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Quadratic decomposition of Laguerre polynomials via lowering operators

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Abstract

A Laguerre polynomial sequence of parameter $\varepsilon/2$ was previously characterized in a recent work [Ana F. Loureiro and P. Maroni (2008) [28]] as an orthogonal $\mathcal{F}_\varepsilon$-Appell sequence, where $\mathcal{F}_\varepsilon$ represents a lowering (or annihilating) operator depending on the complex parameter $\varepsilon \neq -2n$ for any integer $n \geqslant 0$. Here, we proceed to the quadratic decomposition of an $\mathcal{F}_\varepsilon$-Appell sequence, and we conclude that the four sequences obtained by this approach are also Appell but with respect to another lowering operator consisting of a Fourth-order linear differential operator $\mathcal{G}_{\varepsilon,\mu}$, where μ is either 1 or -1. Therefore, we introduce and develop the concept of the $\mathcal{G}_{\varepsilon,\mu}$-Appell sequences and we prove that they cannot be orthogonal. Finally, the quadratic decomposition of the non-symmetric sequence of Laguerre polynomials (with parameter $\varepsilon/2$) is fully accomplished.

Keywords: Orthogonal polynomials; Laguerre polynomials; Appell polynomials; Lowering operator; Genocchi numbers

1. Introduction and preliminary results

We denote by $\mathscr{P}$ the vector space of the polynomials with coefficients in $\mathbb{C}$ (the field of complex numbers) and by $\mathscr{P}'$ its dual space, whose elements are *forms*. The action of $u \in \mathscr{P}'$

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on $f \in \mathscr{P}$ is denoted as $\langle u, f\rangle$. In particular, we denote by $(u)_n := \langle u, x^n\rangle, n \geqslant 0$, the moments of u. A linear operator $T : \mathscr{P} \to \mathscr{P}$ has a transpose ${}^tT : \mathscr{P}' \to \mathscr{P}'$ defined by

$$\langle {}^tT(u), f\rangle = \langle u, T(f)\rangle, \quad u \in \mathscr{P}',\ f \in \mathscr{P}. \tag{1.1}$$

For example, for any form u, any polynomial g, let $Du = u'$ and gu be the forms defined as usually

$$\langle u', f\rangle := -\langle u, f'\rangle, \qquad \langle gu, f\rangle := \langle u, gf\rangle,$$

where D is the derivative operator.

Let $\{B_n\}_{n\geqslant 0}$ is a sequence of monic polynomials with $\deg B_n = n, n \geqslant 0$ (monic polynomial sequence: MPS) and let $\{u_n\}_{n\geqslant 0}$ be the corresponding dual sequence, $u_n \in \mathscr{P}'$, defined by $\langle u_n, P_k\rangle := \delta_{n,k}, n, k \geqslant 0$ (where $\delta_{n,k}$ represents the Kronecker symbol). We recall from [31], that any form $u \in \mathscr{P}'$ may be represented through

$$u = \sum_{n\geqslant 0}\langle u, B_n\rangle u_n. \tag{1.2}$$

A form u is said to be *regular* whenever there is a MPS $\{B_n\}_{n\geqslant 0}$ such that $\langle u, B_n B_m\rangle = k_n\,\delta_{n,m}$ with $k_n \neq 0$ for any $n, m \geqslant 0$. In this case, $\{B_n\}_{n\geqslant 0}$ is called a monic orthogonal polynomial sequence—hereafter MOPS—and it is also characterized by

$$\begin{cases} B_0(x) = 1; \qquad B_1(x) = x - \beta_0 \\ B_{n+2}(x) = (x - \beta_{n+1})B_{n+1}(x) - \gamma_{n+1}B_n(x), \quad n \geqslant 0, \end{cases} \tag{1.3}$$

$$u_n = \left(\langle u_0, B_n^2\rangle\right)^{-1} B_n\, u_0, \quad n \geqslant 0, \tag{1.4}$$

where $\{u_n\}_{n\geqslant 0}$ represents the corresponding dual sequence and $(\beta_n, \gamma_{n+1})_{n\geqslant 0}$ are known as the recurrence coefficients with $\gamma_{n+1} \neq 0, n \geqslant 0$.

When $u \in \mathscr{P}'$ is regular, let Φ be a polynomial such that $\Phi u = 0$; then, $\Phi = 0$, [32].

Entailed in the problem of the symmetrization of sequences of polynomials, comes out the quadratic decomposition (as well as the cubic decomposition) of a polynomial sequence. Within this context, many authors have dealt with symmetrization problems of orthogonal polynomial sequences. Among them we quote [4,12–15,18,24,30,33]. More specifically, in [12,13,15] a symmetric orthogonal polynomial sequence is decomposed into two nonsymmetric sequences. A generalisation of this idea was revealed in [30,33]: to a given MPS $\{B_n\}_{n\geqslant 0}$, we associate two other MPS, $\{P_n\}_{n\geqslant 0}$ and $\{R_n\}_{n\geqslant 0}$, and two sequences of polynomials, $\{a_n\}_{n\geqslant 0}$ and $\{b_n\}_{n\geqslant 0}$, such that

$$B_{2n}(x) = P_n(x^2) + xa_{n-1}(x^2), \quad n \geqslant 0, \tag{1.5}$$

$$B_{2n+1}(x) = b_n(x^2) + xR_n(x^2), \quad n \geqslant 0 \tag{1.6}$$

where $0 \leqslant \deg a_n, \deg b_n \leqslant n$ for any integer $n \geqslant 0$ and $a_{-1}(\cdot) = 0$, [15,30]. Under the assumption that $\{B_n\}_{n\geqslant 0}$ is orthogonal, it is not possible to conclude that $\{P_n\}_{n\geqslant 0}$ and $\{R_n\}_{n\geqslant 0}$ are also orthogonal, up to some supplementary conditions. For instance, $a_n = 0 = b_n, n \geqslant 0$, if and only if the MPS $\{B_n\}_{n\geqslant 0}$ is symmetric (that is $B_n(-x) = (-1)^n B_n(x), n \geqslant 0$) and its orthogonality supplies the orthogonality of both sequences $\{P_n\}_{n\geqslant 0}$ and $\{R_n\}_{n\geqslant 0}$ [30].

Recently, in [28], the two authors have proceeded to the quadratic decomposition (hereafter QD) of an *Appell polynomial sequence* (that is, a MPS $\{B_n\}_{n\geqslant 0}$ such that $B_n(\cdot) = (n+1)^{-1}B'_{n+1}(\cdot), n \geqslant 0$) [3]. The four associated sequences obtained by this approach are also Appell sequences but with respect to another differential operator:

$$\mathscr{F}_\varepsilon := 2DxD + \varepsilon D = 2xD^2 + (2+\varepsilon)D, \tag{1.7}$$

where ε is either 1 or -1, and $D := \frac{\mathrm{d}}{\mathrm{d}x}$. This operator $\mathscr{F}_\varepsilon$, as well as the differential operator D, decreases in one unit the degree of a polynomial. They are indeed simple examples of the so-called *lowering operators*: a linear mapping $\mathscr{O}$ of $\mathscr{P}$ into itself is called *lowering operator* when $\mathscr{O}(1) = 0$ and $\deg(\mathscr{O}(x^n)) = n-1, n \geqslant 1$. The Appell character of a PS may be generalised in a natural way to other *lowering operators* $\mathscr{O}$ rather than D.

Definition 1.1. A MPS $\{B_n\}_{n\geqslant 0}$ is called an $\mathscr{O}$-Appell sequence with respect to a lowering (or annihilating) operator $\mathscr{O}$ if $B_n(\cdot) = B_n^{[1]}(\cdot, \mathscr{O})$: for any integer $n \geqslant 0$, with

$$B_n^{[1]}(x;\mathscr{O}) := \rho_n(\mathscr{O}B_{n+1})(x), \quad n \geqslant 0,$$

where $\rho_n \in \mathbb{C} - \{0\},\ n \geqslant 0$, is chosen for making $B_n^{[1]}(x;\mathscr{O})$ monic [7,8].

