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Quadratic decomposition of Appell sequences☆

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Abstract

We proceed to the quadratic decomposition of Appell sequences and we characterise the four derived sequences obtained by this approach. We prove that the two monic polynomial sequences associated to such quadratic decomposition are also Appell sequences with respect to another (lowering) operator, which we call as $\mathscr{F}_\varepsilon$, where either $\varepsilon = 1$ or -1. Thus, we introduce and develop the concept of the Appell polynomial sequences with respect to the operator $\mathscr{F}_\varepsilon$ (where, ε is a parameter belonging to the field of the complex numbers): the $\mathscr{F}_\varepsilon$-Appell sequences. The orthogonal polynomial sequences that are also $\mathscr{F}_\varepsilon$-Appell correspond to the Laguerre sequences with parameter $\varepsilon/2$. Indeed, this brings an entirely new characterisation of the Laguerre sequences.

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1. Preliminaries

We denote by $\mathscr{P}$ the vector space of the polynomials with coefficients in $\mathbb{C}$ (the field of complex numbers) and by $\mathscr{P}'$ its dual space, whose elements are *forms*. The action of $u \in \mathscr{P}'$ on $f \in \mathscr{P}$ is denoted as $\langle u, f\rangle$. In particular, we denote by $(u)_n := \langle u, x^n\rangle, n\geqslant 0$ the moments of u. Recall that a linear operator $T : \mathscr{P} \to \mathscr{P}$ has a transpose ${}^tT : \mathscr{P}' \to \mathscr{P}'$ defined by

$$\langle {}^tT(u), f\rangle = \langle u, T(f)\rangle, \quad u \in \mathscr{P}', \ f \in \mathscr{P}. \tag{1.1}$$

For example, for any form u, any polynomial g, let $Du = u'$ and gu be the forms defined as usually

$$\langle u', f\rangle := -\langle u, f'\rangle, \quad \langle gu, f\rangle := \langle u, gf\rangle,$$

where D is the differential operator. Thus, the differentiation operator D on forms is minus the transpose of the differentiation operator D on polynomials.

By PS we mean a sequence of polynomials $\{B_n\}_{n\geqslant 0}$ such that $\deg B_n = n, n\geqslant 0$. When any polynomial B_n, $n\geqslant 0$, is monic (that is, $B_n(x) = x^n + b_n(x)$, with $\deg b_n \leqslant n-1$ when $n\geqslant 1$), then we will call it a *monic polynomial sequence* (MPS), [8,9]. The dual sequence $\{u_n\}_{n\geqslant 0}$, $u_n \in \mathscr{P}'$, of a MPS $\{B_n\}_{n\geqslant 0}$ is defined by $\langle u_n, B_k\rangle = \delta_{n,k}$, $n, k\geqslant 0$, where $\delta_{n,k}$ denotes the Kronecker symbol. We will denote as $\{B_n^{[1]}\}_{n\geqslant 0}$ the MPS obtained from a given MPS by a single differentiation $B_n^{[1]}(x) := 1/(n+1)B'_{n+1}(x)$, $n\geqslant 0$.

The form u is called *regular* if we can associate with it a PS $\{B_n\}_{n\geqslant 0}$ such that $\langle u, B_nB_m\rangle = k_n\delta_{n,m}$ with $k_n \neq 0$, for all the integers $n, m\geqslant 0$, [5,8,9]. The PS $\{B_n\}_{n\geqslant 0}$ is then said to be orthogonal with respect to u. If u is a regular form we can assume that the system (of orthogonal polynomials) is monic. Then there exists a dual sequence $\{u_n\}_{n\geqslant 0}$ and the original form u is proportional to u_0. Furthermore, we have

$$u_n = (\langle u_0, B_n^2\rangle)^{-1} B_n u_0, \quad n\geqslant 0. \tag{1.2}$$

The sequence $\{B_n\}_{n\geqslant 0}$ is then called a *monic orthogonal polynomial sequence* (MOPS) and it fulfils the second-order recurrence relation given by

$$B_0(x) = 1, \quad B_1(x) = x - \beta_0, \tag{1.3}$$

$$B_{n+2}(x) = (x - \beta_{n+1})B_{n+1}(x) - \gamma_{n+1}B_n(x), \quad n\geqslant 0. \tag{1.4}$$

with $\beta_n = \langle u_0, xB_n^2\rangle/\langle u_0, B_n^2\rangle$ and $\gamma_{n+1} = \langle u_0, B_{n+1}^2\rangle/\langle u_0, B_n^2\rangle \neq 0, n\geqslant 0$.

