Logarithmic stability estimates for a Robin coefficient in the two-dimensional Stokes system

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Abstract

In this paper, we consider the Stokes equations and we are concerned with the inverse problem of identifying a Robin coefficient on some non accessible part of the boundary from available data on the other part of the boundary. We first study the identifiability of the Robin coefficient and then we establish a stability estimate of logarithm type thanks to a Carleman inequality due to Bukhgeim [9].

Keywords: Inverse boundary coefficient problem, Stokes system, Robin boundary condition, Identifiability, Carleman inequality, Logarithmic stability estimate.

1 Introduction

Let us consider some open Lipschitz bounded connected set domain Ω of $\mathbb{R}^d$, $d \geq 2$. We assume that the boundary $\Gamma = \partial\Omega$ is composed of two parts Γ_0 and Γ_e such that $\Gamma_e \cup \Gamma_0 = \Gamma$ and $\overline{\Gamma}_e \cap \overline{\Gamma}_0 = \emptyset$ (Figure 1 gives an example of such a geometry). We introduce the following boundary problem:

$$\left\{ \begin{array}{rcll} u_t(t,x) - \Delta u(t,x) + \nabla p(t,x) & = & 0, & \forall x \in \Omega, \forall\, t > 0, \\ \nabla \cdot u(t,x) & = & 0, & \forall x \in \Omega, \forall\, t > 0, \\ \nabla u(t,x) \cdot n(x) - p(t,x) n(x) & = & g(t,x), & \forall x \in \Gamma_e, \forall\, t > 0, \\ \nabla u(t,x) \cdot n(x) - p(t,x) n(x) + q(x) u(t,x) & = & 0, & \forall x \in \Gamma_0, \forall\, t > 0, \\ u(0,x) & = & u_0(x), & \forall x \in \Omega. \end{array} \right. \tag{1.1}$$

Notice that we assume that the Robin coefficient q defined on Γ_0 only depends on the space variable. Our objective is to determine the coefficient q from the values of u and p on Γ_e.

Such kinds of systems naturally appear in the modeling of biological problems like, for example, blood flow in the cardiovascular system (see [20] and [23]) or airflow in the lungs (see [3]). The part of the boundary Γ_e represents a physical boundary on which measurements are available and Γ_0 represents an artificial boundary on which Robin boundary conditions or mixed boundary conditions involving the fluid stress tensor and its flux at the outlet are prescribed.

Similar inverse problems have been widely studied for the Laplace equation [2], [4], [10], [11], [12] and [22]. This kind of problems arises in general in corrosion detection which consists in determining a Robin coefficient on the inaccessible portion of the boundary thanks to electrostatic measurements performed on the accessible boundary. Most of these papers prove a logarithmic stability estimate ([2], [4], [10] and [12]). S. Chaabane and M. Jaoua obtained in [11] both local and monotone global Lipschitz stability for regular Robin coefficient and under the assumption that the flux g is non negative. Relaxing this constraint, they obtained in [2] a logarithmic stability estimate. More recently, E. Sincich has obtained in [22] a Lipschitz stability estimate under the further *a priori* assumption that the Robin coefficient is piecewise constant. To prove the stability estimates, different approaches are developped in these papers. A first approach consists in using the harmonic functions properties (see [2], [10]). A characteristic of this method is that it is only valid in dimension 2. Another classical approach is based on Carleman estimates (see [4] and [12]). In [4], the authors use a result proved by K.D. Phung in [19] to obtain a logarithmic stability estimate which is valid in any dimension for an open set Ω of class $\mathcal{C}^\infty$. This result has been generalized in [5] and [6] to $\mathcal{C}^{1,1}$

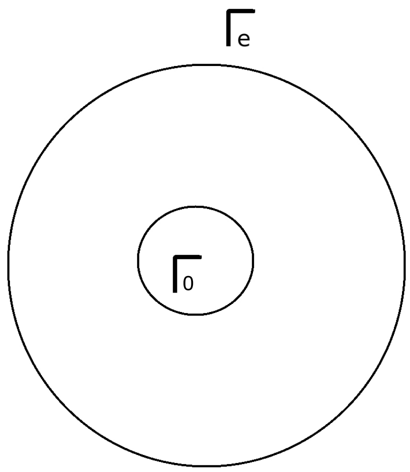

Figure 1: Example of an open set Ω such that $\Gamma_e \cup \Gamma_0 = \Gamma$ and $\overline{\Gamma}_e \cap \overline{\Gamma}_0 = \emptyset$.

and Lipschitz domains. Moreover, in [4], the authors use semigroup theory to obtain a stability estimate in long time for the heat equation from the stability estimate for the Laplace equation.

In this article, we prove an identifiability result and a logarithmic stability estimate for the Stokes equations with Robin boundary conditions (1.1). The paper is organized as follows.

The second section contains preliminary results on the regularity of the solution. For this purpose, we introduce Besov spaces: they appear as the natural space to which the Robin coefficient q belongs.

In the third section, we are interested in the identifiability of the Robin coefficient q. Under some regularity hypotheses and using the theorem of unique continuation for the Stokes equations proved in [15], we prove that if two measurements of the velocity are equal on the boundary $(0, T) \times \Gamma_e$, then the two corresponding Robin coefficients are also equal.

Section 4 corresponds to the main part of our article. The results of this section are only valid in dimension 2. We prove a logarithmic stability estimate, first for the stationary problem and then for the evolution problem. To do this, we use a Carleman inequality due to Bukhgeim which is only valid in dimension 2 (see [9]). The stability estimate for the unsteady problem is deduced from the stability estimate for the stationary problem thanks to the semigroup theory. We end Section 4 by concluding remarks and perspectives to this work.

When we are not more specific, C is a generic constant, whose value may change and which only depends on the geometry of the open set Ω and of the boundaries Γ_e and Γ_0.

We are going to start with some preliminary results which will be useful in the subsequent sections.

2 Preliminary results

2.1 Besov spaces

We introduce the following functional spaces:

$$V = \{v \in H^1(\Omega)^d / \nabla \cdot v = 0 \text{ on } \Omega\},$$

and

$$H = \overline{V}^{L^2(\Omega)}.$$

We also need to introduce the space to which the Robin coefficient q belongs: we are going to assume enough regularity on q in order to give a sense in suitable spaces to the trace product qu. We refer to [16] for more general spaces. Let $s \in \mathbb{R}$, $1 \leq p \leq \infty$ and:

$$B_{s,p}(\mathbb{R}^d) = \{w \in \mathcal{S}'(\mathbb{R}^d)/(1+|\xi|^2)^{\frac{s}{2}}\mathcal{F}w \in L^p(\mathbb{R}_\xi{}^d)\},$$

where $\mathcal{S}'(\mathbb{R}^d)$ is the space of tempered distribution on $\mathbb{R}^d$ and $\mathcal{F}w$ is the Fourier transform of w. Equipped with the norm

$$\|w\|_{B_{s,p}(\mathbb{R}^d)} = \|(1+|\xi|^2)^{\frac{s}{2}}\mathcal{F}w\|_{L^p(\mathbb{R}^d_\xi)},$$

$B_{s,p}(\mathbb{R}^d)$ is a Banach space. We observe that $B_{s,2}(\mathbb{R}^d) = H^s(\mathbb{R}^d)$. In the following, we will need the following properties:

Proposition 2.1. *Let $s > 0$. We have $B_{s,1}(\mathbb{R}^d) \hookrightarrow \mathcal{C}^0(\mathbb{R}^d)$.*

Proof of Proposition 2.1. Let $u \in B_{s,1}(\mathbb{R}^d)$. We start by proving that $\mathcal{F}u \in L^1(\mathbb{R}^d)$:

$$\int_{\mathbb{R}^d} |\mathcal{F}u(\xi)|d\xi = \int_{\mathbb{R}^d} (1+|\xi|^2)^{\frac{s}{2}}|\mathcal{F}u(\xi)|\frac{1}{(1+|\xi|^2)^{\frac{s}{2}}}d\xi,$$

and from the Cauchy-Schwarz inequality, we have:

$$\int_{\mathbb{R}^d} |\mathcal{F}u(\xi)|d\xi \leq \|(1+|\xi|^2)^{\frac{s}{2}}\mathcal{F}u\|_{L^1(\mathbb{R}^d)} \left\|\frac{1}{(1+|\xi|^2)^{\frac{s}{2}}}\right\|_{L^\infty(\mathbb{R}^d)} \leq C\|u\|_{B_{s,1}(\mathbb{R}^d)}. \tag{2.1}$$

Since for almost every $x \in \mathbb{R}^d$,

$$u(x) = \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} e^{ix\cdot\xi}\mathcal{F}u(\xi)d\xi,$$

we deduce the continuity of u from the Lebesgue continuity theorem and according to (2.1), we have:

$$\|u\|_{\mathcal{C}^0(\mathbb{R}^d)} \leq C\|u\|_{B_{s,1}(\mathbb{R}^d)}.$$

□

Using a local map and partition of unity, we build $B_{s,p}(\Gamma)$ from $B_{s,p}(\mathbb{R}^{d-1})$ in the same way that $H^s(\Gamma)$ is built from $H^s(\mathbb{R}^{d-1})$ (see [17]). From the above proposition, we deduce the following corollary:

Corollary 2.2. *Let $s > 0$. We have $B_{s,1}(\Gamma) \hookrightarrow \mathcal{C}^0(\Gamma)$ and the injection is continuous.*

Proposition 2.3. *Let $s \in \mathbb{R}^+$, $u \in H^s(\Gamma)$ and $v \in B_{s,1}(\Gamma)$. Then $uv \in H^s(\Gamma)$ and*

$$\|uv\|_{H^s(\Gamma)} \leq \frac{2^{\frac{s}{2}}}{(2\pi)^{d-1}}\|u\|_{H^s(\Gamma)}\|v\|_{B_{s,1}(\Gamma)}.$$

We refer to [13] for a proof of this proposition.

2.2 Regularity of the stationary problem

First, we are interested in the stationary case:

$$\begin{cases} -\Delta u + \nabla p &= f, & \text{in } \Omega, \\ \nabla \cdot u &= 0, & \text{in } \Omega, \\ \nabla u \cdot n - pn &= g, & \text{on } \Gamma_e, \\ \nabla u \cdot n - pn + qu &= 0, & \text{on } \Gamma_0. \end{cases} \tag{2.2}$$

Let $g \in H^{-\frac{1}{2}}(\Gamma_e)^d$ and $v \in H^{\frac{1}{2}}(\Gamma_e)^d$, we denote $< g, v >_{-\frac{1}{2},\frac{1}{2},\Gamma_e}$ the image of v by the linear form g.

Proposition 2.4. *Let $\alpha > 0$, $f \in L^2(\Omega)^d$, $g \in H^{-\frac{1}{2}}(\Gamma_e)^d$ and $q \in L^\infty(\Gamma_0)$ such that $q \geq \alpha$ on Γ_0. System* (2.2) *admits a unique solution $(u,p) \in V \times L^2(\Omega)$. Moreover, there exists a constant $C(\alpha) > 0$ such that*

$$\|u\|_{H^1(\Omega)^d} \leq C(\alpha)(\|g\|_{H^{-\frac{1}{2}}(\Gamma_e)^d} + \|f\|_{L^2(\Omega)^d}). \tag{2.3}$$

Proof of Proposition 2.4. The variational formulation of the problem is: find $u \in V$ such that for every $v \in V$,

$$\int_\Omega \nabla u : \nabla v + \int_{\Gamma_0} qu \cdot v =< g, \mathbb{1}_{\Gamma_e} v >_{-\frac{1}{2},\frac{1}{2},\Gamma_e} + \int_\Omega f \cdot v.$$

We note $\forall (u,v) \in V^d$,

$$a_q(u,v) = \int_\Omega \nabla u : \nabla v + \int_{\Gamma_0} qu \cdot v, \tag{2.4}$$

and $\forall v \in V$,

$$L_1(v) =< g, \mathbb{1}_{\Gamma_e} v >_{-\frac{1}{2},\frac{1}{2},\Gamma_e} + \int_\Omega f \cdot v.$$

We easily verify that a_q is a continuous symmetric bilinear form. Since $q \geq \alpha > 0$, according to the generalized Poincaré inequality, the bilinear form a_q is coercive. On the other hand, L_1 is a continuous linear form on V. Thus we prove the existence and uniqueness of $u \in V$ solution of equations (2.2) using the Lax-Milgram Theorem. We obtain simultaneously estimate (2.3). We prove the existence and uniqueness of $p \in L^2(\Omega)$ in a classical way, using De Rham Theorem (we refer to [7] for the case of Neumann boundary condition). □

Let us recall existence and regularity results for the Stokes problem with Neumann boundary condition proved in [7].