This concept is not new. As a matter of fact some authors have considered Appell sequences with respect to other operators like the q-derivative [38], operators reducing or augmenting the degree of a polynomial by k units, with $k \geqslant 1$. Among them we quote [10,11,16,17,25,26]. However, such considerations are not useful here.

The primary purpose of this work is to characterize the four sequences associated with the QD of an $\mathscr{F}_\varepsilon$-Appell sequence, in which $\mathscr{F}_\varepsilon$ is the operator given in (1.7) with $\varepsilon \neq -2n, n \geqslant 1$. First, in Section 2, we show that the four polynomial sequences obtained by this approach are also Appell sequences with respect to a fourth-order linear differential operator, denoted by $\mathscr{G}_{\varepsilon,\mu}$, where μ is either 1 or -1. Subsequently, regarding a more accurate information about the arisen $\mathscr{G}_{\varepsilon,\mu}$-Appell sequences, in Section 3 we study these MPS through a functional point of view, where the range for the parameter μ was broadened to a subset of $\mathbb{C}$ (the set of complex numbers) satisfying (3.3). The unique D-Appell and $\mathscr{F}_\varepsilon$-Appell orthogonal sequences are, respectively, the Hermite (a result given by Angelescu [2] and later on by other authors [13,36] but further references may be found in [1]) and the Laguerre polynomial sequences of parameter $\varepsilon/2$, up to a linear change of variable (achieved in [28]). However, in Section 4 we conclude that a $\mathscr{G}_{\varepsilon,\mu}$-Appell sequence cannot be orthogonal. In spite of this negative result, in the last section we successfully reach the complete description of the QD of the nonsymmetric sequence of Laguerre polynomials, by means of the *Genocchi numbers*.

2. The quadratic decomposition of $\mathscr{F}_\varepsilon$-Appell sequences

Pursuing the idea of the quadratic decomposition of an Appell sequence, we explore the $\mathscr{F}_\varepsilon$-Appell sequences. To accomplish so, it is useful to summarize some properties of the operator $\mathscr{F}_\varepsilon$; namely for any $f, g \in \mathscr{P}$, we have

$$\mathscr{F}_\varepsilon(f(x)g(x)) = f(x)\mathscr{F}_\varepsilon(g(x)) + g(x)\mathscr{F}_\varepsilon(f(x)) + 4xf'(x)g'(x),$$

$$\mathscr{F}_\varepsilon(f(t^2))(x) = x\{8x^2f''(x^2) + 2(4+\varepsilon)f'(x^2)\}, \tag{2.1}$$

$$\mathscr{F}_\varepsilon(tf(t^2))(x) = x^2\{8x^2f''(x^2) + 2(8+\varepsilon)f'(x^2)\} + (2+\varepsilon)f(x^2). \tag{2.2}$$

Theorem 2.1. *Consider the QD of a monic sequence* $\{B_n\}_{n\geqslant 0}$ *as in (1.5)–(1.6). If* $\{B_n\}_{n\geqslant 0}$ *is an* $\mathscr{F}_\varepsilon$*-Appell sequence with* $\varepsilon \neq -2(n+1), n \geqslant 0$*, then the four sequences* $\{P_n\}_{n\geqslant 0}$, $\{R_n\}_{n\geqslant 0}$, $\{a_n\}_{n\geqslant 0}$ *and* $\{b_n\}_{n\geqslant 0}$ *satisfy*

$$P_n(x) = \frac{1}{\eta_{n+1}(\varepsilon,-1)}(\mathscr{G}_{\varepsilon,-1}P_{n+1})(x), \quad n \geqslant 0, \tag{2.3}$$

$$R_n(x) = \frac{1}{\eta_{n+1}(\varepsilon,1)}(\mathscr{G}_{\varepsilon,1}R_{n+1})(x), \quad n \geqslant 0, \tag{2.4}$$

$$a_n(x) = \frac{1}{\eta_{n+2}(\varepsilon,-1)}(\mathscr{G}_{\varepsilon,1}a_{n+1})(x), \quad n \geqslant 0, \tag{2.5}$$

$$b_n(x) = \frac{1}{\eta_{n+1}(\varepsilon,1)}(\mathscr{G}_{\varepsilon,-1}b_{n+1})(x), \quad n \geqslant 0, \tag{2.6}$$

where the operators $\mathscr{G}_{\varepsilon,1}$ *and* $\mathscr{G}_{\varepsilon,-1}$ *and the nonzero sequences* $\{\eta_{n+1}(\varepsilon,1)\}_{n\geqslant 0}$ *and* $\{\eta_{n+1}(\varepsilon,-1)\}_{n\geqslant 0}$ *are respectively given by*

$$\mathscr{G}_{\varepsilon,1} = (4DxD + \varepsilon D)(2xD + \mathbb{I})(4xD + (2+\varepsilon)D) \tag{2.7}$$

$$\mathscr{G}_{\varepsilon,-1} = (4DxD + \varepsilon D)(2xD - \mathbb{I})(4xD - (2-\varepsilon)D) \tag{2.8}$$

and

$$\eta_{n+1}(\varepsilon,1) = (n+1)(4(n+1)+\varepsilon)(2n+3)\big[2(2n+3)+\varepsilon\big], \quad n \geqslant 0, \tag{2.9}$$

$$\eta_{n+1}(\varepsilon,-1) = (n+1)(4(n+1)+\varepsilon)(2n+1)\big[2(2n+1)+\varepsilon\big], \quad n \geqslant 0, \tag{2.10}$$

where $D := \frac{\mathrm{d}}{\mathrm{d}x}$ *and* $\mathbb{I}$ *represents the identity on* $\mathscr{P}$.

Proof. Consider $\rho_{n+1} = (n+1)(2(n+1)+\varepsilon)$. Operating with $\mathscr{F}_\varepsilon$ on both members of (1.5) and (1.6) with n replaced by $n+1$, then, under the assumption and by virtue of (2.1)–(2.2), we obtain

$$\begin{aligned}\rho_{2n+2}\{b_n(x^2) + xR_n(x^2)\} &= x\{2(4+\varepsilon)P'_{n+1}(x^2) + 8x^2P''_{n+1}(x^2)\} + (2+\varepsilon)a_n(x^2)\\ &\quad + 2(8+\varepsilon)x^2a'_n(x^2) + 8x^4a''_n(x^2), \quad n \geqslant 0,\\ \rho_{2n+1}\{P_n(x^2) + xa_{n-1}(x^2)\} &= x\{2(4+\varepsilon)b'_n(x^2) + 8x^2b''_n(x^2)\} + (2+\varepsilon)R_n(x^2)\\ &\quad + 2(8+\varepsilon)x^2R'_n(x^2) + 8x^4R''_n(x^2), \quad n \geqslant 0,\end{aligned}$$

which consists of polynomials with only even or odd powers. As a result, we necessarily obtain

$$\rho_{2n+2}R_n(x) = \{2(4+\varepsilon)D + 8xD^2\}(P_{n+1}(x)), \quad n \geqslant 0, \tag{2.11}$$

$$\rho_{2n+1}P_n(x) = \{(2+\varepsilon)\mathbb{I} + 2(8+\varepsilon)xD + 8x^2D^2\}(R_n(x)), \quad n \geqslant 0, \tag{2.12}$$

$$\rho_{2n+2}b_n(x) = \{(2+\varepsilon)\mathbb{I} + 2(8+\varepsilon)xD + 8x^2D^2\}(a_n(x)), \quad n \geqslant 0, \tag{2.13}$$

$$\rho_{2n+1}a_{n-1}(x) = \{2(4+\varepsilon)D + 8xD^2\}(b_n(x)), \quad n \geqslant 0. \tag{2.14}$$

