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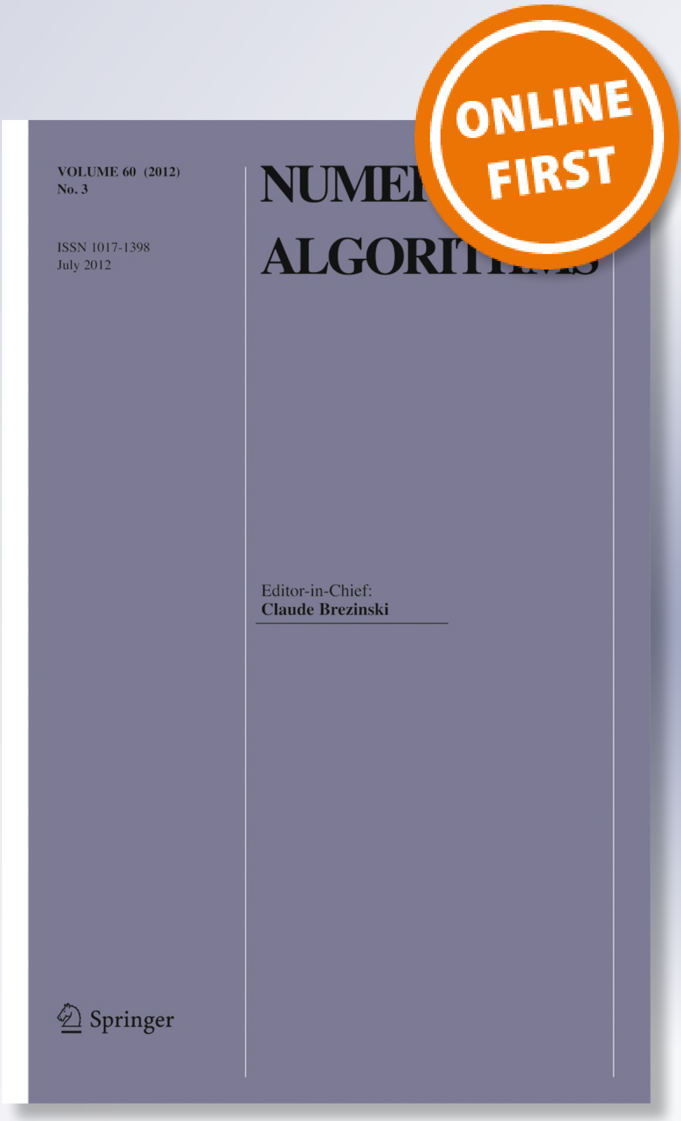

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ORIGINAL PAPER

Connection coefficients for orthogonal polynomials: symbolic computations, verifications and demonstrations in the *Mathematica* language

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Abstract We deal with the problem of obtaining closed formulas for the connection coefficients between orthogonal polynomial sequences and, also, the canonical sequence, using a recursive methodology based on symbolic computations, verifications and demonstrations in the *Mathematica*® language. We present the corresponding software that is available in NETLIB and, with it, we derive new formulas for the connection coefficients for some semi-classical of class 1 families.

Keywords Connection coefficients · Orthogonal polynomials · Symbolic computations · Automatic demonstrations · *Mathematica*®(version 8)

Mathematics Subject Classifications (2010) 33C45 · 33D45 · 42C05 · 33F10 · 68W30 · 68-04

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1 Introduction

The aim of this article is to present the software *CCOP—Connection Coefficients for Orthogonal Polynomials* written in *Mathematica*® language (version 8), and some new formulas for the connection coefficients (in the sequel indicated as CC) obtained by mean of this implementation. The software *CCOP* is available in the library NUMERALGO of NETLIB (http://www.netlib.org/numeralgo/) as na34 package. All the explanations concerning the programming of the commands have been inserted in the tutorial provided with the package. This software has already produced the results presented in [14, 15] for obtaining closed formulas for the CC between two orthogonal polynomials sequences, or between the canonical sequence and an orthogonal one. In this work, we follow the same methodology employed in [14, 15], which is based on symbolic computations, verifications and demonstrations, and we explore a new topic on automatic proofs.

We proceed with the simplest method based only on the recurrence relation fulfilled by any orthogonal sequence, which leads to a general recurrence relation with two indexes satisfied by the CC. The implementation of this relation allows always the recursive computation of the first CC up to a certain order. The capabilities of simplification and factorization of *Mathematica*® are crucial to get these coefficients written in a convenient form that enable us to infer the corresponding closed formulas. This task is not always possible and could be more or less difficult depending on the complexity of the examples. For many families the recurrence coefficients are factorized polynomial or rational functions in n. So the *Mathematica*® algebraic manipulation commands work pretty well. This software is mostly intended for those cases. If we identify the model for the closed formulas, then they can be translated in *Mathematica*® commands and we can easily verify the first results obtained. The final goal of the work is to provide a demonstration of the fact that the closed formulas are really true, that is, the model is a solution of the general recurrence relation. In principle, this demonstration can also be achieved uniquely by implementation depending on the success of the simplifications abilities of *Mathematica*®. When this is not possible, we can always try the procedures employed in [14, 15] doing only a part of the demonstration with the help of that symbolic language.

