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Polynomial sequences associated with the classical linear functionals

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Abstract This work in mainly devoted to the study of polynomial sequences, not necessarily orthogonal, defined by integral powers of certain first order differential operators in deep connection to the classical polynomials of Hermite, Laguerre, Bessel and Jacobi. This connection is streamed from the canonical element of their dual sequences. Meanwhile new Rodrigues-type formulas for the Hermite and Bessel polynomials are achieved.

Keywords Orthogonal polynomials · Classical polynomials · Rodrigues-type formulas · Generalized Stirling numbers

Mathematics Subject Classifications (2010) Primary 35C45 · 42C05; Secondary 11B73

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1 Introduction and preliminary results

Throughout the text, $\mathbb{N}$ will denote the set of all positive integers, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, whereas $\mathbb{R}$ and $\mathbb{C}$ the field of the real and complex numbers, respectively. The notation $\mathbb{R}_+$ corresponds to the set of all positive real numbers. The present investigation is primarily targeted at analysis of sequences of polynomials whose degrees equal its order, which will be shortly called as PS. Whenever the leading coefficient of each of its polynomials equals 1, the PS is said to be a MPS (*monic polynomial sequence*). A PS or a MPS forms a basis of the vector space of polynomials with coefficients in $\mathbb{C}$, here denoted as $\mathscr{P}$. Further notations are introduced as needed.

The dual sequence $\{u_n\}_{n\geqslant 0}$ of a given MPS $\{P_n(x)\}_{n\geqslant 0}$, whose elements are called forms (or *linear functionals*) belong to the dual space $\mathscr{P}'$ of $\mathscr{P}$ and are defined according to

$$\langle u_n, P_k \rangle := \delta_{n,k},\ n, k \geqslant 0,$$

where $\delta_{n,k}$ represents the *Kronecker delta* function. Its first element, u_0, earns the special name of *canonical form* of the MPS. Here, by $\langle u, f \rangle$ we mean the action of $u \in \mathscr{P}'$ over $f \in \mathscr{P}$, but a special notation is given to the action over the elements of the canonical sequence $\{x^n\}_{n\geqslant 0}$—the *moments of* $u \in \mathscr{P}'$: $(u)_n := \langle u, x^n \rangle$, $n \in \mathbb{N}_0$. Any element u of $\mathscr{P}'$ can be written in a series of any dual sequence $\{u_n\}_{n\geqslant 0}$ of a MPS $\{P_n\}_{n\geqslant 0}$ [10]:

$$u = \sum_{n\geqslant 0} \langle u, P_n \rangle\, u_n\ . \tag{1.1}$$

Concerning the recursive relation of any MPS, we have [10]:

$$P_{n+2}(x) = (x - \beta_{n+1}) P_{n+1}(x) - \sum_{\nu=0}^{n} \chi_{n,\nu} P_\nu(x),\ n \in \mathbb{N}_0, \tag{1.2}$$

where

$$\beta_n = \langle u_n, x P_n \rangle,\ n \in \mathbb{N}_0, \tag{1.3}$$

$$\chi_{n,\nu} = \langle u_\nu, x P_{n+1} \rangle,\ 0 \leqslant \nu \leqslant n\ ,\ n \in \mathbb{N}_0. \tag{1.4}$$

Differential equations or other kind of linear relations realized by the elements of the dual sequence can be deduced by transposition of those relations fulfilled by the elements of the corresponding MPS, insofar as a linear operator $T : \mathscr{P} \to \mathscr{P}$ has a transpose ${}^tT : \mathscr{P}' \to \mathscr{P}'$ defined by

$$\langle {}^tT(u), f \rangle = \langle u, T(f) \rangle\ , \quad u \in \mathscr{P}',\ f \in \mathscr{P}. \tag{1.5}$$

For example, for any form u and any polynomial g, let $Du = u'$ and gu be the forms defined as usual by $\langle u', f \rangle := -\langle u, f' \rangle$, $\langle gu, f \rangle := \langle u, gf \rangle$, where D is the differential operator [10]. Thus, D on forms is minus the transpose of the differential operator D on polynomials.

The investigation about the orthogonality of a MPS can be performed in a purely algebraic point of view. Precisely, a form $v \in \mathscr{P}'$ is said to be *regular* if we can associate a PS $\{Q_n\}_{n\geqslant 0}$ such that $\langle v, Q_n Q_m\rangle = k_n \delta_{n,m}$ with $k_n \neq 0$ for all $n, m \in \mathbb{N}_0$ [10, 11]. The PS $\{Q_n\}_{n\geqslant 0}$ is then said to be orthogonal with respect to v and we can assume the system (of orthogonal polynomials) to be monic. Therefore, we can set $v = v_0$ and the remaining elements of the corresponding dual sequence $\{v_n\}_{n\geqslant 0}$ are represented by

$$v_{n+1} = \left(\langle v_0, Q_{n+1}^2(\cdot)\rangle\right)^{-1} \; Q_{n+1}(x) v_0 \; , \; n \in \mathbb{N}_0. \tag{1.6}$$

When $v \in \mathscr{P}'$ is regular and Φ is a polynomial such that $\Phi v = 0$, then $\Phi = 0$ [12].

This unique MOPS $\{Q_n(x)\}_{n\geqslant 0}$ with respect to the regular form v_0 can be characterized by the popular second order recurrence relation

$$\begin{cases} Q_0(x) = 1 \quad ; \quad Q_1(x) = x - \beta_0 \\ Q_{n+2}(x) = (x - \beta_{n+1}) Q_{n+1}(x) - \gamma_{n+1}\, Q_n(x) \; , \quad n \in \mathbb{N}_0. \end{cases} \tag{1.7}$$

The MPS $\{\mathrm{e}^x \mathscr{A}^n \mathrm{e}^{-x}\}_{n\in\mathbb{N}_0}$ where $\mathscr{A} = x^2 - x\frac{d}{dx}x\frac{d}{dx}$, was recently investigated in [19]. It triggered the study of a wider class of polynomial sequences $\{\mathrm{e}^x x^{-\alpha} \mathscr{A}^n x^{\alpha} \mathrm{e}^{-x}\}_{n\in\mathbb{N}_0}$ which, despite not being orthogonal (in the usual sense), the canonical element of their corresponding dual form is regular as long as $\alpha > 0$, and the existence of a MOPS with respect to this canonical form is ensured. The characterization of such a MOPS is undoubtedly an issue, that we could not settle, mainly because of the inherent difficulties of dealing with regular forms fulfilling second order differential equations.