Operating with the equalities (2.11) and (2.12), we deduce

$$\rho_{2n+2}\rho_{2n+3}R_n(x) = \{2\varepsilon D + 8DxD\}\cdot\{(2+\varepsilon)\mathbb{I} + 2(4+\varepsilon)xD + 8xDxD\}(R_{n+1}(x)),\quad n \geqslant 0,$$

$$\begin{aligned}\rho_{2n+1}\rho_{2n+2}P_n(x) &= \{(2+\varepsilon)\mathbb{I} + 2(8+\varepsilon)xD + 8x^2D^2\}\cdot\{2(4+\varepsilon)D + 8xD^2\}\\ &\quad \times (P_{n+1}(x)), \quad n \geqslant 0.\end{aligned}$$

Using the identities

$$Dx = xD - \mathbb{I}; \qquad x^2D^2 = xDxD - xD \quad \text{and} \quad x^2D^2 = DxDx - 3Dx + 2\mathbb{I} \tag{2.15}$$

in the right-hand side of the first and second previous relations respectively, we obtain

$$\rho_{2n+2}\rho_{2n+3}R_n(x) = \{2\varepsilon D + 8DxD\} \cdot \{(2+\varepsilon)\mathbb{I} + 2(4+\varepsilon)xD + 8xDxD\}(R_{n+1}(x)),\quad n \geqslant 0,$$

$$\rho_{2n+1}\rho_{2n+2}P_n(x) = \{(2-\varepsilon)\mathbb{I} - 2(4-\varepsilon)Dx + 8DxDx\} \cdot \{2\varepsilon D + 8DxD\}(P_{n+1}(x)),\quad n \geqslant 0,$$

which correspond to (2.4), under the definitions (2.7) and (2.9), and to (2.3) under the definitions (2.8) and (2.10), respectively.

Likewise, by means of simple manipulations, the system of equalities (2.13) and (2.14) gives rise to another system of two equalities: one involving exclusively elements of the set of polynomials $\{b_n\}_{n\geqslant 0}$ and the other having only elements of the set of polynomials $\{a_n\}_{n\geqslant 0}$, which, on account of the identities (2.15), may be transformed into the following equalities:

$$\rho_{2n+2}\rho_{2n+3}b_n(x) = \{(2-\varepsilon)\mathbb{I} - 2(4-\varepsilon)Dx + 8DxDx\} \cdot \{2\varepsilon D + 8DxD\}(b_{n+1}(x)),\quad n \geqslant 0, \tag{2.16}$$

$$\rho_{2n+1}\rho_{2n+2}a_{n-1}(x) = \{2\varepsilon D + 8DxD\} \cdot \{(2+\varepsilon)\mathbb{I} + 2(4+\varepsilon)xD + 8xDxD\}(a_n(x)),\quad n \geqslant 0, \tag{2.17}$$

where $a_{-1}(\cdot) = 0$. The relation (2.16) provides (2.6), whereas the relation (2.17) with n replaced by $n+1$ leads to (2.5), under the definitions (2.7)–(2.10). □

More information about the polynomial sequences is provided in the next result.

Proposition 2.1. *Let $\{B_n\}_{n\geqslant 0}$ be a $\mathscr{F}_\varepsilon$-Appell sequence and consider its QD according to (1.5)–(1.6). Then, either $\{B_n\}_{n\geqslant 0}$ is symmetric or there exists an integer $p \geqslant 0$ such that $a_p(\cdot) \neq 0$ (respectively, $b_p(\cdot) \neq 0$). In this case, we have*

$$a_n(x) = 0, \qquad b_n(x) = 0, \qquad 0 \leqslant n \leqslant p-1, \quad \text{when } p \geqslant 1, \tag{2.18}$$

$$a_{p+n}(x) = \binom{n+p+1}{n} \frac{\left(p+\frac{3}{2}\right)_n \left(p+\frac{3}{2}+\frac{\varepsilon}{4}\right)_n \left(p+2+\frac{\varepsilon}{4}\right)_n}{\left(\frac{3}{2}\right)_n \left(\frac{3}{2}+\frac{\varepsilon}{4}\right)_n \left(1+\frac{\varepsilon}{4}\right)_n} a_p \widehat{a}_n(x), \tag{2.19}$$

$$b_{p+n}(x) = \binom{n+p}{n} \frac{\left(p+\frac{3}{2}\right)_n \left(p+\frac{3}{2}+\frac{\varepsilon}{4}\right)_n \left(p+1+\frac{\varepsilon}{4}\right)_n}{\left(\frac{1}{2}\right)_n \left(\frac{1}{2}+\frac{\varepsilon}{4}\right)_n \left(1+\frac{\varepsilon}{4}\right)_n} b_p \widehat{b}_n(x), \qquad n \geqslant 0, \tag{2.20}$$

where $\widehat{a}_n$ and $\widehat{b}_n$ are two monic polynomials fulfilling $\deg \widehat{a}_n(x) = n$, $\deg \widehat{b}_n(x) = n$, for $n \geqslant 0$, and $(y)_n = y(y+1)\ldots(y+n-1)$: represents the Pochhammer symbol.

Proof. If $\{B_n\}_{n\geqslant 0}$ is a symmetric sequence, then $a_n(\cdot) = 0, n \geqslant 0$, and also $b_n(\cdot) = 0, n \geqslant 0$. Conversely, if $a_n(\cdot) = 0, n \geqslant 0$ (respectively, $b_n(\cdot) = 0, n \geqslant 0$), then from (2.13) $b_n(\cdot) = 0, n \geqslant 0$ (respectively $a_n(\cdot) = 0, n \geqslant 0$, from (2.14)).

When $\{B_n\}_{n\geqslant 0}$ is not a symmetric sequence, let $p \geqslant 0$ be the smallest integer such that $a_p(\cdot) \neq 0$ and $a_n(\cdot) = 0, 0 \leqslant n \leqslant p-1$ when $p \geqslant 1$. From (2.14), we have $b_n(\cdot) =$ constant $= b_n, 0 \leqslant n \leqslant p$ and by virtue of (2.13), $b_n(\cdot) = 0$ for $0 \leqslant n \leqslant p-1$, $\rho_{2p+2}b_p(x) =$

$(2+\varepsilon)a_p(x)+2(8+\varepsilon)xa_p'(x)+8x^2a_p''(x)$, which implies $a_p(\cdot)=\text{constant}=a_p\neq 0$. Thus, $(2+\varepsilon)a_p = \rho_{2p+2}b_p$. Proceeding by finite induction, then, based on (2.13)–(2.14), we get $\deg(a_{n+p})=n$ and $\deg(b_{n+p})=n$, $n\geqslant 0$. Therefore, we may consider two nonzero sequences $\{\lambda_n\}_{n\geqslant 0}$ and $\{\mu_n\}_{n\geqslant 0}$ such that

$$a_{n+p}(x)=\lambda_n\widehat{a}_n(x) \quad \text{and} \quad b_{n+p}(x)=\mu_n\widehat{b}_n(x), \quad n\geqslant 0, \tag{2.21}$$

where $\widehat{a}_n(\cdot)$ and $\widehat{b}_n(\cdot)$ represent two monic polynomials of degree $n\geqslant 0$, $\mu_0=b_p$ and $\lambda_0=a_p$. Replacing in (2.13) and (2.14) n by $n+p$ and taking into account (2.21), we obtain

$$\rho_{2n+2p+2}\mu_n\widehat{b}_n(x)=(2+\varepsilon)\lambda_n\widehat{a}_n(x)+2(8+\varepsilon)x\lambda_n\widehat{a}_n'(x)+8x^2\lambda_n\widehat{a}_n''(x), \quad n\geqslant 0,$$
$$\rho_{2n+2p+1}\lambda_{n-1}\widehat{a}_{n-1}(x)=2(4+\varepsilon)\mu_n\widehat{b}_n'(x)+8x\mu_n\widehat{b}_n''(x), \quad n\geqslant 0.$$

Therefore, the nonzero sequences $\{\lambda_n\}_{n\geqslant 0}$ and $\{\mu_n\}_{n\geqslant 0}$ satisfy the system

$$\begin{cases}\rho_{2n+2p+2}\mu_n = 8\left(n+\frac{1}{2}\right)\left(n+\frac{1}{2}+\frac{\varepsilon}{4}\right)\lambda_n, & n\geqslant 0,\\ \rho_{2n+2p+1}\lambda_{n-1}=8n\left(n+\frac{\varepsilon}{4}\right)\mu_n, & n\geqslant 0.\end{cases}$$