Proposition 2.5. *Let $k \in \mathbb{N}$. Assume that Ω is a bounded and connected open set in $\mathbb{R}^d$ of class $C^{k+1,1}$. We assume that:*

$$(f,h) \in H^k(\Omega)^d \times H^{k+\frac{1}{2}}(\Gamma)^d.$$

Then the solution (u,p) of the problem:

$$\left\{ \begin{array}{rcll} -\Delta u + \nabla p & = & f & \text{in } \Omega, \\ \nabla \cdot u & = & 0 & \text{in } \Omega, \\ \nabla u \cdot n - pn & = & h & \text{on } \Gamma. \end{array} \right.$$

belongs to $H^{k+2}(\Omega)^d \times H^{k+1}(\Omega)$ and there exists a constant $C > 0$ such that:

$$\|u\|_{H^{k+2}(\Omega)^d} + \|p\|_{H^{k+1}(\Omega)} \leq C(\|h\|_{H^{\frac{1}{2}+k}(\Gamma)^d} + \|f\|_{H^k(\Omega)^d}).$$

From the previous proposition we deduce the following result:

Proposition 2.6. *Let $k \in \mathbb{N}$. Assume that Ω is a bounded and connected open set in $\mathbb{R}^d$ of class $C^{k+1,1}$. Let $\alpha > 0$, $M > 0$, $f \in H^k(\Omega)^d$, $g \in H^{\frac{1}{2}+k}(\Gamma_e)^d$ and $q \in B_{\frac{1}{2}+k,1}(\Gamma_0)$ such that $q \geq \alpha$ on Γ_0. Then the solution (u,p) of system* (2.2) *belongs to $H^{k+2}(\Omega)^d \times H^{k+1}(\Omega)$. Moreover, if $\|q\|_{B_{\frac{1}{2}+k,1}(\Gamma_0)} \leq M$, there exists a constant $C(\alpha, M) > 0$ such that*

$$\|u\|_{H^{k+2}(\Omega)^d} + \|p\|_{H^{k+1}(\Omega)} \leq C(\alpha,M)(\|g\|_{H^{k+\frac{1}{2}}(\Gamma_e)^d} + \|f\|_{H^k(\Omega)^d}).$$

Proof of Proposition 2.6. Let us prove the result for $k = 0$. Let $h = -qu\mathbb{1}_{\Gamma_0} + g\mathbb{1}_{\Gamma_e}$. According to Proposition 2.4, u belongs to $H^1(\Omega)^d$. Thus, we obtain from Proposition 2.3 that $qu \in H^{\frac{1}{2}}(\Gamma_e)^d$, which implies, since $g \in H^{\frac{1}{2}}(\Gamma_e)^d$ and $\overline{\Gamma}_e \cap \overline{\Gamma}_0 = \emptyset$, that $h \in H^{\frac{1}{2}}(\Gamma)^d$. Using Proposition 2.5 with $k = 0$ we obtain that $(u,p) \in H^2(\Omega)^d \times H^1(\Omega)$ and:

$$\|u\|_{H^2(\Omega)^d} + \|p\|_{H^1(\Omega)} \leq C(\|h\|_{H^{\frac{1}{2}}(\Gamma)^d} + \|f\|_{L^2(\Omega)^d}).$$

But, we have from Proposition 2.4, that:

$$\|h\|_{H^{\frac{1}{2}}(\Gamma)^d} \leq C(\|u\|_{H^{\frac{1}{2}}(\Gamma)^d}\|q\|_{B_{\frac{1}{2},1}(\Gamma_0)} + \|g\|_{H^{\frac{1}{2}}(\Gamma_e)^d}),$$

and since by hypothesis, $\|q\|_{B_{\frac{1}{2},1}(\Gamma_0)} \leq M$, we obtain:

$$\|u\|_{H^2(\Omega)^d} + \|p\|_{H^1(\Omega)} \leq C(M)(\|g\|_{H^{\frac{1}{2}}(\Gamma_e)^d} + \|u\|_{H^1(\Omega)^d} + \|f\|_{L^d(\Omega)^d}).$$

Thus we obtain the result for $k = 0$ using the inequality of Proposition 2.4. We then proceed by induction to prove the result for any $k \in \mathbb{N}$. We emphasize that the key argument is that if $u \in H^{1+k}(\Omega)^d$, then $u \in H^{k+\frac{1}{2}}(\Gamma_0)^d$ which implies that $qu_{|\Gamma_0} \in H^{\frac{1}{2}+k}(\Gamma_0)^d$ thanks to Proposition 2.3. Thus, we can apply the regularity result given by Proposition 2.5 and conclude. □

2.3 Regularity of the evolution problem.

Concerning the initial problem (1.1), we can prove, using Galerkin method, the following regularity results. For completeness, the proof of Theorem 2.7 is given in the appendix.

Theorem 2.7. *Assume that Ω is a bounded and connected open set in $\mathbb{R}^d$ of class $C^{1,1}$. Let $T > 0$, $M > 0$, $\alpha > 0$ and $u_0 \in V$. We assume that $g \in H^1(0,T; H^{\frac{1}{2}}(\Gamma_e)^d)$, $q \in B_{\frac{1}{2},1}(\Gamma_0)$ such that $\|q\|_{B_{\frac{1}{2},1}(\Gamma_0)} \leq M$ and $q \geq \alpha$ on Γ_0. Then problem (1.1) admits a unique solution $(u,p) \in L^2(0,T; H^2(\Omega)^d) \cap H^1(0,T; L^2(\Omega)^d) \cap L^\infty(0,T;V) \times L^2(0,T; H^1(\Omega))$.*

The following corollary will be useful when we will prove logarithmic stability estimate for the evolution problem (1.1).

Corollary 2.8. *Assume that Ω is a bounded and connected open set in $\mathbb{R}^d$ of class $C^{2,1}$. Let $M > 0$, $T > 0$, $\alpha > 0$ and $u_0 \in H^3(\Omega)^d \cap H$. We assume that $g \in H^2(0,T; H^{\frac{3}{2}}(\Gamma_e)^d)$, $q \in B_{\frac{3}{2},1}(\Gamma_0)$ such that $\|q\|_{B_{\frac{3}{2},1}(\Gamma_0)} \leq M$ and $q \geq \alpha$ on Γ_0. Then, problem (1.1) admits a unique solution $(u,p) \in L^\infty(0,T; H^3(\Omega)^d) \cap H^1(0,T; H^2(\Omega)^d) \cap H^2(0,T; L^2(\Omega)^d) \times L^\infty(0,T; H^2(\Omega)) \cap H^1(0,T; H^1(\Omega))$.*

The proof of the previous corollary consists in applying Theorem 2.7 to u_t. Let us prove it.

Proof of Corollary 2.8. Let (u,p) the solution of (1.1). Let us consider the following system:

$$\begin{cases} v_t - \Delta v + \nabla\tau = 0, & \text{in } (0,T)\times\Omega, \\ \nabla\cdot v = 0, & \text{in } (0,T)\times\Omega, \\ \nabla v\cdot n - \tau n = g_t, & \text{on } (0,T)\times\Gamma_e, \\ \nabla v\cdot n - \tau n + qv = 0, & \text{on } (0,T)\times\Gamma_0, \\ v(0) = \Delta u_0 - \nabla p(0), & \text{in } \Omega, \end{cases} \tag{2.5}$$

where $p(0) \in H^2(\Omega)^d$ is defined as the solution of the following elliptic boundary problem:

$$\begin{cases} \Delta p(0) = 0, & \text{in } \Omega, \\ p(0) = (\nabla u_0\cdot n)\cdot n - g(0)\cdot n, & \text{on } \Gamma_e, \\ p(0) = (\nabla u_0\cdot n)\cdot n + q(0)u_0\cdot n, & \text{on } \Gamma_0. \end{cases}$$

According to Theorem 2.7, we obtain that (v,τ) belongs to $L^2(0,T; H^2(\Omega)^d) \cap H^1(0,T; L^2(\Omega)^d) \cap L^\infty(0,T;V) \times L^2(0,T; H^1(\Omega))$. Remark that (u_t,p_t) is solution of equations (2.5) in the distribution sense on $(0,T)$. Thus, by uniqueness, $(v,\tau) = (u_t,p_t)$. Then, since $q \in B_{\frac{3}{2},1}(\Gamma_e)$, we deduce from Proposition 2.6 that the linear map:

$$\begin{array}{rcl} H^1(\Omega)^d \times H^{\frac{3}{2}}(\Gamma_e)^d & \to & H^3(\Omega)^d \times H^2(\Omega) \\ (u_t(t), g(t)) & \to & (u(t), p(t)) \end{array}$$

is continuous. Since $(u_t, g) \in L^\infty(0,T;V) \times L^\infty(0,T; H^{\frac{3}{2}}(\Gamma_e)^d)$, we deduce that $(u,p) \in L^\infty(0,T; H^3(\Omega)^d) \times L^\infty(0,T; H^2(\Omega))$. □

3 Identifiability

3.1 Unique continuation

We start by recalling a unique continuation result for the Stokes equations proved in [15].

Theorem 3.1. *Let Ω be an open connected set in $\mathbb{R}^d$, $d \geq 1$. We note $Q = (0,T) \times \Omega$ and let O be an open set in Q. The horizontal component of O is*

$$C(O) = \{(t,x) \in Q, \exists x_0 \in \Omega, (t,x_0) \in O\}.$$

Let $(u,p) \in L^2(0,T;H^1_{loc}(\Omega))^d \times L^2_{loc}(Q)$ be a weak solution of

$$\begin{cases} u_t - \Delta u + \nabla p &= 0, \quad \text{in } (0,T)\times\Omega, \\ \nabla\cdot u &= 0, \quad \text{in } (0,T)\times\Omega, \end{cases}$$

satisfying $u = 0$ in O then $u = 0$ and p is constant in $C(O)$.

We easily deduce the following result from the previous result. It will be very useful in the next subsection.

Corollary 3.2. *Let $\delta > 0$, $x_0 \in \Gamma$ and $r > 0$ such that $\gamma = (t_0-\delta, t_0+\delta) \times (\mathcal{B}(x_0,r)\cap\Gamma)$ is an open set in $(0,T)\times\Gamma$. Let $(u,p) \in L^2(0,T;H^2(\Omega)^d) \times L^2(0,T;H^1(\Omega))$ be solution of:*

$$\begin{cases} u_t - \Delta u + \nabla p &= 0, \quad \text{in } (0,T)\times\Omega, \\ \nabla\cdot u &= 0, \quad \text{in } (0,T)\times\Omega, \end{cases}$$

satisfying $u = 0$ and $\nabla u\cdot n - pn = 0$ on γ. Then $u = 0$ and $p = 0$ in $(t_0-\delta,t_0+\delta)\times\Omega$.

Proof of Corollary 3.2. We extend u and p by 0 on $(t_0-\delta,t_0+\delta)\times(\mathcal{B}(x_0,r)\cap\Omega^c)$:

$$\tilde{u} \text{ (resp } \tilde{p}) = \begin{cases} u \text{ (resp } p) & \text{in } (t_0-\delta,t_0+\delta)\times\Omega \\ 0 & \text{in } (t_0-\delta,t_0+\delta)\times(\mathcal{B}(x_0,r)\cap\Omega^c) \end{cases}$$

and we denote $\tilde{\Omega} = \Omega\cup\mathcal{B}(x_0,r)$. Let us verify that $(\tilde{u},\tilde{p}) \in L^2(0,T;H^1(\Omega)^d)\times L^2(0,T;L^2(\Omega))$ is still a solution of the Stokes equations in $\tilde{\Omega}$. Let $v \in \mathcal{D}(\tilde{\Omega})^d$. We check by integration by parts in space that almost everywhere in $t \in (t_0-\delta,t_0+\delta)$:

$$\int_{\tilde{\Omega}} \tilde{u}_t\cdot v + \int_{\tilde{\Omega}} \nabla\tilde{u}\cdot\nabla v - \int_{\tilde{\Omega}} \tilde{p}\nabla\cdot v = 0.$$

Moreover $\nabla\cdot\tilde{u} = 0$ in $(t_0-\delta,t_0+\delta)\times\tilde{\Omega}$. Therefore, we can apply Theorem 3.1 to $(\tilde{u},\tilde{p})$: $(\tilde{u},\tilde{p}) = (0,0)$ in $(t_0-\delta,t_0+\delta)\times\tilde{\Omega}$ which implies that $u = 0$ and p is constant in $(t_0-\delta,t_0+\delta)\times\Omega$. At last, the fact that $\nabla u\cdot n - pn = 0$ over γ implies that $p = 0$ in $(t_0-\delta,t_0+\delta)\times\Omega$. □