When $u \in \mathscr{P}'$ is regular, let Φ be a polynomial such that $\Phi u = 0$, then $\Phi = 0$, [9]. For $a, b \in \mathbb{C}$, $a \neq 0$, the affine function $ax + b$ gives rise to an isomorphism $T : \mathscr{P} \to \mathscr{P}$ defined by $T(p)(x) = p(ax + b)$ for $p \in \mathscr{P}$, and the inverse operator T^{-1} is associated to the affine function $x/a - b/a$. If $\{B_n\}_{n\geqslant 0}$, is a MOPS with respect to the regular form u, then $\{\widetilde{B}_n\}_{n\geqslant 0}$ (with $\widetilde{B}_n(x) := a^{-n} B_n(ax + b)$) is a MOPS with respect to the regular form ${}^t(T^{-1})(u)$ as is easily seen.

Whenever $\{B_n\}_{n\geqslant 0}$ and $\{B_n^{[1]}\}_{n\geqslant 0}$ are both orthogonal, we say that $\{B_n\}_{n\geqslant 0}$ is a classical sequence (*Hermite*, *Laguerre*, *Bessel or Jacobi*), [5,6,9]. Moreover, if $\{B_n\}_{n\geqslant 0}$ is orthogonal with respect to u_0 then there exists a monic polynomial Φ with deg $\Phi \leqslant 2$ and a polynomial Ψ with deg $\Psi = 1$ such that [9]

$$D(\Phi u_0) + \Psi u_0 = 0.$$

In this case we say that u_0 is a classical form. In particular, a Laguerre form fulfils

$$D(xu_0) + (x - \alpha - 1)u_0 = 0, \tag{1.5}$$

where $\alpha \neq -(n+1)$, $n \geqslant 0$. When $\Re(\alpha + 1) > 0$ one may write

$$\langle u_0, f\rangle = \frac{1}{\Gamma(\alpha+1)} \int_0^{+\infty} \mathrm{e}^{-x} x^{\alpha} f(x)\,\mathrm{d}x.$$

2. The quadratic decomposition of Appell sequences

Our goal is to proceed to the quadratic decomposition of Appell sequences. Let us recall the following facts. An Appell sequence is a MPS $\{B_n\}_{n\geqslant 0}$ such that $B_n^{[1]} = B_n$, $n \geqslant 0$, [2,5,11]. Given a MPS $\{B_n\}_{n\geqslant 0}$ it is always possible to associate with it two MPS $\{P_n\}_{n\geqslant 0}$ and $\{R_n\}_{n\geqslant 0}$ and two sequences $\{a_n\}_{n\geqslant 0}$, $\{b_n\}_{n\geqslant 0}$ such that

$$B_{2n}(x) = P_n(x^2) + xa_{n-1}(x^2), \quad n \geqslant 0, \tag{2.6}$$

$$B_{2n+1}(x) = b_n(x^2) + xR_n(x^2), \quad n \geqslant 0, \tag{2.7}$$

where deg $a_n \leqslant n$, deg $b_n \leqslant n$, $n \geqslant 0$, $a_{-1}(x) = 0$, [5,7].

As a consequence, a MPS $\{B_n\}_{n\geqslant 0}$ is symmetric, that is $B_n(-x) = (-1)^n B_n(x)$, $n \geqslant 0$, if and only if $a_n(x) = b_n(x) = 0$, $n \geqslant 0$, [7,10].

