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NEW RESULTS ABOUT ORTHOGONALITY PRESERVING MAPS

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Abstract. The Alaway's theorem on orthogonality preserving maps [1] is revisited and we provide a new proof of this result, through an original separation property involving regular forms. In fact, we show a light more general result concerning weakly orthogonal sequences(see section 3).

0. Introduction

In this paper, we derive an improved version of the well known Alaway's result on orthogonality preserving maps [1, 2, 3], first by carrying out a new proof without using Laguerre form and secondly by weakening the assumptions. Incidentally, we give a separation property of the set of regular forms which is interesting for itself(see Theorem 1.2). Surprisingly, the methods which have been used possess a completely elementary character.

1. Preliminaries and notations

Let us recall some definitions and general results. Let $\mathcal{P}$ be the vector space of polynomials with coefficients in $\mathbb{C}$ and let $\mathcal{P}'$ be its dual. We denote by $< u, f >$ the effect of $u \in \mathcal{P}'$ on $f \in \mathcal{P}$. In particular, we denote by $(u)_n := < u, x^n >$, $n \geq 0$ the moments of u. For any form u, any polynomial g let $Du = u'$ and gu, be the forms defined by duality

$$< u', f > := - < u, f' >, \quad < gu, f > := < u, gf >, \quad f, g \in \mathcal{P} .$$

We define $(x-c)^{-1}u$ by $< (x-c)^{-1}u, f > := < u, \theta_c f >$, $f \in \mathcal{P}$ where $\big(\theta_c f\big)(x) := \frac{f(x)-f(c)}{x-c}$. Next, we define the right multiplication of a form by a polynomial

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$$\big(uf\big)(x) := < u, \frac{xf(x) - \xi f(\xi)}{x - \xi} >, \; f \in \mathcal{P}, \; u \in \mathcal{P}'.$$

This leads to the Cauchy product of two forms $< vu, f > := < v, uf >$, $u,\ v \in \mathcal{P}'$, $f \in \mathcal{P}$. The translation $\tau_b u$ and the dilatation $h_a u$ of a form u are respectively defined by

$$< \tau_b u, f > := < u, \tau_{-b} f > = < u, f(x+b) >, \; b \in \mathbb{C}, \; u \in \mathcal{P}', \; f \in \mathcal{P}.$$

$$< h_a u, f > := < u, h_a f > = < u, f(ax) >, \; a \in \mathbb{C} - \{0\}, \; u \in \mathcal{P}', \; f \in \mathcal{P}.$$

Let $\{P_n\}_{n\geq 0}$ be a sequence of monic polynomials with deg $P_n = n$, $n \geq 0$ (monic polynomial sequence : MPS) and let $\{u_n\}_{n\geq 0}$ be its dual sequence, $u_n \in \mathcal{P}'$ defined by $< u_n, P_m > := \delta_{n,m}$, $n,\ m \geq 0$.The following result is basic [8].

LEMMA 1.1 *For any $u \in \mathcal{P}'$ and any integer $m \geq 1$, the following statements are equivalent*

i) $< u, P_{m-1} > \neq 0, \quad < u, P_n > = 0, \quad n \geq m$,

ii) $\exists\, \lambda_\nu \in \mathbb{C},\ 0 \leq \nu \leq m-1,\ \lambda_{m-1} \neq 0$ *such that*

$$u = \sum_{\nu=0}^{m-1} \lambda_\nu u_\nu \, .$$

As usual, we denote by $\{P_n^{(1)}\}_{n\geq 0}$ the associated sequence of $\{P_n\}_{n\geq 0}$ with respect to u_0 defined by

$$P_n^{(1)}(x) := < u_0, \frac{P_{n+1}(x) - P_{n+1}(\xi)}{x - \xi} > = \big(u_0 \theta_0 P_{n+1}\big)(x), \; n \geq 0.$$

The form u is called *regular* if we can associate with it a sequence $\{P_n\}_{n\geq 0}$ such that

$$< u, P_m P_n > = r_n \delta_{n,m}, \quad n,\ m \geq 0; \quad r_n \neq 0\,,\ n \geq 0\,.$$