3.2 Application

Proposition 3.3. *Let $T > 0$, $\alpha > 0$, $x_e \in \Gamma_e$, $r > 0$, $g \in H^1(0,T;H^{\frac{1}{2}}(\Gamma_e)^d)$ non identically zero, $u_0 \in V$ and $q_j \in B_{\frac{1}{2},1}(\Gamma_0)$ such that $q_j \geq \alpha$ on Γ_0 for $j = 1,2$. Let (u_j,p_j) be the weak solutions of (1.1) with $q = q_j$ for $j = 1,2$. We assume that $u_1 = u_2$ on $(0,T)\times(\mathcal{B}(x_e,r)\cap\Gamma_e)$. Then $q_1 = q_2$ on Γ_0.*

Proof of Proposition 3.3. We are going to prove Proposition 3.3 by contradiction: we assume that q_1 is not identically equal to q_2.
We have $(u_j,p_j) \in L^2(0,T;H^2(\Omega)^d)\times L^2(0,T;H^1(\Omega))$ for $j = 1,2$ thanks to Theorem 2.7. We define $u = u_1 - u_2$ and $p = p_1 - p_2$. Let us notice that (u,p) is the solution of the following problem:

$$\begin{cases} u_t - \Delta u + \nabla p &= 0, \quad \text{in } (0,T)\times\Omega, \\ \nabla\cdot u &= 0, \quad \text{in } (0,T)\times\Omega, \\ \nabla u\cdot n - pn &= 0, \quad \text{on } (0,T)\times\Gamma_e, \\ \nabla u\cdot n - pn + q_1u_1 - q_2u_2 &= 0, \quad \text{on } (0,T)\times\Gamma_0. \end{cases}$$

By hypothesis, $u = 0$ and $\nabla u \cdot n - pn = 0$ on $(0,T) \times (\mathcal{B}(x_e, r) \cap \Gamma_e)$. Thus, according to Corollary 3.2, $u_1 = u_2$ and $p_1 = p_2$ in $(0,T) \times \Omega$. Consequently, we deduce from

$$\begin{array}{rcll} \nabla u_1 \cdot n - p_1 n + q_1 u_1 & = & 0, & \text{on } (0,T) \times \Gamma_0, \\ \nabla u_1 \cdot n - p_1 n + q_2 u_1 & = & 0, & \text{on } (0,T) \times \Gamma_0, \end{array}$$

that

$$u_1(q_1 - q_2) = 0 \text{ on } (0,T) \times \Gamma_0. \tag{3.1}$$

By hypothesis, q_1 is not identically equal to q_2. Thanks to Corollary 2.2, q_1 and q_2 are continuous. Then, we can find an open set $\kappa \subset \Gamma_0$ with a positive measure such that:

$$(q_1 - q_2)(x) \neq 0, \; \forall x \in \kappa.$$

Equation (3.1) implies that $u_1 \equiv 0$ on $(0,T) \times \kappa$ and then u_1 is the solution of

$$\left\{ \begin{array}{rcll} u_{1t} - \Delta u_1 + \nabla p_1 & = & 0, & \text{in } (0,T) \times \Omega, \\ \nabla \cdot u_1 & = & 0, & \text{in } (0,T) \times \Omega, \\ u_1 & = & 0, & \text{on } (0,T) \times \kappa, \\ \nabla u_1 \cdot n - p_1 n & = & 0, & \text{on } (0,T) \times \kappa. \end{array} \right.$$

Applying again Corollary 3.2, we obtain that $u_1 = 0$ and $p_1 = 0$ in $(0,T) \times \Omega$. This is in contradiction with the assumption that g is non identically zero. □

4 Logarithmic stability estimates

In this section, we assume that $d = 2$ and that the open set $\Omega \subset \mathbb{R}^2$ is of class $\mathcal{C}^{3,1}$ in order to obtain regular solutions of problem (2.2) from Proposition 2.6. We are going to prove logarithmic stability estimates using a Carleman inequality which is stated in Lemma 4.1. First, in Theorem 4.3, we state a logarithmic stability estimate for the stationary problem. Then we deduce from this Theorem two logarithmic stability estimates for the evolution problem using the analytic semigroup theory. These estimates are given in Theorem 4.16 when the flux g is stationary and in Theorem 4.18 when g depends on time.

4.1 Carleman inequality

Let us first state a Carleman inequality proved by Bukhgeim [9]:

Lemma 4.1. *Let $\Psi \in \mathcal{C}^2(\overline{\Omega})$. We have:*

$$\int_\Omega (\Delta\Psi |u|^2 + (\Delta\Psi - 1)|\nabla u|^2) e^\Psi \leq \int_\Omega |\Delta u|^2 e^\Psi + \int_\Gamma \nabla\Psi \cdot n(|u|^2 + |\nabla u|^2 + 2|\partial_\tau |\nabla u|^2|) e^\Psi \tag{4.1}$$

for all $u \in \mathcal{C}^2(\overline{\Omega})$.

The proof of this result, which is only valid in dimension 2, uses computational properties of function defined on $\mathbb{C}$ (in particular, the fact that $4\partial_{\overline{z}}\partial_z = \Delta$).

Remark 4.2. *The result is still true for $u \in H^3(\Omega)$. Indeed, for all $u \in H^3(\Omega)$, there exists $(u_n)_{n\in\mathbb{N}} \in \mathcal{C}^2(\overline{\Omega})^{\mathbb{N}}$ such that*

$$u_n \to u \text{ in } H^3(\Omega). \tag{4.2}$$

We can apply Lemma 4.1 to u_n, for all $n \in \mathbb{N}$. Let us prove that:

$$\lim_{n\to\infty} \int_\Gamma \nabla\Psi \cdot n |\partial_\tau |\nabla u_n|^2| e^\Psi = \int_\Gamma \nabla\Psi \cdot n |\partial_\tau |\nabla u|^2| e^\Psi. \tag{4.3}$$

Note first that $\int_\Gamma \nabla\Psi \cdot n|\partial_\tau|\nabla u|^2|e^\Psi$ has a meaning for $u \in H^3(\Omega)$:

$$\int_\Gamma \nabla\Psi \cdot n |\partial_\tau |\nabla u|^2| e^\Psi \leq 2\|\Psi\|_{\mathcal{C}^1(\overline{\Omega})} \|e^\Psi\|_{\mathcal{C}^0(\overline{\Omega})} \left(\sum_{i=1}^{2} \int_\Gamma |\nabla u| \cdot |\nabla \partial_i u| \right) < \infty.$$

We have:

$$\int_\Gamma |\nabla\Psi\cdot n|\,\big||\partial_\tau|\nabla u_n|^2|-|\partial_\tau|\nabla u|^2|\big|\,e^\Psi$$
$$\leq C\|\Psi\|_{\mathcal{C}^1(\overline{\Omega})}\|e^\Psi\|_{\mathcal{C}^0(\overline{\Omega})}\left(\sum_{i,j=1}^2\left(\int_\Gamma|\partial_j u|^2\right)^{\frac12}\left(\int_\Gamma|\partial_{ij}u_n-\partial_{ij}u|^2\right)^{\frac12}+\left(\int_\Gamma|\partial_{ij}u_n|^2\right)^{\frac12}\left(\int_\Gamma|\partial_j u_n-\partial_j u|^2\right)^{\frac12}\right).$$

According to (4.2), the sequence $(\partial_{ij}u_n)_{n\in\mathbb{N}}$ converges in $L^2(\Gamma)$ towards $\partial_{ij}u$ and $\|\partial_{ij}u_n\|_{L^2(\Gamma)}$ is bounded by a constant independent of n. Then, equality (4.3) follows from (4.2).

4.2 The stationary case

For the stationary problem:

$$\left\{\begin{array}{rcll} -\Delta u+\nabla p &=& 0, & \text{in } \Omega,\\ \nabla\cdot u &=& 0, & \text{in } \Omega,\\ \nabla u\cdot n-pn &=& g, & \text{on } \Gamma_e,\\ \nabla u\cdot n-pn+qu &=& 0, & \text{on } \Gamma_0. \end{array}\right. \tag{4.4}$$

we have the following stability estimate.

Theorem 4.3. *Let $\alpha>0$, $M_1>0$, $M_2>0$, $(g,q_j)\in H^{\frac52}(\Gamma_e)^2\times B_{\frac52,1}(\Gamma_0)$ for $j=1,2$ such that g is not identically zero, $\|g\|_{H^{\frac52}(\Gamma_e)}\leq M_1$, $q_j\geq\alpha$ on Γ_0 and $\|q_j\|_{B_{\frac52,1}(\Gamma_0)}\leq M_2$. We note (u_j,p_j) the solution of (4.4) associated to q_j for $j=1,2$. Let K be a compact subset of $\{x\in\Gamma_0\ u_1\neq0\}$ and let $m>0$ be a constant such that $|u_1|\geq m$ on K. Then there exist positive constants $C(m,M_1,M_2,\alpha)$ and C_1 such that*

$$\|q_1-q_2\|_{L^2(K)}\leq\frac{C(m,M_1,M_2,\alpha)}{\left(\ln\left(\frac{C_1}{\|u_1-u_2\|_{L^2(\Gamma_e)^2}+\|\nabla(u_1-u_2)\cdot n\|_{L^2(\Gamma_e)^2}+\|p_1-p_2\|_{L^2(\Gamma_e)}+\|\nabla(p_1-p_2)\cdot n\|_{L^2(\Gamma_e)}}\right)\right)^{\frac12}}.$$

Remark 4.4. *Note that, thanks to Corollary 3.2, we ensures that $\{x\in\Gamma_0\ u_1\neq0\}$ is not empty. Then, thanks to the continuity of u_1, we obtain the existence of a compact K and a constant m as in Theorem 4.3.*

Remark 4.5. *In [12], the same kind of inequality is proved for the Laplacian problem with Robin boundary conditions under the hypothesis that the measurements are small enough. Here, we free ourselves from this smallness assumption on the measurements.*

Remark 4.6. *Remark that, thanks to the boundary condition on Γ_e, inequality in Theorem 4.3 can be reduced to:*

$$\|q_1-q_2\|_{L^2(K)}\leq\frac{C(m,M_1,M_2,\alpha)}{\left(\ln\left(\frac{C_1}{\|u_1-u_2\|_{L^2(\Gamma_e)^2}+\|p_1-p_2\|_{L^2(\Gamma_e)}+\|\nabla(p_1-p_2)\cdot n\|_{L^2(\Gamma_e)}}\right)\right)^{\frac12}}.$$

Remark 4.7. *Comparing this result with Proposition 3.3, we can emphasize some differences. In Proposition 3.3, we only need that $u_1=u_2$ on γ where γ is a part of the boundary included in Γ_e in order to recover the Robin coefficient everywhere on Γ_0. However, the proof uses the fact that the constraints are equal on γ. Here, we need to have measurements on the whole part of the boundary Γ_e, and the constraint is divided into two terms: $\|\nabla(u_1-u_2)\cdot n\|_{L^2(\Gamma_e)^2}$ in one hand and $\|p_1-p_2\|_{L^2(\Gamma_e)}$ in the other hand. Moreover, we have an additional term: $\|\nabla(p_1-p_2)\cdot n\|_{L^2(\Gamma_e)}$.*

Let us begin by proving this intermediate result which gives us a logarithmic estimate of the traces of u, ∇u, p, ∇p over Γ_0 with respect to the ones over Γ_e.

Lemma 4.8. *Let $(u,p)\in H^4(\Omega)^2\times H^3(\Omega)$ be the solution in Ω of*

$$\left\{\begin{array}{rcl} -\Delta u+\nabla p &=& 0,\\ \nabla\cdot u &=& 0. \end{array}\right.$$

We assume that there exists $A>0$ such that

$$\|u\|_{H^4(\Omega)^2}+\|p\|_{H^3(\Omega)}\leq A. \tag{4.5}$$

Then there exist $C(A)$ and C_1 such that:

$$\|u\|_{L^2(\Gamma_0)^2}+\|\nabla u\|_{L^2(\Gamma_0)^4}+\|p\|_{L^2(\Gamma_0)}+\|\nabla p\|_{L^2(\Gamma_0)^2} \le \frac{C(A)}{\left(\ln\left(\frac{C_1}{\|u\|_{L^2(\Gamma_e)^2}+\|\nabla u\cdot n\|_{L^2(\Gamma_e)^2}+\|p\|_{L^2(\Gamma_e)}+\|\nabla p\cdot n\|_{L^2(\Gamma_e)}}\right)\right)^{\frac{1}{2}}}.$$

Proof of Lemma 4.8. The proof is based on the Carleman inequality of Lemma 4.1 for an appropriate choice of Ψ. Note that we will use (4.1) twice: one time for the velocity u and one time for the pressure p. The weight function Ψ is chosen in order to estimate the traces over Γ_0 with respect to the ones on Γ_e.

