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Cubic Decomposition of 2-Orthogonal Polynomial Sequences

Pascal Maroni and Teresa A. Mesquita*

Abstract. In the present work, we study the general cubic decomposition (CD) of a 2-orthogonal polynomial sequence, beginning with a characterization of all the elements involved in such CD. The recurrence coefficients of the 2-orthogonal sequences which admit a diagonal CD are described and we prove that the correspondent principal components are also 2-orthogonal. Finally, we analyse the CD of a 2-symmetric and 2-orthogonal sequence.

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Introduction

In 1992, while studying the classical orthogonal polynomials of dimension d, Douak and Maroni [7] obtained a natural and complete cubic decomposition of a d-symmetric and d-orthogonal sequence (where d is a positive integer), proving some properties fulfilled by the polynomial sequences involved. More precisely, given such sequence $\{W_n\}_{n\geq 0}$, there are three other polynomial sequences $\{P_n\}_{n\geq 0}$, $\{Q_n\}_{n\geq 0}$ and $\{R_n\}_{n\geq 0}$, so that $W_{3n}(x) = P_n(x^3)$, $W_{3n+1}(x) = xQ_n(x^3)$ and $W_{3n+2}(x) = x^2R_n(x^3)$, and which are also d-orthogonal. The relation between $W_{3n}(x)$ and $P_n(x)$ reminds us the problem first proposed by Chihara in 1964 [6], of finding a pair of orthogonal polynomial sequences $\{W_n\}_{n\geq 0}$ and $\{P_n\}_{n\geq 0}$ so that $W_{3n}(x) = P_n(z)$, where z is a cubic in x. The first answer was presented in 1966 by Barrucand and Dickinson [1]. Further investigations, regarding this same problem, were carried out trough the last decades, namely, in 1993 by Marcellán and Sansigre [14], in 2000 and 2001 by Marcellán and Petronilho

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[17, 18]. All these works deal with a pair of orthogonal sequences fulfilling an identity analogous to one of the three indicated above, and the remaining two of the three subsequences $\{W_{3n}(x)\}_{n\geq 0}$, $\{W_{3n+1}(x)\}_{n\geq 0}$ and $\{W_{3n+2}(x)\}_{n\geq 0}$, are written as rational fractions. Wider researches, concerning polynomial transformations of measures, sieved polynomials, polynomial mappings and positive-definite linear functionals touched this theme, as, for instance, [3, 11, 9, 20, 4, 5, 16, 15, 13, 12].

Outside the context of orthogonality, the single definition of a 2-symmetric sequence raises a special case of a full polynomial cubic decomposition (CD) which was introduced and studied for orthogonal sequences in 1989 by Rodriguez and Tasis [10], and more recently, completely generalized by Maroni et al. [23]. With respect to this last general CD, a 2-symmetric sequence has the most simple CD, which we will call diagonal CD.

With regard to d-symmetric sequences, we must refer that, in 2001 [2], Ben Cheikh proved several results concerning orthogonal $(n-1)$-symmetric curves achieving a unified treatment of some problems related to the symmetrization of sequences of orthogonal polynomials on a real line or on a unit circle. For this matter, and as expected, some of the functional relations which characterize a diagonal CD are already proved in this work of 2001, although under the orthogonality hypothesis.

The aim of the present work is twofold. On one hand, we present basic results concerning the CD of a 2-orthogonal polynomial sequences, and, on the other hand, we study the diagonal CD (considering all parameters) for this kind of polynomial sequence.

In the first section we present the basic definitions and results needed in the sequel, giving a special attention to the presentation of the CD in study and recalling an important theorem which characterizes the nine sequences that constitute such CD. The second section is reserved to the proof of a characterization of all the elements involved in a CD of a 2-orthogonal polynomial sequence and a corollary from which we can establish sufficient conditions for the 2-orthogonality of the principal components. Moreover, the recurrence coefficients of the 2-orthogonal sequences which admit a diagonal CD are described and we prove that the correspondent principal components are also 2-orthogonal, generalizing the result obtained by Douak and Maroni. In the third section, we focus our attention in 2-orthogonal and 2-symmetric sequences.

1. Preliminaries and notations

1.1. Basic definitions and results

Let $\mathcal{P}$ be the vector space of polynomials with coefficients in $\mathbb{C}$ and let $\mathcal{P}'$ be its dual. We denote by $\langle u, p\rangle$ the action of the form or linear functional $u \in \mathcal{P}'$ on $p \in \mathcal{P}$. In particular, $(u)_n = \langle u, x^n\rangle$, $n \geq 0$, are called the moments of u. In the following, we will call polynomial sequence (PS) to any sequence $\{W_n\}_{n\geq 0}$ such that $\deg W_n = n$, $\forall n \geq 0$. We will also call monic polynomial sequence

(MPS) a PS so that in each polynomial the leading coefficient is equal to one. Given a MPS $\{W_n\}_{n\geq 0}$, there are complex sequences, $\{\chi_{n,\nu}\}_{0\leq\nu\leq n,\ n\geq 0}$ and $\{\beta_n\}_{n\geq 0}$ such that

$$W_0(x) = 1, \quad W_1(x) = x - \beta_0, \tag{1.1}$$

$$W_{n+2}(x) = (x - \beta_{n+1})W_{n+1}(x) - \sum_{\nu=0}^{n} \chi_{n,\nu} W_\nu(x). \tag{1.2}$$

This relation is called the structure relation of $\{W_n\}_{n\geq 0}$, and $\{\beta_n\}_{n\geq 0}$ and $\{\chi_{n,\nu}\}_{0\leq\nu\leq n,\ n\geq 0}$ are called the structure coefficients.

Also, there exists a unique sequence $\{w_n\}_{n\geq 0}$, $w_n \in \mathcal{P}'$, called the dual sequence of $\{W_n\}_{n\geq 0}$, such that $\langle w_n, W_m\rangle = \delta_{n,m}$, $n, m \geq 0$, where $\delta_{n,m}$ denotes the Kronecker symbol. Moreover, we can prove that [21]

$$\beta_n = \langle w_n, xW_n(x)\rangle, \;\; \chi_{n,\nu} = \langle w_\nu, xW_{n+1}(x)\rangle, \;\; 0 \leq \nu \leq n, \;\; n \geq 0. \tag{1.3}$$

Lemma 1.1. [21] *For each $u \in \mathcal{P}'$ and each $m \geq 1$, the two following propositions are equivalent.*

- **a.** $\langle u, W_{m-1}\rangle \neq 0$, $\langle u, W_n\rangle = 0$, $n \geq m$.
- **b.** $\exists \lambda_\nu \in \mathbb{C}$, $0 \leq \nu \leq m-1$, $\lambda_{m-1} \neq 0$ *such that* $u = \sum_{\nu=0}^{m-1} \lambda_\nu w_\nu$.

Definition 1.2. [7, 8, 19, 25] Given $\Gamma^1, \Gamma^2, \ldots, \Gamma^d \in \mathcal{P}'$, $d \geq 1$, the polynomial sequence $\{W_n\}_{n\geq 0}$ is called d-orthogonal polynomial sequence (d-OPS) with respect to $\Gamma = (\Gamma^1, \ldots, \Gamma^d)$ if it fulfils

$$\langle \Gamma^\alpha, W_m W_n\rangle = 0, \;\; n \geq md + \alpha, \;\; m \geq 0, \tag{1.4}$$

$$\langle \Gamma^\alpha, W_m W_{md+\alpha-1}\rangle \neq 0, \;\; m \geq 0, \tag{1.5}$$

for each integer $\alpha = 1, \ldots, d$. The conditions (1.4) are called the d-orthogonality conditions and the conditions (1.5) are called the regularity conditions. In this case, the functional Γ, of dimension d, is said regular.

Remark 1.3. If $d = 1$, then we meet again the notion of regular orthogonality. As a further matter, the d-dimensional functional Γ is not unique. Nevertheless, from Lemma 1.1, we have: $\Gamma^\alpha = \sum_{\nu=0}^{\alpha-1} \lambda_\nu^\alpha u_\nu$, $\lambda_{\alpha-1}^\alpha \neq 0$, $1 \leq \alpha \leq d$.

Therefore, since $w = (w_0, \ldots, w_{d-1})$ is unique, from now on, we will only consider the canonical functional of dimension d, $w = (w_0, \ldots, w_{d-1})$, saying that $\{W_n\}_{n\geq 0}$ is d-OPS ($d \geq 1$) with respect to $w = (w_0, \ldots, w_{d-1})$ if

$$\langle w_\nu, W_m W_n\rangle = 0, \;\; n \geq md + \nu + 1, \;\; m \geq 0, \;\; \langle w_\nu, W_m W_{md+\nu}\rangle \neq 0, \;\; m \geq 0,$$

for each integer $\nu = 0, 1, \ldots, d-1$.

A d-MOPS satisfies also a recurrence relation of order $(d+1)$ as the following result establishes.

Theorem 1.4. [19] *Let $\{W_n\}_{n\geq 0}$ be a MPS. The following assertions are equivalent:*

- **a.** $\{W_n\}_{n\geq 0}$ *is d-orthogonal with respect to* $w = (w_0, \ldots, w_{d-1})$.

b. $\{W_n\}_{n\geq 0}$ *satisfies a* $(d+1)$*-order recurrence relation* $(d \geq 1)$*:*

$$W_{m+d+1}(x) = (x-\beta_{m+d})W_{m+d}(x) - \sum_{\nu=0}^{d-1} \gamma_{m+d-\nu}^{d-1-\nu} W_{m+d-1-\nu}(x), \ \ m \geq 0,$$

with initial conditions $W_0(x) = 1, \ W_1(x) = x - \beta_0$ *and if* $d \geq 2:$

$$W_n(x) = (x-\beta_{n-1})W_{n-1}(x) - \sum_{\nu=0}^{n-2} \gamma_{n-1-\nu}^{d-1-\nu} W_{n-2-\nu}(x), \ \ 2 \leq n \leq d,$$

and regularity conditions: $\gamma_{m+1}^0 \neq 0, \ \ m \geq 0.$

c. *For each* (n,ν)*,* $n \geq 0$*,* $0 \leq \nu \leq d-1$*, there are d polynomials* $\Lambda^\mu(n,\nu)$*,* $0 \leq \mu \leq d-1$ *such that*

$$w_{nd+\nu} = \sum_{\mu=0}^{d-1} \Lambda^\mu(n,\nu) w_\mu, \ \ n \geq 0, \ \ 0 \leq \nu \leq d-1, \quad \textit{and also fulfilling}$$

$$\begin{aligned} &\deg \Lambda^\mu(n,\nu) = n, \ \ 0 \leq \nu \leq d-1, \\ &\deg \Lambda^\mu(n,\nu) \leq n, \ \ 0 \leq \mu \leq \nu - 1, \ \ \textit{if } 1 \leq \nu \leq d-1, \\ &\deg \Lambda^\mu(n,\nu) \leq n-1, \ \ \nu+1 \leq \mu \leq d-1, \ \ \textit{if } 0 \leq \nu \leq d-2. \end{aligned}$$