The sequence $\{P_n\}_{n\geq 0}$ is then said orthogonal with respect to u. Necessarily $u = \lambda u_0$, $\lambda \neq 0$ and $\{P_n\}_{n\geq 0}$ is a polynomial sequence whose any polynomial can be supposed monic.
In this case, we have $u_n = (< u_0, P_n^2 >)^{-1} P_n u_0$, $n \geq 0$ and reciprocally. The sequence $\{P_n\}_{n\geq 0}$ is then called a monic orthogonal polynomial sequence (MOPS). It satisfies the following second–order recurrence relation

$$(1.1) \qquad \begin{aligned} &P_0(x) = 1\,, \qquad P_1(x) = x - \beta_0\,, \\ &P_{n+2}(x) = (x - \beta_{n+1}) P_{n+1}(x) - \gamma_{n+1} P_n(x)\,, \;\; n \geq 0\,. \end{aligned}$$

When u is regular, let ϕ be a polynomial such that $\phi u = 0$, then $\phi = 0$.

Moreover, it is well–known that the form $v = (x - c) u_0$ is regular if and only if $P_n(c) \neq 0$, $n \geq 1$. In this case, the sequence $\{Q_n\}_{n\geq 0}$ orthogonal with respect to v is given by [4]

$$(1.2) \qquad (x-c)Q_n(x) = P_{n+1}(x) - \frac{P_{n+1}(c)}{P_n(c)}P_n(x), \ \ n \geq 0.$$

This implies

$$(1.3) \qquad P_{n+1}(x) = Q_{n+1}(x) - \frac{P_n(c)}{P_{n+1}(c)}\gamma_{n+1}Q_n(x), \ \ n \geq 0.$$

The sequence $\{Q_n\}_{n\geq 0}$ fulfills

$$(1.4) \qquad \begin{aligned} &Q_0(x) = 1, \qquad Q_1(x) = x - \xi_0, \\ &Q_{n+2}(x) = (x-\xi_{n+1})Q_{n+1}(x) - \alpha_{n+1}Q_n(x), \ \ n \geq 0, \end{aligned}$$

where

$$(1.5) \qquad \begin{aligned} \xi_n &= \ \ \beta_{n+1} + \frac{P_{n+2}(c)}{P_{n+1}(c)} - \frac{P_{n+1}(c)}{P_n(c)}, \\ \alpha_{n+1} &= \ \ \frac{P_{n+2}(c)P_n(c)}{P^2_{n+1}(c)}\gamma_{n+1}, \quad n \geq 0. \end{aligned}$$

As a consequence of these results, we have the following so called *separation theorem.*

THEOREM 1.2. *Let $f \in \mathcal{P}$ such that $< u, f >= 0$ for any u regular. Then necessarily $f = 0$.*

Proof. Suppose $n := \deg f \geq 1$. Following assumptions f is in the kernel of u

$$f = \sum_{\nu=1}^{n} \lambda_\nu P_\nu.$$

Now, there exists c_1 such that $u(1) = (x - c_1)u$ is regular. Then

$$0 =< u(1), f >= \lambda_1 < u, P_1^2(x) >,$$

which implies $\lambda_1 = 0$.
Thus, there exists c_μ such that $u(\mu) = (x - c_\mu)u(\mu - 1)$ is regular with $\lambda_\nu = 0$, $1 \leq \nu \leq \mu \leq n-1$ and

$$f = \sum_{\nu=\mu+1}^{n} \lambda_\nu P_\nu.$$

When $\mu = n-1$, there exists c_n such that $u(n) = (x - c_n)u(n-1) = \prod_{\nu=1}^{n}(x - c_\nu)u$ is regular and $0 =< u(n), f >= \lambda_n < u, P_n^2 >$. Thus

$\lambda_n = 0$, this is contradictory with the assumption $n \geq 1$. Necessarily $n = 0$, therefore $f = 0$. □