Theorem 1. *Consider the quadratic decomposition of a monic sequence* $\{B_n\}_{n\geqslant 0}$ *as in Eqs.* (2.6)–(2.7). *If* $\{B_n\}_{n\geqslant 0}$ *is an Appell sequence, then the four associated sequences* $\{P_n\}_{n\geqslant 0}$, $\{R_n\}_{n\geqslant 0}$, $\{a_n\}_{n\geqslant 0}$ *and* $\{b_n\}_{n\geqslant 0}$ *are given by*

$$P_n(x) = \frac{1}{(n+1)(2n+1)}(\mathscr{F}_{-1}P_{n+1})(x), \quad n \geqslant 0, \tag{2.8}$$

$$R_n(x) = \frac{1}{(n+1)(2n+3)}(\mathscr{F}_{1}R_{n+1})(x), \quad n \geqslant 0, \tag{2.9}$$

$$a_n(x) = \frac{1}{(n+2)(2n+3)}(\mathscr{F}_{1}a_{n+1})(x), \quad n \geqslant 0, \tag{2.10}$$

$$b_n(x) = \frac{1}{(n+1)(2n+3)}(\mathscr{F}_{-1}b_{n+1})(x), \quad n \geqslant 0, \tag{2.11}$$

where the operator $\mathscr{F}_{\varepsilon}$ *(with* $\varepsilon = 1$ *or* $\varepsilon = -1$*) is given by*

$$\mathscr{F}_{\varepsilon} = 2\mathscr{F} + \varepsilon D \quad \textit{with } \mathscr{F} = DxD. \tag{2.12}$$

Proof. Indeed, by differentiating (2.6) and (2.7) with n replaced by $n+1$, then, under the assumption, we obtain

$$(2n+2)\{b_n(x^2)+xR_n(x^2)\}=2(n+1)x\,P_n^{[1]}(x^2)+a_n(x^2)+2x^2a_n'(x^2), \quad n\geqslant 0,$$

$$(2n+1)\{P_n(x^2)+xa_{n-1}(x^2)\}=2x\,b_n'(x^2)+R_n(x^2)+2n\,x^2\,R_{n-1}^{[1]}(x^2), \quad n\geqslant 0,$$

which consists of polynomials with only even or odd powers. As a result, we necessarily get

$$P_n^{[1]}(x)=R_n(x), \quad n\geqslant 0, \tag{2.13}$$

$$(2n+1)P_n(x)=R_n(x)+2nxR_{n-1}^{[1]}(x), \quad n\geqslant 0, \tag{2.14}$$

$$(2n+2)b_n(x)=a_n(x)+2xa_n'(x), \quad n\geqslant 0, \tag{2.15}$$

$$(2n+1)a_{n-1}(x)=2\,b_n'(x), \quad n\geqslant 0. \tag{2.16}$$

In (2.14), making $n\to n+1$, by differentiating on both sides and using (2.13), we obtain

$$(n+1)(2n+3)R_n(x)=2(xR_{n+1}'(x))'+R_{n+1}'(x), \quad n\geqslant 0. \tag{2.17}$$

On the other hand, we may express (2.14) only in terms of elements of $\{P_n\}_{n\geqslant 0}$ and its derivatives, by taking into consideration (2.13). Thus, we get

$$(n+1)(2n+1)P_n(x)=2(xP_{n+1}'(x))'-P_{n+1}'(x), \quad n\geqslant 0. \tag{2.18}$$

Hence, relations (2.17) and (2.18) may be respectively expressed as follows:

$$R_n(x)=\frac{1}{(n+1)(2n+3)}(2DxD+D)R_{n+1}(x), \quad n\geqslant 0, \tag{2.19}$$

and

$$P_n(x)=\frac{1}{(n+1)(2n+1)}(2DxD-D)P_{n+1}(x), \quad n\geqslant 0. \tag{2.20}$$

In addition, we may express (2.15) only in terms depending on b_n and its derivatives by taking into account (2.16). Indeed, we obtain, in a simplified way,

$$b_n(x)=\frac{1}{(n+1)(2n+3)}(2DxD-D)b_{n+1}(x), \quad n\geqslant 0. \tag{2.21}$$

From (2.16) and on account of (2.15), we get

$$a_n(x)=\frac{1}{(n+2)(2n+3)}(2DxD+D)a_{n+1}(x), \quad n\geqslant 0. \qquad \square \tag{2.22}$$

Proposition 2. *Let $\{B_n\}_{n\geqslant 0}$ be an Appell sequence and let Eqs.* (2.6)–(2.7) *be its quadratic decomposition. Then either $\{B_n\}_{n\geqslant 0}$ is symmetric or there exists an integer $p\geqslant 0$ such that*