Definition 1.5. [7] A PS $\{W_n\}_{n\geq 0}$ is d-symmetric if it fulfils

$$W_n(\xi_k x) = \xi_k^n W_n(x), \ \ n \geq 0, \ \ k = 1, 2, \ldots, d,$$

where $\xi_k = \exp\left(\frac{2ik\pi}{d+1}\right), \ \ k = 1, \ldots, d, \ \ \xi_k^{d+1} = 1.$

If $d = 1$, then $\xi_1 = -1$ and we meet the definition of a symmetric PS in which we have the following property $W_n(-x) = (-1)^n W_n(x), \ \ n \geq 0.$

Lemma 1.6. [7] *A PS* $\{W_n\}_{n\geq 0}$ *is d-symmetric if and only if it fulfils*

$$W_m(x) = x^\mu \sum_{p=0}^{n} a_{m,(d+1)p+\mu} x^{(d+1)p},$$

where $m = (d+1)n + \mu, \ \ 0 \leq \mu \leq d, \ n \geq 0.$

Definition 1.7. [7] The functional $(\Gamma^1, \ldots, \Gamma^d)$ is d-symmetric if, for $n \geq 0$,

$$(\Gamma^\nu)_{(d+1)n+\mu-1} = \langle \Gamma^\nu, x^{(d+1)n+\mu-1} \rangle = 0, \ \ 1 \leq \nu \leq d, \ 1 \leq \mu \leq d+1, \ \nu \neq \mu.$$

In particular, the functional $w = (w_0, \ldots, w_{d-1})$ is d-symmetric if for each integer $0 \leq j \leq d-1$: $(w_j)_{(d+1)n+i} = 0, \ i = 0, 1, \ldots, d, \ \ i \neq j, \ n \geq 0$. If $d = 1$, then we meet the definition of a symmetric form in which we have the following property $(\Gamma)_{(2n+1)} = 0, \ n \geq 0.$

Theorem 1.8. [7] *Let* $\{W_n\}_{n\geq 0}$ *be a d-orthogonal MPS with respect to the functional* $w = (w_0, \ldots, w_{d-1})$*. The following statements are equivalent:*

a. *The functional* Γ *is d-symmetric.*

b. $\{W_n\}_{n\geq 0}$ *is d-symmetric.*

c. $\{W_n\}_{n\geq 0}$ *fulfils the following recurrence relation:*

$$W_{n+d+1}(x) = xW_{n+d}(x) - \gamma_{n+1}^0 W_n(x), \ \ n \geq 0, \quad W_n(x) = x^n, \ \ 0 \leq n \leq d.$$

Definition 1.9. [22] We say that a PS $\{W_n\}_{n\geq 0}$ is compatible with Φ if $\Phi w_n \neq 0,\ \ n \geq 0$.

Remark 1.10. Any OPS is compatible with any monic polynomial and any MPS is compatible with $\Phi = 1$.

Definition 1.11. [22] Given two MPSs $\{Q_n\}_{n\geq 0}$ and $\{R_n\}_{n\geq 0}$, if there is an integer $s \geq 0$ such that

$$\Phi(x)Q_n(x) = \sum_{\nu=n-s}^{n+t} \lambda_{n,\nu} R_\nu(x),\ \ n \geq s, \tag{1.6}$$

$$\exists r \geq s:\ \ \lambda_{r,r-s} \neq 0, \tag{1.7}$$

we say that (1.6)-(1.7) is a finite-type relation between sequences $\{R_n\}_{n\geq 0}$ and $\{Q_n\}_{n\geq 0}$, with respect to $\Phi(x)$. When, instead of (1.7), we take

$$\lambda_{n,n-s} \neq 0,\ \ n \geq s, \tag{1.8}$$

we shall say that (1.6)-(1.8) is a strictly finite-type relation.

Theorem 1.12. [22] *Let $\{Q_n\}_{n\geq 0}$ and $\{R_n\}_{n\geq 0}$ be two MPS and $\{v_n\}_{n\geq 0}$ and $\{r_n\}_{n\geq 0}$ their dual sequences, respectively.*

Let $\{R_n\}_{n\geq 0}$ be compatible with a polynomial Φ. The following properties are equivalent.

i. *There is an integer $s \geq 0$ such that*

$$\Phi(x)Q_n(x) = \sum_{\nu=n-s}^{n+t} \lambda_{n,\nu} R_\nu(x),\ \ n \geq s, \quad \exists r \geq s:\ \ \lambda_{r,r-s} \neq 0.$$

ii. *There are an integer $s \geq 0$ and an application from $\mathbb{N}$ into $\mathbb{N} : m \mapsto \mu_m$ satisfying:* $\max(0, m-t) \leq \mu_m \leq m+s$, $m \geq 0$, $\exists m_0 \geq 0 : \mu_{m_0} = m_0+s$, *and such that*

$$\Phi r_m = \sum_{\nu=m-t}^{\mu_m} \lambda_{\nu,m} v_\nu,\ \ m \geq t, \quad \lambda_{\mu_m,m} \neq 0,\ \ m \geq 0.$$

Remark 1.13. When the relation between $\{R_n\}_{n\geq 0}$ and $\{Q_n\}_{n\geq 0}$ is strictly of finite-type, we have $\mu_m = m + s,\ \ m \geq 0$.

1.2. Cubic decomposition of a monic polynomial sequence

Choosing a cubic polynomial

$$\varpi(x) = x^3 + px^2 + qx + r;\ \ p, q, r \in \mathbb{C} \tag{1.9}$$

and three constants a, b and c, for any MPS $\{W_n\}_{n\geq 0}$, we obtain by Euclidean division [23], three MPSs $\{P_n\}_{n\geq 0}$, $\{Q_n\}_{n\geq 0}$ and $\{R_n\}_{n\geq 0}$, and further six sequences in $\mathcal{P}$, so that

$$W_{3n}(x) = P_n(\varpi(x)) + (x-a)a^1_{n-1}(\varpi(x)) + (x-b)(x-c)a^2_{n-1}(\varpi(x)), \tag{1.10}$$

$$W_{3n+1}(x) = b^1_n(\varpi(x)) + (x-a)Q_n(\varpi(x)) + (x-b)(x-c)b^2_{n-1}(\varpi(x)), \tag{1.11}$$

$$W_{3n+2}(x) = c^1_n(\varpi(x)) + (x-a)c^2_n(\varpi(x)) + (x-b)(x-c)R_n(\varpi(x)), \tag{1.12}$$

with $\deg a^1_{n-1} \le n-1$, $\deg a^2_{n-1} \le n-1$, $\deg b^1_n \le n$, $\deg b^2_{n-1} \le n-1$, $\deg c^1_n \le n$ and $\deg c^2_n \le n$. Organizing the nine component sequences in a matrix, we introduce the following notation, as have been done previously in [23].

$$M_n(x) = \begin{pmatrix} P_n(x) & a^1_{n-1}(x) & a^2_{n-1}(x) \\ b^1_n(x) & Q_n(x) & b^2_{n-1}(x) \\ c^1_n(x) & c^2_n(x) & R_n(x) \end{pmatrix} \tag{1.13}$$

Indeed, (1.10-1.12) is the most general cubic decomposition (CD) of a given MPS $\{W_n\}_{n\ge0}$ and it was presented in [23]. In this CD of $\{W_n\}_{n\ge0}$, the sequences $\{P_n\}_{n\ge0}$, $\{Q_n\}_{n\ge0}$, $\{R_n\}_{n\ge0}$ are called the principal components, and the remaining six $\{a^1_{n-1}\}_{n\ge0}$, $\{a^2_{n-1}\}_{n\ge0}$, $\{b^1_n\}_{n\ge0}$, $\{b^2_{n-1}\}_{n\ge0}$, $\{c^1_n\}_{n\ge0}$, $\{c^2_n\}_{n\ge0}$ are called the secondary components. Notice that these latest are nor necessarily (free) polynomial sequences, neither monic. We will denote by $\{w_n\}_{n\ge0}$, $\{u_n\}_{n\ge0}$, $\{v_n\}_{n\ge0}$ and $\{r_n\}_{n\ge0}$ the dual sequences of $\{W_n\}_{n\ge0}$, $\{P_n\}_{n\ge0}$, $\{Q_n\}_{n\ge0}$ and $\{R_n\}_{n\ge0}$, respectively.

The following result characterizes the nine component sequences of a CD, for any given MPS, that is, for any set of structure coefficients.

Theorem 1.14. [23] *A MPS* $\{W_n\}_{n\ge0}$, *with structure coefficients* (1.1) *and* (1.2), *admits the CD* (1.10)–(1.12) *if and only if the following relations are fulfilled for* $n \ge 0$,

(Z_0) $b^1_0(x) = a - \beta_0$,

(Z_1) $c^1_n(x) = -\sum_{\nu=0}^{n-1} \chi_{3n,3\nu+1}\, b^1_\nu(x) - (\beta_{3n+1} - a)b^1_n(x) + \Theta(x)b^2_{n-1}(x)$
$\quad - \sum_{\nu=0}^{n-1} \chi_{3n,3\nu+2}\, c^1_\nu(x) - \sum_{\nu=0}^{n} \chi_{3n,3\nu}\, P_\nu(x) - (a-b)(a-c)Q_n(x)$,

(Z_2) $c^2_n(x) = -\sum_{\nu=0}^{n} \chi_{3n,3\nu}\, a^1_{\nu-1}(x) + b^1_n(x) + Lb^2_{n-1}(x)$
$\quad - \sum_{\nu=0}^{n-1} \chi_{3n,3\nu+2}\, c^2_\nu(x) - \sum_{\nu=0}^{n-1} \chi_{3n,3\nu+1}\, Q_\nu(x) - (\beta_{3n+1} + a - b - c)Q_n(x)$,

(Z_3) $R_n(x) = -\sum_{\nu=0}^{n} \chi_{3n,3\nu}\, a^2_{\nu-1}(x) - \sum_{\nu=0}^{n-1} \chi_{3n,3\nu+1}\, b^2_{\nu-1}(x)$
$\quad -(\beta_{3n+1} + b + c + p)b^2_{n-1}(x) + Q_n(x) - \sum_{\nu=0}^{n-1} \chi_{3n,3\nu+2}\, R_\nu(x)$,

(Z_4) $P_{n+1}(x) = -\sum_{\nu=0}^{n} \chi_{3n+1,3\nu} P_\nu(x) - (\beta_{3n+2} - a)c^1_n(x) - (a-b)(a-c)c^2_n(x)$
$\quad - \sum_{\nu=0}^{n-1} \chi_{3n+1,3\nu+2}\, c^1_\nu(x) - \sum_{\nu=0}^{n} \chi_{3n+1,3\nu+1}\, b^1_\nu(x) + \Theta(x)R_n(x)$,