We also consider the co-recursive form $u_0(\mu)$ and its orthogonal sequence $\{P_n(\mu;\cdot)\}_{n\geq 0}$ fulfilling

$$(1.6)\qquad \begin{aligned} &P_0(\mu;x) = 1, \quad P_1(\mu;x) = x - \beta_0 - \mu, \\ &P_{n+2}(\mu;x) = (x-\beta_{n+1})P_{n+1}(\mu;x) - \gamma_{n+1}P_n(\mu;x),\ n \geq 0 . \end{aligned}$$

We have [4,7]

$$(1.7)\qquad P_n(\mu;x) = P_n(x) - \mu P^{(1)}_{n-1}(x), \quad n \geq 0 ,$$

$$(1.8)\qquad u_0(\mu) = u_0 + \mu x^{-1} u_0 u_0(\mu).$$

Further, the form u is called *symmetric* when $(u)_{2n+1} = 0$, $n \geq 0$ and called *quasi antisymmetric* when $(u)_{2n} = 0$, $n \geq 1$. When u is regular, the symmetricalness implies $\beta_n = 0$, $n \geq 0$ ([4]) and the quasi antisymmetricalness implies $\beta_n = \alpha_{n-1} - \alpha_n, \alpha_{-1} = 0$, $\gamma_{n+1} = -\alpha_n^2$, $n \geq 0$ ([6]). Finally, let us recall some fundamental properties of linear mappings from $\mathcal{P}$ into $\mathcal{P}$. Let J be a linear mapping from $\mathcal{P}$ into $\mathcal{P}$ and let $\{B_n\}_{n\geq 0}$ be an (MPS). The further results hold.

LEMMA 1.3. *The following statements are equivalent.*

a) J *is injective (one–to–one) and* $\deg J(B_n) \leq n$, $n \geq 0$.

b) The sequence $\{J(B_n)\}_{n\geq 0}$ *is linearly independant and* $\deg J(B_n) \leq n$, $n \geq 0$.

c) $\deg J(B_n) = n$, $n \geq 0$.

d) J *is an isomorphism of* $\mathcal{P}$.

Proof. Evident. □

Now, consider the following linear mapping

$$(1.9)\qquad J := \sum_{\nu\geq 0} \frac{a_\nu(x)}{\nu!} D^\nu,$$

where

$$(1.10)\qquad a_n(x) = \sum_{\nu=0}^{n} a_\nu^n x^\nu, \quad \deg a_n \leq n, \quad n \geq 0.$$

We easily obtain

$$(1.11)\qquad J(\xi^n)(x) = \sum_{\tau=0}^{n} \left(\sum_{\mu+\nu=\tau} \binom{n}{n-\nu} a_\mu^{n-\nu} \right) x^\tau, \quad n \geq 0.$$

This application does not increase the degree.

LEMMA 1.4. *For any linear mapping J, not increasing the degree, there exists $\{a_n\}_{n\geq 0}$ given by (1.10) and such that J is read as in (1.9). This representation is unique. Further, the linear mapping J is an isomorphism of $\mathcal{P}$ if and only if*

$$\lambda_n := \sum_{\mu=0}^{n} \binom{n}{\mu} a_\mu^\mu \neq 0, \quad n \geq 0\,. \tag{1.12}$$

Proof. The linear mapping J is well–determined by the data of the images $J(\xi^n) = D_n$, $n \geq 0$. Writing $D_n(x) = \sum_{\tau=0}^{n} d_\tau^n x^\tau$, from (1.11) we have $\sum_{\mu+\nu=\tau} \binom{n}{n-\nu} a_\mu^{n-\nu} = d_\tau^n$, $0 \leq \tau \leq n$ Therefore

$$a_0^n = d_0^n,\ n \geq 0\,,\ a_\tau^n = d_\tau^n - \sum_{\nu=1}^{\tau} \binom{n}{n-\nu} a_{\tau-\nu}^{n-\nu},\ 1 \leq \tau \leq n. \tag{1.13}$$

Thus, for any $n \geq 0$, a_τ^n, $0 \leq \tau \leq n$ is uniquely determined. The condition c) of Lemma 1.3 implies (1.12). □