On the other hand, this raised the problem of characterizing polynomial sequences generated by integral composite powers of a first order differential operator, whose canonical form we are able to fully characterize, like the classical forms, that is, regular forms u_0 fulfilling

$$(\phi u_0)' + \psi u_0 = 0 \tag{1.8}$$

with $\deg \phi \leqslant 2$, $\deg \psi = 1$ and $\frac{n}{2}\phi''(0) - \psi'(0) \neq 0$, $n \in \mathbb{N}_0$. There are essentially four different equivalence classes depending on the nature of the polynomial ϕ [11, 12], whose representatives are summarized in the next table.

After setting all the required properties of these polynomial sequences generated by integral composite powers of a first order differential operator on Section 2, we seek those possessing orthogonality, where, as it will be pointed in Remark 2.4, the solution is reduced to the Hermite polynomials. Afterwards, the characterization of all the polynomial sequences whose canonical form is classical will be handled on Section 3, where we will unravel the aforementioned MPSs, for each of the four arisen possibilities, either by determining the coefficients or by seeking the connection with the well known classical polynomial sequences. Meanwhile, the procedure will permit to infer new Rodrigues type formulas for the classical polynomials of Hermite and Bessel. At last, some problems remain open in the case of Jacobi form.

2 Polynomials generated by integral powers of a first order differential operator

Lemma 2.1 *Let E be a complex-valued smooth function not identically equal to zero and let ϕ and ψ be two polynomials. The sequence of functions*

$$p_n(x) = \frac{1}{E(x)} \mathscr{A}^n E(x)\ ,\ n \in \mathbb{N}_0, \tag{2.1}$$

with

$$\mathscr{A} = -\phi(x)\frac{d}{dx} + \psi(x) + \phi(x)\frac{E'(x)}{E(x)} \tag{2.2}$$

represent a polynomial sequence whose elements have degree n (in short, PS) if and only if

$$\deg \phi \leqslant 2\ ,\ \deg \psi = 1 \ \ and \ \ \frac{n}{2}\phi''(0) - \psi'(0) \neq 0 \quad , \quad n \in \mathbb{N}_0. \tag{2.3}$$

Moreover, such PS $\{p_n\}_{n\geqslant 0}$ can be equivalently represented by

$$\begin{aligned} p_0(x) \quad &= 1\ , \\ p_{n+1}(x) &= -\phi(x)p_n'(x) + \psi(x)p_n(x)\ ,\ n \in \mathbb{N}_0. \end{aligned} \tag{2.4}$$

Proof Under the assumptions evoked for the function $E(x)$, let us consider the sequence of functions $\{p_n\}_{n\geqslant 0}$ defined by (2.1). The first element of this sequence is the constant function $p_0(x) = 1$. Insofar as the identity

$$\frac{1}{E(x)}\mathscr{A}\big(E(x)g(x)\big) = -\phi(x)g'(x) + \psi(x)g(x) \tag{2.5}$$

holds for any analytic function $g(x)$, it readily follows from (2.1) that the sequence of functions $\{p_n\}_{n\geqslant 0}$ can be represented by (2.4), because we successively have

$$\begin{aligned} p_{n+1}(x) &= \frac{1}{E(x)}\mathscr{A}\Big(E(x)\frac{1}{E(x)}\mathscr{A}^n(E(x))\Big) = \frac{1}{E(x)}\mathscr{A}(E(x)p_n(x)) \\ &= -\phi(x)p_n'(x) + \psi(x)p_n(x),\ \ , n \in \mathbb{N}_0. \end{aligned}$$

Conversely, if a sequence of functions $\{p_n\}_{n\geqslant 0}$ is defined by (2.4), we then have $p_1(x) = \frac{1}{E(x)}\mathscr{A}(E(x))$ and performing analogous steps as the ones made above, by induction, we conclude that necessarily $\{p_n\}_{n\geqslant 0}$ is also given by (2.1).

Now, it remains to show that $\{p_n\}_{n\geqslant 0}$ is actually a sequence of polynomials of exactly degree n if and only if (2.3) hold. Indeed if these constraints for the pair of polynomials (ϕ, ψ) are realized, then (2.4) (as well as (2.1)) ensures that $p_1(x) = \psi(x)$ is a one degree polynomial. Through a finite induction procedure, the relation (2.4) enables the result.

Conversely, if each $p_n(x)$ is a polynomial of exactly degree n, the condition (2.4) implies the claimed constraints for (ϕ, ψ), because they do not depend on n. □

Corollary 2.2 *The polynomial sequence* $\{p_n\}_{n\geqslant 0}$ *given by* (2.1) *under the constraints* (2.3) *does not depend on the choice of the smooth function* E.

Proof The result is a mere consequence of the equivalence shown between (2.1) and (2.4), as long as (ϕ, ψ) satisfy (2.3). □

This paper aims to describe the objects given by (2.1), which under certain choices for the pair (ϕ, ψ) represent polynomials of degree n. From this point forth the pair (ϕ, ψ) will be considered to be a pair of polynomials fulfilling (2.3).

For a question of normalization, we are primarily interested in dealing with monic polynomial sequences. For this reason, we will deal instead with the MPS $\{P_n\}_{n\geqslant 0}$ obtained from $\{p_n\}_{n\geqslant 0}$ after the division by its leading coefficient, say λ_n. Without loss of generality, we will assume that ϕ is a monic polynomial.

To set things more in concrete, we consider $\{P_n\}_{n\geqslant 0}$ such that

$$\lambda_n P_n(x) = \frac{1}{E(x)} \mathscr{A}^n E(x)\ ,\ n \in \mathbb{N}_0, \tag{2.6}$$

where λ_n is a nonzero constant compelling P_n to be monic. According to (2.4), it naturally follows that

$$P_{n+1}(x) = \frac{\lambda_n}{\lambda_{n+1}} (-\phi(x) P'_n(x) + \psi(x) P_n(x))\ ,\ n \in \mathbb{N}_0. \tag{2.7}$$

which, in turn, provides the equalities

$$\lambda_n = \begin{cases} (\psi'(0))^n & ,\ \deg \phi \leqslant 1 \\ (-1)^n(-\psi'(0))_n & ,\ \deg \phi = 2 \end{cases}\ ,\ n \in \mathbb{N}_0, \tag{2.8}$$

where $(x)_n$ denotes the Pochhammer symbol defined by $(x)_0 := 1,\ \ (x)_n = x(x+1)\ldots(x+n-1)$ for $n \in \mathbb{N}$.

Now, we redirect the problem of characterizing this MPS to the study of the corresponding dual sequence, to which we will refer to as $\{u_n\}_{n\geqslant 0}$.