We would like to refer the NAVIMA software [2, 3, 7, 16] which implements a similar recursive approach to connection problems in *Mathematica*®. NAVIMA algorithm generates in a systematic way a linear recurrence relation in one index only, using some additional proprieties of the orthogonal families, like structure relations or differential equations among others. Then that recurrence relation is solved in the following way: computation of the first few CC from the recurrence in order to guess its general expression; afterwards, NAVIMA verify by substitution that this guess satisfies the recurrence relation [3, pp. 768 and 773]. In the present work, any specified character of the sequences like classical, semi-classical or other (which is translated by structure relations, differential equations or others proprieties) are not directly employed, the general recurrence relation that we use is based only on the orthogonality of the sequences. Concerning expansion of multivariable polynomials with some applications to connection problems following the Navima principles see [17]. Other recursive methods to compute recurrence relations for the connection coefficients are also considered in [8, 9] and a computer algebra based method using Zeilberger's algorithm was proposed in [11] taking into account several structure formulas of the classical systems. See also [5] with a implementation in *Maple*™.

We remark the fact that the mathematical literature on this subject is extremely vast and a wide variety of methods have been developed using several other techniques. The reader can find some of the main references in [15]. Furthermore, nowadays, there are symbolic implementations in the domain of orthogonal polynomials. We refer, in particular, to the package *CAOP* [10] for calculating formulas for orthogonal polynomials belonging to the Askey scheme by *Maple*™ and available on internet.

Let us present the summary of this work. In Section 2 we recall the basic definitions and mathematical results needed to understand the subject, and in Section 3, we present briefly the main commands of *CCOP* giving their names and a small description. The commands corresponding to the symmetrical case and the ones concerning the cases of Sections 5.2.1 and 5.2.2, that need a specific treatment, are considered directly in the package.

Section 4 is dedicated to the presentation of a very simple test example: the Charlier polynomials. We follow all the steps of the methodology giving the implementation and the results. Other test examples can be found in the software, namely the cases of the canonical polynomials in terms of the Laguerre ones and the symmetric generalized Hermite family.

In the last but one section, we present some examples of CC derived with our method corresponding to a symmetric semi-classical of class 1 sequence and two non-symmetric semi-classical of class 1 families. All these cases are included in the software. The formulas corresponding to these last examples are new.

We would like to recall that we have already tested this methodology in the cases of Laguerre and Bessel [14], Gegenbauer and Jacobi [15], generalized Hermite [14] (see also the tutorial), and, also, with other semi-classical of class 1 example [15]. We believe that this method allows to explore other cases and derive new closed formulas for CC.

Conclusions and commentaries on the work presented here and in [14, 15] end the paper.

2 Connection coefficients for orthogonal polynomials

Let $\mathcal{P}$ be the vector space of polynomials with coefficients in $\mathbb{C}$ and let $\mathcal{P}'$ be its dual. We denote by $\langle u, p\rangle$ the effect of $u \in \mathcal{P}'$ on $p \in \mathcal{P}$. In particular, $\langle u, x^n\rangle := (u)_n \, , n \geqslant 0$ represent the moments of u.

Let $\{P_n\}_{n\geqslant 0}$ be a monic polynomial sequence (MPS) with deg $P_n = n,\ n \geq 0$, that is, $P_n(x) = x^n + \ldots$. A form u is said regular [12, 13] if and only if there exists a MPS $\{P_n\}_{n\geqslant 0}$, such that:

$$\langle u, P_n P_m \rangle = 0\ ,\ n \neq m\ ,\ n, m \geq 0\ , \tag{1}$$

$$\left\langle u, P_n^2 \right\rangle \neq 0\ ,\ n \geq 0\ . \tag{2}$$

In this case, $\{P_n\}_{n\geqslant 0}$ is said regularly orthogonal with respect to u and is called a monic orthogonal polynomial sequence (MOPS). The orthogonality conditions are given by (1) and (2) corresponds to the regularity conditions.

The sequence $\{P_n\}_{n\geqslant 0}$ is regularly orthogonal with respect to u if and only if [12, 13] there exist two sequences of coefficients $\{\beta_n\}_{n\geqslant 0}$ and $\{\gamma_{n+1}\}_{n\geqslant 0}$, with $\gamma_{n+1} \neq 0,\ n \geqslant 0$, such that, $\{P_n\}_{n\geqslant 0}$ satisfies the following initial conditions and recurrence relation of order 2:

$$P_0(x) = 1, \quad P_1(x) = x - \beta_0, \tag{3}$$

$$P_{n+2}(x) = (x - \beta_{n+1}) P_{n+1}(x) - \gamma_{n+1} P_n(x), \quad n \geqslant 2. \tag{4}$$

Furthermore, the recurrence coefficients $\{\beta_n\}_{n\geqslant 0}$ and $\{\gamma_{n+1}\}_{n\geqslant 0}$ satisfy:

$$\beta_n = \frac{\left\langle u, x P_n^2(x) \right\rangle}{\left\langle u, P_n^2(x) \right\rangle}, \quad n \geqslant 0,$$

$$\gamma_{n+1} = \frac{\left\langle u, P_{n+1}^2(x) \right\rangle}{\left\langle u, P_n^2(x) \right\rangle}, \quad n \geqslant 0. \tag{5}$$

We remark that, from (3) and (5), the regularity conditions (2) are equivalent to the conditions $\gamma_{n+1} \neq 0, n \geqslant 0$.