$a_p(\cdot) \neq 0$ (*respectively*, $b_p(\cdot) \neq 0$). *In this case, we have*

$$a_n(x) = 0, \quad b_n(x) = 0, \quad 0 \leqslant n \leqslant p-1, \quad \text{when } p \geqslant 1, \tag{2.23}$$

$$a_{p+n}(x) = \binom{n+p+1}{n} \frac{\left(p+\frac{3}{2}\right)_n}{\left(\frac{3}{2}\right)_n} a_p \, \widehat{a}_n(x), \quad n \geqslant 0, \tag{2.24}$$

$$b_{p+n}(x) = \binom{n+p}{n} \frac{\left(p+\frac{3}{2}\right)_n}{\left(\frac{1}{2}\right)_n} b_p \, \widehat{b}_n(x), \quad n \geqslant 0, \tag{2.25}$$

where $\widehat{a}_n$ *and* $\widehat{b}_n$ *are two monic polynomials fulfilling* $\deg \widehat{a}_n(x) = n$ *and* $\deg \widehat{b}_n(x) = n$, $n \geqslant 0$ *and the* $(a)_n = a(a+1)\dots(a+n-1)$ *represents the Pochhammer symbol.*

Proof. If $\{B_n\}_{n\geqslant 0}$ is a symmetric sequence then $a_n(\cdot) = 0$, $n \geqslant 0$, and also $b_n(\cdot) = 0$, $n \geqslant 0$. Reciprocally, if $a_n(\cdot) = 0$, $n \geqslant 0$ (respectively, $b_n(\cdot) = 0$, $n \geqslant 0$), then from (2.15) $b_n(\cdot) = 0$, $n \geqslant 0$ (respectively, $a_n(\cdot) = 0$, $n \geqslant 0$, from (2.16)).

When $\{B_n\}_{n\geqslant 0}$ is not a symmetric sequence, let $p \geqslant 0$ be the smallest integer such that $a_p(\cdot) \neq 0$ and $a_n(\cdot) = 0$, $0 \leqslant n \leqslant p-1$ when $p \geqslant 1$. From (2.16), we have $b_n(\cdot) =$ constant, $0 \leqslant n \leqslant p$ and by virtue of (2.15), $b_n(\cdot) = 0$ for $0 \leqslant n \leqslant p-1$, $(2p+2)b_p = a_p(x) + 2xa'_p(x)$, which implies $a_p(\cdot) = \text{constant} = a_p \neq 0$. Thus, $a_p = (2p+2)b_p$.

Proceeding by finite induction, then, by taking into account (2.15)–(2.16), we achieve the conclusion that $\deg(a_{n+p}) = n$ and $\deg(b_{n+p}) = n$, $n \geqslant 0$. Therefore, we may consider two nonzero sequences $\{\lambda_n\}_{n\geqslant 0}$ and $\{\mu_n\}_{n\geqslant 0}$ such that

$$a_{n+p}(x) = \lambda_n \, \widehat{a}_n(x),$$

$$b_{n+p}(x) = \mu_n \, \widehat{b}_n(x), \quad n \geqslant 0,$$

where $\widehat{a}_n(\cdot)$ and $\widehat{b}_n(\cdot)$ are two monic polynomials of degree n, $n \geqslant 0$, $\mu_0 = b_p$ and $\lambda_0 = 2(p+1)b_p$. Due to Eqs. (2.15)–(2.16) we deduce that

$$\lambda_n = \binom{n+p+1}{n} \frac{\left(p+\frac{3}{2}\right)_n}{\left(\frac{3}{2}\right)_n} \lambda_0,$$

$$\mu_n = \frac{n+\frac{1}{2}}{n+p+1} \lambda_n, \quad n \geqslant 0,$$

whence the result. $\square$

Let us recall the definition of a so-called *lowering operator*. A linear mapping $\mathscr{O}$ of $\mathscr{P}$ into itself is called *lowering operator* when $\mathscr{O}(1) = 0$ and $\deg(\mathscr{O}(x^n)) = n-1$, $n \geqslant 1$. Clearly, D

and $\mathscr{F}_\varepsilon$ satisfy these conditions, the latter for $\varepsilon \neq -2, -4, \ldots$. It is possible to introduce lowering operators reducing the degree by $q \geqslant 2$, but it is not useful here.