(Z_5) $a^1_n(x) = -\sum_{\nu=0}^{n} \chi_{3n+1,3\nu}\, a^1_{\nu-1}(x) + c^1_n(x) - \sum_{\nu=0}^{n-1} \chi_{3n+1,3\nu+2}\, c^2_\nu(x)$
$\quad -(\beta_{3n+2} + a - b - c)c^2_n(x) - \sum_{\nu=0}^{n} \chi_{3n+1,3\nu+1}\, Q_\nu(x) + LR_n(x)$,

(Z_6) $a^2_n(x) = -\sum_{\nu=0}^{n} \chi_{3n+1,3\nu}\, a^2_{\nu-1}(x) - \sum_{\nu=0}^{n} \chi_{3n+1,3\nu+1}\, b^2_{\nu-1}(x)$
$\quad +c^2_n(x) - \sum_{\nu=0}^{n-1} \chi_{3n+1,3\nu+2}\, R_\nu(x) - (\beta_{3n+2} + b + c + p)R_n(x)$,

(Z_7) $b^1_{n+1}(x) = -(a-b)(a-c)a^1_n(x) + \Theta(x)a^2_n(x) - \sum_{\nu=0}^{n} \chi_{3n+2,3\nu+1}\, b^1_\nu(x)$
$\quad - \sum_{\nu=0}^{n} \chi_{3n+2,3\nu+2}\, c^1_\nu(x) - \sum_{\nu=0}^{n} \chi_{3n+2,3\nu}\, P_\nu(x) - (\beta_{3n+3} - a)P_{n+1}(x)$,

(Z_8) $Q_{n+1}(x) = -\sum_{\nu=0}^{n} \chi_{3n+2,3\nu}\, a^1_{\nu-1}(x) - (\beta_{3n+3} + a - b - c)a^1_n(x)$
$\quad +La^2_n(x) - \sum_{\nu=0}^{n} \chi_{3n+2,3\nu+2}\, c^2_\nu(x) + P_{n+1}(x) - \sum_{\nu=0}^{n} \chi_{3n+2,3\nu+1}\, Q_\nu(x)$,

(Z_9) $b^2_n(x) = a^1_n(x) - \sum_{\nu=0}^{n} \chi_{3n+2,3\nu}\, a^2_{\nu-1}(x) - (\beta_{3n+3} + b + c + p)a^2_n(x)$
$\quad - \sum_{\nu=0}^{n} \chi_{3n+2,3\nu+1}\, b^2_{\nu-1}(x) - \sum_{\nu=0}^{n} \chi_{3n+2,3\nu+2}\, R_\nu(x)$,

where by convention $\sum_{\nu=0}^{-1} . = 0$ *and*

$$\Theta(x) = x - r + aL + bc(b+c+p), \tag{1.14}$$

$$L = bc - q - (b+c+p)(b+c). \tag{1.15}$$

Finally, we present a result with several equivalent characterizations of the most simple CD, where the six secondary components vanish, that is, where the matrix $M_n(x)$ is a diagonal matrix. This situation will be referred both as a diagonal CD and according to the next definition.

Definition 1.15. A MPS for which the CD (1.10)-(1.12) has all the secondary components equal to zero will be called $\begin{pmatrix} a & b & c \\ p & q & r \end{pmatrix}$- symmetric.

Remark 1.16. Regarding Lemma 1.6, a $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$- symmetric MPS is in fact a 2-symmetric MPS.

Theorem 1.17. [23] *Let* $\{W_n\}_{n\geq 0}$ *be a MPS. The following assertions are equivalent, where* L *is defined by* (1.15) *and* $n \geq 0$.

a. $a^1_m = a^2_m = b^1_m = b^2_m = c^1_m = c^2_m = 0,\ m \geq 0,$

or $\{W_n\}_{n\geq 0}$ *is* $\begin{pmatrix} a & b & c \\ p & q & r \end{pmatrix}$- *symmetric.*

b. $\sigma_\varpi(w_{3n+1}) = 0, \quad \sigma_\varpi((x-a)w_{3n}) = 0, \quad \sigma_\varpi((x-b)(x-c)w_{3n}) = 0,$

$\sigma_\varpi(w_{3n+2}) = 0, \quad \sigma_\varpi((x-a)w_{3n+2}) = 0, \quad \sigma_\varpi((x-b)(x-c)w_{3n+1}) = 0.$

c. *The content of b. and*

$u_n = \sigma_\varpi(w_{3n}),\ \ v_n = \sigma_\varpi((x-a)w_{3n+1}),\ \ r_n = \sigma_\varpi((x-b)(x-c)w_{3n+2}).$

d. **(d.1)** $\beta_{3n} = a,$ **(d.2)** $\beta_{3n+1} = b + c - a,$ **(d.3)**; $\beta_{3n+2} = -(b + c + p),$

(d.4) $\chi_{3n,3\nu} = -(a-b)(a-c)\langle u_\nu, Q_n\rangle,\ 0 \leq \nu \leq n,$

(d.5) $\chi_{3n,3\nu+1} = 0,$ **(d.6)** $\chi_{3n,3\nu+2} = \langle r_\nu, Q_n\rangle,\ 0 \leq \nu < n,$

(d.7) $\chi_{3n+1,3\nu} = \langle u_\nu, xR_n(x)\rangle + [aL + bc(b + c + p) - r]\langle u_\nu, R_n(x)\rangle,$

(d.8) $\chi_{3n+1,3\nu+1} = L\langle v_\nu, R_n(x)\rangle,\ 0 \leq \nu \leq n,$

(d.9) $\chi_{3n+1,3\nu+2} = 0,\ 0 \leq \nu < n,$ **(d.10)** $\chi_{3n+2,3\nu} = 0,\ 0 \leq \nu \leq n,$

(d.11) $\chi_{3n+2,3\nu+1} = \langle v_\nu, P_{n+1}(x)\rangle,$ **(d.12)** $\chi_{3n+2,3\nu+2} = 0,\ 0 \leq \nu \leq n.$

2. Cubic decomposition of a 2-orthogonal sequence

Given a 2-orthogonal MPS $\{W_n\}_{n\geq 0}$, we know that it fulfils a third order recurrence relation, which can be written as follows, where the correspondent recurrence coefficients are $\beta_n,\ \gamma^1_{n+1},\ \gamma^0_{n+1},\ n \geq 0$, with $\gamma^0_{n+1} \neq 0,\ \ n \geq 0.$

$$W_{n+3}(x) = (x - \beta_{n+2})W_{n+2}(x) - \gamma^1_{n+2}W_{n+1}(x) - \gamma^0_{n+1}W_n(x),\ \ n \geq 0,$$

$$W_0(x) = 1,\ \ W_1(x) = x - \beta_0,\ \ W_2(x) = (x - \beta_1)W_1(x) - \gamma^1_1.$$

Taking into consideration Theorem 1.14, we begin to give necessary and sufficient relations concerning the decomposed MPS 2-orthogonality.

Theorem 2.1. *A MPS defined by* (1.10)–(1.12) *is* 2*-orthogonal if and only if the following relations are met, for* $n \geq 0$*, where* $\Theta(x)$ *and* L *are defined by* (1.14) *and* (1.15)*, and* $c^1_{-1}(x) = c^2_{-1}(x) = R_{-1}(x) = 0$.

$$(B_0)\ b^1_0(x) = a - \beta_0,$$

(B_1) $c_n^1(x) = -(\beta_{3n+1} - a)b_n^1(x) + \Theta(x)b_{n-1}^2(x) - \gamma_{3n}^0 c_{n-1}^1(x) - \gamma_{3n+1}^1 P_n(x) - (a-b)(a-c)Q_n(x),$

(B_2) $c_n^2(x) = -\gamma_{3n+1}^1 a_{n-1}^1(x) + b_n^1(x) + Lb_{n-1}^2(x) - \gamma_{3n}^0 c_{n-1}^2(x) - (\beta_{3n+1} + a - b - c)Q_n(x),$

(B_3) $R_n(x) = -\gamma_{3n+1}^1 a_{n-1}^2(x) - (\beta_{3n+1} + b + c + p)b_{n-1}^2(x) + Q_n(x) - \gamma_{3n}^0 R_{n-1}(x),$

(B_4) $P_{n+1}(x) = -\gamma_{3n+1}^0 P_n(x) - (\beta_{3n+2} - a)c_n^1(x) - \gamma_{3n+2}^1 b_n^1(x) - (a-b)(a-c)c_n^2(x) + \Theta(x)R_n(x),$

(B_5) $a_n^1(x) = -\gamma_{3n+1}^0 a_{n-1}^1(x) + c_n^1(x) - (\beta_{3n+2} + a - b - c)c_n^2(x) - \gamma_{3n+2}^1 Q_n(x) + LR_n(x),$

(B_6) $a_n^2(x) = -\gamma_{3n+1}^0 a_{n-1}^2(x) - \gamma_{3n+2}^1 b_{n-1}^2(x) + c_n^2(x) - (\beta_{3n+2} + b + c + p)R_n(x),$

(B_7) $b_{n+1}^1(x) = -(a-b)(a-c)a_n^1(x) + \Theta(x)a_n^2(x) - \gamma_{3n+2}^0 b_n^1(x) - \gamma_{3n+3}^1 c_n^1(x) - (\beta_{3n+3} - a)P_{n+1}(x),$

(B_8) $Q_{n+1}(x) = -(\beta_{3n+3} + a - b - c)a_n^1(x) + La_n^2(x) - \gamma_{3n+3}^1 c_n^2(x) + P_{n+1}(x) - \gamma_{3n+2}^0 Q_n(x),$

(B_9) $b_n^2(x) = a_n^1(x) - (\beta_{3n+3} + b + c + p)a_n^2(x) - \gamma_{3n+2}^0 b_{n-1}^2(x) - \gamma_{3n+3}^1 R_n(x).$

Proof. The sequence $\{W_n\}_{n\geq 0}$ is 2-orthogonal if and only if its structure coefficients are: $\chi_{n+1,n+1} = \gamma_{n+2}^1$, $\chi_{n+1,n} = \gamma_{n+1}^0 \neq 0$, and $\chi_{n+1,\nu} = 0$, $0 \leq \nu < n$. Then, Theorem 1.14 concludes the proof. □

In the next corollary, we present three relations that begin as recurrence relations of third order (for each principal component) and after they are completed with a linear combination of elements of only two secondary component sequences. They establish conditions that assure the 2-orthogonality of the principal components.