Step 1: choice of Ψ.
We choose Ψ as in [13]. There exists $\Psi_0 \in \mathcal{C}^2(\overline{\Omega})$ non identically zero such that:

$$\Delta\Psi_0 = 0 \text{ in } \Omega, \quad \Psi_0 = 0 \text{ on } \Gamma_0, \quad \Psi_0 \ge 0 \text{ on } \Gamma_e, \quad \nabla\Psi_0\cdot n < 0 \text{ on } \Gamma_0.$$

Indeed, let $\chi \in \mathcal{C}^2(\Gamma)$ such that

$$\chi = 0 \text{ on } \Gamma_0, \quad \chi \ge 0 \text{ on } \Gamma_e,$$

and χ non identically zero on Γ_e.The boundary value problem :

$$\begin{cases} \Delta\Psi_0 &= 0, & \text{in } \Omega,\\ \Psi_0 &= \chi, & \text{on } \Gamma,\end{cases}$$

has a unique solution $\Psi_0 \in \mathcal{C}^2(\overline{\Omega})$. Note that Ψ_0 is not constant because χ is non identically zero. So, from the strong maximum principle, $\Psi_0 > 0$ in Ω. According to Hopf Lemma, we have $\nabla\Psi_0\cdot n < 0$ on Γ_0.

Let $\lambda > 0$. Denote $\Psi_1 \in \mathcal{C}^2(\overline{\Omega})$ the unique solution of the boundary value problem:

$$\begin{cases} \Delta\Psi_1 &= \lambda, & \text{in } \Omega,\\ \Psi_1 &= 0, & \text{on } \Gamma.\end{cases}$$

From the comparison principle and the strong maximum principle, we have $\Psi_1 < 0$ in Ω. Moreover, according to the Hopf Lemma, we have $\nabla\Psi_1\cdot n > 0$ on Γ.

Let us consider $\Psi = \Psi_1 + s\Psi_0$ for $s > 0$. To summarize, the function Ψ has the following properties:

$$\Delta\Psi = \lambda \text{ in } \Omega, \quad \Psi = 0 \text{ on } \Gamma_0, \quad \Psi \ge 0 \text{ on } \Gamma_e, \text{ and } \quad s\nabla\Psi_0\cdot n \le \nabla\Psi\cdot n \le \nabla\Psi_1\cdot n \text{ on } \Gamma_0.$$

Step 2: We first apply Lemma 4.1 to u. Using the fact that $\Delta u = \nabla p$, we have:

$$\int_\Omega (\Delta\Psi|u|^2 + (\Delta\Psi-1)|\nabla u|^2)e^\Psi \le \int_\Omega |\nabla p|^2 e^\Psi + \int_\Gamma \nabla\Psi\cdot n(|u|^2+|\nabla u|^2+2|\partial_\tau|\nabla u|^2|)e^\Psi. \tag{4.6}$$

We then apply once again Lemma 4.1 to p, it yields:

$$\int_\Omega (\Delta\Psi|p|^2 + (\Delta\Psi-1)|\nabla p|^2)e^\Psi \le \int_\Omega |\Delta p|^2 e^\Psi + \int_\Gamma \nabla\Psi\cdot n(|p|^2+|\nabla p|^2+2|\partial_\tau|\nabla p|^2|)e^\Psi. \tag{4.7}$$

We have $\Delta p = \mathrm{div}(\Delta u) = 0$ hence $\int_\Omega |\Delta p|^2 e^\Psi = 0$. We now choose $\lambda \ge 2$. By summing up inequalities (4.6) and (4.7) and eliminating the integrals over Ω in the left hand side which are positive terms, we obtain:

$$\int_\Gamma \nabla\Psi\cdot n(|u|^2+|\nabla u|^2+2|\partial_\tau|\nabla u|^2|)e^\Psi + \int_\Gamma \nabla\Psi\cdot n(|p|^2+|\nabla p|^2+2|\partial_\tau|\nabla p|^2|)e^\Psi \ge 0.$$

We now specify the dependence with respect to s. We denote $\theta = \min_{\Gamma_0} |\nabla\Psi_0 \cdot n|$. We note that on Γ_0, $e^{\Psi} = 1$. Consequently:

$$-s\theta\int_{\Gamma_0}(|u|^2+|\nabla u|^2+|p|^2+|\nabla p|^2)+\int_{\Gamma_0}\nabla\Psi_1\cdot n(|u|^2+|\nabla u|^2+|p|^2+|\nabla p|^2)$$
$$+2\int_{\Gamma_0}\nabla\Psi\cdot n(|\partial_\tau|\nabla p|^2|+|\partial_\tau|\nabla u|^2|)+2\int_{\Gamma_e}\nabla\Psi\cdot n(|\partial_\tau|\nabla p|^2|+|\partial_\tau|\nabla u|^2|)e^{\Psi}$$
$$+\int_{\Gamma_e}\nabla\Psi\cdot n(|u|^2+|\nabla u|^2+|p|^2+|\nabla p|^2)e^{\Psi}\geq 0. \quad (4.8)$$

Let us study each of the terms. We have:

$$\int_{\Gamma_0}\nabla\Psi_1\cdot n(|u|^2+|\nabla u|^2+|p|^2+|\nabla p|^2)\leq C(\|u\|^2_{H^3(\Omega)^2}+\|p\|^2_{H^3(\Omega)}).$$

Moreover, since $\nabla\Psi\cdot n\leq\nabla\Psi_1\cdot n$ on Γ_0 and using hypothesis (4.5), we obtain:

$$2\int_{\Gamma_0}\nabla\Psi\cdot n(|\partial_\tau|\nabla p|^2|+|\partial_\tau|\nabla u|^2|)\leq C(A)(\|u\|^2_{H^3(\Omega)^2}+\|p\|^2_{H^3(\Omega)}).$$

Since, on Γ_e, $|\nabla\Psi\cdot n|\leq sC$ for $s\geq 1$, using hypothesis (4.5) and Cauchy-Schwarz inequality, we obtain:

$$2\int_{\Gamma_e}\nabla\Psi\cdot n(|\partial_\tau|\nabla p|^2|+|\partial_\tau|\nabla u|^2|)e^{\Psi}\leq sC(A)\left(\int_{\Gamma_e}(|\nabla p|^2+|\nabla u|^2)e^{2\Psi}\right)^{\frac{1}{2}}.$$

Similarly, for $s\geq 1$ we have:

$$\int_{\Gamma_e}\nabla\Psi\cdot n(|u|^2+|\nabla u|^2+|p|^2+|\nabla p|^2)e^{\Psi}\leq Cs\int_{\Gamma_e}(|u|^2+|\nabla u|^2+|p|^2+|\nabla p|^2)e^{\Psi}.$$

Note that e^{Ψ} depends on s over Γ_e. Hence, reassembling these inequalities, inequality (4.8) becomes:

$$\theta\int_{\Gamma_0}(|u|^2+|\nabla u|^2+|p|^2+|\nabla p|^2)\leq C(A)K_s+\frac{1}{s}(\|u\|^2_{H^3(\Omega)^2}+\|p\|^2_{H^3(\Omega)}),$$

where $K_s=\int_{\Gamma_e}(|u|^2+|\nabla u|^2+|p|^2+|\nabla p|^2)e^{\Psi}+\left(\int_{\Gamma_e}(|\nabla p|^2+|\nabla u|^2)e^{2\Psi}\right)^{\frac{1}{2}}$. Remark that, thanks to classical interpolation inequalities (see [1]), there exists $C>0$ such that for all $f\in H^2(\Gamma_e)$:

$$\|\nabla f\cdot\tau\|_{L^2(\Gamma_e)}\leq\|f\|_{H^1(\Gamma_e)}\leq C\|f\|^{\frac{1}{2}}_{L^2(\Gamma_e)}\|f\|^{\frac{1}{2}}_{H^2(\Gamma_e)}.$$

Applying the previous inequality and the assumption (4.5), there exists $C(A)>0$ such that:

$$\int_{\Gamma_e}|\nabla u\cdot\tau|^2\leq C(A)\|u\|_{L^2(\Gamma_e)^2},\quad\text{and}\quad\int_{\Gamma_e}|\nabla p\cdot\tau|^2\leq C(A)\|p\|_{L^2(\Gamma_e)}. \quad (4.9)$$

In order to precise the dependence with respect to s of K_s, we denote:

$$\delta=\int_{\Gamma_e}|u|^2+|p|^2+|\nabla u\cdot n|^2+|\nabla p\cdot n|^2. \quad (4.10)$$

We obtain, using the fact that $\nabla u=\nabla u\cdot n+\nabla u\cdot\tau$ on Γ_e and inequality (4.9), that there exists $C(A)>0$ such that:

$$K_s\leq e^{ks}\left(\int_{\Gamma_e}|u|^2+|\nabla u|^2+|p|^2+|\nabla p|^2+\left(\int_{\Gamma_e}|\nabla p|^2+|\nabla u|^2\right)^{\frac{1}{2}}\right)\leq C(A)e^{ks}(\delta+\delta^{\frac{1}{2}}+\delta^{\frac{1}{4}}),$$

where $k=\max_{\Gamma_e}\Psi_0$. To summarize, using again assumption (4.5), we have that there exists $C(A)>0$ such that:

$$K_s\leq C(A)e^{ks}\delta^{\frac{1}{4}}. \quad (4.11)$$

Hence we get for all $s \geq 1$:

$$\int_{\Gamma_0} (|u|^2 + |\nabla u|^2 + |p|^2 + |\nabla p|^2) \leq C(A) e^{ks} \delta^{\frac{1}{4}} + \frac{C}{s} (\|u\|^2_{H^3(\Omega)^2} + \|p\|^2_{H^3(\Omega)}).$$

Remark that this inequality is trivially verified for $0 < s \leq 1$ by continuity of the trace mapping. To summarize, we have proved:

$$\int_{\Gamma_0} (|u|^2 + |\nabla u|^2 + |p|^2 + |\nabla p|^2) \leq C(A) \left(e^{ks} \delta^{\frac{1}{4}} + \frac{1}{s} \right), \forall s > 0.$$

We now optimize the upper bound with respect to s. We denote $f(s) = e^{ks}\delta^{\frac{1}{4}} + \frac{1}{s}$. Let us study the function f in $\mathbb{R}^*_+$. We have:

$$\begin{cases} \lim_{s\to 0} f(s) = +\infty, \\ \lim_{s\to\infty} f(s) = +\infty. \end{cases}$$

So since f is continuous on $\mathbb{R}^+_*$, f reaches its minimum at a point $s_0 > 0$. In this point,

$$f'(s_0) = 0 \Leftrightarrow \delta^{\frac{1}{4}} = \frac{e^{-ks_0}}{k{s_0}^2}, \text{ thus } f(s_0) = \frac{1}{ks_0^2} + \frac{1}{s_0}.$$

Hence:

$$\int_{\Gamma_0} (|u|^2 + |\nabla u|^2 + |p|^2 + |\nabla p|^2) \leq \frac{C(A)}{s_0^\beta} \left(\frac{1}{k} + 1 \right),$$

where $\beta = 1$ if $s_0 \geq 1$ and $\beta = 2$ otherwise. But:

$$\frac{1}{\delta^{\frac{1}{4}}} = k{s_0}^2 e^{ks_0} \leq k e^{(k+2)s_0},$$

that is to say:

$$\frac{1}{s_0} \leq \frac{k+2}{ln\left(\frac{1}{k\delta^{\frac{1}{4}}}\right)}.$$

In the same way, when $s_0 < 1$, we obtain:

$$\frac{1}{s_0^2} \leq \frac{k+1}{ln\left(\frac{1}{ke^k\delta^{\frac{1}{4}}}\right)}.$$

Using the fact that $ln\left(x^{\frac{1}{2}}\right) = \frac{1}{2} ln(x)$ for all $x > 0$ and remembering the definition (4.10) of δ, the desired result follows. □

Let us now prove Theorem 4.3.