Now, given a MPS $\{B_n\}_{n\geqslant 0}$, we construct the sequence $\{B_n^{[1]}(\cdot;\mathscr{O})\}_{n\geqslant 0}$ defined by

$$B_n^{[1]}(x;\mathscr{O}) = \rho_n(\mathscr{O}B_{n+1})(x), \quad n\geqslant 0, \tag{2.26}$$

where $\rho_n \in \mathbb{C} - \{0\}$, $n\geqslant 0$, is chosen for making $B_n^{[1]}(x;\mathscr{O})$ monic. Thus, we have

$$B_n^{[1]}(x;\mathscr{F}_\varepsilon) = \frac{1}{(n+1)(2(n+1)+\varepsilon)}(\mathscr{F}_\varepsilon B_{n+1})(x), \quad n\geqslant 0. \tag{2.27}$$

where $\varepsilon \neq -2(n+1)$, $n\geqslant 0$. Consequently, relations (2.8)–(2.9) become

$$P_n(x) = P_n^{[1]}(x;\mathscr{F}_{-1}), \quad n\geqslant 0,$$

$$R_n(x) = R_n^{[1]}(x;\mathscr{F}_{1}), \quad n\geqslant 0.$$

Definition 3. A MPS $\{B_n\}_{n\geqslant 0}$ is called an $\mathscr{O}$-Appell sequence with respect to a lowering operator $\mathscr{O}$ if $B_n(\cdot) = B_n^{[1]}(\cdot,\mathscr{O})$ for all integers $n\geqslant 0$, [3,4].

After this definition, the Theorem 1 allows us to conclude that $\{P_n\}_{n\geqslant 0}$ is $\mathscr{F}_{-1}$-Appell and $\{R_n\}_{n\geqslant 0}$ is $\mathscr{F}_1$-Appell.

Moreover, on account of relations (2.10)–(2.11) and (2.24)–(2.25) given in Proposition 2, we may say that the sequences $\{\widehat{a}_n\}_{n\geqslant 0}$ and $\{\widehat{b}_n\}_{n\geqslant 0}$ are $\mathscr{F}_1$ and $\mathscr{F}_{-1}$-Appell, respectively.

3. The $\mathscr{F}_\varepsilon$-Appell sequences

Let $\{B_n\}_{n\geqslant 0}$ be a MPS with dual sequence $\{u_n\}_{n\geqslant 0}$.

Before characterizing $\mathscr{F}_\varepsilon$-Appell sequences, we must determine the dual sequence of $\{B_n^{[1]}(\cdot;\mathscr{F}_\varepsilon)\}_{n\geqslant 0}$, denoted as $\{u_n^{[1]}(\mathscr{F}_\varepsilon)\}_{n\geqslant 0}$. For this purpose we have to know the transpose ${}^t\mathscr{F}_\varepsilon$ defined according to (1.1):

$$\begin{aligned}\langle {}^t\mathscr{F}_\varepsilon u, f\rangle &= \langle u, \mathscr{F}_\varepsilon f\rangle = \langle u, (2\mathscr{F}+\varepsilon D)f\rangle \\ &= \langle (2\,{}^t\mathscr{F} - \varepsilon D)u, f\rangle,\end{aligned}$$

therefore ${}^t\mathscr{F}_\varepsilon = (2\,{}^t\mathscr{F} - \varepsilon D)$. However, the convention on D (${}^tD = -D$) permits to write ${}^t\mathscr{F} = \mathscr{F}$, leaving out a light abuse of notation without consequence. Thus, ${}^t\mathscr{F}_\varepsilon := \mathscr{F}_{-\varepsilon}$ and $\mathscr{F}_\varepsilon$ is defined on $\mathscr{P}$ and $\mathscr{P}'$.

Now it is easy to prove

$$\mathscr{F}_\varepsilon(pf) = f(\mathscr{F}_\varepsilon p) + p(\mathscr{F}_\varepsilon f) + 4xp'f', \quad p, f \in \mathscr{P}, \tag{3.28}$$

$$\mathscr{F}_\varepsilon(pu) = p(\mathscr{F}_\varepsilon u) + (\mathscr{F}_\varepsilon p)u + 4xp'u', \quad p\in\mathscr{P},\ \ u\in\mathscr{P}'. \tag{3.29}$$

Lemma 4. *The sequence* $\{u_n^{[1]}(\mathscr{F}_\varepsilon)\}_{n\geqslant 0}$ *fulfils*

$$\mathscr{F}_{-\varepsilon}(u_n^{[1]}(\mathscr{F}_\varepsilon)) = (n+1)(2(n+1)+\varepsilon)u_{n+1}, \quad n\geqslant 0. \tag{3.30}$$