Corollary 2.2. *A* 2*-orthogonal MPS with CD given by* (1.10)–(1.12) *fulfils the following relations, where* $\Theta(x)$ *and* L *are defined by* (1.14) *and* (1.15)*.*

$$P_{n+3}(x) = \big(\Theta(x) - \bar{A}_{3n+3}\big)P_{n+2}(x) - \bar{B}_{3n+3}P_{n+1}(x) - \bar{C}_{3n+3}P_n(x) - \bar{M}_{3n+3}b_n^1(x) - \bar{K}_{3n+3}b_{n+1}^1(x) - \bar{H}_{3n+3}b_{n+2}^1(x) - \bar{N}_{3n+3}c_n^1(x) - \bar{V}_{3n+3}c_{n+1}^1(x) - \bar{S}_{3n+3}c_{n+2}^1(x), \tag{2.1}$$

$$Q_{n+3}(x) = \big(\Theta(x) - \bar{A}_{3n+4}\big)Q_{n+2}(x) - \bar{B}_{3n+4}Q_{n+1}(x) - \bar{C}_{3n+4}Q_n(x) - \bar{M}_{3n+4}c_n^2(x) - \bar{K}_{3n+4}c_{n+1}^2(x) - \bar{H}_{3n+4}c_{n+2}^2(x) - \bar{N}_{3n+4}a_n^1(x) - \bar{V}_{3n+4}a_{n+1}^1(x) - \bar{S}_{3n+4}a_{n+2}^1(x), \tag{2.2}$$

$$\begin{aligned} R_{n+3}(x) &= \big(\Theta(x) - \bar{A}_{3n+5}\big)R_{n+2}(x) - \bar{B}_{3n+5}R_{n+1}(x) - \bar{C}_{3n+5}R_n(x) \quad (2.3)\\ &\quad - \bar{M}_{3n+5}a_n^2(x) - \bar{K}_{3n+5}a_{n+1}^2(x) - \bar{H}_{3n+5}a_{n+2}^2(x)\\ &\quad - \bar{N}_{3n+5}b_n^2(x) - \bar{V}_{3n+5}b_{n+1}^2(x) - \bar{S}_{3n+5}b_{n+2}^2(x), \quad \textit{where} \end{aligned}$$

$$\begin{aligned} \bar{A}_n &= \gamma_{n+2}^0 + \gamma_{n+3}^0 + \gamma_{n+4}^0 + \gamma_{n+3}^1(\beta_{n+2} + 2\beta_{n+3} + p)\\ &\quad + \gamma_{n+4}^1(2\beta_{n+3} + \beta_{n+4} + p)\\ &\quad + (\beta_{n+3} - a)(\beta_{n+3} + a - b - c)(\beta_{n+3} + b + c + p)\\ &\quad - (\beta_{n+3} - a)L + (a-b)(a-c)(\beta_{n+3} + b + c + p);\\ \bar{B}_n &= \gamma_{n+1}^1\gamma_{n+2}^1\gamma_{n+3}^1 + \gamma_n^0\gamma_{n+2}^0 + \gamma_{n+1}^0(\gamma_{n+2}^0 + \gamma_{n+3}^0)\\ &\quad + \gamma_{n+1}^0\gamma_{n+3}^1(\beta_n + \beta_{n+2} + \beta_{n+3} + p) + \gamma_{n+2}^0\gamma_{n+1}^1(\beta_n + \beta_{n+1} + \beta_{n+3} + p);\\ \bar{C}_n &= \gamma_{n-2}^0\gamma_n^0\gamma_{n+2}^0;\\ \bar{M}_n &= \gamma_{n-1}^0(\gamma_{n+1}^0\gamma_{n+3}^1 + \gamma_{n+2}^0\gamma_{n+1}^1) + \gamma_n^0\gamma_{n+2}^0\gamma_{n-1}^1;\\ \bar{K}_n &= \gamma_{n+1}^0\gamma_{n+3}^1 + \gamma_{n+3}^0\gamma_{n+2}^1 + \gamma_{n+2}^1\gamma_{n+3}^1(\beta_{n+1} + \beta_{n+2} + \beta_{n+3} + p)\\ &\quad + \gamma_{n+2}^0\Big(\gamma_{n+1}^1 + \gamma_{n+2}^1 + \gamma_{n+3}^1 + \gamma_{n+4}^1 + (a-b)(a-c) - L\\ &\quad + (\beta_{n+3} - a)(\beta_{n+1} + \beta_{n+3} + a + p)\\ &\quad + (\beta_{n+1} + a - b - c)(\beta_{n+1} + b + c + p)\Big);\\ \bar{H}_n &= \gamma_{n+3}^1 + \gamma_{n+4}^1 + \gamma_{n+5}^1 - L + (a-b)(a-c)\\ &\quad + (\beta_{n+4} - a)(\beta_{n+3} + \beta_{n+4} + a + p)\\ &\quad + (\beta_{n+3} + a - b - c)(\beta_{n+3} + b + c + p);\\ \bar{N}_n &= \gamma_n^1(\gamma_{n+1}^0\gamma_{n+3}^1 + \gamma_{n+2}^0\gamma_{n+1}^1) + \gamma_n^0\gamma_{n+2}^1\gamma_{n+3}^1\\ &\quad + \gamma_n^0\gamma_{n+2}^0(\beta_{n-1} + \beta_{n+1} + \beta_{n+3} + p);\\ \bar{V}_n &= \gamma_{n+2}^0(\beta_{n+1} + \beta_{n+2} + \beta_{n+3} + p) + \gamma_{n+3}^0(\beta_{n+2} + \beta_{n+3} + \beta_{n+4} + p)\\ &\quad + \gamma_{n+3}^1\Big(\gamma_{n+2}^1 + \gamma_{n+3}^1 + \gamma_{n+4}^1 + (a-b)(a-c) - L\\ &\quad + (\beta_{n+2} + a - b - c)(\beta_{n+2} + b + c + p)\\ &\quad + (\beta_{n+3} - a)(\beta_{n+2} + \beta_{n+3} + p + a)\Big);\\ \bar{S}_n &= \beta_{n+3} + \beta_{n+4} + \beta_{n+5} + p. \end{aligned}$$

Remark 2.3. The initial conditions of relations (2.1)-(2.3) can be written explicitly through relations (B_0)-(B_9) of Theorem 2.1 with $n = 0, 1, 2$. This will not be done here due to their extensive expressions.

Proof. In order to obtain the indicated relations, we apply the elimination method to the list of identities of Theorem 2.1. With respect to relation (2.1), we begin to consider (B_4), with $n \to n+1$,

$$\begin{aligned} P_{n+2}(x) &= \Theta(x)R_{n+1}(x) - (a-b)(a-c)c_{n+1}^2(x) - \gamma_{3n+5}^1 b_{n+1}^1(x)\\ &\quad - (\beta_{3n+5} - a)c_{n+1}^1(x) - \gamma_{3n+4}^0 P_{n+1}(x), \end{aligned}$$

and we need to substitute $\Theta(x)R_{n+1}(x) - (a-b)(a-c)c^2_{n+1}(x)$ by an equivalent expression written in terms of elements of the sequences $\{P_n(x)\}_{n\geq 0}$, $\{b^1_n(x)\}_{n\geq 0}$ and $\{c^1_n(x)\}_{n\geq 0}$. Having the purpose of deducing such expression, let us consider relation (B_3), with $n \to n+1$, where $Q_{n+1}(x)$ is replaced by the expression given by (B_8), yielding:

$$R_{n+1}(x) = P_{n+1}(x) - \gamma^0_{3n+2}Q_n(x) - (\beta_{3n+3} + a - b - c)a^1_n(x)$$
$$+ (L - \gamma^1_{3n+4})a^2_n(x) - (\beta_{3n+4} + b + c + p)b^2_n(x) - \gamma^1_{3n+3}c^2_n(x) - \gamma^0_{3n+3}R_n(x).$$

Replacing, also, $Q_n(x)$ by the expression given by (B_3), we get:

$$\begin{aligned} R_{n+1}(x) &= P_{n+1}(x) - \gamma^0_{3n+2}\gamma^1_{3n+1}a^2_{n-1}(x) + (L - \gamma^1_{3n+4})a^2_n(x) \\ &- \gamma^0_{3n+2}(\beta_{3n+1} + b + c + p)b^2_{n-1}(x) - (\beta_{3n+4} + b + c + p)b^2_n(x) \\ &- \gamma^0_{3n}\gamma^0_{3n+2}R_{n-1}(x) - (\gamma^0_{3n+2} + \gamma^0_{3n+3})R_n(x) \\ &- (\beta_{3n+3} + a - b - c)a^1_n(x) - \gamma^1_{3n+3}c^2_n(x). \end{aligned} \tag{2.4}$$

To obtain a suitable expression for $\Theta(x)R_{n+1}(x)$ as explained above, we will have to take identity (2.4) multiplied by $\Theta(x)$, therefore, it is time to write $\Theta(x)a^2_n(x)$, $\Theta(x)b^2_{n-1}(x)$, $\Theta(x)R_n(x)$, $\Theta(x)a^1_n(x)$ and $\Theta(x)c^2_n(x)$ as linear combinations of elements of the sequences $\{P_n(x)\}_{n\geq 0}$, $\{b^1_n(x)\}_{n\geq 0}$ and $\{c^1_n(x)\}_{n\geq 0}$ plus an additional term with coefficient $-(a-b)(a-c)$. In fact, by (B_7), (B_1) and (B_4) we have (respectively):

$$\begin{aligned} \Theta(x)a^2_n(x) &= (a-b)(a-c)a^1_n(x) + \gamma^0_{3n+2}b^1_n(x) + b^1_{n+1}(x) \\ &\quad + \gamma^1_{3n+3}c^1_n(x) + (\beta_{3n+3} - a)P_{n+1}(x), \end{aligned} \tag{2.5}$$

$$\begin{aligned} \Theta(x)b^2_{n-1}(x) &= (\beta_{3n+1} - a)b^1_n(x) + \gamma^0_{3n}c^1_{n-1}(x) + c^1_n(x) + \gamma^1_{3n+1}P_n(x) \\ &\quad + (a-b)(a-c)Q_n(x), \end{aligned} \tag{2.6}$$

$$\begin{aligned} \Theta(x)R_n(x) &= \gamma^1_{3n+2}b^1_n(x) + (\beta_{3n+2} - a)c^1_n(x) + (a-b)(a-c)c^2_n(x) \\ &\quad + \gamma^0_{3n+1}P_n(x) + P_{n+1}(x). \end{aligned} \tag{2.7}$$