As usual, we suppose that

$$\beta_n = 0,\ \gamma_{n+1} = 0,\ P_n(x) = 0,\ n < 0.$$

A form u is said symmetric if and only if $(u)_{2n+1} = 0,\ n \geq 0$. A polynomial sequence, $\{P_n\}_{n\geqslant 0}$, is said symmetric if and only if $P_n(-x) = (-1)^n P_n(x),\ n \geqslant 0$.

Let $\{P_n\}_{n\geqslant 0}$ be a MOPS with respect to u. The following statements are equivalent [6]:

(a) u is symmetric.
(b) $\{P_n\}_{n\geqslant 0}$ is symmetric.
(c) $\beta_n = 0,\ n \geqslant 0$.

The canonical sequence $\{X_n\}_{n\geqslant 0},\ X_n(x) = x^n$, is orthogonal with respect to the Dirac measure $\delta_0,\ \langle \delta_0, f \rangle = f(0)$, defined by the moments $(\delta_0)_n = \delta_{0,n},\ n \geqslant 0$, where δ is the Dirac symbol. This sequence is not regularly orthogonal, since its recurrence coefficients are

$$\beta_n = 0\ ,\ \gamma_{n+1} = 0\ ,\ n \geqslant 0\ .$$

Given two MPS $\{P_n\}_{n\geqslant 0}$ and $\{\tilde{P}_n\}_{n\geqslant 0}$ the coefficients that satisfy the equality

$$P_n(x) = \sum_{m=0}^{n} \lambda_{n,m} \tilde{P}_m(x), \ n \geqslant 0, \tag{6}$$

are called the CC: $\lambda_{n,m} := \lambda_{n,m}^{P\tilde{P}} := \lambda_{n,m}(P \leftarrow \tilde{P})$.

It is obvious that these coefficients exist and are unique, because the polynomials are linearly independent.

Let us suppose that the two monic polynomial sequences $\{P_n\}_{n\geqslant 0}$ and $\{\tilde{P}_n\}_{n\geqslant 0}$ are orthogonal and are given by their recurrence coefficients $\{\beta_n\}_{n\geqslant 0}$, $\{\gamma_{n+1}\}_{n\geqslant 0}$ and $\{\tilde{\beta}_n\}_{n\geqslant 0}$, $\{\tilde{\gamma}_{n+1}\}_{n\geqslant 0}$, respectively, let us consider the problem of computing and determining closed formulas for the CC.

As demonstrated in [14, 15], the CC fulfill the following boundary and initial conditions and general recurrence relation

$$\lambda_{n,m} = 0 \ , \ n < 0 \text{ or } m < 0 \text{ or } m > n \ , \tag{7}$$

$$\lambda_{n,n} = 1 \ , \ n \geqslant 0 \ , \tag{8}$$

$$\lambda_{1,0} = \tilde{\beta}_0 - \beta_0 \ , \tag{9}$$

$$\lambda_{n,m} = \left(\tilde{\beta}_m - \beta_{n-1}\right) \lambda_{n-1,m} - \gamma_{n-1}\lambda_{n-2,m} + \tilde{\gamma}_{m+1}\lambda_{n-1,m+1} + \lambda_{n-1,m-1}, \quad 0 \leqslant m \leqslant n-1 \ , \ n \geqslant 2 \ . \tag{10}$$

All the work developed in this article is based on this recurrence relation.

If $\{P_n\}_{n\geq 0}$ and $\{\tilde{P}_n\}_{n\geq 0}$ are symmetric, then

$$\lambda_{2n-1,2m} = 0 \ , \ \lambda_{2n,2m+1} = 0 \ , \ 0 \leqslant m \leqslant n-1 \ , \ n \geqslant 1 \ ,$$

and the relation (10) is equivalent to

$$\begin{aligned} \lambda_{2n,2m} &= \tilde{\gamma}_{2m+1}\lambda_{2n-1,2m+1} + \lambda_{2n-1,2m-1} - \gamma_{2n-1}\lambda_{2n-2,2m} \ , \\ \lambda_{2n+1,2m+1} &= \tilde{\gamma}_{2m+2}\lambda_{2n,2m+2} + \lambda_{2n,2m} - \gamma_{2n}\lambda_{2n-1,2m+1} \ , \\ & 0 \leqslant m \leqslant n-1 \ , \ n \geqslant 1 \ . \end{aligned}$$

3 Commands for symbolic computation of connection coefficients

In this section, we furnish a list with the description of the main commands implemented in the software *CCOP*: *CC*, *MOP*, *verificationRCC*, *verificationDCC* and *demonstrationDCC*.