Lemma 2.3 *The dual sequence of the MPS* $\{P_n\}_{n\geqslant 0}$ *above defined in* (2.6) *(or by* (2.7)*) fulfills*

$$\left(\phi u_0\right)' + \psi u_0 = 0 \tag{2.9}$$

$$\left(\phi u_{n+1}\right)' + \psi u_{n+1} = \frac{\lambda_{n+1}}{\lambda_n} u_n\ ,\ n \in \mathbb{N}_0. \tag{2.10}$$

Proof The action of u_0 over (2.7) is given by

$$\langle u_0, -\phi P'_n + \psi P_n \rangle = 0\ ,\ n \in \mathbb{N}_0,$$

which, by transposition, on account of (1.5), is equivalent to

$$\langle \left(\phi u_0\right)' + \psi u_0, P_n \rangle = 0\ ,\ n \in \mathbb{N}_0,$$

providing (2.9). Likewise, the action of u_{k+1} over (2.7) yields

$$\langle u_{k+1}, -\phi P_n' + \psi P_n \rangle = \frac{\lambda_{n+1}}{\lambda_n}\delta_{k,n}\ ,\ n,k \in \mathbb{N}_0,$$

and again, due to (1.5), we may write this latter as

$$\langle \big(\phi u_{k+1}\big)' + \psi u_{k+1}, P_n \rangle = \frac{\lambda_{n+1}}{\lambda_n}\delta_{k,n}\ ,\ n,k \in \mathbb{N}_0.$$

Considering (1.1), the relation (2.10) is then a consequence of this latter equality. □

Additionally, the moments of the dual sequence fulfill

$$\begin{aligned}
&\Big(\psi'(0) - \frac{k}{2}\phi''(0)\Big)(u_{n+1})_{k+1} + \Big(\psi(0) - k\phi'(0)\Big)(u_{n+1})_k - k\phi(0)(u_{n+1})_{k-1} \\
&\quad = \frac{\lambda_{n+1}}{\lambda_n}(u_n)_k\ ,\ n \in \mathbb{N}_0, \\
&\Big(\psi'(0) - \frac{k}{2}\phi''(0)\Big)(u_0)_{k+1} + \Big(\psi(0) - k\phi'(0)\Big)(u_0)_k - k\phi(0)(u_0)_{k-1} \\
&\quad = 0\ ,\ k \in \mathbb{N}_0,
\end{aligned} \tag{2.11}$$

with the initial conditions that $(u_n)_k = \delta_{n,k}$ whenever $0 \leqslant k \leqslant n$.

It is worth to notice that indeed

$$x^n = \sum_{\nu=0}^{n} (u_\nu)_n\ P_\nu(x)\ ,\ n \in \mathbb{N}_0. \tag{2.12}$$

All the regular forms fulfilling equations like in (2.9) under the constraints (2.3) have been deeply explored and they are essentially classical forms—see Table 1.

Remark 2.4 We notice that if $\{P_n\}_{n\geqslant 0}$ defined in (2.6) is a MOPS, then it is the Hermite MOPS.

Indeed, the orthogonality assumption over the MPS $\{P_n\}_{n\geqslant 0}$ ensures the elements of the corresponding dual sequence $\{u_n\}_{n\geqslant 0}$ to be given by (1.6). The combination of this information with (2.10) leads to

$$P_{n+1}\Big((\phi u_0)' + \psi u_0\Big) + P_{n+1}'\phi u_0 = \frac{\lambda_{n+1}}{\lambda_n}\frac{\langle u_0, P_{n+1}^2\rangle}{\langle u_0, P_n^2\rangle}P_n u_0\ ,\ n \geqslant 0,$$

Table 1 Expressions for the polynomials ϕ andψ for each classical family

Regularity conditions $n \in \mathbb{N}_0$	Hermite	Laguerre $\alpha \neq -(n+1)$	Bessel $\alpha \neq -\frac{n}{2}$	Jacobi $\alpha, \beta \neq -(n+1)$ $\alpha+\beta \neq -(n+2)$
$\phi(x)$	1	x	x^2	$x(x-1)$
$\psi(x)$	$2x$	$x-\alpha-1$	$-2\,(\alpha x+1)$	$-(\alpha+\beta+2)\,x+(\alpha+1)$

which, on account of (2.9) together with the regularity of the canonical form u_0, enables

$$\phi(x)\,P'_{n+1}(x) = \frac{\lambda_{n+1}}{\lambda_n}\frac{\langle u_0,\, P^2_{n+1}\rangle}{\langle u_0,\, P^2_n\rangle}\,P_n(x)\ ,\ n \geqslant 0.$$

A mere comparison of the leading coefficients shows that $\deg\phi = 0$ and, because ϕ was assumed to be monic, $\phi(x) = 1$ and, thus, $\frac{\lambda_{n+1}}{\lambda_n}\frac{\langle u_0, P^2_{n+1}\rangle}{\langle u_0, P^2_n\rangle} = (n+1)$. Therefore, $P'_{n+1}(x) = (n+1)\,P_n(x)$, for $n \in \mathbb{N}_0$, and, consequently, $\{P_n\}_{n\geqslant 0}$ coincides with the Hermite polynomial sequence.

Since the $\deg\phi \leqslant 2$ and $\deg\psi = 1$ there are only four possible arising cases, better to say, the analysis shall then be split into four different classes. On the other hand, looking at the equation (2.9) fulfilled by the form u_0 we readily come to the conclusion that u_0 is necessarily a regular form. Needless to say that this does not imply $\{P_n\}_{n\geqslant 0}$ to be orthogonal (ergo, classical). Actually, as pointed in Remark 2.4, whenever $\deg\phi \geqslant 1$ the MPS $\{P_n\}_{n\geqslant 0}$ cannot be orthogonal (in the usual sense).

3 Analysis of the four possible situations

The analysis taken throughout this section will be drawn according to the nature of the polynomial ϕ and therefore split into four different cases. Representatives ϕ and ψ for each one of the four possible situations will be chosen from Table 1. Other choices can be considered as long as they realize the required conditions (2.3), guaranteeing the admissibility of regular solutions u_0 of (2.9) [11].