These identities permit to write, through (B_9) multiplied by $\Theta(x)$, the following, for $n \geq 0$:

$$\begin{aligned} \Theta(x)a^1_n(x &= (\beta_{3n+3} + b + c + p)\Theta(x)a^2_n(x) + \gamma^0_{3n+2}\Theta(x)b^2_{n-1}(x) + \Theta(x)b^2_n(x) \\ &\quad + \gamma^1_{3n+3}\Theta(x)R_n(x) \end{aligned}$$

$$\Rightarrow \Theta(x)a^1_n(x) = \Big(\gamma^0_{3n+2}(\beta_{3n+1} + \beta_{3n+3} - a + b + c + p) + \gamma^1_{3n+2}\gamma^1_{3n+3}\Big)b^1_n(x)$$
$$+ (\beta_{3n+3} + \beta_{3n+4} - a + b + c + p)b^1_{n+1}(x) + \gamma^0_{3n}\gamma^0_{3n+2}c^1_{n-1}(x)$$
$$+ \Big(\gamma^1_{3n+3}(\beta_{3n+2} + \beta_{3n+3} - a + b + c + p) + \gamma^0_{3n+2} + \gamma^0_{3n+3}\Big)c^1_n(x) + c^1_{n+1}(x)$$
$$+ \Big(\gamma^1_{3n+3} + \gamma^1_{3n+4} + (\beta_{3n+3} - a)(\beta_{3n+3} + b + c + p)\Big)P_{n+1}(x) \tag{2.8}$$
$$+ \Big(\gamma^0_{3n+2}\gamma^1_{3n+1} + \gamma^0_{3n+1}\gamma^1_{3n+3}\Big)P_n(x) + (a-b)(a-c)\phi_1(x),$$

with $\phi_1(x) = (\beta_{3n+3}+b+c+p)a_n^1(x)+\gamma_{3n+3}^1 c_n^2(x)+\gamma_{3n+2}^0 Q_n(x)+Q_{n+1}(x)$.
Using (B_6) multiplied by $\Theta(x)$, we obtain the following, for $n \geq 1$:

$$\begin{aligned}
\Theta(x)c_n^2(x) &= \gamma_{3n+1}^0\Theta(x)a_{n-1}^2(x) + \Theta(x)a_n^2(x) + \gamma_{3n+2}^1\Theta(x)b_{n-1}^2(x) \\
&\quad + (\beta_{3n+2}+b+c+p)\Theta(x)R_n(x) \\
&\Rightarrow \Theta(x)c_n^2(x) = \gamma_{3n-1}^0\gamma_{3n+1}^0 b_{n-1}^1(x) \\
&+ \Big(\gamma_{3n+1}^0 + \gamma_{3n+2}^0 + \gamma_{3n+2}^1(\beta_{3n+1}+\beta_{3n+2}-a+b+c+p)\Big)b_n^1(x) \\
&+ b_{n+1}^1(x) + \Big(\gamma_{3n+1}^0\gamma_{3n}^1 + \gamma_{3n}^0\gamma_{3n+2}^1\Big)c_{n-1}^1(x) \\
&+ \Big(\gamma_{3n+2}^1 + \gamma_{3n+3}^1 + (\beta_{3n+2}-a)(\beta_{3n+2}+b+c+p)\Big)c_n^1(x) \\
&+ \Big(\gamma_{3n+1}^0(\beta_{3n}+\beta_{3n+2}-a+b+c+p) + \gamma_{3n+1}^1\gamma_{3n+2}^1\Big)P_n(x) \\
&+ (\beta_{3n+2}+\beta_{3n+3}-a+b+c+p)P_{n+1}(x) + (a-b)(a-c)\phi_2(x),
\end{aligned} \tag{2.9}$$

with $\phi_2(x) = \gamma_{3n+1}^0 a_{n-1}^1(x) + a_n^1(x) + (\beta_{3n+2}+b+c+p)c_n^2(x) + \gamma_{3n+2}^1 Q_n(x)$.

Thus, we can now multiply (2.4) by $\Theta(x)$ and introduce (2.5)-(2.9). Afterwards, we add, in both members, the term $-(a-b)(a-c)c_{n+1}^2(x)$ and after several algebraic simplifications where the identities (B_2), (B_5), (B_8) and (B_9) are used, we obtain the following:

$$\begin{aligned}
&\Theta(x)R_{n+1}(x) - (a-b)(a-c)c_{n+1}^2(x) = \big(\Theta(x) - \bar{A}_{3n} + \gamma_{3n+4}^0\big)P_{n+1}(x) \\
&- \bar{B}_{3n}P_n(x) - \bar{C}_{3n}P_{n-1}(x) - \bar{M}_{3n}b_{n-1}^1(x) - \bar{K}_{3n}b_n^1(x) \\
&+ \big(-\bar{H}_{3n} + \gamma_{3n+5}^1\big)b_{n+1}^1(x) - \bar{N}_{3n}c_{n-1}^1(x) - \bar{V}_{3n}c_n^1(x) \\
&+ \big(-\bar{S}_{3n} + \beta_{3n+5} - a\big)c_{n+1}^1(x), \quad n \geq 1.
\end{aligned} \tag{2.10}$$

Finally, identity (2.1) is established when we insert (2.10) information in (B_4). The relations (2.2) and (2.3) can be obtained by similar calculations. □

Remark 2.4.
- $b_n^1 = c_n^1 = 0,\ n \geq 0 \ \Rightarrow\ \{P_n\}_{n\geq 0}$ is 2-orthogonal.
- $c_n^2 = a_n^1 = 0,\ n \geq 0 \ \Rightarrow\ \{Q_n\}_{n\geq 0}$ is 2-orthogonal.
- $a_n^2 = b_n^2 = 0,\ n \geq 0 \ \Rightarrow\ \{R_n\}_{n\geq 0}$ is 2-orthogonal.
- If $\bar{M}_n = \bar{K}_n = \bar{H}_n = \bar{N}_n = \bar{V}_n = \bar{S}_n = 0,\ n \geq 0$, then the principal components are 2-orthogonal.
- In [24] it was proved a converse of corollary 2.2.

Example 2.5. Taking, as an example, the 2-Chebyshev MOPS defined by $\beta_n = 0$, $\gamma_{n+1}^0 = \gamma$, $\gamma_{n+1}^1 = \alpha,\ n \geq 0$, with $\gamma \neq 0$ [8], we obtain:

$$\bar{M}_n = 3\alpha\gamma^2,\ \ \bar{K}_n = p\alpha^2 + q\gamma + 6\alpha\gamma,\ \ \bar{H}_n = q + 3\alpha,$$
$$\bar{N}_n = \gamma\left(3\alpha^2 + p\gamma\right),\ \ \bar{V}_n = q\alpha + 3\alpha^2 + 2p\gamma,\ \ \bar{S}_n = p.$$

Thus, $\bar{M}_n = \bar{K}_n = \bar{H}_n = \bar{N}_n = \bar{V}_n = \bar{S}_n = 0,\ n \geq 0$, if and only if $p = q = 0$ and $\alpha = 0$, that is, the given 2-Chebyshev MOPS is 2-symmetric. In that case, the three principal components fulfill the following recurrence relation $B_{n+3}(x) = (x - r - 3\gamma)B_{n+2}(x) - 3\gamma^2 B_{n+1}(x) - \gamma^3 B_n(x),\ n \geq 0$, with the initial conditions: $P_0(x) = Q_0(x) = R_0(x) = 1$,

$P_1(x) = x - r - \gamma,\ \ Q_1(x) = x - r - 2\gamma,\ \ R_1(x) = x - r - 3\gamma,$
$P_2(x) = x^2+r^2-2x(r+2\gamma)+4r\gamma+\gamma^2,\ Q_2(x) = x^2+r^2-x(2r+5\gamma)+5r\gamma+3\gamma^2,$
$R_2(x) = x^2 + r^2 - 2x(r + 3\gamma) + 6r\gamma + 6\gamma^2.$

2.1. The diagonal cubic decomposition of a 2-orthogonal sequence

As mentioned before, in reference [7] we find a natural cubic decomposition for a d-orthogonal and d-symmetric MPS, when $a = b = c = 0$ and $\varpi(x) = x^3$. It is in fact what we called above a diagonal cubic decomposition, since all the secondary components are trivial. Therefore, unlike the orthogonal case (see [23]), we know particular examples of diagonal CDs of 2-orthogonal MPSs, more specifically, of 2-symmetric sequences (with $a = b = c = p = q = r = 0$). The next result describes the 2-orthogonal sequences admitting a diagonal CD, as well as all the elements of that CD.

Theorem 2.6. *Let* $\{W_n\}_{n\geq 0}$ *be a MPS defined by* (1.10)–(1.12)*, such that*

$$a^1_n = a^2_n = b^1_n = b^2_n = c^1_n = c^2_n = 0,\ \ n \geq 0.$$

Then $\{W_n\}_{n\geq 0}$ *is* 2*-orthogonal if and only if the following relations are met, where* $\Theta(x)$ *and* L *are defined by* (1.14) *and* (1.15)*.*

(b_1) $L = 0,$ (b_2) $(a - b)(a - c) = 0,$
(b_3) $\beta_{3n} = a,\ \ n \geq 0,$ (b_4) $\beta_{3n+1} = b + c - a,\ \ n \geq 0,$
(b_5) $\beta_{3n+2} = -(b + c + p),\ \ n \geq 0,$ (b_6) $\gamma^1_n = 0,\ \ n \geq 1,$
(b_7) $P_{n+1}(x) = Q_{n+1}(x) + \gamma^0_{3n+2}Q_n(x),\ \ n \geq 0,$
(b_8) $Q_{n+1}(x) = R_{n+1}(x) + \gamma^0_{3n+3}R_n(x),\ \ n \geq 0,$
(b_9) $\Theta(x)R_n(x) = P_{n+1}(x) + \gamma^0_{3n+1}P_n(x),\ \ n \geq 0.$

Proof. Let us consider the relations of Theorem 2.1, with $a^1_n = a^2_n = b^1_n = b^2_n = c^1_n = c^2_n = 0,\ \ n \geq 0.$

$(\tilde{B}_0)$ $\beta_0 = a,$ $(\tilde{B}_1)$ $\gamma^1_{3n+1}P_n(x) + (a - b)(a - c)Q_n(x) = 0,$

$(\tilde{B}_2)$ $\beta_{3n+1} + a - b - c = 0,$ $(\tilde{B}_3)$ $Q_n(x) = \gamma^0_{3n}R_{n-1}(x) + R_n(x),$

$(\tilde{B}_4)$ $P_{n+1}(x) + \gamma^0_{3n+1}P_n(x) = \Theta(x)R_n(x),$ $(\tilde{B}_5)$ $\gamma^1_{3n+2}Q_n(x) = LR_n(x),$

$(\tilde{B}_6)$ $\beta_{3n+2} + b + c + p = 0,$ $(\tilde{B}_7)$ $\beta_{3n+3} = a,$

$(\tilde{B}_8)$ $P_{n+1}(x) = \gamma^0_{3n+2}Q_n(x) + Q_{n+1}(x),$ $(\tilde{B}_9)$ $\gamma^1_{3n+3} = 0,\ \ n \geq 0.$

Notice that the relations $(\tilde{B}_2)$, $(\tilde{B}_3)$, $(\tilde{B}_4)$, $(\tilde{B}_6)$ and $(\tilde{B}_8)$ correspond to identities (b_4), (b_8), (b_9), (b_5) and (b_7). Also, $(\tilde{B}_7)$ and $(\tilde{B}_0)$ correspond to (b_3). From $(\tilde{B}_1)$, we obtain $\gamma^1_{3n+1} + (a-b)(a-c) = 0, n \geq 0$. Let us consider $(\tilde{B}_1)$, with $n \to n + 1$, and let us substitute $P_{n+1}(x)$ by the expression given by $(\tilde{B}_8)$, as follows.

$$\gamma^1_{3n+4}\Big(\gamma^0_{3n+2}Q_n(x) + Q_{n+1}(x)\Big) + (a - b)(a - c)Q_{n+1}(x) = 0.$$

We conclude immediately that $\gamma^1_{3n+4} = 0$ and thus $(a - b)(a - c) = 0$, which corresponds to (b_2). Moreover, we have $\gamma^1_{3n+1} = 0, n \geq 0$, due to $(\tilde{B}_1)$. Also, replacing the term $Q_n(x)$ of $(\tilde{B}_5)$ by the expression given by $(\tilde{B}_3)$, we get: $\gamma^1_{3n+2}\gamma^0_{3n}R_{n-1}(x) + \big(\gamma^1_{3n+2} - L\big)R_n(x) = 0$, which implies $\gamma^1_{3n+2} - L = 0$ and

$\gamma^1_{3n+2} = 0, n \geq 1$. Hence, $L = 0 = \gamma^1_{3n+2}$, $n \geq 0$. Therefore, we have obtained (b_1) and taking into consideration $(\tilde{B}_9)$ we get (b_6).