Proof of Theorem 4.3. We have on Γ_0:

$$(q_2 - q_1)u_1 = q_2(u_1 - u_2) + \nabla(u_1 - u_2) \cdot n - (p_1 - p_2)n,$$

which leads to

$$\|q_1 - q_2\|_{L^2(K)} \leq C(m, M_2) \left(\|u_1 - u_2\|_{L^2(\Gamma_0)^2} + \|\nabla(u_1 - u_2) \cdot n\|_{L^2(\Gamma_0)^2} + \|p_1 - p_2\|_{L^2(\Gamma_0)} \right).$$

Let $j = 1, 2$. According to Proposition 2.6 the solution (u_j, p_j) of problem (4.4) associated to q_j belongs to $H^4(\Omega)^2 \times H^3(\Omega)$ and moreover there exists $C(\alpha, M_1, M_2) > 0$ such that

$$\|u_1 - u_2\|_{H^4(\Omega)^2} \leq C(\alpha, M_1, M_2) \text{ and } \|p_1 - p_2\|_{H^3(\Omega)} \leq C(\alpha, M_1, M_2).$$

Consequently, we can apply Lemma 4.8 and we obtain:

$$\|q_1 - q_2\|_{L^2(K)} \leq \frac{C(\alpha, m, M_1, M_2)}{\left(\ln \left(\frac{C_1}{\|u_1-u_2\|_{L^2(\Gamma_e)^2} + \|\nabla(u_1-u_2)\cdot n\|_{L^2(\Gamma_e)^2} + \|p_1-p_2\|_{L^2(\Gamma_e)} + \|\nabla(p_1-p_2)\cdot n\|_{L^2(\Gamma_e)}} \right) \right)^{\frac{1}{2}}}.$$

□

4.3 Evolution problem

In order to use semigroup properties, we begin by introducing the Stokes operator associated with the Robin boundary conditions on Γ_0.

4.3.1 Properties of the Stokes operator

We recall that the bilinear form a_q is defined by (2.4).

Definition 4.9. *We define the set $\mathcal{D}(A_q)$ as follows:*

$$\mathcal{D}(A_q) = \{u \in V / \exists C > 0, \forall v \in V, |a_q(u,v)| \leq C\|v\|_{L^2(\Omega)^2}\},$$

and the operator $A_q : \mathcal{D}(A_q) \subset H \to H$ by:

$$\forall u \in \mathcal{D}(A_q),\ a_q(u,v) = (A_q u, v)_{L^2(\Omega)^2}, \forall v \in V.$$

Proposition 4.10. *Let $\alpha > 0$ and $q \in L^\infty(\Gamma_0)$ such that $q \geq \alpha$ almost everywhere on Γ_0. The operator A_q has the following properties:*

1. *$A_q \in \mathcal{L}(\mathcal{D}(A_q), H)$ is invertible and its inverse is compact on H.*
2. *A_q is selfadjoint.*

As a consequence, A_q admits a family of eigenvalues ϕ_q^l

$$A_q \phi_q^l = \lambda_q^l \phi_q^l \text{ with } 0 < \lambda_q^1 \leq \lambda_q^2 \leq ... \leq \lambda_q^j \text{ and } \lim_{j \to \infty} \lambda_q^j = +\infty,$$

which is complete and orthogonal both in H and V.

Proof of Proposition 4.10. It relies on classical arguments for which we refer to [8] or [21]. □

Corollary 4.11. *The operator $A_q^{\frac{1}{2}}$ is an isometry from $(V, \| \ \|_{H^1(\Omega)})$ to $(H, \| \ \|_{L^2(\Omega)})$.*

According to the min-max Theorem, since $a_q(u,u) \geq a_\alpha(u,u)$ for all $u \in V$, we have the following lower bound: there exists $\mu > 0$ such that for all $l \geq 1$

$$\lambda_q^l \geq \mu. \tag{4.12}$$

Proposition 4.12. *Let $\alpha > 0$ and $q \in L^\infty(\Gamma_0)$ such that $q \geq \alpha$ almost everywhere on Γ_0. The operator $-A_q$ generates an analytic semigroup on H. This analytic semigroup is explicitly given by:*

$$e^{-tA_q} f = \sum_{l \geq 1} e^{-t\lambda_q^l} (\phi_q^l, f)_{L^2(\Omega)^2} \phi_q^l, \tag{4.13}$$

for all $f \in H$.

Proof of Proposition 4.12. It follows from the construction of the operator A_q. We refer to [18] and [14] for details. □

Proposition 4.13. *Let $k \in \mathbb{N}$. We assume that $q \in B_{\frac{1}{2}+k,1}(\Gamma_0)$ is such that on Γ_0, $q \geq \alpha$. Then for each $f \in H \cap H^k(\Omega)^2$, there exists $u \in H^{k+2}(\Omega)^2$ solution of $A_q u = f$ if and only if there exists $p \in H^{k+1}(\Omega)$ such that (u,p) is solution of the following problem:*

$$\begin{cases} -\Delta u + \nabla p &= f \quad \text{in } \Omega \\ \nabla \cdot u &= 0 \quad \text{in } \Omega \\ \nabla u \cdot n - pn &= 0 \quad \text{on } \Gamma_e \\ \nabla u \cdot n - pn + qu &= 0 \quad \text{on } \Gamma_0 \end{cases} \tag{4.14}$$

Moreover, if we assume that there exists $M > 0$, $\|q\|_{B_{\frac{1}{2}+k,1}(\Gamma_0)} \leq M$, then there exists $C(\alpha, M) > 0$ such that $\|u\|_{H^{2+k}(\Omega)^2} \leq C(\alpha,M)\|f\|_{H^k(\Omega)^2}$.

Proof of Proposition 4.13. It follows from the construction of the operator A_q and from Proposition 2.6. $\square$

Corollary 4.14. *Let $k \in \mathbb{N}^*$ and $q \in B_{\frac{1}{2}+2(k-1),1}(\Gamma_0)$ such that $q \geq \alpha$ on Γ_0. Then $\mathcal{D}(A_q^k) \hookrightarrow H^{2k}(\Omega)^2 \cap H$.*

Proof of Corollary 4.14. For $k = 1$, it is clear. Take now $k = 2$. Let $u \in \mathcal{D}(A_q^2)$. We have

$$A_q^2 u = f \Leftrightarrow \begin{cases} A_q u = v \\ A_q v = f \end{cases}$$

But $v \in \mathcal{D}(A_q) \subset H^2(\Omega)^2 \cap H$ by assumption, so $u \in H^4(\Omega)^2 \cap H$ thanks to the regularity properties of the solution of the Stokes problem summarize in Proposition 2.6. We conclude by induction on k. $\square$

Remark 4.15. *Let us remark that $\mathcal{D}(A_q)$ is not equal to $H^2(\Omega)^2 \cap H$: it comes from the boundary conditions.*

4.3.2 The flux g does not depend on t

In this paragraph, we consider the evolution problem (1.1) given in the introduction. We assume in this part that g does not depend on time. Let $\alpha > 0$, $M_1 > 0$ and $M_2 > 0$. In the following, we assume that

$$g \in H^{\frac{5}{2}}(\Gamma_e)^2 \text{ is non identically zero and } \|g\|_{H^{\frac{5}{2}}(\Gamma_e)^2} \leq M_1, \tag{4.15}$$

$$q \in B_{\frac{5}{2},1}(\Gamma_0) \text{ is such that } \|q\|_{B_{\frac{5}{2},1}(\Gamma_0)} \leq M_2 \text{ and , } q \geq \alpha \text{ on } \Gamma_0. \tag{4.16}$$

Let us prove the following theorem:

Theorem 4.16. *Let $\alpha > 0$, $M_1 > 0$, $M_2 > 0$ and $u_0 \in H \cap H^3(\Omega)^2$. We assume that g satisfies* (4.15) *and for $j = 1, 2$, q_j satisfies* (4.16)*. We note (u_j, p_j) the solution of* (1.1) *associated to q_j. Let K be a compact subset of $\{x \in \Gamma_0\ v_1 \neq 0\}$ where (v_1, τ_1) is the solution of* (4.4) *with $q = q_1$ and let $m > 0$ be a constant such that $|v_1| \geq m$ on K. Then there exist $C(\alpha, m, M_1, M_2) > 0$ and $C_1 > 0$ such that*

$$\|q_1 - q_2\|_{L^2(K)}$$
$$\leq \frac{C(\alpha, m, M_1, M_2)}{\left(\ln\left(\frac{C_1}{\|u_1-u_2\|_{L^\infty(0,+\infty;L^2(\Gamma_e)^2)} + \|\nabla(u_1-u_2)\cdot n\|_{L^\infty(0,+\infty;L^2(\Gamma_e)^2)} + \|p_1-p_2\|_{L^\infty(0,+\infty;L^2(\Gamma_e))} + \|\nabla(p_1-p_2)\cdot n\|_{L^\infty(0,+\infty;L^2(\Gamma_e))}}\right)\right)^{\frac{1}{2}}}.$$

Remark 4.17. *Due the method which relies on semigroup theory, we need measurements during an infinite time.*

Proof of Theorem 4.16. Let $j = 1, 2$ and (v_j, τ_j) be the solution of the stationary problem (2.2) with $q = q_j$. According to Proposition 2.6, (v_j, τ_j) belongs to $H^4(\Omega)^2 \times H^3(\Omega)$ and moreover, thanks to assumptions (4.15) and (4.16), there exists a constant $C(\alpha, M_1, M_2) > 0$ such that

$$\|v_j\|_{H^4(\Omega)^2} + \|\tau_j\|_{H^3(\Omega)} \leq C(\alpha, M_1, M_2). \tag{4.17}$$

We denote $(w_j, \pi_j) = (u_j - v_j, p_j - \tau_j)$. Thanks to Theorem 4.3, we are able to estimate $\|q_1 - q_2\|_{L^2(K)}$ with respect to an increasing function of $(v_1 - v_2)_{|\Gamma_e}$ and $(\tau_1 - \tau_2)_{|\Gamma_e}$ and their respective gradients in L^2 norm. Our objective is now to compare the asymptotic behavior of $u_1 - u_2$ and $p_1 - p_2$ to the solution of the stationary problem $v_1 - v_2$ and $\tau_1 - \tau_2$. More precisely, we are going to prove that:

$$\|w_j(t, .)\|_{H^3(\Omega)^2} + \|\pi_j(t, .)\|_{H^2(\Omega)} \leq g(t),$$

where g is a function which tends to 0 when t goes to $+\infty$. This inequality, combined with Theorem 4.3, will allow us to conclude the proof of Theorem 4.16.

We have that (w_j, π_j) is the solution of the following problem: for $t > 0$

$$\begin{cases} \partial_t w - \Delta w + \nabla \pi &= 0, & \text{in } \Omega, \\ \nabla \cdot w &= 0, & \text{in } \Omega, \\ \nabla w \cdot n - \pi n &= 0, & \text{on } \Gamma_e, \\ \nabla w \cdot n - \pi n + q_j w &= 0, & \text{on } \Gamma_0. \end{cases}$$

completed with the initial condition $w(0) = u_0 - v_j$. Let $t > 0$. We have from the theory of analytic semigroup that:

$$w_j(t,.) = e^{-tA_{q_j}} w_j(0,.). \tag{4.18}$$

Let $\eta > 0$. There exists a constant $C > 0$ independent of q_j such that:

$$\|A_{q_j}^{\eta} e^{-tA_{q_j}}\| \le C\frac{e^{-\mu t}}{t^{\eta}},\ t > 0,\ \eta > 0, \tag{4.19}$$

where μ is given by (4.12) and where $\|\ \|$ is the norm operator. We obtain from Proposition 4.13, that:

$$\|w_j(t,.)\|_{H^3(\Omega)^2} \le C(\alpha, M_2)\|A_q w_j(t,.)\|_{H^1(\Omega)^2}.$$

Then, since $w_j(t,.)$ is given by (4.18), and using Corollary 4.11 plus estimates (4.19) with $\eta = \frac{3}{2}$ and (4.17), it follows:

$$\begin{aligned}\|w_j(t,.)\|_{H^3(\Omega)^2} &\le C(\alpha, M_2)\|A_q^{\frac{3}{2}} e^{-tA_{q_j}} w_j(0,.)\|_{L^2(\Omega)^2} \le C(\alpha, M_2)\frac{e^{-\mu t}}{t^{\frac{3}{2}}}(\|u_0\|_{L^2(\Omega)^2} + \|v_j\|_{L^2(\Omega)^2})\\ &\le C(\alpha, u_0, M_1, M_2)\frac{e^{-\mu t}}{t^{\frac{3}{2}}}.\end{aligned} \tag{4.20}$$

Using the regularity result for the stationary case given in Proposition 2.6, we have that:

$$\|\pi_j(t,.)\|_{H^2(\Omega)} \le C(\alpha, M_2)\|\partial_t w_j(t,.)\|_{L^2(\Omega)^2}. \tag{4.21}$$

Note that, thanks to Proposition 4.13 we have:

$$\|\partial_t w_j(t,.)\|_{L^2(\Omega)^2} = \|A_{q_j} w_j(t,.)\|_{L^2(\Omega)^2}.$$

Thus, since $w_j(t,.)$ is given by (4.18), we deduce from estimates (4.19) with $\eta = 1$ and (4.17) that:

$$\|\pi_j(t,.)\|_{H^2(\Omega)} \le C(\alpha, u_0, M_1, M_2)\frac{e^{-\mu t}}{t}. \tag{4.22}$$

Remark that $(u_j, p_j) \in L^\infty(0, +\infty; H^3(\Omega)^2) \times L^\infty(0, +\infty; H^2(\Omega))$. Let $\nu > 0$. In fact, thanks to equations (4.20) and (4.22), we obtain that $(w_j, \pi_j) \in L^\infty(\nu, +\infty; H^3(\Omega)^2) \times L^\infty(\nu, +\infty; H^2(\Omega))$ and since $u_j = w_j + v_j$ and $p_j = \pi_j + \tau_j$, we deduce that $(u_j, p_j) \in L^\infty(\nu, +\infty; H^3(\Omega)^2) \times L^\infty(\nu, +\infty; H^2(\Omega))$. Moreover, thanks to Corollary 2.8, we have $(u_j, p_j) \in L^\infty(0, \nu; H^3(\Omega)^2) \times L^\infty(0, \nu; H^2(\Omega))$.