Among the characterization properties are: an explicit expansion in terms of the monomial basis, a generating function and a recursive relation (1.2). This latter is important not only for computational reasons but also because it permits to know whether a MPS can be or not d-orthogonal [9, 18], which, roughly speaking, means that the elements of the MPS would then fulfill a recursive relation of order $d+1$ (a constant value, independent of the degree of the element). In this case more specific expressions for the β and χ coefficients presented in (1.2) and defined in (1.3)–(1.4) can be straightforwardly obtained from (2.7)–(2.10):

$$\beta_{n+1} = \beta_n + \frac{\lambda_n}{\lambda_{n+1}}\langle u_{n+1}, \phi P_n\rangle,\ n \in \mathbb{N}_0, \tag{3.1}$$

$$\chi_{n+1,\nu+1} = \frac{\lambda_{\nu+1}\lambda_{n+1}}{\lambda_\nu\lambda_{n+2}}\chi_{n,\nu} + \frac{\lambda_{n+1}}{\lambda_{n+2}}\langle u_{\nu+1}, \phi P_{n+1}\rangle\ ,\ 0 \leqslant \nu \leqslant n,\ n \in \mathbb{N}_0, \tag{3.2}$$

with the initial conditions $\beta_0 = (u_0)_1$, determined from the analysis of the moments of order 0 of (2.9), so that

$$\beta_0 = -\frac{\psi(0)}{\psi'(0)} \tag{3.3}$$

and

$$\chi_{n,0} = \frac{\lambda_n}{\lambda_{n+1}} \left\langle u_0, x\left(-\phi P_n' + \psi P_n\right)\right\rangle = \frac{\lambda_n}{\lambda_{n+1}} \left\langle x\left((\phi u_0)' + \psi u_0\right) + \phi u_0, P_n\right\rangle$$
$$= \frac{\lambda_n}{\lambda_{n+1}} \left\langle u_0, \phi P_n\right\rangle \ , \ n \in \mathbb{N}_0. \tag{3.4}$$

From this point we need to split the analysis into the four possible cases.

3.1 Hermite case

So far, we have seen that when the orthogonality of the MPS $\{P_n\}_{n\geqslant 0}$ is assumed, we are necessarily handling with the Hermite polynomial sequence, ergo the Hermite form. Now, we are willing to find all polynomial sequences $\{P_n\}_{n\geqslant 0}$ defined in (2.6) such that u_0 is the regular form of Hermite. As representatives for this case we consider $\phi(x) = 1$ and $\psi(x) = 2x$ (see Table 1).

As a matter of fact, (3.1) together with (3.3) implies $\beta_n = 0$ for all $n \in \mathbb{N}_0$, while (3.2) provides

$$\chi_{n+1,\nu+1} = \chi_{n,\nu} \ , \ 0 \leqslant \nu \leqslant n-1, \tag{3.5}$$

$$\chi_{n+1,n+1} = \chi_{n,n} + \frac{1}{2} \ , \ n \in \mathbb{N}_0. \tag{3.6}$$

According to (3.4) it follows $\chi_{n,0} = \frac{1}{2}\delta_{n,0}$, $n \in \mathbb{N}_0$. Thus, $\chi_{n,n} = \frac{n+1}{2}$ and $\chi_{n,\nu} = 0$ for $0 \leqslant \nu \leqslant n-1$. We achieve therefore the conclusion that the canonical form of an MPS $\{P_n\}_{n\geqslant 0}$ defined by (2.7) with $\deg\phi = 0$ is regular if and only if $\{P_n\}_{n\geqslant 0}$ is the Hermite polynomial sequence.

The latter result permits to obtain many Rodrigues type formulas for the Hermite polynomials, since they are represented by

$$P_n(x) = 2^{-n}\frac{1}{E(x)}\left(-\frac{d}{dx} + 2x + \frac{E'(x)}{E(x)}\right)^n E(x) \ , \ n \in \mathbb{N}_0, \tag{3.7}$$

rather than the well known one, which could be recovered from upon the choice $E(x) = \mathrm{e}^{-x^2/2}$. Other possible choices for $E(x)$ could be, for instance, $E(x) = \mathrm{e}^{\mathrm{e}^x}$.

3.2 Laguerre case

Here we consider the case where $\deg\phi = 1$, we make use of the Laguerre form, which is the unique regular form, up to an affine transformation, that is solution of (2.9) under the assumed conditions. As a representative of this class, we set $\phi(x) = x$ and $\psi(x) = x - \alpha - 1$, and, therefore, (2.7) becomes

$$P_{n+1}(x;\alpha) = -xP_n'(x;\alpha) + (x - \alpha - 1)P_n(x;\alpha)$$

Expressing

$$P_n(x;\alpha) = \sum_{\nu=0}^{n} (-1)^{n+\nu} c_{n,\nu}(\alpha) x^\nu \ , \ n \in \mathbb{N}_0, \tag{3.8}$$

from (2.7) we derive

$$\sum_{\nu=0}^{n+1}(-1)^{n+\nu+1}c_{n+1,\nu}x^{\nu} = -\sum_{\tau=0}^{n}\nu(-1)^{n+\nu}c_{n,\nu}x^{\nu} + \sum_{\nu=0}^{n}(-1)^{n+\nu}c_{n,\nu}\left(x^{\nu+1}-(\alpha+1)x^{\nu}\right) \,,\ n\in\mathbb{N}_0,$$

(under the notation $c_{n,\nu} := c_{n,\nu}(\alpha)$) and, therefore,

$$\begin{cases} c_{n,n}(\alpha)=1 \quad , \quad c_{n,0}(\alpha) = (\alpha+1)^n \quad , \quad c_{n,n+\nu+1}(\alpha)=0\,,\ n,\nu\in\mathbb{N}_0, \\ c_{n+1,\nu}(\alpha) = c_{n,\nu-1}(\alpha)+(\nu+\alpha+1)c_{n,\nu}(\alpha)\,,\ 0\leqslant\nu\leqslant n\,,\ n\in\mathbb{N}_0, \end{cases} \tag{3.9}$$

under the convention $c_{n,-1}(\alpha)=0$.

These correspond to the non-central Stirling numbers of second kind (or simply, the generalized Stirling numbers) treated in [6], where it was also pointed out the denomination of *non-central Lah numbers* proposed in [5]. Without entering into further considerations, their explicit formula is

$$c_{n,\nu}(\alpha) = \frac{1}{\nu!}\sum_{i=0}^{\nu}\binom{\nu}{i}(-1)^{\nu-i}(i+\alpha+1)^n = \frac{1}{\nu!}\left(\Delta_{\alpha+1}^{\nu}x^n\right)\Big|_{x=0}\,,\ n\in\mathbb{N}_0, \tag{3.10}$$

where $\Delta_{\alpha+1}f(x) = f(x+\alpha+1)-f(x)$. Moreover, these coefficients are the bridge to connect the canonical MPS with the (factorial) polynomial sequences $\{(-1)^n(-x+\alpha+1)_n\}_{n\geqslant 0}$ because for $x\neq 0$

$$x^n = \sum_{k=0}^{n}c_{n,k}(\alpha)(-1)^k(-x+\alpha+1)_k \quad \text{or}$$

$$(x+\alpha+1)^n = \sum_{k=0}^{n}c_{n,k}(\alpha)(-1)^k(-x)_k\,,\ n\in\mathbb{N}_0, \tag{3.11}$$

and

$$\sum_{k=0}^{n}c_{n,k}(\alpha)(-1)^{n+k}(\alpha+1)_k = 1\,,\ n\in\mathbb{N}_0.$$

Conversely, regarding the moments of the dual sequence, we can consider the inverse relation of (3.8). Thus, from (2.11), we have

$$\begin{cases} (u_{n+1})_{k+1} = (u_n)_k + (\alpha+1+k)(u_{n+1})_k\,,\ n\in\mathbb{N}_0, \\ (u_0)_k = (\alpha+1)_k\,,\ k\in\mathbb{N}_0, \end{cases} \tag{3.12}$$

with the initial conditions $(u_n)_k = \delta_{n,k}$ whenever $0\leqslant k\leqslant n$. Thus, the set $\{(u_n)_k\}$ mimics the set of Stirling numbers of first kind, and differ from it by the decentralizing factor $(\alpha+1)$. They are actually the non-central Stirling numbers of first kind, pointed in [6]. Finally, we have (2.12), where $(u_n)_k$ satisfies (3.12).