Inasmuch as $\rho_{n+1}=(n+1)(2(n+1)+\varepsilon)$, $n\geqslant 0$, this latter implies

$$\lambda_n=\binom{n+p+1}{n}\frac{\left(p+\frac{3}{2}\right)_n\left(p+\frac{3}{2}+\frac{\varepsilon}{4}\right)_n\left(p+2+\frac{\varepsilon}{4}\right)_n}{\left(\frac{3}{2}\right)_n\left(\frac{3}{2}+\frac{\varepsilon}{4}\right)_n\left(1+\frac{\varepsilon}{4}\right)_n}\lambda_0;$$
$$\mu_n=\frac{\left(n+\frac{1}{2}\right)\left(n+\frac{1}{2}+\frac{\varepsilon}{4}\right)}{(n+p+1)\left(n+p+1+\frac{\varepsilon}{4}\right)}\lambda_n, \quad n\geqslant 0,$$

whence the result. □

The two MPS emerged with the QD of an $\mathscr{F}_\varepsilon$-Appell sequence, are also Appell sequences with respect to the lowering operators $\mathscr{G}_{\varepsilon,1}$ and $\mathscr{G}_{\varepsilon,-1}$, in the light of Definition 1.1. Analogously, on account of the relations (2.5)–(2.6) and (2.19)–(2.20) given in Proposition 2.1, we may say that the sequences $\{\widehat{a}_n\}_{n\geqslant 0}$ and $\{\widehat{b}_n\}_{n\geqslant 0}$ are, respectively, $\mathscr{G}_{\varepsilon,1}$ and $\mathscr{G}_{\varepsilon,-1}$-Appell. From this point on, we pose the problem of characterizing such Appell sequences with respect to these two arisen operators. To accomplish this goal, we will consider the more general operator

$$\mathscr{G}_{\varepsilon,\mu}:=(4DxD+\varepsilon D)(8(xD)^2+2\varepsilon xD+2\mathbb{I}+\mu(8xD+\varepsilon\mathbb{I}))$$

with the convention: $(xD)^{k+1}=xD(xD)^k$ for any integer $k\geqslant 0$, which matches $\mathscr{G}_{\varepsilon,-1}$ or $\mathscr{G}_{\varepsilon,1}$ as long as we consider $\mu=-1$ or $\mu=1$, respectively. Naturally, it is possible to express

$$\begin{aligned}\mathscr{G}_{\varepsilon,\mu}:=\;&32D(xD)^3+16\varepsilon D(xD)^2+2(4+\varepsilon^2)DxD+2\varepsilon D\\&+\mu\{32D(xD)^2+12\varepsilon DxD+\varepsilon^2D\}.\end{aligned}\tag{2.22}$$

The forthcoming developments will be made from a functional point of view, requiring the characterization of the associated dual sequence, which will be carried out in the next section.

3. The $\mathscr{G}_{\varepsilon,\mu}$-Appell sequences

Let $\{B_n\}_{n\geqslant 0}$ be a MPS with dual sequence $\{u_n\}_{n\geqslant 0}$. Consider the sequence $\{B_n^{[1]}(\cdot;\mathscr{G}_{\varepsilon,\mu})\}_{n\geqslant 0}$ given by

$$B_n^{[1]}(x;\mathscr{G}_{\varepsilon,\mu}) = \frac{1}{\widehat{\rho}_{n+1}}(\mathscr{G}_{\varepsilon,\mu}B_{n+1})(x), \quad n\geqslant 0, \tag{3.1}$$

where $\mathscr{G}_{\varepsilon,\mu}$ is given by (2.22) and

$$\widehat{\rho}_{n+1} := \widehat{\rho}_{n+1}(\varepsilon,\mu) = (n+1)(4(n+1)+\varepsilon)(2+2(n+1)(4(n+1)+\varepsilon) + (8+8n+\varepsilon)\mu) \tag{3.2}$$

for $n\geqslant 0$. Necessarily the parameters ε and μ must be chosen so that $\widehat{\rho}_{n+1}\neq 0$, for all the integers $n\geqslant 0$; therefore, ε and μ are two complex parameters such that

$$\varepsilon \neq -4(n+1) \quad \text{and} \quad \mu \neq -\frac{2+2(n+1)(4n+4+\varepsilon)}{8(n+1)+\varepsilon}, \quad n\geqslant 0. \tag{3.3}$$

Whenever $\mu\in\{-1,1\}$, then $\widehat{\rho}_{n+1}(\varepsilon,\mu)$ equals $\eta_{n+1}(\varepsilon,\mu)$, given by (2.9)–(2.10), for any integer $n\geqslant 0$.

Before characterizing $\mathscr{G}_{\varepsilon,\mu}$-Appell sequences, we must determine the dual sequence of $\{B_n^{[1]}(\cdot;\mathscr{G}_{\varepsilon,\mu})\}_{n\geqslant 0}$, denoted as $\{u_n^{[1]}(\mathscr{G}_{\varepsilon,\mu})\}_{n\geqslant 0}$. For this purpose, we need to know the transpose ${}^t\mathscr{G}_{\varepsilon,\mu}$ defined according to (1.1):

$$\langle {}^t\mathscr{G}_{\varepsilon,\mu}u, f\rangle = \langle u, \mathscr{G}_{\varepsilon,\mu}f\rangle = \langle u, \{32D(xD)^3 + 16(\varepsilon+2\mu)D(xD)^2 + 2(4+\varepsilon^2+6\varepsilon\mu)DxD + \varepsilon(2+\varepsilon\mu)D\}f\rangle;$$

therefore,

$${}^t\mathscr{G}_{\varepsilon,\mu} = 32D(xD)^3 - 16(\varepsilon+2\mu)D(xD)^2 + 2(4+\varepsilon^2+6\varepsilon\mu)DxD - \varepsilon(2+\varepsilon\mu)D.$$

However, the convention on D (${}^tD = -D$) permits to write ${}^t\alpha_\nu := (-1)^{\nu+1}D(xD)^\nu$, with $\alpha_\nu := D(xD)^\nu$, leaving out a slight abuse of notation without consequence. Thus, ${}^t\mathscr{G}_{\varepsilon,\mu} := \mathscr{G}_{-\varepsilon,-\mu}$ and $\mathscr{G}_{\varepsilon,\mu}$ is defined on $\mathscr{P}$ and $\mathscr{P}'$.

For the sequel, it is worth to express $\mathscr{G}_{\varepsilon,\mu}$ in terms of x^kD^{k+1}: instead of $D(xD)^k$: (with $k=0,1,2,3$). Based on the identities

$$DxD = xD^2 + D, \qquad D(xD)^2 = x^2D^3 + 3xD^2 + D \quad \text{and}$$
$$D(xD)^3 = x^3D^4 + 6x^2D^3 + 7xD^2 + D,$$

the operator $\mathscr{G}_{\varepsilon,\mu}$ given by (2.22) may be expressed as follows:

$$\mathscr{G}_{\varepsilon,\mu} = 32x^3D^4 + 16(12+\varepsilon)x^2D^3 + 2(116+\varepsilon(24+\varepsilon))xD^2 + 2(4+\varepsilon)(5+\varepsilon)D + \mu\{32x^2D^3 + 12(8+\varepsilon)xD^2 + (4+\varepsilon)(8+\varepsilon)D\}. \tag{3.4}$$