We are now able to prove Theorem 4.16. We have from (4.20):

$$\|v_1 - v_2\|_{L^2(\Gamma_e)^2} \le C(\alpha, u_0, M_1, M_2)\frac{e^{-\mu t}}{t^{\frac{3}{2}}} + \|u_1 - u_2\|_{L^\infty(0,+\infty;L^2(\Gamma_e)^2)}.$$

Then, passing to the limit when t goes to infinity, we get:

$$\|v_1 - v_2\|_{L^2(\Gamma_e)^2} \le \|u_1 - u_2\|_{L^\infty(0,+\infty;L^2(\Gamma_e)^2)}.$$

We prove similarly:

$$\|\nabla(v_1 - v_2)\cdot n\|_{L^2(\Gamma_e)^2} \le \|\nabla(u_1 - u_2)\cdot n\|_{L^\infty(0,+\infty;L^2(\Gamma_e)^2)}.$$

In the same way, but using now (4.22), we obtain:

$$\|\tau_1 - \tau_2\|_{L^2(\Gamma_e)} \le \|p_1 - p_2\|_{L^\infty(0,+\infty;L^2(\Gamma_e))},$$

and

$$\|\nabla(\tau_1 - \tau_2)\cdot n\|_{L^2(\Gamma_e)} \le \|\nabla(p_1 - p_2)\cdot n\|_{L^\infty(0,+\infty;L^2(\Gamma_e))}.$$

To summarize, we have obtained:

$$\begin{aligned}&\|v_1 - v_2\|_{L^2(\Gamma_e)^2} + \|\nabla(v_1 - v_2)\cdot n\|_{L^2(\Gamma_e)^2} + \|\tau_1 - \tau_2\|_{L^2(\Gamma_e)} + \|\nabla(\tau_1 - \tau_2)\cdot n\|_{L^2(\Gamma_e)}\\ &\le \|u_1 - u_2\|_{L^\infty(0,+\infty;L^2(\Gamma_e)^2)} + \|\nabla(u_1 - u_2)\cdot n\|_{L^\infty(0,+\infty;L^2(\Gamma_e)^2)} + \|p_1 - p_2\|_{L^\infty(0,+\infty;L^2(\Gamma_e))} + \|\nabla(p_1 - p_2)\cdot n\|_{L^\infty(0,+\infty;L^2(\Gamma_e))}\end{aligned}$$

Applying Theorem 4.3 to (v_j, τ_j) for $j = 1, 2$, we obtain the existence of positive constants $C(m, M_1, M_2, \alpha)$ and C_1 such that

$$\|q_1 - q_2\|_{L^2(K)} \le \frac{C(m, M_1, M_2, \alpha)}{\left(\ln\left(\frac{C_1}{\|v_1-v_2\|_{L^2(\Gamma_e)^2} + \|\nabla(v_1-v_2)\cdot n\|_{L^2(\Gamma_e)^2} + \|\tau_1-\tau_2\|_{L^2(\Gamma_e)} + \|\nabla(\tau_1-\tau_2)\cdot n\|_{L^2(\Gamma_e)}}\right)\right)^{\frac{1}{2}}}.$$

We conclude by using the fact that the function $x \to \frac{1}{ln(\frac{1}{x})}$ increases on $\mathbb{R}_+^*$. □

4.3.3 The flux g depends on t

We restrict our study to the case where g is colinear to the outgoing normal n: $g = \kappa\, n$.
Let $\alpha > 0$, $M_1 > 0$ and $M_2 > 0$. We assume that:

$$\kappa \in H^2_{loc}(0,+\infty; H^{\frac{3}{2}}(\Gamma_e)), \tag{4.23}$$

and

$$q \in B_{\frac{5}{2},1}(\Gamma_0) \text{ is such that } \|q\|_{B_{\frac{5}{2},1}(\Gamma_0)} \leq M_2 \text{ and , } q \geq \alpha \text{ on } \Gamma_0. \tag{4.24}$$

Let us introduce h such that:

$$h \in H^{\frac{5}{2}}(\Gamma_e) \text{ is non identically zero and } \|h\|_{H_{\frac{5}{2}}(\Gamma_0)} \leq M_1. \tag{4.25}$$

We suppose that:

$$\lim_{t\to\infty}\left(\|\kappa(t,.)-h\|_{H^{\frac{3}{2}}(\Gamma_e)} + \left\{\int_0^t e^{-\mu(t-s)}\|\partial_t\kappa(s,.)\|^2_{H^{\frac{3}{2}}(\Gamma_e)}ds\right\}^{\frac{1}{2}}\right) = 0, \tag{4.26}$$

where μ is given by equation (4.12).

Theorem 4.18. *Let $\alpha > 0$, $M_1 > 0$, $M_2 > 0$ and $u_0 \in H^3(\Omega)^2 \cap H$. We assume that h and κ satisfy respectively (4.25) and (4.23) and for $j = 1, 2$, q_j satisfies (4.24). We denote by (u_j, p_j) the solution of (1.1) associated to q_j. Let K be a compact subset of $\{x \in \Gamma_0\ v_1 \neq 0\}$ where (v_1, τ_1) is the solution of*

$$\begin{cases} -\Delta v + \nabla\tau &= 0, & \text{in } \Omega,\\ \nabla\cdot v &= 0, & \text{in } \Omega,\\ \nabla v\cdot n - \tau n &= hn, & \text{on } \Gamma_e,\\ \nabla v\cdot n - \tau n + q_1 v &= 0, & \text{on } \Gamma_0.\end{cases}$$

We assume that (4.26) is verified. Then there exist $C(\alpha, m, M_1, M_2) > 0$ and $C_1 > 0$ such that

$$\|q_1 - q_2\|_{L^2(K)} \leq \frac{C(\alpha,m,M_1,M_2)}{\left(\ln\left(\frac{C_1}{\|u_1-u_2\|_{L^\infty(0,+\infty;L^2(\Gamma_e)^2)}+\|\nabla(u_1-u_2)\cdot n\|_{L^\infty(0,+\infty;L^2(\Gamma_e)^2)}+\|p_1-p_2\|_{L^\infty(0,+\infty;L^2(\Gamma_e))}+\|\nabla(p_1-p_2)\cdot n\|_{L^\infty(0,+\infty;L^2(\Gamma_e))}}\right)\right)^{\frac{1}{2}}}.$$

Proof of Theorem 4.18. For $j = 1, 2$, we decompose u_j into $u_j = v_j + w_j$ where $(v_j, \tau_j) \in H^4(\Omega)^2 \times H^3(\Omega)$ is the solution of the stationary problem:

$$\begin{cases} -\Delta v + \nabla\tau &= 0, & \text{in } \Omega,\\ \nabla\cdot v &= 0, & \text{in } \Omega,\\ \nabla v\cdot n - \tau n &= hn, & \text{on } \Gamma_e,\\ \nabla v\cdot n - \tau n + q_j v &= 0, & \text{on } \Gamma_0.\end{cases}$$

and (w_j, π_j) is solution of the following problem:

$$\begin{cases} \partial_t w - \Delta w + \nabla\pi &= 0, & \text{in } (0,+\infty)\times\Omega,\\ \nabla\cdot w &= 0, & \text{in } (0,+\infty)\times\Omega,\\ \nabla w\cdot n - \pi n &= (\kappa - h)n, & \text{on } (0,+\infty)\times\Gamma_e,\\ \nabla w\cdot n - \pi n + q_j w &= 0, & \text{on } (0,+\infty)\times\Gamma_i,\\ w(0,x) &= u_0(x) - v_j(x), & \text{in } \Omega.\end{cases}$$

We would like to perform the same reasoning as in Theorem 4.16. That is to say, using the fact that we are able to estimate $\|q_1 - q_2\|_{L^2(K)}$ with respect to an increasing function of $(v_1 - v_2)_{|\Gamma_e}$ and $(\tau_1 - \tau_2)_{|\Gamma_e}$ in H^1 norm, we want to compare the asymptotic behavior of $u_1 - u_2$ and $p_1 - p_2$ to the solution of the stationary problem $v_1 - v_2$ and $\tau_1 - \tau_2$. More precisely, we are going to prove that:

$$\|w_j(t,.)\|_{H^3(\Omega)^2} + \|\nabla\pi_j(t,.)\|_{H^1(\Omega)^2} \leq g(t),$$

where g is a function which tends to 0 when t goes to $+\infty$. Since the function κ depends on t, there will be one more step than in Theorem 4.16 and that is why we assume (4.26).

We divide (w_j,π_j) into two terms: $w_j = u_j^0 + \tilde{w}_j$ and $\pi_j = p_j^0 + \tilde{p}_j$, where (u_j^0, p_j^0) is solution of

$$\left\{\begin{array}{ccll} \partial_t u^0 - \Delta u^0 + \nabla p^0 & = & 0, & \text{in } (0,+\infty)\times\Omega, \\ \nabla\cdot u^0 & = & 0, & \text{in } (0,+\infty)\times\Omega, \\ \nabla u^0\cdot n - p^0 n & = & (\kappa-h)n, & \text{on } (0,+\infty)\times\Gamma_e, \\ \nabla u^0\cdot n - p^0 n + q_j u^0 & = & 0, & \text{on } (0,+\infty)\times\Gamma_i, \\ u^0(0,x) & = & 0, & \text{in } \Omega, \end{array}\right.$$

and $(\tilde{w}_j,\tilde{p}_j)$ is solution of

$$\left\{\begin{array}{ccll} \partial_t \tilde{w} - \Delta \tilde{w} + \nabla \tilde{p} & = & 0, & \text{in } (0,+\infty)\times\Omega, \\ \nabla\cdot \tilde{w} & = & 0, & \text{in } (0,+\infty)\times\Omega, \\ \nabla \tilde{w}\cdot n - \tilde{p} n & = & 0, & \text{on } (0,+\infty)\times\Gamma_e, \\ \nabla \tilde{w}\cdot n - \tilde{p} n + q_j \tilde{w} & = & 0, & \text{on } (0,+\infty)\times\Gamma_i, \\ \tilde{w}(0,x) & = & u_0(x) - v_j(x), & \text{in } \Omega. \end{array}\right.$$

Let $t>0$. Using the same arguments as in the previous subsection, we prove that:

$$\|\tilde{w}_j(t,.)\|_{H^3(\Omega)^2} \le C(\alpha,u_0,M_1,M_2)\frac{e^{-\mu t}}{t^2}, \tag{4.27}$$

and

$$\|\tilde{p}_j(t,.)\|_{H^2(\Omega)} \le C(\alpha,u_0,M_1,M_2)\frac{e^{-\mu t}}{t}. \tag{4.28}$$

It remains for us to bound $\|u_j^0(t,.)\|_{H^3(\Omega)^2}$ and $\|\nabla p_j^0(t,.)\|_{H^1(\Omega)^2}$. Let $t>0$. We are going to prove that there exists a constant $C(\alpha,M_2)$ such that:

$$\begin{aligned} &\|u_j^0(t,.)\|_{H^3(\Omega)^2} + \|\nabla p_j^0(t,.)\|_{H^1(\Omega)^2} \\ &\quad\le C(\alpha,M_2)\left(\|\kappa(t,.)-h\|_{H^{\frac{3}{2}}(\Gamma_e)} + e^{-\mu t}\|\kappa(0,.)-h\|_{H^{\frac{3}{2}}(\Gamma_e)} + \left\{\int_0^t e^{-\mu(t-s)}\|\partial_t\kappa(s,.)\|^2_{H^{\frac{3}{2}}(\Gamma_e)}ds\right\}^{\frac{1}{2}}\right). \end{aligned} \tag{4.29}$$

If inequality (4.29) is satisfied, we can end the proof of Theorem 4.18:

$$\|w_1(t,.)-w_2(t,.)\|_{H^3(\Omega)^2} \le \|u_1^0(t,.)-u_2^0(t,.)\|_{H^3(\Omega)^2} + \|\tilde{w}_1(t,.)-\tilde{w}_2(t,.)\|_{H^3(\Omega)^2},$$

$$\|\nabla\pi_1(t,.)-\nabla\pi_1(t,.)\|_{H^1(\Omega)^2} \le \|\nabla p_1^0(t,.)-\nabla p_2^0(t,.)\|_{H^1(\Omega)^2} + \|\nabla\tilde{p}_1(t,.)-\nabla\tilde{p}_2(t,.)\|_{H^1(\Omega)^2},$$

and in the following two estimates, the right hand side tends to 0 when t goes to infinity thanks to inequalities (4.27), (4.28) and assumption (4.26).

