\Psi_u$, we obtain $\deg\Psi_v = \max(\deg\Psi_u, \deg\Phi_u - 1)$. Then $\deg\Psi_v - 1 = s$, from Theorem 3.2.

If $\deg\phi = \deg\Psi_u$, we have $\deg\Psi_v \leq \deg\phi$. Then $\deg\Phi_v - 2 = \deg\phi - 1 \geq \deg\Psi_v - 1$. This gives that $s_v = s$.

(ii) If $\phi(0) = 0$. From (3.12), we write $\theta_0^2\Phi_v = \theta_0\phi$, $\theta_0\Psi_v = \theta_0\Psi_u - \theta_0\phi$.

Then

$$\left\langle v, \theta_0^2\Phi_v + \theta_0\Psi_v\right\rangle = \frac{1}{\gamma_1}(\langle u, \Psi_u\rangle - \Psi_u(0)) = 0,$$

since $\langle u, \Psi_u \rangle = 0$ and $\Psi_u(0) = -(1/2)\phi(0) = 0$, from (3.4). Therefore, equation (3.11) can be simplified by x and becomes $(\tilde{\Phi}_v v)' + \tilde{\Psi}_v v = 0$, with

$$\tilde{\Phi}_v(x) = \phi(x) \text{ and } \tilde{\Psi}_v(x) = \frac{1}{2}\big(\psi(x) - (\theta_0\phi)(x)\big). \tag{3.19}$$

Next, from (3.19) we write

$$\tilde{\Phi}'_v(0) + \tilde{\Psi}_v(0) = \frac{1}{2}\big(\phi'(0) + \psi(0)\big) \tag{3.20}$$

and

$$\big\langle v, \theta_0^2\tilde{\Phi}_v + \theta_0\tilde{\Psi}_v \big\rangle = \frac{1}{2\gamma_1}\big\{\langle u, \theta_0\phi + \psi\rangle - \big(\phi'(0) + \psi(0)\big)\big\}. \tag{3.21}$$

Assuming that $\phi'(0) + \psi(0) = 0$. Then

$$\big\langle v, \theta_0^2\tilde{\Phi}_v + \theta_0\tilde{\Psi}_v \big\rangle = \frac{1}{2\gamma_1}\langle u, \theta_0\phi + \psi\rangle, \tag{3.22}$$

according to (3.20) and (3.21). On the other hand, writing (3.10) with $c = 0$, we get

$$\big\langle u, \theta_0^2\Phi_u + \theta_0\Psi_u \big\rangle = \frac{1}{2}\langle u, \theta_0\phi + \psi\rangle. \tag{3.23}$$

Formulae (3.22) and (3.23) imply

$$\big\langle v, \theta_0^2\tilde{\Phi}_v + \theta_0\tilde{\Psi}_v \big\rangle = \frac{1}{\gamma_1}\big\langle u, \theta_0^2\Phi_u + \theta_0\Psi_u \big\rangle. \tag{3.24}$$

Because $(\Phi_u)'(0) + \Psi_u(0) = 0$, then necessarily $\langle u, \theta_0^2\Phi_u + \theta_0\Psi_u\rangle \neq 0$ from Theorem 3.2.

It follows from (3.24) that $\langle v, \theta_0^2\tilde{\Phi}_v + \theta_0\tilde{\Psi}_v\rangle \neq 0$ and so equation (3.11) is only simplified once by x. It is easy to see that $\deg\tilde{\Phi}_v = \deg\Phi_u - 1$ and $\deg\tilde{\Psi}_v = \deg\Psi_u - 1$. Hence, form v is of class $s - 1$ satisfying (3.19).

4. An example from the case $s = 2$

4.1 Canonical cases

In this part, we quote the canonical cases of the symmetric semi-classical form of class $s = 2$, when $\deg\Phi \leq 4$ and $1 \leq \deg\Psi \leq 3$. In accordance with the degree of the even polynomial Φ, we establish three situations: $\deg\Phi = 4$, $\deg\Phi = 2$ and $\Phi(x) = 1$. By a suitable shifting and considering the multiplicity order of 0 of Φ, we get the following canonical cases.