Lemma 3.1 *The MPS* $\{P_n(\cdot;\alpha)\}_{n\geqslant 0}$ *has the following generating function*

$$G(x,t)=\mathrm{e}^{-(\alpha+1)t-x(\mathrm{e}^{-t}-1)}=\sum_{n\geqslant 0}P_n(x;\alpha)\frac{t^n}{n!}\ .$$

Proof Indeed, a generating function $G(x,t)=\sum_{n\geqslant 0}P_n(x;\alpha)\frac{t^n}{n!}$ must be a solution of the partial differential equation

$$\frac{\partial}{\partial t}G(x,t)=-x\frac{\partial}{\partial x}G(x,t)+(x-\alpha-1)G(x,t)$$

satisfying the boundary conditions $\lim_{t\to 0}G(x,t)=1$ and $\lim_{x\to 0}G(x,t)=\mathrm{e}^{-(\alpha+1)t}$. Thus, the function $\Phi(x,t)=\mathrm{e}^{-(\alpha+1)t-x(\mathrm{e}^{-t}-1)}$ is a solution of the problem. □

The latter result brings the status of Sheffer-type for the underlying MPS $\{P_n(\cdot;\alpha)\}_{n\geqslant 0}$ [15].

Meanwhile, if we bring into discussion the generating function for the (monic) Laguerre polynomials, $\{Q_n(\cdot;\alpha)\}_{n\geqslant 0}$,

$$H(x,t)=(t+1)^{-(\alpha+1)}\mathrm{e}^{x\frac{t}{t+1}}=\sum_{n\geqslant 0}Q_n(x;\alpha)\frac{t^n}{n!}\ ,$$

we readily observe that the generating function of the aforementioned MPS $\{P_n\}_{n\geqslant 0}$ can be expressed as

$$G(x,t)=H\left(x,\mathrm{e}^t-1\right)$$

Hence, by recalling [2, p.51]

$$\frac{(\mathrm{e}^t-1)^k}{k!}=\sum_{n\geqslant k}S(n,k)\frac{t^n}{n!}\ ,\ n\in\mathbb{N}_0,$$

we deduce the identity

$$P_n(x;\alpha)=\sum_{\nu=0}^{n}S(n,\nu)Q_\nu(x;\alpha)\ ,\ n\in\mathbb{N}_0.$$

Conversely, the identity [2, p.51]

$$\frac{(\log(t+1))^k}{k!}=\sum_{n\geqslant k}s(n,k)\frac{t^n}{n!}\ ,\ n\in\mathbb{N}_0,$$

provides the reciprocal

$$Q_n(x;\alpha)=\sum_{\nu=0}^{n}s(n,\nu)P_\nu(x;\alpha)\ ,\ n\in\mathbb{N}_0.$$

An alternative way to obtain the latter identities, but rather less intuitive, would be via Faa di Bruno's formula.

Lemma 3.2 *The structure relation of the MPS* $\{P_n(\cdot;\alpha)\}_{n\geqslant 0}$ *is*

$$P_{n+2}(x;\alpha) = (x-n-\alpha-2)\,P_{n+1}(x;\alpha) - \sum_{\nu=0}^{n}\binom{n+1}{\nu}\left(\alpha+\frac{n+2}{n+1-\nu}\right)P_\nu(x;\alpha)\ ,\ n\geqslant 0.$$

Proof In this case $\beta_0=\alpha+1$ and therefore the remaining ones are

$$\beta_n = n+\alpha+1\ ,\ n\in\mathbb{N}_0,$$

(which match the βs of the second order recursive relation of the Laguerre polynomials). Besides, according to (3.4)

$$\chi_{n,0}=\langle u_0, xP_n\rangle = \chi_{n-1,0}=\chi_{0,0}=\alpha+1$$

because $\chi_{0,0}=\alpha+1$, which, in particular, guarantees this sequence is not d-orthogonal. The remaining coefficients satisfy the recurrence relation

$$\begin{cases}\chi_{n+1,n+1}=\chi_{n,n}+\beta_{n+1}\ ,\ n\in\mathbb{N}_0,\\ \chi_{n+1,\nu+1}=\chi_{n,\nu}+\chi_{n,\nu+1}\ ,\ 0\leqslant\nu\leqslant n-1\ ,\ n\in\mathbb{N},\\ \chi_{n,0}=(\alpha+1)\ ,\ n\in\mathbb{N}_0,\\ \chi_{0,n}=(\alpha+1)\delta_{0,n}\ ,\ n\in\mathbb{N}_0.\end{cases}$$

Thus $\chi_{n,\nu}=\binom{n+2}{\nu}+\binom{n+1}{\nu}\alpha\ ,\ 0\leqslant\nu\leqslant n\ ,\ n\in\mathbb{N}_0.$ □

The fact that $\chi_{n,0}\neq 0$ for all $n\in\mathbb{N}_0$ discards the possibility of $\{P_n(\cdot;\alpha)\}_{n\geqslant 0}$ to be orthogonal or d-orthogonal.

The example obtained under the choices of $E(x)=x^\alpha e^{-x}$, $\phi(x)=x$ and $\psi(x)=x-\alpha$, has received a special attention, as we may read in [4, pp.254–255] (or in the references therein) after the work taken in [16, 17].