REMARKS 1. When the sequence $\{a_n\}_{n\geq 0}$ is an (MPS), the map J given by (1.9) is an isomorphism, since in this case $\lambda_n = 2^n$, $n \geq 0$.
2. In particular, we have

$$\begin{cases} J(1)(x) = a_0^0\,, \\ J(\xi)(x) = (a_0^0 + a_1^1)x + a_0^1\,, \\ J(\xi^2)(x) = (a_0^0 + 2a_1^1 + a_2^2)x^2 + (2a_0^1 + a_1^2)x + a_0^2\,. \end{cases} \tag{1.14}$$

2. Characterizing orthogonality preserving isomorphisms

Let J be an isomorphism and let $\{P_n\}_{n\geq 0}$ an (MPS). Consider the following (MPS)$\{\widehat{P}_n\}_{n\geq 0}$ where $\widehat{P}_n := \lambda_n^{-1} J(P_n)$, $n \geq 0$. Denoting by $\{\hat{u}_n\}_{n\geq 0}$ the dual sequence of $\{\widehat{P}_n\}_{n\geq 0}$, by virtue of Lemma 1.1 , we have

$$J(\hat{u}_n) = \lambda_n u_n,\ n \geq 0. \tag{2.1}$$

where

$$J(u) := {}^t J(u) = \sum_{n\geq 0} \frac{(-1)^n}{n!} D^n(a_n u),\ u \in \mathcal{P}'\,.$$

When $\{P_n\}_{n\geq 0}$ and $\{\widehat{P}_n\}_{n\geq 0}$ are respectively orthogonal sequences, we have

$$u_n = (< u_0, P_n^2 >)^{-1} P_n u_0, \quad \hat{u}_n = (< \hat{u}_0, \widehat{P}_n^2 >)^{-1} \widehat{P}_n \hat{u}_0, \quad n \geq 0 .$$

Moreover $\{P_n\}_{n\geq 0}$ fulfils (1.1) and $\{\widehat{P}_n\}_{n\geq 0}$ fulfils

$$\begin{aligned} &\widehat{P}_0(x) = 1, \quad \widehat{P}_1(x) = x - \hat{\beta}_0 , \\ &\widehat{P}_{n+2}(x) = (x - \hat{\beta}_{n+1})\widehat{P}_{n+1}(x) - \hat{\gamma}_{n+1}\widehat{P}_n(x), \quad n \geq 0 . \end{aligned} \tag{2.2}$$

Since from (2.1) we have $< u_n, P_m >= \lambda_n^{-1} < \hat{u}_n, J(P_m) >$, $n, m \geq 0$, then in particular

$$\begin{aligned} &< u_0, P_m > = \lambda_0^{-1} < \hat{u}_0, J(P_m) >, \ m \geq 0 , \\ &< u_0, P_1 P_m > = \lambda_1^{-2}\gamma_1\hat{\gamma}_1^{-1} < \hat{u}_0, J(P_1)J(P_m) >, \ m \geq 0 . \end{aligned} \tag{2.3}$$

This leads to

$$\begin{aligned} &< u_0, xP_m(x) > -\lambda_1^{-2}\gamma_1\hat{\gamma}_1^{-1} < \hat{u}_0, \big(J\xi\big)(x)\big(JP_m\big)(x) > \\ &= \beta_0 < u_0, P_m > \{1 - \lambda_0^2\lambda_1^{-2}\gamma_1\hat{\gamma}_1^{-1}\}, \ m \geq 0 . \end{aligned}$$

Consequently for any $f \in \mathcal{P}$

$$\begin{aligned} &< u_0, xf(x) - \lambda_0\lambda_1^{-2}\gamma_1\hat{\gamma}_1^{-1}\Big(J^{-1}\big(J(\xi)J(f)\big)\Big)(x) > \\ &= \beta_0 < u_0, f > \{1 - \lambda_0^2\lambda_1^{-2}\gamma_1\hat{\gamma}_1^{-1}\}. \end{aligned} \tag{2.4}$$

Now suppose that J preserves orthogonality for any orthogonal sequence. Then the following result holds [1].