I. $\deg\Phi = 4$ and $\Psi(x) = x\big(b_1x^2 + b_2\big)$ with $|b_1| + |b_2| \neq 0$.
(I_a) $\Phi(x) = (x^2 - 1)(x^2 - a^2)$ with $a \neq 0$ and $a^2 \neq 1$,
(I_b) $\Phi(x) = x^2(x^2 - 1)$,
(I_c) $\Phi(x) = (x^2 - 1)^2$ and
(I_d) $\Phi(x) = x^4$.
II. $\deg\Phi = 2$ and $\Psi(x) = x\big(b_1x^2 + b_2\big)$ with $b_1 \neq 0$.
(II_a) $\Phi(x) = x^2 - 1$ and
(II_b) $\Phi(x) = x^2$.
III. $\Phi(x) = 1$ and $\Psi(x) = x\big(b_1x^2 + b_2\big)$ with $b_1 \neq 0$. Thus, we have produced seven canonical symmetric semi-classical forms of class $s = 2$.

It follows, from Theorems 3.2 and 3.4, that the linear form $u = \sigma(w)$ is of class $s = 1$ and the form $v = \gamma_1^{-1} x u$ is of class $s = 0$ in the cases $\mathrm{I_b}$, $\mathrm{I_d}$ and $\mathrm{II_b}$ when $\Phi(0) = 0$ and of class $s = 1$ in the other cases.

4.2 Study of the case II_b

In the sequel, we focus attention on case $\mathrm{II_b}$, just studied in other contexts, see [2,6] and also [13], where the authors use another method. Because w is of class $s = 2$, we must establish condition (3.2) with $\Phi(x) = x^2$ and $\Psi(x) = x(b_1 x^2 + b_2)$, $b_1 \neq 0$, this leads to

$$b_1 \gamma_1 + 1 + b_2 \neq 0, \quad \text{where } \gamma_1 = (w)_2. \tag{4.1}$$

From a suitable transformation with the linear operator $h_{\sqrt{2/b_1}}$, we can choose $b_1 = 2$, then while denoting $b_2 = -2\alpha - 1$, the functional equation fulfilled by w becomes

$$\begin{aligned} &(x^2 w)' + x(2x^2 - 2\alpha - 1)w = 0, \\ &(w)_0 = 1, (w)_1 = 0, (w)_2 = \gamma_1, \quad \text{with } \gamma_1 \neq \alpha. \end{aligned} \tag{4.2}$$

Following Theorems 3.2 and 3.4, the form $u = \sigma(w)$ is of class $s = 1$ fulfilling (3.4) with

$$\Phi_u(x) = x^2; \quad \Psi_u(x) = x(x - \alpha - 1). \tag{4.3}$$

Moreover, v is a classical form satisfying (3.11) with

$$\Phi_v(x) = x; \quad \Psi_v(x) = x - \alpha - 1. \tag{4.4}$$

We observe that $v = \mathcal{L}(\alpha)$ is the Laguerre form with parameter α, where $\alpha \neq -n$, $n \geq 1$. The elements of the second-order recurrence relation of its corresponding MOPS $\{R_n\}_{n \geq 0}$ are given by Refs [3,9,11]

$$\beta_n^R = 2n + \alpha + 1; \quad \gamma_{n+1}^R = (n+1)(n + \alpha + 1), \quad n \geq 0. \tag{4.5}$$

Next, we need to study the regularity of the linear form $u = \gamma_1 x^{-1} v + \delta_0$, where δ_0 is the Dirac form at origin defined by $\langle \delta_0, f \rangle = f(0)$.

The inverse problem of the product of a form by a polynomial of degree 1 has been treated in Ref. [7]. Let us recall the following proposition.

PROPOSITION 4.1. *Let v be a regular form. The linear form $u = \gamma_1 x^{-1} v + \delta_0$ is regular if and only if $\gamma_1 \neq \lambda_n$, $n \geq 0$, where*

$$\lambda_0 = 0, \quad \lambda_{n+1} = -\frac{R_{n+1}(0)}{R_n^{(1)}(0)}, \quad n \geq 0, \tag{4.6}$$

with $R_n^{(1)}(0) = \langle v, \theta_0 R_{n+1}(x) \rangle$.