- *CC*[*rc*, *rct*][*p*][*pt*][*n*, *m*]
  CC computes recursively the connection coefficient $\lambda_{n,m} := \lambda_{n,m}(P \leftarrow \tilde{P})$ defined in (6), using the conditions (7)–(9) and the recurrence relation (10).

- $MOP[rc][p][n, x]$
  MOP computes recursively the monic orthogonal polynomial $P_n(x)$ of degree n in the variable x using the initial conditions (2) and the recurrence relation (3).
- $verificationRCC[rc, rct][p][pt][nmax]$
  verificationRCC makes a verification of the first connection coefficients $\lambda_{n,m} := \lambda_{n,m}(P \leftarrow \tilde{P})$ computed recursively by the *CC* command up to an index $n = nmax$. *verificationRCC* is based on the definition (6) of the connection coefficients.
- $verificationDCC[dcc][rc, rct][p][pt][nmax]$
  verificationDCC makes a comparison between the values of the connection coefficients $\lambda_{n,m} := \lambda_{n,m}(P \leftarrow \tilde{P})$ computed by the *CC* command and the ones computed using direct closed formulas, this, up to the index $n = nmax$.
- $demonstrationDCC[dcc][rc, rct][p][pt][n, m]$
  demonstrationDCC tries to demonstrate the direct closed formulas for the connection coefficients $\lambda_{n,m} := \lambda_{n,m}(P \leftarrow \tilde{P})$ for every integers n and m such that $0 \leqslant m \leqslant n-1, n \geqslant 1$.
- Arguments *rc*, *rct*, *p*, *pt* and *dcc* of the preceding commands.
  - *rc* and *rct* are the names of the commands that define the recurrence coefficients $\{\beta_n, \gamma_n\}$ and $\{\tilde{\beta}_n, \tilde{\gamma}_n\}$ of the two polynomials sequences P and $\tilde{P}$ with parameters p and pt respectively.
  - *dcc* is the name of the command that implements the direct closed formulas for the connection coefficients.

4 A complete case study

Let us see how the commands we have developed work and what results they produce in the simple case of the classical discrete monic Charlier polynomials $\{P_n(\alpha, .)\}_{n\geq 0}$ with parameter α [6]. The Charlier recurrence coefficients

$$\beta_n(\alpha) = n + \alpha \ , \ n \geq 0\,; \ \gamma_n(\alpha) = n\alpha \ , \ n \geq 1 \ , \ \alpha \neq 0 \,,$$

are implement in the following command.

$$CharlierC[\alpha\_][n\_] := CharlierC[\alpha][n] =$$
$$If[\ And[NumericQ[n], n < 0],\ Return[\{0, 0\}],\ Return[\{n + \alpha, \alpha * n\}]\ ];$$

The monic Charlier polynomials are defined, using the *MOP* command, by

$$CharlierP[\alpha\_][n\_, x\_] := CharlierP[\alpha][n, x] = MOP[CharlierC][\alpha][n, x];$$

The connection coefficients $\lambda_{n,m} := \lambda_{n,m}(P(\alpha; -) \leftarrow P(\tilde{\alpha}; -))$ are computed recursively up to $n = 6$, for example, by the next calling statement of the *CC* command.

$In[\ ] := Table[\ CC[CharlierC, CharlierC][\alpha][\tilde{\alpha}][n, m], \{n, 0, 6\}, \{m, 0, n\}\ ]//$
$TableForm$

$Out[\ ]//TableForm =$

1						
$-\alpha + \tilde{\alpha}$	1					
$(\alpha - \tilde{\alpha})^2$	$-2(\alpha - \tilde{\alpha})$	1				
$-(\alpha - \tilde{\alpha})^3$	$3(\alpha - \tilde{\alpha})^2$	$-3(\alpha - \tilde{\alpha})$	1			
$(\alpha - \tilde{\alpha})^4$	$-4(\alpha - \tilde{\alpha})^3$	$6(\alpha - \tilde{\alpha})^2$	$-4(\alpha - \tilde{\alpha})$	1		
$-(\alpha - \tilde{\alpha})^5$	$5(\alpha - \tilde{\alpha})^4$	$-10(\alpha - \tilde{\alpha})^3$	$10(\alpha - \tilde{\alpha})^2$	$-5(\alpha - \tilde{\alpha})$	1	
$(\alpha - \tilde{\alpha})^6$	$-6(\alpha - \tilde{\alpha})^5$	$15(\alpha - \tilde{\alpha})^4$	$-20(\alpha - \tilde{\alpha})^3$	$15(\alpha - \tilde{\alpha})^2$	$-6(\alpha - \tilde{\alpha})$	1

Now, we can verify these results and the next ones up to $nmax = 20$, for example, calling

$$In[\ ] := Timing[\ verificationRCC[CharlierC, CharlierC][\alpha, \tilde{\alpha}][20]\ ]$$

and we get the answer

$Out[\ ] = \{75.0289, True\}$

Note that the *Mathematica*® command *Timing[expr]* evaluates *expr* and returns a list of the time in seconds used, together with the result obtained.