After making a few number of computations, we are able to deduce the $\mathscr{G}_{\varepsilon,\mu}$-derivative of the product of two polynomials:

$$\mathscr{G}_{\varepsilon,\mu}\big(fp\big)(x) = f(x)(\mathscr{G}_{\varepsilon,\mu}p) + (\mathscr{G}_{\varepsilon,\mu}f)p(x) + 128x^3f'(x)p^{(3)}(x) + 48\{(\varepsilon+12+2\mu)f'(x) + 4xf''(x)\}x^2p''(x)$$

$$+\{(116+\varepsilon^2+48\mu+6\varepsilon(4+\mu))f'(x) + 12(\varepsilon+2(6+\mu))xf''(x)+32x^2f^{(3)}(x)\}4xp'(x) \tag{3.5}$$

for any $p, f \in \mathscr{P}$. By transposition, we may also compute the $\mathscr{G}_{\varepsilon,\mu}$-derivative of the product of a polynomial by a form:

$$(\mathscr{G}_{-\varepsilon,-\mu} fu) = f(\mathscr{G}_{-\varepsilon,-\mu}u) - (\mathscr{G}_{\varepsilon,\mu} f)u + f'(x)L_3(u) + f''(x)L_2(u) + f^{(3)}(x)L_1(u) + 2^6x^3f^{(4)}(x)u, \quad f \in \mathscr{P},\ u \in \mathscr{P}', \tag{3.6}$$

where

$$\begin{aligned} L_3(u) &= \tau_{3,0}u + \tau_{3,1}xu' + \tau_{3,2}x^2u'' + 2^7 \cdot x^3(u)^{(3)} \\ L_2(u) &= \tau_{2,0}xu + \tau_{2,1}x^2u' + 3\cdot 2^6x^3u'' \\ L_1(u) &= \tau_{1,0}x^2u + 2^7x^3u' \end{aligned} \tag{3.7}$$

with

$$\tau_{3,0} = 4(20+\varepsilon^2+6\varepsilon\mu); \qquad \tau_{3,1} = 2^2(116+\varepsilon^2+6\varepsilon(\mu-4)-48\mu);$$
$$\tau_{3,2} = -2^4\cdot 3(\varepsilon-12+2\mu); \qquad \tau_{2,0} = 2^2(116+\varepsilon^2+6\varepsilon\mu);$$
$$\tau_{2,1} = 2^4\cdot 3(12-\varepsilon-2\mu); \qquad \tau_{1,0} = 2^7\cdot 3.$$

Lemma 3.1. *The dual sequence of* $\{B_n^{[1]}(\cdot;\mathscr{G}_{\varepsilon,\mu})\}_{n\geqslant 0}$ *denoted as* $\{u_n^{[1]}(\mathscr{G}_{\varepsilon,\mu})\}_{n\geqslant 0}$ *fulfils*

$$\mathscr{G}_{-\varepsilon,-\mu}(u_n^{[1]}(\mathscr{G}_{\varepsilon,\mu})) = \widehat{\rho}_{n+1}u_{n+1}, \quad n \geqslant 0, \tag{3.8}$$

where $\widehat{\rho}_{n+1}, n \geqslant 0$, *is given by* (3.2).

Proof. Following the definition of a dual sequence, $\langle u_n^{[1]}(\mathscr{G}_{\varepsilon,\mu}), B_m^{[1]}(x;\mathscr{G}_{\varepsilon,\mu})\rangle = \delta_{n,m}$ for any integers $n, m \geqslant 0$, which corresponds to $(\widehat{\rho}_{n+1})^{-1}\langle u_n^{[1]}(\mathscr{G}_{\varepsilon,\mu}), \mathscr{G}_{\varepsilon,\mu}(B_{m+1})\rangle = \delta_{n,m}$ for n, $m \geqslant 0$, that is

$$\langle \mathscr{G}_{-\varepsilon,-\mu}(u_n^{[1]}(\mathscr{G}_{\varepsilon,\mu})), B_{m+1}\rangle = \widehat{\rho}_{n+1}\delta_{n,m}, \quad n, m \geqslant 0. \tag{3.9}$$

In particular, from the latter we have $\langle \mathscr{G}_{-\varepsilon,-\mu}(u_n^{[1]}(\mathscr{G}_{\varepsilon,\mu})), B_{m+1}\rangle = 0$, for any integers $m \geqslant n+1, n \geqslant 0$, which implies [34,35]

$$\mathscr{G}_{-\varepsilon,-\mu}\big(u_n^{[1]}(\mathscr{G}_{\varepsilon,\mu})\big) = \sum_{\nu=0}^{n+1}\lambda_{n,\nu}u_\nu, \quad n \geqslant 0,$$

with $\lambda_{n,\nu} = \langle \mathscr{G}_{-\varepsilon,-\mu}(u_n^{[1]}(\mathscr{G}_{\varepsilon,\mu})), B_\nu\rangle, 0 \leqslant \nu \leqslant n+1$. Consequently, due to (3.9), we obtain (3.8). □

This last result enables us to characterize all the $\mathscr{G}_{\varepsilon,\mu}$-Appell sequences through the elements of its dual sequence.

Proposition 3.1. *The MPS* $\{B_n\}_{n\geqslant 0}$ *is a* $\mathscr{G}_{\varepsilon,\mu}$*-Appell sequence if and only if its dual sequence* $\{u_n\}_{n\geqslant 0}$ *fulfils*

$$u_n = \frac{1}{\alpha_n}\mathscr{G}^n_{-\varepsilon,-\mu}(u_0), \quad n \geqslant 0, \tag{3.10}$$

where $\alpha_n = 32^n n!(1+\frac{\varepsilon}{4})_n(\frac{8+\varepsilon+4\mu-\Delta_{\varepsilon,\mu}}{8})_n(\frac{8+\varepsilon+4\mu+\Delta_{\varepsilon,\mu}}{8})_n$, *with* $\Delta_{\varepsilon,\mu} = \sqrt{\varepsilon^2+16(\mu^2-1)}$, *and* $\mathscr{G}^n_{-\varepsilon,-\mu}$ *representing the nth power of the operator* $\mathscr{G}_{-\varepsilon,-\mu}$.

Proof. The condition is necessary. From (3.8), the sequence $\{u_n\}_{n\geqslant 0}$ satisfies

$$\mathscr{G}_{-\varepsilon,-\mu}(u_n) = \widehat{\rho}_{n+1}(\varepsilon,\mu)u_{n+1}, \quad n \geqslant 0, \tag{3.11}$$

with $\widehat{\rho}_{n+1}(\varepsilon,\mu)$ as given in (3.2). In particular, for $n=0$,

$$u_1 = \frac{1}{(4+\varepsilon)(10+8\mu+\varepsilon(2+\mu))}\mathscr{G}_{-\varepsilon,-\mu}u_0.$$

Proceeding by finite induction, we easily get (3.10).

The condition is sufficient. From (3.10), it is easy to see that (3.11) is fulfilled. Therefore by comparing it with (3.8), we obtain

$$\mathscr{G}_{-\varepsilon,-\mu}(u_n^{[1]}(\mathscr{G}_{\varepsilon,\mu})) = \mathscr{G}_{-\varepsilon,-\mu}u_n, \quad n \geqslant 0.$$

The lowering operator $\mathscr{G}_{-\varepsilon,-\mu}$ satisfies $\mathscr{G}_{-\varepsilon,-\mu}(\mathscr{P}) = \mathscr{P}$; therefore, $\mathscr{G}_{-\varepsilon,-\mu}$ is one-to-one on $\mathscr{P}'$ whence $u_n^{[1]}(\mathscr{G}_{\varepsilon,\mu}) = u_n,\ n \geqslant 0$. □

Remark. Naturally, any $\mathscr{O}$-Appell sequence (see Definition 1.1) may be characterized in a similar way:

$$u_n = \frac{1}{\widetilde{\alpha}_n}({}^t\mathscr{O})^n(u_0), \quad n \geqslant 0,$$

where $\{\widetilde{\alpha}_n\}_{n\geqslant 0}$ represents a sequence of nonzero complex numbers to be determined according to the lowering operator $\mathscr{O}$.

4. About the orthogonality of a $\mathscr{G}_{\varepsilon,\mu}$-Appell sequence

While seeking all the orthogonal polynomial sequences possessing the $\mathscr{G}_{\varepsilon,\mu}$-Appell character, not even a single example appears as is shown in the next result.