Conversely, if we suppose the relations (b_0)-(b_9) and $a^1_n = a^2_n = b^1_n = b^2_n = c^1_n = c^2_n = 0, \quad n \geq 0$, we can easily prove that $(\tilde{B}_0)$-$(\tilde{B}_9)$ are fulfilled, which correspond to the list of relations of Theorem 2.1 for a diagonal CD. □

When a 2-orthogonal MPS admits a diagonal CD, it fulfils the list of conditions of Theorem 2.6, which imply that the coefficients $\bar{M}_n$, $\bar{K}_n$, $\bar{H}_n$, $\bar{N}_n$, $\bar{V}_n$, $\bar{S}_n$ of Corollary 2.2 vanish for $n \geq 0$. Hence, we know that the three principal component sequences fulfil the following recurrence relations of third order, where $n \geq 0$.

$$\begin{aligned} &P_{n+3}(x) = \{\Theta(x) - \gamma^0_{3n+5} - \gamma^0_{3n+6} - \gamma^0_{3n+7}\}P_{n+2}(x) \\ &-\{\gamma^0_{3n+3}\gamma^0_{3n+5} + \gamma^0_{3n+4}(\gamma^0_{3n+5} + \gamma^0_{3n+6})\}P_{n+1}(x) - \gamma^0_{3n+1}\gamma^0_{3n+3}\gamma^0_{3n+5}P_n(x), \\ &P_0(x) = 1, \quad P_1(x) = \Theta(x) - \gamma^0_1, \\ &P_2(x) = \{\Theta(x) - \gamma^0_2 - \gamma^0_3 - \gamma^0_4\}P_1(x) - \gamma^0_1(\gamma^0_2 + \gamma^0_3); \end{aligned} \tag{2.11}$$

$$\begin{aligned} &Q_{n+3}(x) = \{\Theta(x) - \gamma^0_{3n+6} - \gamma^0_{3n+7} - \gamma^0_{3n+8}\}Q_{n+2}(x) \\ &-\{\gamma^0_{3n+4}\gamma^0_{3n+6} + \gamma^0_{3n+5}(\gamma^0_{3n+6} + \gamma^0_{3n+7})\}Q_{n+1}(x) - \gamma^0_{3n+2}\gamma^0_{3n+4}\gamma^0_{3n+6}Q_n(x), \\ &Q_0(x) = 1, \quad Q_1(x) = \Theta(x) - \gamma^0_1 - \gamma^0_2, \\ &Q_2(x) = \{\Theta(x) - \gamma^0_3 - \gamma^0_4 - \gamma^0_5\}Q_1(x) - \gamma^0_1\gamma^0_3 - \gamma^0_2(\gamma^0_3 + \gamma^0_4); \end{aligned} \tag{2.12}$$

$$\begin{aligned} &R_{n+3}(x) = \{\Theta(x) - \gamma^0_{3n+7} - \gamma^0_{3n+8} - \gamma^0_{3n+9}\}R_{n+2}(x) \\ &-\{\gamma^0_{3n+5}\gamma^0_{3n+7} + \gamma^0_{3n+6}(\gamma^0_{3n+7} + \gamma^0_{3n+8})\}R_{n+1}(x) - \gamma^0_{3n+3}\gamma^0_{3n+5}\gamma^0_{3n+7}R_n(x), \\ &R_0(x) = 1, \quad R_1(x) = \Theta(x) - \gamma^0_1 - \gamma^0_2 - \gamma^0_3, \\ &R_2(x) = \{\Theta(x) - \gamma^0_4 - \gamma^0_5 - \gamma^0_6\}R_1(x) - \gamma^0_2\gamma^0_4 - \gamma^0_3(\gamma^0_4 + \gamma^0_5). \end{aligned} \tag{2.13}$$

In brief, given a 2-orthogonal MPS $\{W_n\}_{n\geq 0}$, with respect to (w_0, w_1), defined by (1.10)-(1.12), such that $a^1_n = a^2_n = b^1_n = b^2_n = c^1_n = c^2_n = 0, \quad n \geq 0$, the principal components are also 2-orthogonal, with respect to (u_0, u_1), (v_0, v_1) and (r_0, r_1), respectively. Furthermore, each one of the following features: 2-orthogonality, diagonal CD and the fact that the principal components fulfil finite-type relations, allow us to achieve relations between the forms w_0, w_1,u_0, u_1, v_0, v_1, r_0 and r_1, as the following result announces.

Theorem 2.7. *Given a 2-orthogonal MPS $\{W_n\}_{n\geq 0}$, with respect to (w_0, w_1), defined by (1.10)–(1.12), so that $a^1_n = a^2_n = b^1_n = b^2_n = c^1_n = c^2_n = 0, \quad n \geq 0$, we have the following relations involving w_0, w_1 and elements of the dual sequences of the principal components.*

$$u_0 = \sigma_\varpi(w_0), \quad u_1 = \sigma_\varpi(w_3), \tag{2.14}$$

$$v_0 = \sigma_\varpi((x-a)w_1), \quad v_1 = \sigma_\varpi((x-a)w_4), \tag{2.15}$$

$$r_0 = \sigma_\varpi((x-b)(x-c)w_2), \quad r_1 = \sigma_\varpi((x-b)(x-c)w_5), \tag{2.16}$$

$$r_m = u_m + \left(\gamma^0_{3m+2} + \gamma^0_{3m+3}\right)u_{m+1} + \gamma^0_{3m+3}\gamma^0_{3m+5}u_{m+2}, \tag{2.17}$$

$$r_m = v_m + \gamma^0_{3m+3}v_{m+1}, \quad m \geq 0; \tag{2.18}$$

and, there are polynomials $\Lambda^\mu(n,\nu)$, $0 \leq \mu \leq 1$, *such that*

$$w_2 = \Lambda^0(1,0)w_0 + \Lambda^1(1,0)w_1, \quad w_3 = \Lambda^0(1,1)w_0 + \Lambda^1(1,1)w_1, \tag{2.19}$$

$$w_4 = \Lambda^0(2,0)w_0 + \Lambda^1(2,0)w_1, \quad w_5 = \Lambda^0(2,1)w_0 + \Lambda^1(2,1)w_1, \tag{2.20}$$

and $\deg\Lambda^0(1,0) = 1$, $\deg\Lambda^1(1,0) = 0$, $\deg\Lambda^0(1,1) \leq 1$, $\deg\Lambda^1(1,1) = 1$, $\deg\Lambda^0(2,0) = 2$, $\deg\Lambda^1(2,0) \leq 1$, $\deg\Lambda^0(2,1) \leq 2$, $\deg\Lambda^1(2,1) = 2$.

Proof. Relations (2.14-2.16) are directly obtained from Theorem 1.17. Furthermore, applying Theorem 1.4, there are polynomials $\Lambda^\mu(n,\nu)$, $0 \leq \mu \leq 1$, fulfilling relations (2.19-2.20) and whose degrees are characterized as announced. Finally, Theorem 2.6 puts in evidence two relations of finite type. More precisely, identities (b_7) and (b_8) yield the following strictly finite-type relation between sequences $\{R_n\}_{n\geq 0}$ and $\{P_n\}_{n\geq 0}$, with respect to $\Phi(x) = 1$ (see definition 1.11): $P_n(x) = \sum_{\nu=n-2}^{n} \lambda_{n,\nu}R_\nu(x)$, $\quad n \geq 2$, where $\lambda_{n,n-2} = \gamma^0_{3n-3}\gamma^0_{3n-1} \neq 0$, $\lambda_{n,n-1} = \gamma^0_{3n-1} + \gamma^0_{3n}$ and $\lambda_{n,n} = 1$.

Similarly, identity (b_8) is the following strictly finite-type relation between sequences $\{R_n\}_{n\geq 0}$ and $\{Q_n\}_{n\geq 0}$, with respect to $\Phi(x) = 1$.

$$Q_n(x) = \sum_{\nu=n-1}^{n} \lambda_{n,\nu}R_\nu(x), \quad n \geq 1, \quad \text{where } \lambda_{n,n-1} = \gamma^0_{3n} \neq 0, \quad \lambda_{n,n} = 1.$$

Thus, Theorem 1.12 establishes relations (2.17-2.18). □

3. The cubic decomposition of a 2-orthogonal and 2-symmetric sequence

Motivated by the structure of a 2-symmetric sequence and its simple cubic decomposition, we will now focus our attention in that type of sequences, aiming to fully understand what further information can our general CD bring to the already known results. The next result describes any CD of a 2-orthogonal and 2-symmetric MPS.

Theorem 3.1. *A MPS* $\{W_n\}_{n\geq 0}$, *defined by* (1.10)–(1.12) *is* 2*-orthogonal and* 2*-symmetric if and only if the following relations are fulfilled for* $n \geq 0$.