The observation of the above results getting by the *CC* commands allows us to infer the following direct closed formula for the connection coefficients

$$\lambda_{n,m} = (-1)^{n-m}\binom{n}{m}(\alpha - \tilde{\alpha})^{n-m},\ 0 \le m \le n-1,\ n \ge 1\ , \tag{11}$$

which can be implement in a command as follows

$$CharlierDCC[\alpha\_][\alpha t\_][n\_, m\_] :=$$
$$(-1)\hat{}(n-m) * Binomial[n, m] * (\alpha - \alpha t)\hat{}(n-m);$$

In order to compare the results given by this command with those produced by *CC* up to $nmax = 20$, for example, we do

$$In[\ ] := Timing[\ verificationDCC[CharlierDCC]$$
$$[CharlierC, CharlierC][\alpha][\tilde{\alpha}][20]\ ]$$
$$Out[\ ] = \{0.009881, True\}$$

The automatic demonstration of the formula (11) is achieved in *Mathematica*® doing

$In[\ ]:=Timing[$
$demonstrationDCC[CharlierDCC][CharlierC, CharlierC][\alpha][\tilde{\alpha}][n, m]\ ]$

$Out[\ ] = \{0.36858, True\}$

We recall that the formula (11) is well known and can be found in several references; see, among others, [2, 9].

5 Some results obtained

In this section, we present closed formulas for the connection coefficients obtained by mean of our methodology for some cases of symmetric and non-symmetric semi-classical of class 1 orthogonal families. The formulas corresponding to the non-symmetric cases are new.

5.1 Symmetric semi-classical of class 1 case

Let us consider a symmetric semi-classical of class 1 sequence given in [1, p. 317] and also in [6, p. 156]. The recurrence coefficients are

$$\beta_n = 0\,, \tag{12}$$

$$\gamma_{2n+1} = \frac{(n+\beta+1)(n+\alpha+\beta+1)}{(2n+\alpha+\beta+1)(2n+\alpha+\beta+2)}\,, \tag{13}$$

$$\gamma_{2n+2} = \frac{(n+1)(n+\alpha+1)}{(2n+\alpha+\beta+2)(2n+\alpha+\beta+3)}\,, \tag{14}$$

for $n \geq 0$, with the regularity conditions $\alpha \neq -(n+1)$, $\beta \neq -(n+1)$, $\alpha+\beta \neq -(n+1)$, $n \geq 0$.

In the sequel, first we present the CC with respect to the canonical sequence, afterward we furnish the CC corresponding to the change of only one parameter and at last we consider the change of the two parameters.

Proposition 5.1 *The* $\lambda_{n,m} := \lambda_{n,m}(P(\alpha,\beta;-) \leftarrow X)$ *corresponding to* (12)–(14) *are given by*

$$\lambda_{2n,2m} = (-1)^{n+m}\binom{n}{m}\frac{\prod_{k=m+1}^{n}(\beta+k)}{\prod_{k=n+m+1}^{2n}(\alpha+\beta+k)}\,,$$

$$\lambda_{2n+1,2m+1} = (-1)^{n+m}\binom{n}{m}\frac{\prod_{k=m+2}^{n+1}(\beta+k)}{\prod_{k=n+m+2}^{2n+1}(\alpha+\beta+k)}\,,$$

for $0 \leq m \leq n$, $n \geq 0$.

Proposition 5.2 *The* $\lambda_{n,m} := \lambda_{n,m}(X \leftarrow P(\tilde{\alpha},\tilde{\beta};-))$ *corresponding to* (12)–(14) *are given by*

$$\lambda_{2n,2m} = \binom{n}{m}\frac{\prod_{k=m+1}^{n}(\tilde{\beta}+k)}{\prod_{k=2m+2}^{n+m+1}(\tilde{\alpha}+\tilde{\beta}+k)}\,,$$

$$\lambda_{2n+1,2m+1} = \binom{n}{m} \frac{\prod_{k=m+2}^{n+1}(\tilde{\beta}+k)}{\prod_{k=2m+3}^{n+m+2}(\tilde{\alpha}+\tilde{\beta}+k)} ,$$

for $0 \le m \le n$*,* $n \ge 0$*.*

Proposition 5.3 *The* $\lambda_{n,m} := \lambda_{n,m}(\alpha, \beta; \tilde{\alpha}, \beta) = \lambda_{n,m}(P(\alpha, \beta; -) \leftarrow P(\tilde{\alpha}, \beta; -))$ *corresponding to* (12)–(14) *are given by*

$$\lambda_{2n,2m}(\alpha, \beta; \tilde{\alpha}, \beta) = \binom{n}{m} \frac{\prod_{k=m+1}^{n}(\beta+k)\prod_{k=0}^{n-m-1}(\alpha-\tilde{\alpha}+k)}{\prod_{k=n+m+1}^{2n}(\alpha+\beta+k)\prod_{k=2m+2}^{n+m+1}(\tilde{\alpha}+\beta+k)} ,$$