We introduce (y_j,ρ_j) the solution of

$$\left\{\begin{array}{ccll} -\Delta y + \nabla\rho & = & 0, & \text{in } \Omega, \\ \nabla\cdot y & = & 0, & \text{in } \Omega, \\ \nabla y\cdot n - \rho n & = & (\kappa-h)n, & \text{on } \Gamma_e, \\ \nabla y\cdot n - \rho n + q_j y & = & 0, & \text{on } \Gamma_i, \end{array}\right.$$

for all $t>0$. We know that $(y_j(t,.),\rho_j(t,.)) \in H^3(\Omega)^2\times H^2(\Omega)$ and satisfies, thanks to Proposition 2.6:

$$\|y_j(t,.)\|_{H^3(\Omega)^2} + \|\rho_j(t,.)\|_{H^2(\Omega)} \le C(\alpha,M_2)\|\kappa(t,.)-h\|_{H^{\frac{3}{2}}(\Gamma_e)}. \tag{4.30}$$

Remark that $y_j(t,.)$ belongs to $\mathcal{D}(A_{q_j}^{\frac{3}{2}})$. Indeed, there exists a unique $\tilde{p}(t,.)\in H^3(\Omega)$ solution of

$$\left\{\begin{array}{ccll} \Delta\tilde{p} & = & 0, & \text{in } \Omega, \\ \tilde{p} & = & \kappa-h, & \text{on } \Gamma_e, \\ \tilde{p} & = & 0, & \text{on } \Gamma_i, \end{array}\right. \tag{4.31}$$

for all $t > 0$ and there exists a constant $C > 0$ such that

$$\|\tilde{p}(t,.)\|_{H^3(\Omega)} \leq C\|\kappa(t,.) - h\|_{H^{\frac{3}{2}}(\Gamma_e)}. \tag{4.32}$$

Then $(y_j, \rho_j + \tilde{p})$ is solution of

$$\begin{cases} -\Delta y + \nabla(\rho + \tilde{p}) & = & \nabla \tilde{p}, & \text{in } \Omega, \\ \nabla \cdot y & = & 0, & \text{in } \Omega, \\ \nabla y \cdot n - (\rho + \tilde{p})n & = & 0, & \text{on } \Gamma_e, \\ \nabla y \cdot n - (\rho + \tilde{p})n + q_j y & = & 0, & \text{on } \Gamma_0, \end{cases}$$

for all $t > 0$. Remark that, since $\nabla\tilde{p} \in L^2(\Omega)$, we have that $y_j(t) \in \mathcal{D}(A_{q_j})$ by definition of $\mathcal{D}(A_{q_j})$. Notice that the fact that g is colinear to n is important here to do the *change of variable* in the pressure. We deduce from $A_{q_j} y_j(t) = \nabla\tilde{p}(t) \in V = \mathcal{D}(A_{q_j}^{\frac{1}{2}})$ that $y_j(t) \in \mathcal{D}(A_{q_j}^{\frac{3}{2}})$. Moreover, using Corollary 4.11 and inequality (4.32), there exists a constant $C > 0$ such that:

$$\|A_{q_j}^{\frac{3}{2}} y_j(t,.)\|_{L^2(\Omega)^2} = \|A_{q_j} y_j(t,.)\|_{H^1(\Omega)^2} = \|\nabla\tilde{p}(t)\|_{H^1(\Omega)^2} \leq C\|\kappa(t,.) - h\|_{H^{\frac{3}{2}}(\Gamma_e)}, \tag{4.33}$$

that is to say, using moreover (4.30):

$$\|y_j(t,.)\|_{\mathcal{D}(A_{q_j}^{\frac{3}{2}})} \leq C(\alpha, M_2)\|\kappa(t,.) - h\|_{H^{\frac{3}{2}}(\Gamma_e)}. \tag{4.34}$$

We can use the same argument, replacing $\kappa - h$ by $\partial_t\kappa$, to prove that $\partial_t y_j(t,.) \in \mathcal{D}(A_{q_j}^{\frac{3}{2}})$ together with the estimate

$$\|\partial_t y_j(t,.)\|_{\mathcal{D}(A_{q_j}^{\frac{3}{2}})} \leq C(\alpha, M_2)\|\partial_t\kappa(t,.)\|_{H^{\frac{3}{2}}(\Gamma_e)}. \tag{4.35}$$

Let us consider $\overline{w}_j = u_j^0 - y_j$ and $\overline{p}_j = p_j^0 - \rho_j$. The couple $(\overline{w}_j, \overline{p}_j)$ is solution of

$$\begin{cases} \partial_t w - \Delta w + \nabla p & = & -\partial_t y_j, & \text{in } (0,+\infty) \times \Omega, \\ \nabla \cdot w & = & 0, & \text{in } (0,+\infty) \times \Omega, \\ \nabla w \cdot n - pn & = & 0, & \text{on } (0,+\infty) \times \Gamma_e, \\ \nabla w \cdot n - pn + q_j w & = & 0, & \text{on } (0,+\infty) \times \Gamma_i, \\ w(0,x) & = & -y_j(0,x), & \text{in } \Omega. \end{cases} \tag{4.36}$$

We know that $\overline{w}_j$ is given by:

$$\overline{w}_j(t,.) = -e^{-tA_{q_j}} y_j(0,.) - \int_0^t e^{-(t-s)A_{q_j}} \partial_t y_j(s,.) ds.$$

Using the family $(\phi_{q_j}^l)_{l\geq 1}$ defined by Proposition 4.10, we have: $\overline{w}_j(t,.) = \sum_{l\geq 1} C_l(t)\phi_{q_j}^l$, with

$$C_l(t) = -e^{-t\lambda_{q_j}^l}(y_j(0,.), \phi_{q_j}^l)_{L^2(\Omega)^2} - \int_0^t e^{-(t-s)\lambda_{q_j}^l}(\partial_t y_j(s,.), \phi_{q_j}^l)_{L^2(\Omega)^2} ds.$$

Thus, recalling that $(\lambda_{q_j}^l)_{l\geq 1}$ satisfies (4.12), there exists $C > 0$ such that:

$$C_l(t)^2 \leq 2e^{-2t\mu}(y_j(0,.), \phi_{q_j}^l)_{L^2(\Omega)^2}^2 + C\int_0^t e^{-(t-s)\mu}(\partial_t y_j(s,.), \phi_{q_j}^l)_{L^2(\Omega)^2}^2 ds.$$

We obtain from estimates (4.34) and (4.35):

$$\|\overline{w}_j(t,.)\|_{\mathcal{D}(A_{q_j}^{\frac{3}{2}})} \leq C(\alpha, M_2)\left(e^{-\mu t}\|\kappa(0,.) - h\|_{H^{\frac{3}{2}}(\Gamma_e)} + \left\{\int_0^t e^{-\mu(t-s)}\|\partial_t\kappa(s,.)\|_{H^{\frac{3}{2}}(\Gamma_e)}^2 ds\right\}^{\frac{1}{2}}\right). \tag{4.37}$$

Remark that, thanks to Proposition 4.13 and Corollary 4.11, we have:

$$\|\overline{w}_j(t,.)\|_{H^3(\Omega)^2} \leq C(\alpha, M_2)\|A\overline{w}_j(t,.)\|_{H^1(\Omega)^2} = C(\alpha, M_2)\|A^{\frac{3}{2}}\overline{w}_j(t,.)\|_{L^2(\Omega)^2} \leq C(\alpha, M_2)\|\overline{w}_j(t,.)\|_{\mathcal{D}(A_{q_j}^{\frac{3}{2}})}. \tag{4.38}$$

To summarize, using (4.38) and (4.37), we obtain the desired estimate:

$$\|\overline{w}_j(t,.)\|_{H^3(\Omega)^2} \leq C(\alpha, M_2)\left(e^{-\mu t}\|\kappa(0,.)-h\|_{H^{\frac{3}{2}}(\Gamma_e)} + \left\{\int_0^t e^{-\mu(t-s)}\|\partial_t\kappa(s,.)\|^2_{H^{\frac{3}{2}}(\Gamma_e)}ds\right\}^{\frac{1}{2}}\right).$$

Using now the regularity result for the stationary problem given in Proposition 2.6, we have:

$$\|\nabla\overline{p}_j(t,.)\|_{H^1(\Omega)^2} \leq C(\alpha, M_2)\left(\|\partial_t y_j(t,.)\|_{H^1(\Omega)^2} + \|\partial_t\overline{w}_j(t,.)\|_{H^1(\Omega)^2}\right).$$

Since $A_{q_j}\overline{w}_j = -\partial_t y_j - \partial_t\overline{w}_j$, we obtain:

$$\|\nabla\overline{p}_j(t,.)\|_{H^1(\Omega)^2} \leq C(\alpha, M_2)\left(\|\partial_t y_j(t,.)\|_{H^1(\Omega)^2} + \|A_{q_j}\overline{w}_j(t,.)\|_{H^1(\Omega)^2}\right).$$

Thanks to Corollary 4.11, we know that $\|A_{q_j}\overline{w}_j(t,.)\|_{H^1(\Omega)^2} = \|A^{\frac{3}{2}}_{q_j}\overline{w}_j(t,.)\|_{L^2(\Omega)^2}$. Therefore, using (4.35) and (4.37), we obtain:

$$\|\nabla\overline{p}_j(t,.)\|_{H^1(\Omega)^2} \leq C(\alpha, M_2)\left(\|\kappa(t,.)-h\|_{H^{\frac{3}{2}}(\Gamma_e)} + \left\{\int_0^t e^{-\mu(t-s)}\|\partial_t\kappa(s,.)\|^2_{H^{\frac{3}{2}}(\Gamma_e)}ds\right\}^{\frac{1}{2}}\right).$$

The estimate (4.29) follows from $u_j^0 = \overline{w}_j + y_j$, $p_j^0 = \overline{w}_j + \rho_j$ and inequality (4.30). □

Remark 4.19. *Let $l \in H^2_{loc}(0,+\infty; H^{\frac{3}{2}}(\Gamma_e))$ and $h \in H^{\frac{3}{2}}(\Gamma_e)$. Assume that there exists $\theta > 0$ such that:*

$$\sup_{t\geq 0} e^{t\theta}(\|l(t,.)\|_{H^{\frac{3}{2}}(\Gamma_e)} + \|\partial_t l(t,.)\|_{H^{\frac{3}{2}}(\Gamma_e)}) < \infty,$$

Then $\kappa = h + l$ satisfies (4.26). *We note that a particular case of function satisfying* (4.26) *is given by $l(t,x) = \omega(t)\rho(x)$ where $\omega \in H^2_{loc}(0,+\infty)$, $\rho \in H^{\frac{3}{2}}(\Gamma_e)$ and $\lim_{t\to\infty} e^{t\theta}\omega(t) = \lim_{t\to\infty} e^{t\theta}\omega'(t) = 0$.*

4.4 Conclusion

To conclude, we have proved, under some regularity assumptions on the open set Ω and on the solution (u,p) of system (1.1), logarithmic stability estimates for the Stokes system with mixed Neumann and Robin boundary conditions. Due to the method which relies on the Carleman inequality proved in [9], these estimates are valid in dimension 2 and we need measurements on the whole boundary part Γ_e.
Our result could be improved in many different ways. In particular, a first concern could be to prove a logarithmic stability estimate which is valid in any dimension. It would be also interesting to know if local measurements on a part of the boundary included in Γ_e still allow to get the same result. Moreover, in our stability estimates, we need measurements on Γ_e of u, p, $\nabla u \cdot n$ and $\nabla p \cdot n$, while the identifiability result given by Proposition 3.3 only requires information on u and $\nabla u \cdot n - pn$ on Γ_e. Therefore, it might be interesting to know if it is possible to obtain a stability inequality with less measurement terms and in particular, if it is possible to get rid of the gradient term $\nabla p \cdot n$.