THEOREM. *When for any orthogonal sequence $\{P_n\}_{n\geq 0}$, the sequence $\{\widehat{P}_n\}_{n\geq 0}$ where $\widehat{P}_n = \lambda_n^{-1}J(P_n)$, $n \geq 0$ is also orthogonal, then necessarily $J = s(h_a \circ \tau_{-b})$, $s \neq 0$.*

Proof. Given $\{P_n\}_{n\geq 0}$ orthogonal, since the co–recursive sequence $\{P_n(\mu;\cdot)\}_{n\geq 0}$ is also orthogonal, then putting

$$\widehat{P}_n(x;\mu) = \lambda_n^{-1}\big(JP_n(\mu;\cdot)\big)(x), \ n \geq 0$$

the sequence $\{\widehat{P}_n(\cdot;\mu)\}_{n\geq 0}$ is orthogonal for any $\mu \in \mathbb{C}$ by virtue of assumption. We only need to calculate $\widehat{P}_1(x;\mu) = x - \hat{\beta}_0(\mu)$ and $\widehat{P}_2(x;\mu) = \big(x - \hat{\beta}_1(\mu)\big)\widehat{P}_1(x;\mu) - \hat{\gamma}_1(\mu)$. From (1.14), we have

$$\big(J(1)\big)(x) = s, \ \big(J(\xi)\big)(x) = s(ax + b),$$

$$\big(J(\xi^2)\big)(x) = \lambda_2 x^2 + (2sb + a_1^2)x + a_0^2,$$

where we have put $\lambda_0 = s, \lambda_1 = sa, a_0^1 = sb$. Taking (1.7) into account, we obtain

$$\widehat{P}_1(x;\mu) = s^{-1}a^{-1}\{\big(J(\xi)\big)(x) - (\beta_0+\mu)s\} = x + a^{-1}\big(b-\beta_0-\mu\big)\ ,$$

whence

$$\hat{\beta}_0(\mu) = a^{-1}(\beta_0 - b + \mu) = \hat{\beta}_0 + a^{-1}\mu\ . \tag{2.5}$$

Likewise, we have

$$\begin{aligned}\widehat{P}_2(x;\mu) &= \lambda_2^{-1}\Big\{\big(J(\xi^2)\big)(x) - (\beta_0+\beta_1)\big(J(\xi)\big)(x) + (\beta_0\beta_1-\gamma_1)s\Big\}\\ &\quad - \mu\lambda_2^{-1}\Big\{\big(J(\xi)\big)(x) - \beta_1 s\Big\},\\ &= x^2 + \lambda_2^{-1}\big\{2sb + a_1^2 - sa(\beta_0+\beta_1+\mu)\big\}x\\ &\quad + \lambda_2^{-1}\big\{a_0^2 - sb(\beta_0+\beta_1+\mu) + s(\beta_0\beta_1 - \gamma_1 + \beta_1\mu)\big\},\end{aligned}$$

which implies

$$\hat{\beta}_0(\mu) + \hat{\beta}_1(\mu) = \lambda_2^{-1}\big\{sa(\beta_0+\beta_1+\mu) - 2sb - a_1^2\big\},$$

$$\begin{aligned}&\hat{\beta}_0(\mu)\hat{\beta}_1(\mu) - \hat{\gamma}_1(\mu)\\ &\qquad = \lambda_2^{-1}\big\{a_0^2 - sb(\beta_0+\beta_1+\mu) + s(\beta_0\beta_1-\gamma_1+\beta_1\mu)\big\}.\end{aligned}$$

Therefore

$$\begin{cases}\hat{\beta}_1(\mu) = A\mu + \hat{\beta}_1\\ \hat{\gamma}_1(\mu) = a^{-1}A\mu^2 + \big\{\hat{\beta}_0 A + a^{-1}\hat{\beta}_1 + s\lambda_2^{-1}(b-\beta_1)\big\}\mu + \hat{\gamma}_1\end{cases}$$

with

$$\hat{\beta}_1 = A(\beta_0+\beta_1) + B + a^{-1}(\beta_1 - b) \tag{2.6}$$

$$\hat{\gamma}_1 = s\lambda_2^{-1}\gamma_1 + a^{-1}(A\beta_0^2 + B\beta_0 + C) \tag{2.7}$$

$$\begin{cases}A = sa\lambda_2^{-1} - a^{-1}\\ B = 2a^{-1}b(1 - sa\lambda_2^{-1}) - \lambda_2^{-1}a_1^2\\ C = -a\lambda_2^{-1}a_0^2 + b(a^{-1}b - B)\end{cases} \tag{2.8}$$