3.3 Bessel case

The choice of $\phi(x)=x^2$ and $\psi(x)=-2(\alpha x+1)$, yields the polynomial sequence

$$P_n(x;\alpha)=\frac{(-1)^n}{(2\alpha)_n}\frac{1}{E(x)}\mathscr{A}^n E(x) \tag{3.13}$$

equivalently defined by the differential relation

$$P_{n+1}(x;\alpha)=\frac{1}{2\alpha+n}\left(x^2P_n'(x;\alpha)+2(\alpha x+1)P_n(x;\alpha)\right)\ ,\ n\in\mathbb{N}_0. \tag{3.14}$$

Lemma 3.3 *The MPS* $\{P_n(\cdot;\alpha)\}_{n\geqslant 0}$ *and the canonical sequence* $\{x^n\}_{n\geqslant 0}$ *realize the following inverse relations:*

$$P_n(x;\alpha)=\sum_{\nu=0}^{n}\binom{n}{\nu}\frac{2^{n-\nu}}{(2\alpha+\nu)_{n-\nu}}x^\nu\ ,\ n\in\mathbb{N}_0, \tag{3.15}$$

whereas

$$x^k = \frac{(-1)^k 2^k}{(2\alpha)_k}\, P_0(x;\alpha) + \sum_{\nu=0}^{k-1}\left(\frac{(-1)^{k-\nu-1}2^{k-\nu-1}(2\alpha)_{\nu+1}}{(2\alpha)_k}\sum_{\mu=\nu}^{k-1}(-1)^{\nu+\mu-1}\binom{\mu}{\nu}\right)$$
$$\times P_{\nu+1}(x;\alpha)\ ,\ k \in \mathbb{N}_0. \tag{3.16}$$

Proof From (3.14), we deduce

$$P_n(x;\alpha) = \sum_{\nu=0}^{n} \frac{2^{n-\nu}(2\alpha)_\nu}{(2\alpha)_n}\widehat{c}_{n,\nu}(\alpha)x^\nu\ ,\ n \in \mathbb{N}_0,$$

where $\widehat{c}_{n,\nu}(\alpha)$ fulfills the triangular relation

$$\widehat{c}_{n+1,\nu}(\alpha) = \widehat{c}_{n,\nu-1}(\alpha) + \widehat{c}_{n,\nu}(\alpha)\ ,\ 0 \leqslant \nu \leqslant n\ ;\ n,\nu \in \mathbb{N}_0,$$
$$\widehat{c}_{n,0}(\alpha) = 1\ ,\ n \in \mathbb{N}_0,$$

which yields $\widehat{c}_{n,\nu} = \binom{n}{\nu}$, $0 \leqslant \nu \leqslant n$, $n, \nu \in \mathbb{N}_0$, whence (3.15).

Regarding the reciprocal relation of (3.15), i.e., to write the monomials in terms of the polynomials $P_n(\cdot;\alpha)$, within the framework of (2.12), we seek an expression for the moments of the dual sequence $\{u_n(\alpha)\}_{n\geqslant 0}$ given in (2.9)–(2.10) also realizing (2.11). In this case,

$$(u_n)_k = \frac{(-1)^{k-n}2^{k-n}(2\alpha)_n}{(2\alpha)_k} d_{k,n}$$

where $d_{n,k}$ fulfills the triangular relation

$$d_{k+1,n+1} = d_{k,n} - d_{k,n+1} \quad ; \quad d_{n,0} = 1 \quad ; \quad d_{0,n} = \delta_{0,n}\ ,\ n \in \mathbb{N}_0.$$

which provides

$$d_{k+1,n+1} = \sum_{\mu=n}^{k}(-1)^{n+\mu-1}\binom{\mu}{n}\ ,\ 0 \leqslant n \leqslant k,\ n,k \in \mathbb{N}_0.$$

For other considerations regarding the set of numbers $\{|d_{k,n}|\}_{k,n}$ we refer to the entry A059260 of [14]. Consequently,

$$(u_0)_k = \frac{(-1)^k 2^k}{(2\alpha)_k} \quad , \quad (u_{\nu+1})_{k+1} = \frac{(-1)^{k-\nu}2^{k-\nu}(2\alpha)_{\nu+1}}{(2\alpha)_{k+1}}\sum_{\mu=\nu}^{k}(-1)^{\nu+\mu-1}\binom{\mu}{\nu} \tag{3.17}$$

and (2.12) becomes (3.16). □

Recalling the expression for the (classical) monic Bessel polynomials

$$B_n(x;\alpha) = \sum_{\nu=0}^{n}\binom{n}{\nu}\frac{2^{n-\nu}}{(2\alpha+n-1+\nu)_{n-\nu}}x^\nu\ ,\ n \in \mathbb{N}_0, \tag{3.18}$$

a simple relation between the nonorthogonal sequence $\{P_n(x;\alpha)\}_{n\geqslant 0}$ and the orthogonal sequence of Bessel polynomials $\{B_n(x;\alpha)\}_{n\geqslant 0}$ comes out:

$$P_n(x;\alpha) = B_n\left(x;\alpha - \frac{n-1}{2}\right)\ ,\ n\in\mathbb{N}_0. \tag{3.19}$$

This fact, actually has the consequence of providing new Rodrigues type formulas for the Bessel polynomials rather than the one already known [3]

$$B_n(x;\alpha) = \frac{(-1)^n}{(-2n-2\alpha+2)_n\, x^{2\alpha-2}\mathrm{e}^{-2/x}}\frac{d^n}{dx^n}\left(x^{2\alpha-2+2n}\mathrm{e}^{-2/x}\right)\ ,\ n\in\mathbb{N}_0.$$

Lemma 3.4 *The Bessel polynomials $\{B_n(\cdot;\alpha)\}_{n\geqslant 0}$ with $\alpha\neq-\frac{n}{2}$, $n\in\mathbb{N}_0$, can be generated by the Rodrigues type formula*

$$B_n(x;\alpha) = \frac{1}{(2\alpha)_n}\frac{1}{E(x)}\left(-x^2\frac{d}{dx}-2\left(\alpha+\frac{n+1}{2}\right)x-2+x^2\frac{E'(x)}{E(x)}\right)^n E(x)\ ,\quad n\geqslant 0.$$

Proof The result is a mere consequence of the definition of the polynomials $\{P_n(\cdot;\alpha)\}_{n\geqslant 0}$ and (3.19) written in the reverse way: $B_n(x;\alpha) = P_n\left(x;\alpha+\frac{n+1}{2}\right)\ ,\ n\in\mathbb{N}_0$. □

Besides, from (3.15) (or by taking into account the identity $B'_{n+1}(x;\alpha) = (n+1)B_n(x;\alpha+1)$) we deduce

$$P'_{n+1}(x;\alpha) = \left(n+1\right)P_n(x;\alpha+\frac{1}{2}\right)\ ,\ n\in\mathbb{N}_0.$$

Combining this latter with (3.14), another structure relation for these polynomials comes out:

$$\frac{n}{2\alpha+n}x^2P_{n-1}\left(x;\alpha+\tfrac{1}{2}\right) = -P_{n+1}(x;\alpha)-\frac{2}{2\alpha+n}(\alpha x+1)P_n(x;\alpha)\ ,\ n\in\mathbb{N}.$$