A Existence and uniqueness for the unsteady problem

We study the regularity of solutions of the unsteady problem:

$$\begin{cases} u_t - \Delta u + \nabla p &= 0, & \text{in } (0,T)\times\Omega, \\ \nabla\cdot u &= 0, & \text{in } (0,T)\times\Omega, \\ \nabla u\cdot n - pn &= g, & \text{on } (0,T)\times\Gamma_e, \\ \nabla u\cdot n - pn + qu &= 0, & \text{on } (0,T)\times\Gamma_0, \\ u(0,\cdot) &= u_0, & \text{in } \Omega. \end{cases}$$

We are going to prove Theorem 2.7. First of all, as a preliminary result, we prove the following existence result:

Proposition A.1. *Let $T>0$, $M>0$, $\alpha>0$ and $u_0\in H$. We assume that $g\in L^2(0,T;L^2(\Gamma_e)^d)$ and that $q\in L^\infty(\Gamma_0)$ such that $q\geq\alpha$ on Γ_0. There exists $u\in L^2(0,T;V)$ such that for all $v\in V$ in the distribution sense on $(0,T)$:*

$$\frac{d}{dt}\int_\Omega u\cdot v + \int_\Omega \nabla u:\nabla v + \int_{\Gamma_0} qu\cdot v = \int_{\Gamma_e} g\cdot v, \tag{A.1}$$

and $\forall v\in V$,

$$\int_\Omega u(0)\cdot v = \int_\Omega u_0\cdot v. \tag{A.2}$$

Proof of Proposition A.1. We begin by proving, using a Galerkin method, that there exists $u\in L^2(0,T;V)$ such that

$$\forall v\in V, \forall\psi\in\mathcal{C}^1(0,T) \text{ such that } \psi(T)=0$$

$$-\int_0^T\int_\Omega u(t,x)\cdot v(x)\psi'(t)dxdt + \int_0^T\int_\Omega \nabla u(t,x):\nabla v(x)\psi(t)dxdt$$

$$+\int_0^T\int_{\Gamma_0} q(x)u(t,x)\cdot v(x)\psi(t)dxdt - \psi(0)\int_\Omega u_0(x)\cdot v(x)dx = \int_0^T\int_{\Gamma_e} g(t,x)\cdot v(x)\psi(t)dxdt. \tag{A.3}$$

Let $(w_i)_{i\in\mathbb{N}}$ be a Hilbert basis of V which is also an orthogonal basis of H. For each $n\in\mathbb{N}$, we define an approximate solution as follows: we search $u_n\in V_n = Vect\{w_i\}_{1\leq i\leq n}$ which satisfies

$$\begin{cases} \displaystyle\int_\Omega u_{n,t}\cdot w_j + \int_\Omega \nabla u_n\cdot\nabla w_j + \int_{\Gamma_0} qu_n\cdot w_j = \int_{\Gamma_e} g\cdot w_j, \forall j\in\{1,\ldots,n\}, \\ \displaystyle u_n(0) = \sum_{k=1}^n (u_0,w_k)_{L^2(\Omega)^d} w_k. \end{cases} \tag{A.4}$$

Note that $u_{n,t}$ denotes $\partial_t u_n$.

Let $t\in[0,T]$. We decompose $u_n(t,.)$ in the Hilbert basis:

$$u_n(t,.) = \sum_{i=1}^n \xi_i(t)w_i.$$

We denote A=$\Big[\int_\Omega w_i(x)\cdot w_j(x)dx\Big]_{1\leq i,j\leq n}$, B=$\Big[\int_\Omega \nabla w_i(x):\nabla w_j(x) + \int_{\Gamma_0} q(x)w_i(x)\cdot w_j(x)dx\Big]_{1\leq i,j\leq n}$, $\xi(t) = (\xi_i(t))_{1\leq i\leq n}$ and $L(t) = (\int_{\Gamma_e} g(t,x)\cdot w_i(x)dx)_{1\leq i\leq n}$. We can rewrite the system (A.4) in the form:

$$\begin{cases} A\xi'(t) + B\xi(t) = L(t), \\ \xi(0) = ((u_0,w_i)_{L^2(\Omega)^d})_{1\leq i\leq n}. \end{cases}$$

Since the matrix A is invertible, the system has a unique global solution $\xi\in H^1(0,T)^d$. We are now going to prove that there exists a constant $C>0$ independent of $n\in\mathbb{N}$ such that:

$$\sup_{0\leq t\leq T}\int_\Omega |u_n|^2 + \int_0^T\int_\Omega |\nabla u_n|^2 + \int_0^T\int_\Omega |u_n|^2 \leq C. \tag{A.5}$$

Let $t \in [0,T]$. Multiplying the first equation of (A.4) by ξ_j and summing over j for $j = 1, \dots, n$ we obtain:

$$\int_0^t \int_\Omega u_{n,t} \cdot u_n + \int_0^t \int_\Omega |\nabla u_n|^2 + \int_0^t \int_{\Gamma_0} q|u_n|^2 = \int_0^t \int_{\Gamma_e} g \cdot u_n \tag{A.6}$$

Let $\epsilon > 0$. Thus:

$$\int_0^t \int_{\Gamma_e} g \cdot u_n \leq C \int_0^T \int_{\Gamma_e} |g|^2 + \epsilon \int_0^t \int_{\Gamma_e} |u_n|^2 \leq C \int_0^T \int_{\Gamma_e} |g|^2 + \epsilon \int_0^t \|u_n\|^2_{H^1(\Omega)^d}.$$

Choosing ϵ small enough and using the fact that $q \geq \alpha$ on Γ_0, we obtain:

$$\sup_{t\in[0,T]} \int_\Omega |u_n|^2 + \int_0^T \int_\Omega |\nabla u_n|^2 + \int_0^T \int_\Omega |u_n|^2 \leq C(\int_0^T \int_{\Gamma_e} |g|^2 + \int_\Omega |u_0|^2). \tag{A.7}$$

This gives (A.5). According to inequality (A.5), there exists $u \in L^2(0,T;V)$ such that, up to a subsequence,

$$u_n \rightharpoonup u \text{ in } L^2(0,T;V).$$

Let $j \in \mathbb{N}$. Multiplying the first equation of (A.4) by $\psi \in \mathcal{C}^1([0,T])$ such that $\psi(T) = 0$ then integrating over $(0,T)$, we get, $\forall n \geq j$:

$$\int_0^T \int_\Omega u_{n,t}(t,x) \cdot w_j(x)\psi(t)dxdt + \int_0^T \int_{\Gamma_0} q(x)u_n(t,x) \cdot w_j(x)\psi(t)dxdt$$
$$+ \int_0^T \int_\Omega \nabla u_n(t,x) : \nabla w_j(x)\psi(t)dxdt = \int_0^T \int_{\Gamma_e} g(t,x) \cdot w_j(x)\psi(t)dxdt. \tag{A.8}$$

Taking into account that:

$$\int_0^T \int_\Omega u_{n,t}(t,x) \cdot w_j(x)\psi(t)dxdt$$
$$= -\int_0^T \int_\Omega u_n(t,x) \cdot w_j(x)\psi'(t)dxdt - \int_\Omega u_n(0,x) \cdot w_j(x)\psi(0)dx,$$

we easily pass to the limit when n goes to infinity in (A.8). Remark that this inequality is still valid replacing w_j by any $v \in V$ by continuity. This ends the proof of the existence of $u \in L^2(0,T;V)$ which satisfies (A.1) in the distribution sense on $(0,T)$.

Let us finish the proof of Proposition A.1 by proving that the initial condition (A.2) is satisfied. We deduce from equality (A.3) that $\frac{d}{dt}(u,v)_{L^2(\Omega)^d} \in L^2(0,T)$. Consequently, the function $t \to (u(t),v)_{L^2(\Omega)}$ is continuous. This gives a sense to $(u(0),v)_{L^2(\Omega)^d}$. Let $\psi \in \mathcal{C}^1(0,T)$ such that $\psi(T) = 0$. Multiplying (A.1) by ψ, we obtain:

$$-\int_0^T (u,v)_{L^2(\Omega)^d}\psi'(t)dt + \int_0^T a_q(u,v)\psi(t)dt = (u(0,.),v)_{L^2(\Omega)^d}\psi(0) + \int_0^T l(v)\psi(t)dt.$$

Comparing with equality (A.3), we obtain $\psi(0)(u(0,.) - u_0, v)_{L^2(\Omega)^d} = 0$. Let ψ be such that $\psi(0) \neq 0$, we have $(u(0) - u_0, v)_{L^2(\Omega)^d} = 0$, $\forall v \in V$. □

We are now able to prove Theorem 2.7.

Proof of Theorem 2.7. We will begin by proving that $u_t \in L^2(0,T;H)$, then we will conclude by using the regularity result for the stationary problem from Proposition 2.6.
Let $t \in [0,T]$. Multiplying the first equation of (A.4) by ξ'_j and summing over j for $j = 1, \dots, n$ we obtain:

$$\int_0^t \int_\Omega |u_{n,t}|^2 + \int_0^t \int_{\Gamma_0} q u_n u_{n,t} + \int_0^t \int_\Omega \nabla u_n : \nabla u_{n,t} = \int_0^t \int_{\Gamma_e} g \cdot u_{n,t}.$$

We have:

$$\int_0^t \int_{\Gamma_e} g \cdot u_{n,t} = -\int_0^t \int_{\Gamma_e} g_t u_n - \int_{\Gamma_e} g(0)u_n(0) + \int_{\Gamma_e} g(t)u_n(t). \tag{A.9}$$

Let $\epsilon > 0$. Thanks to Cauchy-Schwarz inequality and estimate (A.7), there exists $C > 0$:

$$\left|\int_0^t \int_{\Gamma_e} g_t u_n\right| \leq C\left(\int_0^T \int_{\Gamma_e} |g_t|^2 + \sup_{t\in[0,T]} \int_{\Gamma_e} |g|^2 + \|u_0\|^2_{H^1(\Omega)^2} + \epsilon \int_\Omega |\nabla u_n(t)|^2\right).$$

Using successively integration by parts over $(0,T)$ we finally obtain, choosing ϵ small enough:

$$\sup_{t\in[0,T]} \int_\Omega |\nabla u_n|^2 + \int_0^T \int_\Omega |u_{n,t}|^2 \leq C\left(\|u_0\|^2_{H^1(\Omega)^d} + \int_0^T \int_{\Gamma_e} |g_t|^2 + \sup_{t\in[0,T]} \int_{\Gamma_e} |g|^2\right). \tag{A.10}$$

We deduce that $(u_n)_{n\in\mathbb{N}}$ is bounded in $H^1(0,T;H)\cap L^\infty(0,T;V)$ and therefore $u \in H^1(0,T;H)\cap L^\infty(0,T;V)$.

Then we use the regularity result for the stationary problem. For all $t \in [0,T]$, we have $u_t(t) \in L^2(\Omega)^d$ so by Proposition 2.6 the solution $(u(t),p(t))$ belongs to $H^2(\Omega)^d \times H^1(\Omega)$ since the map:

$$\begin{array}{rcl} L^2(\Omega)^d \times H^{\frac{1}{2}}(\Gamma_e)^d & \to & H^2(\Omega)^d \times H^1(\Omega) \\ (u_t(t), g(t)) & \to & (u(t), p(t)) \end{array}$$

is linear and continuous. Since $(u_t, g) \in L^2(0,T;L^2(\Omega)^d) \times L^2(0,T;H^{\frac{1}{2}}(\Gamma_e)^d)$, we deduce that $(u,p) \in L^2(0,T;H^2(\Omega)^d) \times L^2(0,T;H^1(\Omega))$.

Let us now prove its uniqueness. Assume that u_1 and u_2 are two solutions and let $w = u_1 - u_2$. Then $w \in H^1(0,T;H) \cap L^2(0,T;V)$ and we have for all $v \in V$:

$$\int_\Omega w_t(t) \cdot v + \int_\Omega \nabla w(t) : \nabla v + \int_{\Gamma_0} q w(t) \cdot v = 0,\ w(0) = 0. \tag{A.11}$$

Taking $v = w(t)$ in (A.11), we find:

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |w(t)|^2 + \int_\Omega |\nabla w(t)|^2 + \int_{\Gamma_0} q|w(t)|^2 = 0,$$

that is to say

$$\int_\Omega |w(t)|^2 \leq \int_\Omega |w(0)|^2 = 0, \text{ for all } t \in [0,T].$$

So $u_1 = u_2$ on $(0,T)\times\Omega$. To conclude, thanks to system (1.1), we obtain $p_1 = p_2$. □

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