$$\lambda_{2n+1,2m+1}(\alpha, \beta; \tilde{\alpha}, \beta) = \binom{n}{m} \frac{\prod_{k=m+2}^{n+1}(\beta+k)\prod_{k=0}^{n-m-1}(\alpha-\tilde{\alpha}+k)}{\prod_{k=n+m+2}^{2n+1}(\alpha+\beta+k)\prod_{k=2m+3}^{n+m+2}(\tilde{\alpha}+\beta+k)} ,$$

for $0 \le m \le n$*,* $n \ge 0$*.*

Proposition 5.4 *The* $\lambda_{n,m} := \lambda_{n,m}(\alpha, \beta; \alpha, \tilde{\beta}) = \lambda_{n,m}(P(\alpha, \beta; -) \leftarrow P(\alpha, \tilde{\beta}; -))$ *corresponding to* (12)–(14) *are given by*

$$\lambda_{2n,2m}(\alpha, \beta; \alpha, \tilde{\beta}) = (-1)^{n+m} \binom{n}{m} \frac{\prod_{k=m+1}^{n}(\alpha+k)\prod_{k=0}^{n-m-1}(\beta-\tilde{\beta}+k)}{\prod_{k=n+m+1}^{2n}(\alpha+\beta+k)\prod_{k=2m+2}^{n+m+1}(\alpha+\tilde{\beta}+k)} ,$$

$$\lambda_{2n+1,2m+1}(\alpha, \beta; \alpha, \tilde{\beta}) = (-1)^{n+m} \binom{n}{m} \frac{\prod_{k=m+1}^{n}(\alpha+k)\prod_{k=0}^{n-m-1}(\beta-\tilde{\beta}+k)}{\prod_{k=n+m+2}^{2n+1}(\alpha+\beta+k)\prod_{k=2m+3}^{n+m+2}(\alpha+\tilde{\beta}+k)} ,$$

for $0 \le m \le n$*,* $n \ge 0$*.*

Remark 5.5 We remark that $\lambda_{2n,2m}(\alpha, \beta; \alpha, \tilde{\beta}) = (-1)^{n+m}\lambda_{2n,2m}(\beta, \alpha; \tilde{\beta}, \alpha)$.

Proposition 5.6 *The* $\lambda_{n,m} := \lambda_{n,m}(\alpha, \beta; \tilde{\alpha}, \tilde{\beta}) := \lambda_{n,m}(P(\alpha, \beta; -) \leftarrow P(\tilde{\alpha}, \tilde{\beta}; -))$ *corresponding to* (12)–(14) *are given by*

$$\lambda_{2n,2m}\big(\alpha, \beta; \tilde{\alpha}, \tilde{\beta}\big) = \sum_{\nu=m}^{n} \lambda_{2n,2\nu}\big(\alpha, \beta; \alpha, \tilde{\beta}\big)\lambda_{2\nu,2m}\big(\alpha, \tilde{\beta}; \tilde{\alpha}, \tilde{\beta}\big) , \tag{15}$$

$$\lambda_{2n+1,2m+1}\big(\alpha, \beta; \tilde{\alpha}, \tilde{\beta}\big) = \sum_{\nu=m}^{n} \lambda_{2n+1,2\nu+1}\big(\alpha, \beta; \alpha, \tilde{\beta}\big)\lambda_{2\nu+1,2m+1}\big(\alpha, \tilde{\beta}; \tilde{\alpha}, \tilde{\beta}\big) ,$$

where the CC in the right-hand side are furnished by Propositions 5.3 *and* 5.4*.*

Proof Due to the symmetrical character, it holds

$$P_{2n}(\alpha,\beta;x)=\sum_{\nu=0}^{n}\lambda_{2n,2\nu}\big(\alpha,\beta;\alpha,\tilde{\beta}\big)P_{2\nu}\big(\alpha,\tilde{\beta};x\big)\,, \tag{16}$$

$$P_{2\nu}(\alpha,\tilde{\beta};x)=\sum_{m=0}^{\nu}\lambda_{2\nu,2m}\big(\alpha,\tilde{\beta};\tilde{\alpha},\tilde{\beta}\big)P_{2m}\big(\tilde{\alpha},\tilde{\beta};x\big)\,. \tag{17}$$

Replacing (17) in (16), we can deduce that

$$P_{2n}(\alpha,\beta;x)=\sum_{m=0}^{n}P_{2m}\big(\tilde{\alpha},\tilde{\beta};x\big)\sum_{\nu=m}^{n}\lambda_{2n,2\nu}\big(\alpha,\beta;\alpha,\tilde{\beta}\big)\lambda_{2\nu,2m}\big(\alpha,\tilde{\beta};\tilde{\alpha},\tilde{\beta}\big)\,,$$

which implies (15). The demonstration for odd indexes is similar. □

Closed formulas for the CC for a non-symmetric sequence related to this one with the same second recurrence coefficients γ_{n+1} but with $\beta_n=(-1)^n$ are given in [15, pp. 310–313].