Let us put $\vartheta(u_0) := a\hat{\gamma}_1\gamma_1^{-1} - a^{-1}$, then (2.7) becomes

$$\big(\vartheta(u_0) - A\big)\gamma_1 = A\beta_0^2 + B\beta_0 + C. \tag{2.9}$$

When u_0 is a symmetric regular form $u_0 = w_0$, we have $\beta_0 = 0$, consequently

$$\big(\vartheta(w_0) - A\big)\gamma_1 = C.$$

This provides for (2.9)

$$(\vartheta(u_0) - \vartheta(w_0))\gamma_1 = A\beta_0^2 + B\beta_0. \tag{2.10}$$

When u_0 is an antisymmetric regular form $u_0 = z_0$,
we have $\gamma_1 = -\beta_0^2 \neq 0$ and from (2.9), (2.10) respectively this implies

$$-\vartheta(z_0)\beta_0^2 = B\beta_0 + C,$$
$$\vartheta(z_0)\beta_0^2 = (\vartheta(w_0) - A)\beta_0^2 - B\beta_0.$$

Hence $0 = (\vartheta(w_0) - A)\beta_0^2 + C$ for any $\beta_0 \neq 0$ where $\vartheta(w_0), C$ do not depend on β_0. This yields

$$\vartheta(w_0) - A = 0, \quad C = 0. \tag{2.11}$$

But we have not taken into account the regularity of $\hat{u}_0(\mu)$.
For $\hat{\gamma}_1(\mu) \neq 0$, $\forall\mu \in \mathbb{C}$, it is necessary and sufficient that $A = 0$ and $\hat{\beta}_0 A + a^{-1}\hat{\beta}_1 + s\lambda_2^{-1}(b - \beta_1) = 0$ or

$$A = 0, \quad B = 0. \tag{2.12}$$

It follows from (2.8) $\lambda_2 = sa^2$, $a_1^2 = 2sb(a-1)$, $a_0^2 = sb^2$ and from (2.7) : $a^{-2}\gamma_1\hat{\gamma}_1^{-1} = 1$, therefore (2.4) becomes

$$< u_0, xf(x) - s^{-1}\Big(J^{-1}\big(J(\xi)J(f)\big)\Big)(x) >= 0. \tag{2.13}$$

On now using the Theorem 1.2, we finally arrive at

$$xf(x) - s^{-1}\Big(J^{-1}\big(J(\xi)J(f)\big)\Big)(x) = 0,$$

or

$$J\big(\xi f(\xi)\big)(x) = s^{-1}J\big(\xi\big)(x)J\big(f\big)(x), \quad f \in \mathcal{P}.$$

With $f(x) = x^n$, $n \geq 0$, this leads to $J\big(\xi^n\big)(x) = s(ax+b)^n$, $n \geq 0$ and the result is proved. □

3. Generalization

In fact, the previous result is a consequence of the more general following claim. First, a definition.

Definition. The sequence $\{P_n\}_{n\geq 0}$ is said *weakly orthogonal of index* (p,q) with respect to u, when there exist two integers $p, q \geq 1$ such that [5]

$$\begin{aligned} &< u, P_{p-1} >\neq 0, \quad < u, P_n >= 0, \quad n \geq p,\\ &< xu, P_{q-1} >\neq 0, \quad < xu, P_n >= 0, \quad n \geq q. \end{aligned} \tag{3.1}$$

EXAMPLES. An orthogonal sequence is weakly orthogonal of index $(1,2)$, but in general the converse is not true. An interesting case where weak orthogonality is equivalent to regular orthogonality arises when $\{P_n\}_{n\geq 0}$ is orthogonal and its derivative sequence $\{P_n^{[1]}\}_{n\geq 0}$ is weakly orthogonal of index $(1,2)$, with $P_n^{[1]}(x) := (n+1)^{-1}P'_{n+1}(x)$, $n \geq 0$. Then $\{P_n^{[1]}\}_{n\geq 0}$ is orthogonal, in other words $\{P_n\}_{n\geq 0}$ is necessarily classical [5, 8].