Moreover,

$$\begin{aligned} &\beta_0^P = \beta_0^R - a_0; \quad \beta_{n+1}^P = \beta_{n+1}^R + a_n - a_{n+1}, \quad n \geq 0, \\ &\gamma_{n+1}^P = -a_n(a_n - \beta_n^R), \quad n \geq 0, \end{aligned} \tag{4.7}$$

where

$$a_n = -\frac{R_{n+1}(0) + \gamma_1 R_n^{(1)}(0)}{R_n(0) + \gamma_1 R_{n-1}^{(1)}(0)}, \quad n \geq 0; \quad \text{with } R_{-1}^{(1)}(0) = 0. \tag{4.8}$$

In particular, we have the following result.

PROPOSITION 4.2. *We consider v the Laguerre form with parameter α, where $\alpha \neq -n$, $n \geq 1$. The linear form $u = \gamma_1 x^{-1} v + \delta_0$ is regular if and only if $\gamma_1 \neq \lambda_n$, $n \geq 0$, where*

$$\lambda_0 = 0, \quad \lambda_{n+1} = (e_n(\alpha))^{-1}, \quad n \geq 0,$$

with

$$e_n(\alpha) = \begin{cases} \frac{\Gamma(n+\alpha+2)-\Gamma(\alpha+1)(n+1)!}{\alpha\Gamma(n+\alpha+2)}, & \text{if } \alpha \neq 0, \\ \displaystyle\sum_{\nu=1}^{n+1} \frac{1}{\nu}, & \text{if } \alpha = 0, \end{cases} \quad n \geq 0. \tag{4.9}$$

Moreover, the elements of the second-order recurrence relation of its corresponding MOPS $\{P_n\}_{n\geq 0}$ are given by

$$\begin{aligned} &\beta_0^P = \gamma_1; \quad \beta_{n+1}^P = 2n + \alpha + 3 + a_n - a_{n+1}, \quad n \geq 0, \\ &\gamma_{n+1}^P = -a_n(a_n - 2n - \alpha - 1), \quad n \geq 0, \end{aligned} \tag{4.10}$$

where

$$a_n = (n + \alpha + 1)\frac{1 - \gamma_1 e_n(\alpha)}{1 - \gamma_1 e_{n-1}(\alpha)}, \quad n \geq 0, \tag{4.11}$$

with $e_{-1}(\alpha) = 0$.

For the proof, we need the following identity.

LEMMA 4.3. *For the complex number $\alpha \neq -n$, $n \geq 1$, we have*

$$\sum_{\nu=0}^{n} \binom{n+1}{\nu} \frac{(-1)^\nu}{(\alpha + n - \nu + 1)} = (-1)^n e_n(\alpha), \quad n \geq 0. \tag{4.12}$$

Proof. First, for $\alpha \neq 0$, we use the following Gauss identity [5]

$${}_2F_1\big(-(n+1), b; c; 1\big) = \frac{(c-b)_{n+1}}{(c)_{n+1}}, \quad n \geq 0, \tag{4.13}$$

where ${}_2F_1(a, b; c; z)$ is a hypergeometric function and $(a)_n$ is Pochhammer's symbol given by

$$(a)_n = \begin{cases} 1 & \text{if } n = 0 \\ a(a+1)\cdots(a+n-1) & \text{if } n \geq 1. \end{cases}$$

For $a \notin \mathbb{Z}_-$, we can write $(a)_n = \Gamma(a+n)/\Gamma(a), n \geq 0$, where Γ is the Gamma function.

Here, we take $b = -(\alpha + n + 1)$ and $c = -(\alpha + n)$ in identity (4.13). We have

$$(-(n+1))_\nu = \begin{cases} (-1)^\nu \frac{(n+1)!}{(n-\nu+1)!}, & 0 \leq \nu \leq n+1, \\ 0, & \nu \geq n+2, \end{cases} \quad n \geq 0,$$
$$\frac{(b)_\nu}{(c)_\nu} = \frac{n+\alpha+1}{n-\nu+\alpha+1} \quad \text{for } 0 \leq \nu \leq n+1, \tag{4.14}$$

$$(c-b)_{n+1} = (n+1)!, \text{ and } (c)_{n+1} = (-1)^{n+1} \frac{\Gamma(\alpha+n+1)}{\Gamma(\alpha)} \quad n \geq 0. \tag{4.15}$$

According to (4.14) and (4.15), relation (4.13) gives

$$\sum_{\nu=0}^{n+1} \binom{n+1}{\nu} \frac{(-1)^\nu}{(\alpha+n-\nu+1)} = (-1)^{n+1}(n+1)! \frac{\Gamma(\alpha)}{\Gamma(\alpha+n+2)}, \quad n \geq 0. \tag{4.16}$$

This leads to (4.12).