$(\mathbf{IC_1})$ $a^2_0(x) = -p$, $\quad (\mathbf{IC_2})$ $b^2_0(x) = p^2 - q$,

$(\mathbf{i_1})$ $a^1_n(x) = (b+c+p)a^2_n(x) + \gamma^0_{3n+2}b^2_{n-1}(x) + b^2_n(x)$,

$(\mathbf{i_2})$ $b^1_n(x) = \gamma^0_{3n-2}\gamma^0_{3n}a^2_{n-2}(x) + \left(\gamma^0_{3n} + \gamma^0_{3n+1}\right)a^2_{n-1}(x) + a^2_n(x)$

$+\Big((a-b-c)(b+c+p) - L\Big)b^2_{n-1}(x) + (a+p)\gamma^0_{3n}R_{n-1}(x) + (a+p)R_n(x)$,

$$(\mathbf{i_3})\ c_n^1(x) = \gamma_{3n+1}^0(a+p)a_{n-1}^2(x) + (a+p)a_n^2(x) + \gamma_{3n-1}^0\gamma_{3n+1}^0 b_{n-2}^2(x) + \left(\gamma_{3n+1}^0 + \gamma_{3n+2}^0\right) b_{n-1}^2(x) + b_n^2(x) + \Big((a-b-c)(b+c+p) - L\Big) R_n(x),$$

$$(\mathbf{i_4})\ c_n^2(x) = \gamma_{3n+1}^0 a_{n-1}^2(x) + a_n^2(x) + (b+c+p)R_n(x),$$

$$(\mathbf{i_5})\ P_n(x) = \Big((a-b-c)(b+c+p) - L\Big) a_{n-1}^2(x) + \gamma_{3n-1}^0(a+p)b_{n-2}^2(x) + (a+p)b_{n-1}^2(x) + \gamma_{3n-3}^0\gamma_{3n-1}^0 R_{n-2}(x) + \left(\gamma_{3n-1}^0 + \gamma_{3n}^0\right) R_{n-1}(x) + R_n(x),$$

$$(\mathbf{i_6})\ Q_n(x) = (b+c+p)b_{n-1}^2(x) + \gamma_{3n}^0 R_{n-1}(x) + R_n(x),$$

$$(\mathbf{i_7})\ p\gamma_{3n+1}^0\gamma_{3n+3}^0 a_{n-1}^2(x) + p\left(\gamma_{3n+3}^0 + \gamma_{3n+4}^0\right) a_n^2(x) + pa_{n+1}^2(x) + \gamma_{3n-1}^0\gamma_{3n+1}^0\gamma_{3n+3}^0 b_{n-2}^2(x) + \Big(\gamma_{3n+3}^0\left(\gamma_{3n+1}^0 + \gamma_{3n+2}^0\right) + \gamma_{3n+2}^0\gamma_{3n+4}^0\Big) b_{n-1}^2(x) + \left(r - x + \gamma_{3n+3}^0 + \gamma_{3n+4}^0 + \gamma_{3n+5}^0\right) b_n^2(x) + b_{n+1}^2(x) + q\gamma_{3n+3}^0 R_n(x) + qR_{n+1}(x) = 0,$$

$$(\mathbf{i_8})\ q\gamma_{3n+1}^0 a_{n-1}^2(x) + qa_n^2(x) + p\gamma_{3n-1}^0\gamma_{3n+1}^0 b_{n-2}^2(x) + p\left(\gamma_{3n+1}^0 + \gamma_{3n+2}^0\right) b_{n-1}^2(x) + pb_n^2(x) + \gamma_{3n-3}^0\gamma_{3n-1}^0\gamma_{3n+1}^0 R_{n-2}(x) + \Big(\gamma_{3n+1}^0\left(\gamma_{3n-1}^0 + \gamma_{3n}^0\right) + \gamma_{3n}^0\gamma_{3n+2}^0\Big) R_{n-1}(x) + \left(r - x + \gamma_{3n+1}^0 + \gamma_{3n+2}^0 + \gamma_{3n+3}^0\right) R_n(x) + R_{n+1}(x) = 0,$$

$$(\mathbf{i_9})\ \gamma_{3n-2}^0\gamma_{3n}^0\gamma_{3n+2}^0 a_{n-2}^2(x) + \Big(\gamma_{3n+2}^0\left(\gamma_{3n}^0 + \gamma_{3n+1}^0\right) + \gamma_{3n+1}^0\gamma_{3n+3}^0\Big) a_{n-1}^2(x) + \left(r - x + \gamma_{3n+2}^0 + \gamma_{3n+3}^0 + \gamma_{3n+4}^0\right) a_n^2(x) + a_{n+1}^2(x) + q\gamma_{3n+2}^0 b_{n-1}^2(x) + qb_n^2(x) + p\gamma_{3n}^0\gamma_{3n+2}^0 R_{n-1}(x) + p\left(\gamma_{3n+2}^0 + \gamma_{3n+3}^0\right) R_n(x) + pR_{n+1}(x) = 0,$$

where, for any component sequence $\{\zeta_n\}_{n\geq 0}$, $\zeta_i = 0,\ i = -2, -1.$

Proof. Let us consider the relations of Theorem 2.1, with $\beta_n = 0, \quad n \geq 0$, and $\gamma_n^1 = 0$.

$(\tilde{B}_0)$ $b_0^1(x) = a,$

$(\tilde{B}_1)$ $c_n^1(x) = a\, b_n^1(x) + \Theta(x)b_{n-1}^2(x) - \gamma_{3n}^0 c_{n-1}^1(x) - (a-b)(a-c)Q_n(x),$

$(\tilde{B}_2)$ $c_n^2(x) = b_n^1(x) + Lb_{n-1}^2(x) - \gamma_{3n}^0 c_{n-1}^2(x) - (a-b-c)Q_n(x),$

$(\tilde{B}_3)$ $R_n(x) = -(b+c+p)b_{n-1}^2(x) + Q_n(x) - \gamma_{3n}^0 R_{n-1}(x),$

$(\tilde{B}_4)$ $P_{n+1}(x) = -\gamma_{3n+1}^0 P_n(x) + a\, c_n^1(x) - (a-b)(a-c)c_n^2(x) + \Theta(x)R_n(x),$

$(\tilde{B}_5)$ $a_n^1(x) = -\gamma_{3n+1}^0 a_{n-1}^1(x) + c_n^1(x) - (a-b-c)c_n^2(x) + LR_n(x),$

$(\tilde{B}_6)$ $a_n^2(x) = -\gamma_{3n+1}^0 a_{n-1}^2(x) + c_n^2(x) - (b+c+p)R_n(x),$

$(\tilde{B}_7)$ $b_{n+1}^1(x) = -(a-b)(a-c)a_n^1(x) + \Theta(x)a_n^2(x) - \gamma_{3n+2}^0 b_n^1(x) + a\, P_{n+1}(x),$

$(\tilde{B}_8)$ $Q_{n+1}(x) = -(a-b-c)a_n^1(x) + La_n^2(x) + P_{n+1}(x) - \gamma_{3n+2}^0 Q_n(x),$

$(\tilde{B}_9)$ $b_n^2(x) = a_n^1(x) - (b+c+p)a_n^2(x) - \gamma_{3n+2}^0 b_{n-1}^2(x).$

Let us first notice that from $(\tilde{B}_3)$, $(\tilde{B}_6)$ and $(\tilde{B}_9)$, we have immediately (i_6),(i_4) and (i_1), respectively. Considering $(\tilde{B}_2)$ with $n = 0$ and regarding $(\tilde{B}_0)$ we have:

$$c_0^2(x) = b + c, \tag{3.1}$$

and after the transformation $n \to n + 1$ we can introduce (i_4) and (i_6), yielding

$$\begin{aligned} b_{n+1}^1(x) &= \gamma_{3n+1}^0 \gamma_{3n+3}^0 a_{n-1}^2(x) + \left(\gamma_{3n+3}^0 + \gamma_{3n+4}^0\right) a_n^2 + a_{n+1}^2 \\ &+ \Big((a-b-c)(b+c+p) - L\Big) b_n^2(x) + (a+p)\gamma_{3n+3}^0 R_n(x) \\ &+ (a+p)R_{n+1}(x). \end{aligned} \tag{3.2}$$

With respect to the initial conditions, we obtain (IC_1) from $(\tilde{B}_6)$ with $n = 0$ and using (3.1). Also, from $(\tilde{B}_1)$ with $n = 0$, we have:

$$c_0^1(x) = a^2 - (a-b)(a-c). \tag{3.3}$$

Using (3.1) and (3.3), we get from $(\tilde{B}_5)$ with $n = 0$:

$$a_0^1(x) = a^2 - (a-b)(a-c) - (a-b-c)(b+c) + L = -bp - cp - q. \tag{3.4}$$

Thus, we obtain (IC_2) considering $(\tilde{B}_9)$ with $n = 0$ and introducing (IC_1) and (3.4). With this particular information, we can rewrite (3.2) in order to justify identity (i_2), just by decreasing n and taking into consideration $(\tilde{B}_0)$. Let us consider $(\tilde{B}_5)$ with n replaced by $n+1$. Inserting (i_1) and (i_4) we get:

$$\begin{aligned} c_{n+1}^1(x) &= \gamma_{3n+4}^0 (a+p) a_n^2(x) + (a+p) a_{n+1}^2(x) + \gamma_{3n+2}^0 \gamma_{3n+4}^0 b_{n-1}^2 \\ &+ \left(\gamma_{3n+4}^0 + \gamma_{3n+5}^0\right) b_n^2 + b_{n+1}^2 + \Big((a-b-c)(b+c+p) - L\Big) R_{n+1}. \end{aligned} \tag{3.5}$$

Hence, we can establish identity (i_3) attending to $(IC1)$ and $(IC2)$. Proceeding in a similar way, we get (i_5) from $(\tilde{B}_8)$. The remaining three identities (i_7), (i_8) and (i_9) are obtained as the previous ones from relations $(\tilde{B}_1)$, $(\tilde{B}_4)$ and $(\tilde{B}_7)$, where $(\tilde{B}_1)$ is taken with n replaced by $n + 1$.

Finally, supposing the enunciated list of relations, we can perform a straightforward confirmation of Theorem 2.1 relations for a 2-symmetric sequence, which concludes the proof. □

Theorem 3.1 shows that in a CD of a 2-orthogonal and 2-symmetric sequence, the six component sequences of the first and second columns of $M_n(x)$, defined in (1.13), are written in terms of the three component sequences of the third column, more precisely $\{a_n^2\}_{n\geq 0}$, $\{b_n^2\}_{n\geq 0}$ and $\{R_n\}_{n\geq 0}$, where these latest fulfil relations $(i_7) - (i_9)$. For that reason, we were driven to investigate the CD where $a_n^2(x)$ or $b_n^2(x)$ vanish. Indeed, as the next Proposition clarifies, those cases are strictly related to the conditions $p = q = 0$ where the resultant CD has a lower triangular matrix $M_n(x)$, which can also be considered a diagonal CD for the additional choice of parameters $a = b = c = 0$.