Concerning the generating function, $G(x,t)$, for the PS $\{(2\alpha)_nP_n(\cdot;\alpha)\}_{n\geqslant 0}$, its expression can be deduced based on the differential relation (3.14), which implies $G(x,t)$ to be solution of the partial differential equation

$$\frac{\partial}{\partial t}G(x,t) = x^2\frac{\partial}{\partial x}G(x,t)+2(\alpha x+1)G(x,t)$$

satisfying the boundary condition $G(x,0)=1$. As a consequence, we have

$$G(x,t) = \mathrm{e}^{2t}(tx-1)^{-2\alpha} = \sum_{n\geqslant 0}(2\alpha)_nP_n(x;\alpha)\frac{t^n}{n!}\ .$$

Despite the non-orthogonality of the MPS $\{P_n(\cdot) := P_n(\cdot;\alpha)\}_{n\geqslant 0}$, we may envisage whether the d-orthogonality can fit in this MPS, for some positive integer $d\geqslant 2$. However, we have:

Lemma 3.5 *The MPS $\{P_n(\cdot;\alpha)\}_{n\geqslant 0}$ cannot be d-orthogonal because the order of its recursive relation depends on the degree of their elements. More precisely,*

$$P_{n+2}(x) = (x-\beta_{n+1})P_{n+1}(x) - \sum_{\nu=0}^{n} \chi_{n,\nu} P_\nu(x)$$

with

$$\beta_n = -\frac{2(2\alpha-1)}{(n+2\alpha-1)(n+2\alpha)} \quad \textit{and} \quad \chi_{n,0} = \frac{-2^{n+2}(n+1)!}{(2\alpha)_{n+1}(2\alpha)_{n+2}} \neq 0\ ,\ n\in\mathbb{N}_0.$$

Proof From (3.1) with $\lambda_{n+1} = -(2\alpha+n)\lambda_n$ we have in this case

$$\beta_{n+1} = \beta_n - \frac{1}{2\alpha+n}\big\langle u_{n+1}, x^2 P_n(x)\big\rangle\ ,\ n\in\mathbb{N}_0.$$

However, due to (1.2) with $n\to n-1$ we deduce

$$\big\langle u_{n+1}, x^2 P_n(x)\big\rangle = \beta_{n+1}+\beta_n\ ,\ n\in\mathbb{N}_0,$$

whence,

$$\beta_n = \beta_0\frac{(2\alpha-1)_n}{(2\alpha+1)_n} = \beta_0\frac{2\alpha(2\alpha-1)}{(2\alpha-1+n)(2\alpha+n)}\ ,\ n\in\mathbb{N}_0,$$

and, finally, from (3.3) we obtain the expression of β_n.

Instead of following a similar procedure as the one taken to determine the coefficients $\chi_{n,\nu}$ in the Laguerre case, we will use the relations (3.15) together with (3.17) to write

$$\chi_{n,\nu} = \langle u_\nu, xP_{n+1}\rangle = \sum_{\sigma=0}^{n+1}\binom{n+1}{\sigma}\frac{2^{n+1-\sigma}(2\alpha)_\sigma}{(2\alpha)_{n+1}}(u_\nu)_{\sigma+1}$$

The particular choice of $\nu=0$ becomes

$$\chi_{n,0} = \sum_{\sigma=0}^{n+1}\binom{n+1}{\sigma}\frac{2^{n+2}(-1)^{\sigma+1}}{(2\alpha)_{n+1}(2\alpha+\sigma)} = -\frac{2^{n+2}(n+1)!}{(2\alpha)_{n+1}(2\alpha)_{n+2}} \neq 0\ ,\ n\in\mathbb{N}_0,$$

whereas

$$\chi_{n,\nu+1} = \frac{2^{n+1-\nu}(2\alpha)_{\nu+1}}{(2\alpha)_{n+1}}\sum_{\sigma=\nu}^{n+1}\binom{n+1}{\sigma}\frac{(-1)^\sigma}{(2\alpha+\sigma)}\sum_{\mu=\nu}^{\sigma}(-1)^{\mu-1}\binom{\mu}{\nu}\ ,$$

$$0\leqslant\nu\leqslant n-1\ ,\ n\in\mathbb{N}.$$

The condition $\chi_{n,0}\neq 0$ refutes the d-orthogonality of $\{P_n\}_{n\geqslant 0}$. □

3.4 Jacobi case

Proceeding in a similar way as in the previous cases, we consider $\phi(x) = x(x-1)$ and $\psi(x) = -(\alpha+\beta+2)\,x + (\alpha+1)$, so that

$$P_{n+1}(x,\alpha,\beta) = \frac{1}{n+\alpha+\beta+2} \times\Big(x(x-1)\,{P_n}'(x,\alpha,\beta) + \big((\alpha+\beta+2)x-(\alpha+1)\big)P_n(x,\alpha,\beta)\Big)\,, \quad n\in\mathbb{N}_0, \tag{3.20}$$

and the corresponding canonical form, which coincides with the Jacobi form, satisfies

$$\big(x(x-1)u_0\big)' - \big((\alpha+\beta+2)x-(\alpha+1)\big)u_0 = 0\,. \tag{3.21}$$

Lemma 3.6 *The elements of the MPS $\{P_n(\cdot;\alpha,\beta)\}_{n\geqslant 0}$ are explicitly given by*

$$P_n(x,\alpha,\beta) = \sum_{\nu=0}^{n} \frac{(\alpha+\beta+2)_\nu}{(\alpha+\beta+2)_n}(-1)^{n+\nu}c_{n,\nu}(\alpha)x^\nu\ ,\ n\in\mathbb{N}_0, \tag{3.22}$$

where the set of numbers $\{c_{n,\nu}\}_{0\leqslant\nu\leqslant n}$ is defined in (3.9)–(3.10).