Next, the right member of (4.12) depends on the complex number α, having the following limit when α tends to 0

$$\lim_{\alpha \to 0} (-1)^n e_n(\alpha) = (-1)^n (F(1) - F(n+2)), \quad n \geq 0, \tag{4.17}$$

where F is the digamma function defined as the logarithmic derivative of the gamma function: $F(z) = \Gamma'(z)/\Gamma(z)$, $z \in \mathbb{C} \backslash \mathbb{Z}_-$.

According to the following recurrence relation

$$F(z+n) = \sum_{\nu=0}^{n-1} \frac{1}{z+\nu} + F(z), \quad n \geq 1$$

and from (4.17), we easily obtain (4.12) for $\alpha = 0$. □

Proof of Proposition 4.2. Here, we have [3,7]

$$R_n(0) = (-1)^n \frac{\Gamma(n+\alpha+1)}{\Gamma(\alpha+1)}, \quad n \geq 0, \tag{4.18}$$

$$R_n^{(1)}(0) = \frac{\Gamma(n+\alpha+2)(n+1)!}{\Gamma(\alpha+1)} \sum_{\nu=0}^{n} \frac{(-1)^{n-\nu}}{(\alpha+1+\nu)\Gamma(n-\nu+1)(\nu+1)!}, \quad n \geq 0. \tag{4.19}$$

From the previous lemma, the value of $R_n^{(1)}(0)$ for $n \geq 0$ can be expressed as follows:

$$R_n^{(1)}(0) = (-1)^n \frac{\Gamma(n+\alpha+2)}{\Gamma(\alpha+1)} e_n(\alpha), \quad n \geq 0. \tag{4.20}$$

Substituting (4.18) and (4.20) in (4.6) and (4.8), we then obtain (4.9) and (4.11). □

Now, we give the regularity condition of the symmetric linear form w satisfied by (4.2) and the elements of second-order recurrence relation of its MOPS $\{W_n\}_{n \geq 0}$.

PROPOSITION 4.4. *Let* w *be a symmetric linear form satisfying* (4.2). *Then, for all* $\alpha \neq -n$, $n \geq 1$ *and* $\gamma_1 \neq \lambda_n$, $n \geq 0$ *where* λ_n *is given in* (4.9), w *is semi-classical of class* $s = 2$.

Moreover, the elements of its corresponding MOPS $\{W_n\}_{n\geq 0}$ *are given by*

$$\gamma_{2n+2} = (n+\alpha+1)\frac{1-\gamma_1 e_n(\alpha)}{1-\gamma_1 e_{n-1}(\alpha)}, \quad \gamma_{2n+3} = (n+1)\frac{1-\gamma_1 e_{n-1}(\alpha)}{1-\gamma_1 e_n(\alpha)}, \quad n \geq 0, \tag{4.21}$$

with $e_{-1}(\alpha) = 0$.

Proof. We know that the symmetric form w is regular if and only if the forms $u = \sigma(w)$ and $v = \gamma_1^{-1} x u$ are regular [3,8]. Therefore and according to Proposition 4.2, we must have $\alpha \neq -n$, $n \geq 1$ and $\gamma_1 \neq \lambda_n$, $n \geq 0$.

Substituting the polynomials of the quadratic decomposition (2.3) in relation (2.2), we obtain

$$\gamma_{2n+1} + \gamma_{2n+2} = \beta_n^R; \quad \gamma_{2n+2}\gamma_{2n+3} = \gamma_{n+1}^R, \quad n \geq 0, \tag{4.22}$$

$$\gamma_{2n+2} + \gamma_{2n+3} = \beta_{n+1}^P; \quad \beta_0^P = \gamma_1; \quad \gamma_{2n+1}\gamma_{2n+2} = \gamma_{n+1}^P, \quad n \geq 0. \tag{4.23}$$

On the other hand, from (4.7) and (4.22), we have

$$\beta_{n+1}^P = \gamma_{2n+3} + \gamma_{2n+4} + a_n - a_{n+1}, \quad n \geq 0. \tag{4.24}$$

While comparing (4.24) and (4.23), we get

$$\gamma_{2n+4} - \gamma_{2n+2} = a_{n+1} - a_n, \quad n \geq 0.$$

By iteration, this gives

$$\gamma_{2n+2} = a_n, \quad n \geq 0,$$

since $\gamma_2 = \alpha + 1 - \gamma_1 = a_0$, from (4.5), (4.7) and (4.11).