For any (MPS)$\{P_n\}_{n\geq 0}$, we have the following so called structure relation

$$\begin{aligned} &P_0(x) = 1, \quad P_1(x) = x - \beta_0 , \\ &P_{n+2}(x) = (x-\beta_{n+1})P_{n+1}(x) - \sum_{\nu=0}^{n} \chi_{n,\nu}P_\nu(x), \quad n \geq 0 , \end{aligned} \tag{3.2}$$

with

$$\begin{aligned} \beta_n &= < u_n, xP_n(x) >, \quad n \geq 0 , \\ \chi_{n,\nu} &= < u_\nu, xP_{n+1}(x) >, \quad 0 \leq \nu \leq n . \end{aligned} \tag{3.3}$$

Further, the co recursive sequence $\{P_n(\mu;\cdot)\}_{n\geq 0}$ of $\{P_n\}_{n\geq 0}$ is defined by

$$\begin{aligned} &P_0(\mu;x) = 1, \quad P_1(\mu;x) = x - \beta_0 - \mu , \\ &P_{n+2}(\mu;x) = (x-\beta_{n+1})P_{n+1}(\mu;x) - \sum_{\nu=0}^{n} \chi_{n,\nu}P_\nu(\mu;x), \quad n \geq 0 . \end{aligned} \tag{3.4}$$

The associated sequence $\{P_n^{(1)}\}_{n\geq 0}$ fulfils [7]

$$\begin{aligned} &P_0^{(1)}(x) = 1, \quad P_1^{(1)}(x) = x - \beta_0^{(1)} , \\ &P_{n+2}^{(1)}(x) = (x-\beta_{n+1}^{(1)})P_{n+1}^{(1)}(x) - \sum_{\nu=0}^{n} \chi_{n,\nu}^{(1)}P_\nu^{(1)}(x), \quad n \geq 0 , \end{aligned} \tag{3.5}$$

where

$$\beta_n^{(1)} = \beta_{n+1}, \quad n \geq 0, \quad \chi_{n,\nu}^{(1)} = \chi_{n+1,\nu+1}, \quad 0 \leq \nu \leq n . \tag{3.6}$$

LEMMA 3.1. *The relation* (1.7) *remains true.*

Proof. For $n = 1$, the relation (1.7) holds. Suppose it holds for $1 \le n \le m+1$. Then we have

$$\begin{aligned} P_{m+2}(\mu;x) &= (x-\beta_{m+1})P_{m+1}(\mu;x) - \sum_{\nu=0}^{m} \chi_{m,\nu} P_\nu(\mu;x) \\ &= (x-\beta_{m+1})\{P_{m+1}(x) - \mu P_m^{(1)}(x)\} \\ &\quad - \sum_{\nu=0}^{m} \chi_{m,\nu}\{P_\nu(x) - \mu P_{\nu-1}^{(1)}(x)\} \\ &= P_{m+2}(x) - \mu\Big\{(x-\beta_{m+1})P_m^{(1)}(x) \\ &\quad - \sum_{\nu=0}^{m} \chi_{m,\nu} P_{\nu-1}^{(1)}(x)\Big\} \\ &= P_{m+2}(x) - \mu P_{m+1}^{(1)}(x)\ . \end{aligned}$$

□

Obviously when the sequence $\{P_n\}_{n\ge0}$ is weakly orthogonal of index $(1,2)$, then the sequence $\{P_n(\mu;\cdot)\}_{n\ge0}$ is weakly orthogonal of index $(1,2)$.

LEMMA 3.2. For any (MPS)$\{P_n\}_{n\ge0}$, the following assertions are equivalent (see [8])

a) The sequence $\{P_n\}_{n\ge0}$ is weakly orthogonal of index $(1,2)$.

b) $\chi_{0,0} \neq 0, \quad \chi_{n,0} = 0, \quad n \ge 1$.

c) $xu_0 = \beta_0 u_0 + \chi_{0,0} u_1, \quad \chi_{0,0} \neq 0$.