5.2 Non-symmetric semi-classical of class 1 cases

In [4] are presented some generalizations of Jacobi polynomials obtained by a procedure of perturbation of the functional equation. In that way, two new sequences are given explicitly. In this article, we are going to derive the closed formulas for the CC with respect to the canonical sequence in both senses. These formulas are new.

5.2.1 Case 1

Let us consider the case 1 presented in [4]. The recurrence coefficients are

$$\beta_0=-\frac{\mu-1}{\mu-2\alpha-3}\,, \tag{18}$$

$$\beta_{n+1}=(-1)^n\frac{\mu(\mu-2n-2\alpha-4)+(-1)^{n+1}(2\alpha+1)}{(2n+2\alpha+3-\mu)(2n+2\alpha+5-\mu)},\ n\geq 0, \tag{19}$$

$$\gamma_{2n+1}=2\frac{(n+\alpha+1)(2n+1-\mu)}{(4n+2\alpha+3-\mu)^2},\ n\geq 0, \tag{20}$$

$$\gamma_{2n+2}=\frac{(2n+2)(2n+2\alpha+3-\mu)}{(4n+2\alpha+5-\mu)^2},\ n\geq 0\,, \tag{21}$$

with the regularity conditions $\mu \neq 2n+1$, $\mu \neq 2n+2\alpha+1$, $\alpha \neq -(n+1)$, $n \geq 0$. For $\mu = 0$, the classical monic Jacobi polynomials with parameters $(\alpha, \alpha+1)$ are recovered [6]. Next, we present the CC with respect to the canonical sequence.

Proposition 5.7 *The* $\lambda_{n,m} := \lambda_{n,m}(P(\alpha, \mu; -) \leftarrow X)$ *corresponding to* (18)–(21) *are given by*

$$\lambda_{2n,2m} = \binom{n}{m} \frac{\prod_{k=m+1}^{n}(\mu - 2k + 1)}{\prod_{k=n+m+1}^{2n}(2\alpha - \mu + 2k + 1)}, 0 \leq m \leq n,$$

$$\lambda_{2n,2m+1} = 2n\binom{n-1}{m} \frac{\prod_{k=m+2}^{n}(\mu - 2k + 1)}{\prod_{k=n+m+1}^{2n}(2\alpha - \mu + 2k + 1)}, 0 \leq m \leq n-1,$$

$$\lambda_{2n+1,2m} = -\binom{n}{m} \frac{\prod_{k=m+1}^{n+1}(\mu - 2k + 1)}{\prod_{k=n+m+1}^{2n+1}(2\alpha - \mu + 2k + 1)}, 0 \leq m \leq n,$$

$$\lambda_{2n+1,2m+1} = \binom{n}{m} \frac{\prod_{k=m+2}^{n+1}(\mu - 2k + 1)}{\prod_{k=n+m+2}^{2n+1}(2\alpha - \mu + 2k + 1)}, 0 \leq m \leq n,$$

for $n \geq 0$.

Proposition 5.8 *The* $\lambda_{n,m} := \lambda_{n,m}(X \leftarrow P(\tilde{\alpha}, \tilde{\mu}; -))$ *corresponding to* (18)–(21) *are given by*

$$\lambda_{2n,2m} = (-1)^{n+m}\binom{n}{m} \frac{\prod_{k=m+1}^{n}(\tilde{\mu} - 2k + 1)}{\prod_{k=2m+1}^{n+m}(2\tilde{\alpha} - \tilde{\mu} + 2k + 1)}, 0 \leq m \leq n,$$

$$\lambda_{2n,2m+1} = (-1)^{n+m}2n\binom{n-1}{m} \frac{\prod_{k=m+2}^{n}(\tilde{\mu} - 2k + 1)}{\prod_{k=2m+2}^{n+m+1}(2\tilde{\alpha} - \tilde{\mu} + 2k + 1)}, 0 \leq m \leq n-1,$$

$$\lambda_{2n+1,2m} = (-1)^{n+m}\binom{n}{m} \frac{\prod_{k=m+1}^{n+1}(\tilde{\mu} - 2k + 1)}{\prod_{k=2m+1}^{n+m+1}(2\tilde{\alpha} - \tilde{\mu} + 2k + 1)}, 0 \leq m \leq n,$$

$$\lambda_{2n+1,2m+1} = (-1)^{n+m}\binom{n}{m} \frac{\prod_{k=m+2}^{n+1}(\tilde{\mu} - 2k + 1)}{\prod_{k=2m+2}^{n+m+1}(2\tilde{\alpha} - \tilde{\mu} + 2k + 1)}, 0 \leq m \leq n,$$

for $n \geq 0$.