Proposition 3.2. *Let $\{W_n\}_{n\geq 0}$ be a MPS defined by (1.10)–(1.12), 2-orthogonal and 2-symmetric. Then, the following statements are equivalent.*

a. $a_n^1(x) = 0,\ n \geq 0;$
b. $a_n^2(x) = 0,\ n \geq 0;$
c. $b_n^2(x) = 0,\ n \geq 0;$
d. $p = q = 0;$
e. $a_n^1(x) = a_n^2(x) = b_n^2(x) = 0,\ n \geq 0;$
f. *The component sequences fulfil the following (where* $n \geq 0$ *and* $R_i = 0,\ i = -2, -1$*):*

$$a_n^1(x) = a_n^2(x) = b_n^2(x) = 0;\ \ c_n^2(x) = (b+c)R_n(x);$$

$$P_n(x) = \gamma_{3n-3}^0\gamma_{3n-1}^0 R_{n-2}(x) + \left(\gamma_{3n-1}^0 + \gamma_{3n}^0\right) R_{n-1}(x) + R_n(x); \tag{3.6}$$

$$Q_n(x) = \gamma_{3n}^0 R_{n-1}(x) + R_n(x); \tag{3.7}$$

$$b_n^1(x) = aQ_n(x);\ \ c_n^1(x) = \Big(a(b+c) - bc\Big) R_n(x);$$

$$R_{n+1}(x) = \left(x - r - \gamma_{3n+1}^0 - \gamma_{3n+2}^0 - \gamma_{3n+3}^0\right) R_n(x) \tag{3.8}$$
$$- \left(\gamma_{3n+1}^0\left(\gamma_{3n-1}^0 + \gamma_{3n}^0\right) + \gamma_{3n}^0\gamma_{3n+2}^0\right) R_{n-1}(x) - \gamma_{3n-3}^0\gamma_{3n-1}^0\gamma_{3n+1}^0 R_{n-2}(x),$$

g. $W_{3n}(x) = P_n(\varpi(x)),$
$W_{3n+1}(x) = aQ_n(\varpi(x)) + (x-a)Q_n(\varpi(x)),$
$W_{3n+2}(x) = \Big(a(b+c) - bc\Big) R_n(\varpi(x)) + (x-a)(b+c)R_n(\varpi(x))$
$\qquad + (x-b)(x-c)R_n(\varpi(x)),$
where $\varpi(x) = x^3 + r$ *and* $\{P_n\}_{n\geq 0}$, $\{Q_n\}_{n\geq 0}$ *and* $\{R_n\}_{n\geq 0}$ *fulfil relations* (3.6), (3.7) *and* (3.8), *respectively.*

h. $\{W_n\}_{n\geq 0}$ *is* $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & r \end{pmatrix}$*- symmetric.*

Proof. In the following arguments we will use the content of Theorem 3.1 following its notation.
a. $\Rightarrow$ *d.* Let us suppose that $a_n^1(x) = 0,\ n \geq 0$. By (i_1) this means that

$$b_n^2(x) = -(b+c+p)a_n^2(x) - \gamma_{3n+2}^0 b_{n-1}^2(x),\ n \geq 0. \tag{3.9}$$

Reading (3.9) with $n = 0$ and inserting $(IC1)$ and $(IC2)$, we obtain

$$-q = p(b+c). \tag{3.10}$$

Let us now consider (i_7)-(i_9) with $n = 0$:

$$p\left(\gamma_3^0 + \gamma_4^0\right) a_0^2(x) + pa_1^2(x) + \left(r - x + \gamma_3^0 + \gamma_4^0 + \gamma_5^0\right) b_0^2(x)$$
$$+ b_1^2(x) + q\gamma_3^0 + qR_1(x) = 0, \tag{3.11}$$

$$qa_0^2(x) + pb_0^2(x) + \left(r - x + \gamma_1^0 + \gamma_2^0 + \gamma_3^0\right) + R_1(x) = 0, \tag{3.12}$$

$$\left(r - x + \gamma_2^0 + \gamma_3^0 + \gamma_4^0\right) a_0^2(x) + a_1^2(x) + qb_0^2(x) + p\left(\gamma_2^0 + \gamma_3^0\right)$$
$$+ pR_1(x) = 0. \tag{3.13}$$

Using $(IC1)$ and $(IC2)$, we obtain $R_1(x)$ from (3.12), then $a_1^2(x)$ from (3.13) and finally $b_1^2(x)$ from (3.11). The coefficient of x of the resultant $b_1^2(x)$ is $3p^2 - 2q$. On the other hand, we can calculate $b_1^2(x)$ from (3.9) with $n = 1$ and the correspondent coefficient of x is $2(b+c+p)p$. The two equations

$3p^2-2q=2(b+c+p)p$ and (3.10) yield $p=q=0$.
b. $\Rightarrow$ *d.* If $a_n^2(x)=0,\ n\geq 0$, then $(IC1)$ implies $p=0$. Furthermore, relation (i_9) becomes $q\left(b_n^2(x)+\gamma_{3n+2}^0 b_{n-1}^2(x)\right)=0$. If we suppose that $q\neq 0$, then we get

$$b_n^2(x)=(-1)^{n+1}q\prod_{k=1}^{n}\gamma_{3k+2}^0,\ n\geq 1.$$

In particular, $\deg\left(b_n^2(x)\right)=0,\ n\geq 0$. Considering (i_7) with $n=0$:

$$\left(r-x+\gamma_3^0+\gamma_4^0+\gamma_5^0\right)b_0^2(x)+b_1^2(x)+q\gamma_3^0+qR_1(x)=0,$$

and since $b_0^2(x)=-q$ and $b_1^2(x)=q\gamma_5^0$, we obtain $qR_1(x)=\left(x-r-\gamma_4^0\right)(-q)$. Taking (i_7) with $n=1$:

$$\begin{aligned}\Big(\gamma_6^0\left(\gamma_4^0+\gamma_5^0\right)+\gamma_5^0\gamma_7^0\Big)b_0^2(x)+\left(r-x+\gamma_6^0+\gamma_7^0+\gamma_8^0\right)b_1^2(x)+b_2^2(x)\\ +q\gamma_6^0R_1(x)+qR_2(x)=0,\end{aligned}$$

and inserting the information previously obtained for $b_0^2(x)$, $b_1^2(x)$, $b_2^2(x)$ and $qR_1(x)$, we get the following identity which is impossible if $q\neq 0$.

$$\begin{aligned}\Big(\gamma_6^0\left(\gamma_4^0+\gamma_5^0\right)+\gamma_5^0\gamma_7^0\Big)(-q)+\left(r-x+\gamma_6^0+\gamma_7^0+\gamma_8^0\right)q\gamma_5^0-q\gamma_8^0\gamma_5^0\\ +\gamma_6^0\left(x-r-\gamma_4^0\right)(-q)=-qR_2(x).\end{aligned}$$

c. $\Rightarrow$ *d.* If $b_n^2(x)=0,\ n\geq 0$, then $(IC1)$ says that $q=p^2$ and (i_7) and (i_9) correspond to the following two identities:

$$\begin{aligned}&p\gamma_{3n+1}^0\gamma_{3n+3}^0a_{n-1}^2(x)+p\left(\gamma_{3n+3}^0+\gamma_{3n+4}^0\right)a_n^2(x)+pa_{n+1}^2(x)\\ &+p^2\gamma_{3n+3}^0R_n(x)+p^2R_{n+1}(x)=0,\end{aligned}\tag{3.14}$$

$$\begin{aligned}&\gamma_{3n-2}^0\gamma_{3n}^0\gamma_{3n+2}^0a_{n-2}^2(x)+\Big(\gamma_{3n+2}^0\left(\gamma_{3n}^0+\gamma_{3n+1}^0\right)+\gamma_{3n+1}^0\gamma_{3n+3}^0\Big)a_{n-1}^2(x)\\ &+\left(r-x+\gamma_{3n+2}^0+\gamma_{3n+3}^0+\gamma_{3n+4}^0\right)a_n^2(x)+a_{n+1}^2(x)+p\gamma_{3n}^0\gamma_{3n+2}^0R_{n-1}(x)\\ &+p\left(\gamma_{3n+2}^0+\gamma_{3n+3}^0\right)R_n(x)+pR_{n+1}(x)=0.\end{aligned}\tag{3.15}$$

If we suppose that $p\neq 0$, then (3.14) yields

$$\begin{aligned}p\gamma_{3n+3}^0R_n(x)+pR_{n+1}(x)=&-\gamma_{3n+1}^0\gamma_{3n+3}^0a_{n-1}^2(x)\\ &-\left(\gamma_{3n+3}^0+\gamma_{3n+4}^0\right)a_n^2(x)-a_{n+1}^2(x),\end{aligned}\tag{3.16}$$

and replacing $p\gamma_{3n+3}^0R_n(x)+pR_{n+1}(x)$ of (3.15) with this last expression, we obtain:

$$\begin{aligned}&\gamma_{3n-2}^0\gamma_{3n}^0\gamma_{3n+2}^0a_{n-2}^2(x)+\gamma_{3n+2}^0\left(\gamma_{3n}^0+\gamma_{3n+1}^0\right)a_{n-1}^2(x)\\ &+\left(r-x+\gamma_{3n+2}^0\right)a_n^2(x)+p\gamma_{3n}^0\gamma_{3n+2}^0R_{n-1}(x)+p\gamma_{3n+2}^0R_n(x)=0.\end{aligned}$$

In particular, we deduce that $\deg\left(a_n^2(x)\right)<n$, which implies that the right hand of (3.16) has degree less than $n+1$. But indeed, this is impossible since the left hand of (3.16) has degree $n+1$. Hence, we must conclude that $p=0$ and from $(IC2)$ we have also $q=0$.
d. $\Rightarrow$ *f.* and *d.* $\Rightarrow$ *g.* If we suppose $p=q=0$, $(IC1)$ and $(IC2)$ tell us that $a_0^2(x)=b_0^2(x)=0$. Thus, identities (i_7) and (i_9) imply $a_n^2(x)=b_n^2(x)=$

0, $n \geq 0$. Moreover, the remaining list of relations of Theorem 3.1 establish the CD described in item f. which corresponds entirely to the CD of item g.

To finalize the proof of the equivalence between $a. - g.$, we remark that obviously we have: $f. \Rightarrow e. \Rightarrow a.$; $e. \Rightarrow b.$; $e. \Rightarrow c.$; $e. \Rightarrow d.$ and $e. \Leftrightarrow f.$ If we consider the CD presented in $g.$, with $\varpi(x) = x^3 + r$, and we choose $a = b = c = 0$, then we obtain a diagonal CD where the principal components are the ones given by g. Conversely, item $h.$ signifies that for $a = b = c = 0$ and $p = q = 0$, the correspondent CD of $\{W_n\}_{n\geq 0}$ is given by the following: $W_{3n}(x) = P_n(\varpi(x))$, $W_{3n+1}(x) = xQ_n(\varpi(x))$, $W_{3n+2}(x) = x^2R_n(\varpi(x))$, where, due to the conditions $p = q = 0$, the principal components are defined by relations (3.6), (3.7) and (3.8), respectively. Altering the auxiliary polynomials x and x^2 to $x - a$ and $(x - b)(x - c)$, for arbitrary constants a, b, c, we deduce the CD indicated in g. □

Corollary 3.3. *Let $\{W_n\}_{n\geq 0}$ be a 2-orthogonal and 2-symmetric MPS defined by (1.10)–(1.12). Then $\{W_n\}_{n\geq 0}$ is $\begin{pmatrix} a & b & c \\ p & q & r \end{pmatrix}$- symmetric if and only if $a = b = c = p = q = 0$.*

Proof. Let us suppose that we have a diagonal CD. Attending to the relations of Theorem 2.6 and since $\beta_n = 0$, $n \geq 0$, we conclude that $a = b = c = p = q = 0$. Conversely, if we choose $a = b = c = p = q = 0$, then Proposition 3.2 provides that $b^1_n(x) = c^1_n(x) = c^2_n(x) = 0$, $n \geq 0$. □

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