Proof Based on this differential-recursive relation (3.20), we derive the explicit expression of their elements. Indeed, by setting

$$P_n(x,\alpha,\beta) = \sum_{\nu=0}^{n} \frac{\widetilde{c}_{n,\nu}(\alpha,\beta)}{(\alpha+\beta+2)_n}x^\nu \tag{3.23}$$

then, regarding (3.20), the set of coefficients $\widetilde{c}_{n,\nu}$ fulfill the triangular relation

$$\begin{aligned}\widetilde{c}_{n+1,\nu}(\alpha,\beta) &= (\nu+\alpha+1)\widetilde{c}_{n,\nu}(\alpha,\beta) - (\nu+\alpha+\beta+1)\widetilde{c}_{n,\nu-1}(\alpha,\beta)\\ \widetilde{c}_{n,0}(\alpha,\beta) &= (-1)^n(\alpha+1)^n\end{aligned}$$

which can be shrunk to the same coefficients $c_{n,\nu}$, given in (3.9)–(3.10) on the aforementioned Laguerre case, if we consider

$$\widetilde{c}_{n,\nu}(\alpha,\beta) = (-1)^{n+\nu}(\alpha+\beta+2)_\nu\ c_{n,\nu}(\alpha)$$

The new set of coefficients $\{c_{n,\nu}(\alpha)\}$ no longer depends on β. Consequently, we obtain (3.22). □

Notwithstanding the generating function for the MPS $\{P_n(x;\alpha,\beta)\}_{n\geqslant 0}$ seems to be complicate to obtain, we succeeded in determining the following:

Lemma 3.7 *The PS $\{p_n(x;\alpha,\beta)\}_{n\geqslant 0}$ where $p_n(x;\alpha,\beta) = (\alpha+\beta+2)_n\times P_n(x;\alpha,\beta)$ have the following generating function*

$$G(x,t) = \frac{\mathrm{e}^{(\alpha-\beta)t}}{(\cosh(t)+x\sinh(t))^{\alpha+\beta+2}} = \sum_{n\geqslant 0} p_n(x;\alpha,\beta)\frac{t^n}{n!} \tag{3.24}$$

Proof The differential-recursive relation

$$p_{n+1}(x;\alpha,\beta) = -x^2 p_n'(x;\alpha,\beta) + \big(-(\alpha+\beta+2)x+\alpha-\beta\big)p_n(x;\alpha,\beta)\ ,\ n\in\mathbb{N}_0,$$

fulfilled by the PS $\{p_n(x;\alpha,\beta)\}_{n\geqslant 0}$, implies the generating function to be solution of the differential equation

$$\frac{\partial}{\partial t}G(x,t) = -x^2\frac{\partial}{\partial x}G(x,t) + \big(-(\alpha+\beta+2)x+\alpha-\beta\big)G(x,t)$$

satisfying the boundary condition $G(x,0)=1$. The desired solution is (3.24). □

Regarding the structure relation fulfilled by $\{P_n(\cdot;\alpha,\beta)\}_{n\geqslant 0}$ we have the following result:

Lemma 3.8 *The MPS* $\{P_n(\cdot) := P_n(\cdot;\alpha,\beta)\}_{n\geqslant 0}$ *fulfills the recurrence relation*

$$P_{n+2}(x) = (x-\beta_{n+1})P_{n+1}(x) - \sum_{\nu=0}^{n}\chi_{n,\nu}P_\nu(x)$$

with

$$\beta_n = \frac{2(\alpha+\beta+1)(\alpha+1)+n(n+3+2\alpha+2\beta)}{2(n+\alpha+\beta+2)(n+\alpha+\beta+1)}\ ,\ n\in\mathbb{N}_0,$$

and

$$\chi_{n,0} = \frac{1}{(\alpha+\beta+2)_{n+1}}\sum_{\nu=0}^{n+1}\frac{(-1)^{n+\nu+1}c_{n+1,\nu}(\alpha)(\alpha+1)_{\nu+1}}{(\alpha+\beta+2+\nu)}\ ,\ n\in\mathbb{N}_0.$$

Proof The procedure is very similar to the one taken in the Bessel case. Precisely, from (3.1), where $\lambda_{n+1} = (\alpha+\beta+n+2)\lambda_n$, and by taking into consideration (1.2) and (3.3) it follows

$$(n+\alpha+\beta+3)\beta_{n+1} = (n+\alpha+\beta+1)\beta_n + 1\ ,\ n\in\mathbb{N}_0,$$

and $\beta_0 = \frac{\alpha+1}{\alpha+\beta+2}$, which amounts to the same as

$$(n+\alpha+\beta+2)(n+\alpha+\beta+1)\beta_n = (\alpha+\beta+1)(\alpha+1) + \frac{n}{2}(n+3+2\alpha+2\beta)\ ,\ n\in\mathbb{N}_0,$$

and, finally, from (3.3) we obtain the expression of β_n.

Taking into account (3.22) and the moments of u_0,

$$(u_0)_n = \frac{(\alpha+1)_n}{(\alpha+\beta+2)_n}\ ,\ n\in\mathbb{N}_0,$$

we conclude

$$\chi_{n,0} = \langle u_0, xP_{n+1}\rangle = \sum_{\nu=0}^{n+1} \frac{(\alpha+\beta+2)_\nu}{(\alpha+\beta+2)_{n+1}} (-1)^{n+1+\nu} c_{n+1,\nu}(\alpha)(u_0)_{\nu+1}$$

$$= \frac{1}{(\alpha+\beta+2)_{n+1}} \sum_{\nu=0}^{n+1} \frac{(-1)^{n+\nu+1} c_{n+1,\nu}(\alpha)(\alpha+1)_{\nu+1}}{(\alpha+\beta+2+\nu)}\ ,\ n \in \mathbb{N}_0.$$

□

Actually, as far as $\alpha+\beta > -1$, $\chi_{n,0} \neq 0$ for $n \in \mathbb{N}_0$. Indeed, bearing in mind (3.11) with $n \to n+1$, *i.e.*,

$$\sum_{\nu=0}^{n+1}(-1)^{n+\nu+1} c_{n+1,\nu}(\alpha)(\alpha+1)_{\nu+1} = (\alpha+1)(-1)^{n+1} \sum_{\nu=0}^{n+1}(-1)^{\nu} c_{n+1,\nu}(\alpha)(\alpha+2)_\nu$$

$$= \alpha + 1$$

it readily follows

$$\left|\chi_{n,0}\right| > \left|\frac{1}{(\alpha+\beta+2)_{n+1}}\right| \left|\sum_{\nu=0}^{n+1} \frac{(-1)^{n+\nu+1} c_{n+1,\nu}(\alpha)(\alpha+1)_{\nu+1}}{\alpha+\beta+2+\nu}\right|$$

$$= \left|\frac{(\alpha+1)}{(\alpha+\beta+2)_{n+2}}\right| > 0\ ,\ n \in \mathbb{N}_0.$$

Therefore, if $\alpha+\beta > -1$, the MPS $\{P_n(\cdot;\alpha,\beta)\}_{n\geqslant 0}$ cannot be d-orthogonal.

Otherwise, if $\alpha+\beta < -1$ fulfilling the regularity conditions pointed out in Table 1, we conjecture that $\chi_{n,0} \neq 0$.

The determination of the remaining coefficients $\chi_{n,\nu}$ seems to require more laborious computations, that are deferred for a further work. Besides, regarding the nature of this MPS $\{P_n(\cdot;\alpha,\beta)\}_{n\geqslant 0}$, the connection with the well known Jacobi classical polynomial sequences is not as simple to establish as in the precedent cases of Hermite, Laguerre or Bessel. For this reason, we leave this issue as an open problem.

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