5.2.2 Case 2

Now, let us consider the case 2 presented in [4]. The recurrence coefficients are

$$\beta_0=-\frac{\mu-1}{\mu-\alpha-2}\,,\ \beta_{2n+1}=\frac{(\mu+1)(\alpha+1)}{(2n+\alpha+3)(\mu-2n-\alpha-2)},\ n\geq 0, \tag{22}$$

$$\beta_{2n}=-\frac{(\mu-1)(\alpha+1)}{(2n+\alpha+1)(\mu-2n-\alpha-2)},\ n\geq 1, \tag{23}$$

$$\gamma_{2n+1}=\frac{(\mu-2n-1)(\mu-2\alpha-2n-3)(2n+\alpha+1)(2n+\alpha+3)}{(2n+\alpha+2-\mu)^2(\mu-4n-2\alpha-3)(\mu-4n-2\alpha-5)},n\geq 0, \tag{24}$$

$$\gamma_{2n+2}=4(n+1)\frac{(\mu-\alpha-2n-2)(\mu-\alpha-2n-4)(n+\alpha+2)}{(2n+\alpha+3)^2(\mu-4n-2\alpha-5)(\mu-4n-2\alpha-7)},\ n\geq 0, \tag{25}$$

with the regularity conditions $\mu\neq 2n+1$, $\mu\neq 2n+2\alpha+3$, $\alpha\neq-(n+1)$, $n\geq 0$. For $\mu=0$, the classical monic Jacobi polynomials with parameters $(\alpha,\alpha+2)$ are recovered [6]. Next, we present the CC with respect to the canonical sequence.

Proposition 5.9 *The* $\lambda_{n,m}:=\lambda_{n,m}(P(\alpha,\mu;-)\leftarrow X)$ *corresponding to* (22)–(25) *are given by*

$$\lambda_{2n,2m}=\frac{\binom{n}{m}(\alpha+2m+1)}{(\alpha+2n+1)}\frac{\prod_{k=m+1}^{n}(\mu-2k+1)}{\prod_{k=n+m+1}^{2n}(2\alpha-\mu+2k+1)}\,,0\leq m\leq n,$$

$$\lambda_{2n,2m+1}=\frac{2n\binom{n-1}{m}}{(\alpha+2n+1)}\frac{\prod_{k=m+2}^{n}(\mu-2k+1)}{\prod_{k=n+m+2}^{2n}(2\alpha-\mu+2k+1)}\,,0\leq m\leq n-1,$$

$$\lambda_{2n+1,2m}=-\frac{\binom{n}{m}}{(\alpha-\mu+2n+2)}\frac{\prod_{k=m+1}^{n+1}(\mu-2k+1)}{\prod_{k=n+m+2}^{2n+1}(2\alpha-\mu+2k+1)}\,,0\leq m\leq n,$$

$$\lambda_{2n+1,2m+1}=\frac{\binom{n}{m}(\alpha-\mu+2m+2)}{(\alpha-\mu+2n+2)}\frac{\prod_{k=m+2}^{n+1}(\mu-2k+1)}{\prod_{k=n+m+2}^{2n+1}(2\alpha-\mu+2k+1)}\,,0\leq m\leq n,$$

for $n\geq 0$.

Proposition 5.10 *The* $\lambda_{n,m} := \lambda_{n,m}(X \leftarrow P(\tilde{\alpha}, \tilde{\mu}; -))$ *corresponding to* (22)–(25) *are given by*

$$\lambda_{2n,2m} = (-1)^{n+m} \frac{\binom{n}{m}(\alpha-\mu+2n+2)}{(\alpha-\mu+2m+2)} \frac{\prod_{k=m+1}^{n}(\mu-2k+1)}{\prod_{k=2m+2}^{n+m+1}(2\alpha-\mu+2k+1)}, 0 \le m \le n,$$

$$\lambda_{2n,2m+1} = (-1)^{n+m} \frac{2n\binom{n-1}{m}}{(\alpha+2m+3)} \frac{\prod_{k=m+2}^{n}(\mu-2k+1)}{\prod_{k=2m+3}^{n+m+1}(2\alpha-\mu+2k+1)}, 0 \le m \le n-1,$$

$$\lambda_{2n+1,2m} = (-1)^{n+m} \frac{\binom{n}{m}}{(\alpha-\mu+2m+2)} \frac{\prod_{k=m+1}^{n+1}(\mu-2k+1)}{\prod_{k=2m+2}^{n+m+1}(2\alpha-\mu+2k+1)}, 0 \le m \le n,$$

$$\lambda_{2n+1,2m+1} = (-1)^{n+m} \frac{\binom{n}{m}(\alpha+2n+3)}{(\alpha+2m+3)} \frac{\prod_{k=m+2}^{n+1}(\mu-2k+1)}{\prod_{k=2m+3}^{n+m+2}(2\alpha-\mu+2k+1)}, 0 \le m \le n,$$

for $n \ge 0$.

6 Conclusions

There are two difficult steps in the methodology exposed here and in [14, 15]. The first one consists to infer the closed formulas for the CC from enough data produced and treated by a symbolic language like *Mathematica*®. The second one is to accomplish the demonstration of the model, which can be done completely via *Mathematica*® or only in part with the help of that language.

In spite of some limitations, we know that this method can be useful to treat several other examples of CC. Furthermore, the implementation principles studied in the tutorial provided with the package can be applied in a similar way to other situations in the branch of orthogonal polynomials and in mathematics.

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