Theorem 4.1. *There is no regularly orthogonal polynomial sequence being* $\mathscr{G}_{\varepsilon,\mu}$*-Appell.*

Proof. Suppose there is a MOPS $\{B_n\}_{n\geqslant 0}$ which is also a $\mathscr{G}_{\varepsilon,\mu}$-Appell sequence and let $\{\beta_n, \gamma_{n+1}\}_{n\geqslant 0}$ be its recurrence coefficients in accordance with (1.3). From (1.4) and (3.11), we obtain

$$\mathscr{G}_{-\varepsilon,-\mu}(B_n u_0) = \lambda_n B_{n+1}u_0, \quad n \geqslant 0, \tag{4.1}$$

with $\lambda_n := \lambda_n(\varepsilon) = \frac{\widehat{\rho}_{n+1}(\varepsilon,\mu)}{\gamma_{n+1}}$, $n \geqslant 0$, where $\widehat{\rho}_{n+1}, n \geqslant 0$, is defined in (3.2). We recall that, within the range of ε and μ, $\widehat{\rho}_{n+1}$ is always different from zero for any integer $n \geqslant 0$. The particular choice of $n=0$ in (4.1) provides

$$\mathscr{G}_{-\varepsilon,-\mu}u_0 = \lambda_0 B_1 u_0. \tag{4.2}$$

Consider $n+1$ instead of n in (4.1). Following (3.6)–(3.7), because of the $\mathscr{G}_{\varepsilon,\mu}$-Appell character and on account of (4.2), we derive

$$\begin{aligned} &B'_{n+1}L_3(u_0) + B''_{n+1}L_2(u_0) + B^{(3)}_{n+1}L_1(u_0) \\ &\quad = \left\{\lambda_{n+1}B_{n+2} - \lambda_0 B_1 B_{n+1} + \lambda_n\gamma_{n+1}B_n - 2^6x^3B^{(4)}_{n+1}\right\}u_0, \quad n \geqslant 0. \end{aligned}$$

The particular choices of $n = 0$, $n = 1$ and $n = 2$ in this latter permits to deduce the simpler equations

$$L_3(u_0) = U_2(x)u_0 \tag{4.3}$$

$$L_2(u_0) = U_3(x)u_0 \tag{4.4}$$

$$L_1(u_0) = U_4(x)u_0 \tag{4.5}$$

where $L_i(u_0)$, with $i = 1, 2, 3$, are given in (3.7) and

$$U_2(x) = \lambda_1 B_2(x) - \lambda_0 B_1^2(x) + \lambda_0\gamma_1,$$
$$U_3(x) = \frac{1}{2}\{\lambda_2 B_3(x) - \lambda_0 B_1(x)B_2(x) + \lambda_1\gamma_2 B_1(x) - B_2'(x)U_2(x)\},$$
$$U_4(x) = \frac{1}{6}\{\lambda_3 B_4(x) - \lambda_0 B_1(x)B_3(x) + \lambda_2\gamma_3 B_2(x) - B_3'(x)U_2(x) - B_3''(x)U_3(x)\}.$$

Naturally, $\deg U_k \leqslant k$ for $k = 2, 3$ or 4, so there are coefficients $\theta_{k,j}$ with $0 \leqslant j \leqslant k$ such that $U_k(x) = \sum_{j=0}^{k} \theta_{k,j} x^j$, for $k = 2, 3, 4$. Between (4.5) after a single differentiation and (4.4), it is possible to eliminate the term in u_0'', and consequently we have

$$\{(3^2 \cdot 2^8 + 3\tau_{1,0} - 2\tau_{2,1})x^2 - 3U_4(x)\}u_0' = \{3U_4'(x) - 2U_3(x) - 2(3\tau_{1,0} - \tau_{2,0})x\}u_0. \tag{4.6}$$

The elimination of the term u_0' between the equalities (4.6) and (4.5) and the regularity of u_0 leads to $C_3 \equiv 0$ where

$$C_3(x) = -2^7 x^3\{3U_4'(x) - 2U_3(x) - 2(3\tau_{1,0} - \tau_{2,0})x\} + ((3^2 \cdot 2^7 + 3\tau_{1,0} - 2\tau_{2,1})x^2 - 3U_4(x))(U_4(x) - \tau_{1,0}x^2),$$

which implies

$$\theta_{4,4} = \theta_{4,1} = \theta_{4,0} = \theta_{3,0} = 0, \qquad \theta_{3,3} = \frac{3}{2^8}(\theta_{4,3})^2,$$
$$\theta_{3,2} = \frac{1}{2^7}\theta_{4,3}(3\theta_{4,2} - 3\tau_{1,0} + \tau_{2,1})$$
$$\theta_{3,1} = \frac{1}{2^8}\{3(\theta_{4,2})^2 + 2^8\tau_{2,0} - \theta_{4,2}(2^7 \cdot 3 + 6\tau_{1,0} - 2\tau_{2,1}) - \tau_{1,0}(-2^7 \cdot 3 - 3\tau_{1,0} + 2\tau_{2,1})\}. \tag{4.7}$$

Differentiating both sides of (4.4) and then eliminating the term in $u_0^{(3)}$ between the resulting equation and (4.3), we deduce a second-order differential equation in u_0. The elimination of the term in u_0'' between the obtained equation and (4.4) leads us to

$$\{[2^7 \cdot 3\tau_{2,0} - 2^6 \cdot 3^2\tau_{3,1} + \tau_{2,1}(-2^7 \cdot 3 - 2\tau_{2,1} + 3\tau_{3,2})]x - 2^7 \cdot 3U_3(x)\}xu_0'$$
$$= \{\tau_{2,0}(2(2^7 \cdot 3 + \tau_{2,1}) - 3\tau_{3,2})x - (2^7 \cdot 3^2 + 2\tau_{2,1} - 3\tau_{3,2})U_3(x) + 3 \cdot 2^6(3\tau_{3,0} - 3U_2(x) + 2U_3'(x))x\}u_0. \tag{4.8}$$

By eliminating the term in u_0' between (4.8) and (4.5), and by taking into consideration the regularity of u_0, we get the condition $C_2 \equiv 0$ where

$$C_2(x) = -(2^7x^3)\{\tau_{2,0}(2(2^7 \cdot 3 + \tau_{2,1}) - 3\tau_{3,2})x - (2^7 \cdot 3^2 + 2\tau_{2,1} - 3\tau_{3,2})U_3(x)$$

$$+ 3 \cdot 2^6\big(3\tau_{3,0} - 3U_2(x) + 2U_3'(x)\big)x\} + \{[2^7 \cdot 3\tau_{2,0} - 2^6 \cdot 3^2\tau_{3,1} + \tau_{2,1}(-2^7 \cdot 3 - 2\tau_{2,1} + 3\tau_{3,2})]x^2 - 2^7 \cdot 3U_3(x)x\}(U_4(x) - \tau_{1,0}x^2) \tag{4.9}$$

which, on account of (4.7), implies that

$$\theta_{3,0} = \theta_{3,3} = \theta_{3,2} = \theta_{2,2} = \theta_{2,1} = 0$$
$$\theta_{3,1} = \frac{1}{2^8}\{3\theta_{4,2}^2 + 2^8\tau_{2,0} - \theta_{4,2}(2^7 \cdot 3 + 6\tau_{1,0} - 2\tau_{2,1}) - \tau_{1,0}(-2^7 \cdot 3 - 3\tau_{1,0} + 2\tau_{2,1})\}.$$

As a result, $U_2(x) = \theta_{2,0}$, $U_3(x) = \theta_{3,1}x$ and $U_4(x) = \theta_{4,2}x^2$, and, according to (4.5) u_0 fulfils

$$(\tau_{1,0} - \theta_{4,2})x^2u_0 + 2^7x^3u_0' = 0,$$

contradicting the regularity of u_0 because it would imply the moment equation $(\tau_{1,0} - \theta_{4,2} - 2^7(n+3))\langle u_0, x^n\rangle = 0,\ n \geqslant 0.$ □

Notwithstanding this negative result, it might be worth to seek quasi-orthogonal or d-orthogonal sequences (for some integer $d \geqslant 2$, [20,9]) possessing the $\mathcal{G}_{\varepsilon,\mu}$-Appell property.