Proof. *a)* $\Longrightarrow$ *b)* Evident. *b)* $\Longrightarrow$ *c)* Following Lemma 1.1. *c)* $\Longrightarrow$ *a)* Evident. □

Now we are able to prove

PROPOSITION. *If the isomorphism J transforms any orthogonal sequence into a weakly orthogonal sequence of index $(1,2)$, then necessarily $J = s(h_a \circ \tau_{-b})$.*

Proof. By virtue of the property *c)* of the Lemma 3.2, the relation (2.4) remains true

$$(3.7)\qquad \begin{aligned} &< u_0, xf(x) - s^{-1}\Lambda(u_0)\Big(J^{-1}\big(J(\xi)J(f)\big)\Big)(x) > \\ &= \beta_0 < u_0, f > \big\{1 - \Lambda(u_0)\big\}\ , \end{aligned}$$

where

$$(3.8)\qquad \Lambda(u_0) = \lambda_0^2 \lambda_1^{-2} \gamma_1 \widehat{\chi}_{0,0}^{-1}\ .$$

Likewise, the relations (2.6) remain valid

$$\hat{\beta}_1(\mu) = A\mu + \hat{\beta}_1,$$
$$\hat{\chi}_{0,0}(\mu) = a^{-1}A\mu^2 + \{\hat{\beta}_0 A + a^{-1}\hat{\beta}_1 + s\lambda_2^{-1}(b-\beta_1)\}\mu + \hat{\chi}_{0,0} ,$$

and also (2.9)

$$(\vartheta(u_0) - A)\chi_{0,0} = A\beta_0^2 + B\beta_0 + C,$$

with $\vartheta(u_0) = a\hat{\chi}_{0,0}\chi_{0,0}^{-1} - a^{-1}$ and where A, B, C are given by (2.8). A symmetric sequence $\{P_n\}_{n\geq 0}$ is characterized by $\beta_n = 0$, $n \geq 0$; $\chi_{2n+1,2\nu} = 0$, $0 \leq \nu \leq n$; $\chi_{2n,2\nu+1} = 0, 0 \leq \nu \leq n-1$ ([7]). In the case of a quasi-antisymmetric sequence, we have $0 = (u_0)_2 = \beta_0^2 + \chi_{0,0}$ Therefore, we have always $C = 0$ and $A = 0, B = 0$ yielded by $\hat{\chi}_{0,0}(\mu) \neq 0$, $\forall \mu \in \mathbb{C}$. Thus $\vartheta(u_0) = 0$ and this leads to $\Lambda(u_0) = 1$ in accordance with the following identity $a\Lambda(u_0) = (\vartheta(u_0) + a^{-1})^{-1}$. From (3.7), we meet again (2.13), thus we obtain the desired result. □

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References

[1] W. R. Allaway, *Orthogonality preserving maps and the Laguerre functional*, Proc. Amer. Math. Soc. **100** (1987), 82–86.
[2] W. A. Al-Salam and A. Verma, *Some orthogonality preserving operators*, Proc. Amer. Math. Soc. **23** (1969), 136–139.
[3] W.A. Al-Salam and A. Verma, *Orthogonality preserving operators*, Accad. Naz. Lincei **58** (1975), 833–838.
[4] T. S. Chihara, *An introduction to orthogonal polynomials*, Gordon and Breach, New-York, 1978.
[5] P. Maroni, *Une caractérisation des polynômes orthogonaux semi-classiques*, C. R. Acad. Sci. Paris Sér. I, **301** (1985), no. 6, 269–272.
[6] P. Maroni, *Sur la suite de polynômes orthogonaux associée à la forme* $u = \delta_c + \lambda(x-c)^{-1}L$, Period. Math. Hung. **21** (1990), 223–248.
[7] P. Maroni, *Une théorie algébrique des polynômes orthogonaux. Application aux polynômes orthogonaux semi-classiques*, Ann. Comput. Appl. Math. **9** (1991), 95 130.
[8] P. Maroni, *Variations around classical orthogonal polynomials. Connected problems* J. Comput. Appl. Math. **48** (1993), 133–155.

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