Hence, (4.21) holds from (4.11) and (4.22). □

4.3 Integral representation

Regarding the integral representation of the form w, we start with the representation of the Laguerre form $v = \mathcal{L}(\alpha)$ for $\Re(\alpha) > -1$ [3,11], we have

$$\langle v, f\rangle = \frac{1}{\Gamma(\alpha+1)}\int_0^{+\infty} x^\alpha e^{-x} f(x)\,dx, \quad f \in \mathcal{P}. \tag{4.25}$$

Thus, the linear form $u = \gamma_1 x^{-1} v + \delta_0$ possesses the following representation

$$\langle u, f\rangle = \frac{\gamma_1}{\Gamma(\alpha+1)}\int_0^{+\infty} x^{\alpha-1} e^{-x} f(x)\,dx + \left(1 - \frac{\gamma_1}{\alpha}\right) f(0), \quad \Re(\alpha) > 0. \tag{4.26}$$

We first decompose the polynomial f as follows:

$$f(x) = f_1(x^2) + x f_2(x^2). \tag{4.27}$$

Because w is symmetric, we have $\langle w, f\rangle = \langle u, f_1\rangle$ and with the change of the variable $t = \sqrt{x}$ in (4.26), we obtain

$$\langle w, f\rangle = \frac{2\gamma_1}{\Gamma(\alpha+1)}\int_0^{+\infty} t^{2\alpha-1} e^{-t^2} f_1(t^2)\,dt + \left(1 - \frac{\gamma_1}{\alpha}\right) f_1(0), \quad f \in \mathcal{P}. \tag{4.28}$$

Next, we observe that

$$\int_{-\infty}^{+\infty} t|t|^{2\alpha-1}\mathrm{e}^{-t^2}f_2(t^2)\mathrm{d}t = 0, \quad \Re(\alpha) > 0,$$

and from (4.27), relation (4.28) gives the integral representation of the form w for $\Re(\alpha) > 0$, then we get

$$\langle w,f\rangle = \frac{\gamma_1}{\Gamma(\alpha+1)}\int_{-\infty}^{+\infty}|t|^{2\alpha-1}\mathrm{e}^{-t^2}f(t)\mathrm{d}t + \left(1-\frac{\gamma_1}{\alpha}\right)f(0), \quad f\in\mathcal{P}. \tag{4.29}$$

We notice from (4.30) that the form w is a linear combination of the generalized Hermite form and of the Dirac measure delta. We have for $\alpha \neq -n$, $n \geq 0$,

$$w = \frac{\gamma_1}{\alpha}\mathcal{H}\left(\alpha-\frac{1}{2}\right) + \left(1-\frac{\gamma_1}{\alpha}\right)\delta, \quad \gamma_1 \neq \alpha, \tag{4.30}$$

where $\mathcal{H}(\mu)$ is the generalized Hermite symmetric form of parameter μ, with $\mu \neq -n-1/2$, $n \geq 0$ and it is semi-classical of class $s=1$ satisfying [3,6]

$$\big(x\mathcal{H}(\mu)\big)' + \big(2x^2-(2\mu+1)\big)\mathcal{H}(\mu) = 0, \quad (\mathcal{H}(\mu))_0 = 1. \tag{4.31}$$

We see that even moments of the form w can be expressed from the Laguerre form $v=\mathcal{L}(\alpha)$. We have [3,11] for all $\alpha \neq -n$, $n \geq 1$,

$$(v)_n = \frac{\Gamma(n+\alpha+1)}{\Gamma(\alpha+1)}, \quad n \geq 0. \tag{4.32}$$

It follows that

$$(w)_0 = 1, \quad (w)_{2n+2} = \gamma_1\frac{\Gamma(n+\alpha+1)}{\Gamma(\alpha+1)}, \quad n \geq 0, \tag{4.33}$$

from (4.32) and since $v = \gamma_1^{-1}x\sigma(w)$.

Note

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