5. Applications. The quadratic decomposition of a Laguerre sequence

As far as we are concerned, the QD of non-symmetric polynomial sequences was not yet considered, inasmuch as it is not evident. Nonetheless, the combination of the obtained with some already known results (specially those given in [30,33]) permits to describe the associated polynomial sequences to the QD of a Laguerre sequence with complex parameter.

Proposition 5.1. *A Laguerre sequence $\{B_n\}_{n\geqslant 0}$ of parameter $\frac{\varepsilon}{2}$ (with $\varepsilon \neq -2(n+1), n \geqslant 0$) fulfils (1.5)–(1.6) where $\{R_n\}_{n\geqslant 0}$ and $\{P_n\}_{n\geqslant 0}$ are respectively $\mathcal{G}_{\varepsilon,1}$ and $\mathcal{G}_{\varepsilon,-1}$-Appell sequences and $\{a_n\}_{n\geqslant 0}$, $\{b_n\}_{n\geqslant 0}$ are two PS given by*

$$a_n(x) = \sum_{\nu=0}^{n} \lambda_{n,\nu} R_\nu(x), \quad n \geqslant 0 \tag{5.1}$$

$$b_n(x) = \sum_{\nu=0}^{n} \theta_{n,\nu} P_\nu(x), \quad n \geqslant 0, \tag{5.2}$$

with

$$\lambda_{n,\nu} = \binom{2n+2}{2\nu} \frac{(-1)^{n-\nu}2^{2n-2\nu+1}}{2\nu+1} \frac{\left(2+\frac{\varepsilon}{2}\right)_{2n+1}}{\left(2+\frac{\varepsilon}{2}\right)_{2\nu}} \mathfrak{G}_{2n-2\nu+2}, \quad 0 \leqslant \nu \leqslant n,\ n \geqslant 0, \tag{5.3}$$

$$\theta_{n,\nu} = \binom{2n+2}{2\nu} \frac{(-1)^{n-\nu}2^{2n-2\nu}}{n+1} \frac{\left(1+\frac{\varepsilon}{2}\right)_{2n+1}}{\left(1+\frac{\varepsilon}{2}\right)_{2\nu}} \mathfrak{G}_{2n-2\nu+2}, \quad 0 \leqslant \nu \leqslant n,\ n \geqslant 0, \tag{5.4}$$

where $\mathfrak{G}_n$ represent the unsigned Genocchi numbers.

The *Genocchi numbers* were presumably introduced by Edouard Lucas in [29], but they owe the name to the italian mathematician Angelo Genocchi (1817–1889) [23]. E.T. Bell developed intensive studies on these numbers in the 1920s in [5,6]. Such numbers are intimately related to the much more famous *Bernoulli numbers* as will be presented just after the proof of the precedent result. There are many possibilities for computing the values of the Genocchi numbers (see for example [19,22,39], and also the entry A001469 in [37] for further references). The proof of the latter proposition requires the following known result.

Lemma 5.1 (*[30]*). *Given a MPS $\{B_n\}_{n\geqslant 0}$, it is possible to associate two MPS $\{R_n\}_{n\geqslant 0}$ and $\{P_n\}_{n\geqslant 0}$ and two sequences $\{a_n\}_{n\geqslant 0}$ and $\{b_n\}_{n\geqslant 0}$ according to (1.5)–(1.6) and (5.1)–(5.2). If, in addition, $\{B_n\}_{n\geqslant 0}$ is a MOPS fulfilling the second-order recurrence relation (1.3), necessarily the coefficients $\lambda_{n,\nu}, \theta_{n,\nu}, 0 \leqslant \nu \leqslant n, n \geqslant 0$, satisfy the following system:*

$$\lambda_{n,n} = -\sum_{\nu=1}^{n}\{\beta_{2\nu} + \beta_{2\nu+1}\}, \quad n \geqslant 0, \tag{5.5}$$

$$\theta_{n,n} = -\beta_0 - \sum_{\nu=1}^{n}\{\beta_{2\nu-1} + \beta_{2\nu}\}, \quad n \geqslant 0, \tag{5.6}$$

$$\theta_{n+1,\nu} + \gamma_{2n+2}\theta_{n,\nu} = \lambda_{n,\nu-1} + \gamma_{2\nu+1}\lambda_{n,\nu} + \sum_{\mu=\nu}^{n}\lambda_{n,\mu}\theta_{\mu,\nu}\beta_{2\mu+1} \tag{5.7}$$

$$\lambda_{n+1,\nu} + \gamma_{2n+3}\lambda_{n,\nu} = \theta_{n+1,\nu} + \gamma_{2\nu+2}\theta_{n+1,\nu+1} + \sum_{\mu=\nu}^{n}\theta_{n+1,\mu+1}\lambda_{\mu,\nu}\beta_{2\mu+2} \tag{5.8}$$

for $0 \leqslant \nu \leqslant n, n \geqslant 0$, with $\lambda_{n,-1} = 0, n \geqslant 0$.

Proof of Proposition 5.1. Let $\{B_n\}_{n\geqslant 0}$ be a Laguerre sequence of parameter $\frac{\varepsilon}{2}$ with $\varepsilon \neq -2n, n \geqslant 1$. The two authors have shown in [28, theorem 6] that such a sequence to be the unique MOPS being $\mathscr{F}_\varepsilon$-Appell. So, necessarily the second-order recurrence relation (1.3) holds and we recall the well-known expression for its recurrence coefficients:

$$\beta_n = 2n + 1 + \frac{\varepsilon}{2}; \qquad \gamma_{n+1} = (n+1)\left(n + 1 + \frac{\varepsilon}{2}\right), \quad n \geqslant 0. \tag{5.9}$$

Reconsidering the quadratic decomposition of $\{B_n\}_{n\geqslant 0}$ given in (1.5)–(1.6), but this time describing the sequences $\{a_n\}_{n\geqslant 0}$ and $\{b_n\}_{n\geqslant 0}$ by means of the associated MPS $\{P_n\}_{n\geqslant 0}$ and $\{R_n\}_{n\geqslant 0}$, there exist two sets of numbers $\{\lambda_{n,\nu}\}_{0\leqslant\nu\leqslant n}$ and $\{\theta_{n,\nu}\}_{0\leqslant\nu\leqslant n}$ such that (5.1)–(5.2) hold. By virtue of Theorem 2.1, the MPS $\{R_n\}_{n\geqslant 0}$ and $\{P_n\}_{n\geqslant 0}$ are respectivelly $\mathscr{G}_{\varepsilon,1}$ and $\mathscr{G}_{\varepsilon,-1}$-Appell sequences. Just as it was observed in the proof of Theorem 2.1, the conditions (2.11)–(2.14) hold.

Inserting the relations (5.1)–(5.2) into (2.13) and, afterwards, by taking into consideration (2.12), the fact that $\{P_n\}_{n\geqslant 0}$ is a linearly independent sequence provides

$$\theta_{n,\nu} = \frac{\gamma_{2\nu+1}}{\gamma_{2n+2}}\lambda_{n,\nu}, \quad n \geqslant 0,\ 0 \leqslant \nu \leqslant n. \tag{5.10}$$

Likewise, consider the insertion of the expression (5.1) into (2.5). The $\mathscr{G}_{\varepsilon,1}$-Appell character of $\{R_n\}_{n\geqslant 0}$ permits to derive

$$\lambda_{n+1,\nu+1} = \frac{\gamma_{2n+4}\gamma_{2n+3}}{\gamma_{2\nu+3}\gamma_{2\nu+2}}\lambda_{n,\nu}, \quad 0 \leqslant \nu \leqslant n, \tag{5.11}$$

if we take into account that $\{R_n\}_{n\geqslant 0}$ is a linearly independent sequence. Proceeding by finite induction, we deduce

$$\lambda_{n+1,\nu+1} = \left\{\prod_{\tau=0}^{2\nu+1}\frac{\gamma_{2n-2\nu+\tau+3}}{\gamma_{\tau+2}}\right\}\lambda_{n-\nu,0}, \quad 0 \leqslant \nu \leqslant n,$$

which, on account of (5.9), may be expressed as

$$\lambda_{n,\nu} = \frac{1}{2\nu+1}\binom{2n+2}{2\nu}\frac{\left(2+\frac{\varepsilon}{2}\right)_{2n+1}}{\left(2+\frac{\varepsilon}{2}\right)_{2\nu}\left(2+\frac{\varepsilon}{2}\right)_{2(n-\nu)+1}}\lambda_{n-\nu,0}, \quad 0 \leqslant \nu \leqslant n. \tag{5.12}$$

Based on Lemma 5.1, we will carry out the determination of the coefficients $\lambda_{n-\nu,0}$. The particular choice $n = 0$ in (5.5)–(5.6) and on account of (5.9), respectively, provides

$$\lambda_{0,0} = -2\left(2+\frac{\varepsilon}{2}\right), \qquad \theta_{0,0} = -\left(1+\frac{\varepsilon}{2}\right). \tag{5.13}$$

On account of (5.10) and (5.11), the relations (5.7)–(5.8) with $\nu = 0$ may be rewritten like

$$\begin{cases} \dfrac{1}{\gamma_{2n+4}}\lambda_{n+1,0} = \displaystyle\sum_{\mu=0}^{n}\frac{\lambda_{n,\mu}\lambda_{\mu,0}}{\gamma_{2\mu+2}}\beta_{2\mu+1} \\ \lambda_{n+1,0} = \dfrac{\gamma_1}{\gamma_{2n+4}}\lambda_{n+1,0} + \gamma_{2n+3}\displaystyle\sum_{\mu=0}^{n}\frac{\lambda_{n,\mu}\lambda_{\mu,0}}{\gamma_{2\mu+2}}\beta_{2\mu+2}, \quad n \geqslant 0. \end{cases} \tag{5.14}$$

Since, $\beta_{2\mu+2} = \beta_{2\mu+1} + 2$, for $\mu \geqslant 0$, from (5.14) we derive

$$\lambda_{n+1,0} = \frac{\gamma_1}{\gamma_{2n+4}}\lambda_{n+1,0} + \frac{\gamma_{2n+3}}{\gamma_{2n+4}}\lambda_{n+1,0} + 2\gamma_{2n+3}\sum_{\mu=0}^{n}\frac{\lambda_{n,\mu}\lambda_{\mu,0}}{\gamma_{2\mu+2}}, \quad n \geqslant 0,$$

which, considering (5.9) and (5.12), may be expressed like

$$\lambda_{n+1,0} = (n+2)\left(2+\frac{\varepsilon}{2}\right)_{2n+3}\sum_{\mu=0}^{n}\left\{\binom{2n+2}{2\mu} \times \frac{\lambda_{n-\mu,0}\lambda_{\mu,0}}{(2\mu+1)(\mu+1)\left(2+\frac{\varepsilon}{2}\right)_{2\mu+1}\left(2+\frac{\varepsilon}{2}\right)_{2(n-\mu)+1}}\right\}, \quad n \geqslant 0. \tag{5.15}$$

Proceeding by finite induction, we infer that there is a set of positive integers $\{\chi_n\}_{n\geqslant 0}$, not depending on the parameter ε, fulfilling the equality

$$\lambda_{n,0} = (-1)^{n+1}2^{2n+1}\chi_n\left(2+\frac{\varepsilon}{2}\right)_{2n+1}, \quad n \geqslant 0. \tag{5.16}$$

Indeed, on account of (5.13), $\chi_0 = 1$, and, under the assumption, from the relation (5.15) we get

$$\lambda_{n+1,0} = (n+2)(-1)^n 2^{2n+2}\left(2+\frac{\varepsilon}{2}\right)_{2n+3}\sum_{\mu=0}^{n}\left\{\binom{2n+2}{2\mu}\frac{\chi_{n-\mu}\chi_\mu}{(2\mu+1)(\mu+1)}\right\}, \quad n \geqslant 0.$$

Insofar as the integers χ_n, $n \geqslant 0$, do not depend on ε, they are necessarily related by the equality

$$\frac{\chi_{n+1}}{(2n+4)!} = \frac{1}{2n+3}\sum_{\mu=0}^{n}\frac{\chi_{n-\mu}}{(2n-2\mu+2)!}\frac{\chi_\mu}{(2\mu+2)!}, \quad n \geqslant 0. \tag{5.17}$$

Suppose there is an analytic function L defined on an open set of $\mathbb{C}$ such that $L(z) = \sum_{n\geqslant 0}\frac{\chi_n}{(2n+2)!}z^n$. Based upon the relation (5.17), $L(z)$ is a solution of the differential equation

$$(zL(z^2))' = \frac{1}{2} + \frac{1}{2}(zL(z^2))^2,$$

whence the conclusion $zL(z^2) = \tan(\frac{z}{2})$. Following for example [21,40] and denoting by $\mathfrak{G}_{2n}$ the *unsigned Genocchi numbers*, it is possible to write

$$\tan\left(\frac{z}{2}\right) = \sum_{n\geqslant 0} \mathfrak{G}_{2n+2} \frac{z^{2n+1}}{(2n+2)!};$$

Therefore, $\chi_n = \mathfrak{G}_{2n+2}$ and (5.16) becomes

$$\lambda_{n,0} = (-1)^{n+1} 2^{2n+1} \mathfrak{G}_{2n+2} \left(2 + \frac{\varepsilon}{2}\right)_{2n+1}, \quad n \geqslant 0.$$

Inserting in (5.12), this last equality with $n-\mu$ instead of n, we obtain (5.3) and, on account of (5.10), we obtain (5.4). $\square$

The unsigned Genocchi numbers are directly related to the *Bernoulli numbers* $\mathfrak{B}_n$ via $\mathfrak{G}_{2n} = 2(1-2^{2n})\mathfrak{B}_{2n}$, where $\mathfrak{B}_n$ are defined by [21,40]

$$\frac{z}{e^z - 1} = 1 - \frac{1}{2}z + \sum_{n\geqslant 1} (-1)^{n+1} \mathfrak{B}_{2n} \frac{z^{2n}}{(2n)!}.$$

This last result could also be achieved by using the well-known expression of the Laguerre polynomials.

6. Concluding remarks

The $\mathcal{O}$-Appell orthogonal polynomial sequences are part of a wider collection of polynomials satisfying the Hahn's property, the so-called $\mathcal{O}$-classical polynomials: a MOPS $\{B_n\}_{n\geqslant 0}$ is said to be $\mathcal{O}$-classical whenever $\{B_n^{[1]}(\cdot;\mathcal{O})\}_{n\geqslant 0}$ is also a MOPS. For instance, the Laguerre polynomials of parameter $\varepsilon/2$ are $\mathscr{F}_\varepsilon$-Appell and $\mathscr{F}_\varepsilon$-classical. In the first author's thesis [27], all the $\mathscr{F}_\varepsilon$-classical polynomials were determined where, apart from the $\mathscr{F}_\varepsilon$-Appell ones, the Jacobi polynomials of parameters $(\frac{\varepsilon}{2}, \mu-\frac{\varepsilon}{4})$ (with $4\mu-\varepsilon \neq -4(n+1)$ and $4\mu+\varepsilon \neq -4n$ for any integer $n \geqslant 0$) were found. The MPS of the $\mathscr{F}_\varepsilon$-derivatives of these latter, $\{B_n^{[1]}(\cdot;\mathscr{F}_\varepsilon)\}_{n\geqslant 0}$, is also a Jacobi sequence of parameters $(\frac{\varepsilon}{2}, \mu-\frac{\varepsilon}{4}+2)$. In general, the determination and characterization of all the $\mathcal{O}$-classical polynomials have some inherent technical difficulties that sometimes are hard to overcome and the more complicate the operator $\mathcal{O}$ is, the more laborious the resolution becomes. After the proof of the nonexistence of the orthogonal $\mathscr{G}_{\varepsilon,\mu}$-Appell sequences, remains the open question of the determination and characterization of the $\mathscr{G}_{\varepsilon,\mu}$-classical polynomials.

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