Proof. Indeed, successively we have

$$\langle u_n^{[1]}(\mathscr{F}_\varepsilon), B_m^{[1]}(x; \mathscr{F}_\varepsilon)\rangle = \delta_{n,m} \quad n, m \geqslant 0,$$

$$\langle u_n^{[1]}(\mathscr{F}_\varepsilon), \mathscr{F}_\varepsilon(B_{m+1})\rangle = (n+1)(2(n+1)+\varepsilon)\delta_{n,m} \quad n, m \geqslant 0,$$

$$\langle \mathscr{F}_{-\varepsilon}(u_n^{[1]}(\mathscr{F}_\varepsilon)), B_{m+1}\rangle = (n+1)(2(n+1)+\varepsilon)\delta_{n,m} \quad n, m \geqslant 0. \tag{3.31}$$

In particular,

$$\langle \mathscr{F}_{-\varepsilon}(u_n^{[1]}(\mathscr{F}_\varepsilon)), B_{m+1}\rangle = 0, \quad m \geqslant n+1, \quad n \geqslant 0.$$

This implies [11,12]

$$\mathscr{F}_{-\varepsilon}(u_n^{[1]}(\mathscr{F}_\varepsilon)) = \sum_{\nu=0}^{n+1} \lambda_{n,\nu} u_\nu, \quad n \geqslant 0,$$

with $\lambda_{n,\nu} = \langle \mathscr{F}_{-\varepsilon}(u_n^{[1]}(\mathscr{F}_\varepsilon)), B_\nu\rangle, 0 \leqslant \nu \leqslant n+1$. Consequently, on account of (3.31), we obtain (3.30). □

From this we obtain the result:

Proposition 5. *The MPS $\{B_n\}_{n\geqslant 0}$ is a $\mathscr{F}_\varepsilon$-Appell sequence if and only if its dual sequence $\{u_n\}_{n\geqslant 0}$ fulfils*

$$u_n = \frac{1}{n!\,2^n\left(1+\dfrac{\varepsilon}{2}\right)_n} \mathscr{F}_{-\varepsilon}^n(u_0), \quad n \geqslant 0. \tag{3.32}$$

Proof. The condition is necessary. From (3.30), the sequence $\{u_n\}_{n\geqslant 0}$ satisfies

$$\mathscr{F}_{-\varepsilon}(u_n) = (n+1)(2(n+1)+\varepsilon)u_{n+1}, \quad n \geqslant 0. \tag{3.33}$$

In particular, for $n = 0$,

$$u_1 = \frac{1}{2+\varepsilon} \mathscr{F}_{-\varepsilon} u_0.$$

By recurrence, we easily get (3.32).

The condition is sufficient. From (3.32), it is easy to see that (3.33) is fulfilled. Therefore by comparing it with (3.30), we obtain

$$\mathscr{F}_{-\varepsilon}(u_n^{[1]}(\mathscr{F}_\varepsilon)) = \mathscr{F}_{-\varepsilon} u_n, \quad n \geqslant 0.$$

The lowering operator $\mathscr{F}_\varepsilon$ satisfies $\mathscr{F}_\varepsilon(\mathscr{P}) = \mathscr{P}$, and therefore $\mathscr{F}_{-\varepsilon}$ is one-to-one on $\mathscr{P}'$. We then get $u_n^{[1]}(\mathscr{F}_\varepsilon) = u_n n \geqslant 0$, whence the expected result. □

Now, our aim is to find all the polynomial sequences that are both orthogonal and $\mathscr{F}_\varepsilon$-Appell. See [1] for considering the same problem with the operator D.

4. The $\mathscr{F}_\varepsilon$-Appell orthogonal sequences

In view of the characterisation of all the $\mathscr{F}_\varepsilon$-Appell orthogonal sequences, we present the following result.

Theorem 6. *All the $\mathscr{F}_\varepsilon$-Appell orthogonal sequences are the Laguerre sequences with parameter $\alpha = \varepsilon/2$, up to an affine transformation.*

Proof. Assume that the MOPS $\{B_n\}_{n\geqslant 0}$ is also a $\mathscr{F}_\varepsilon$-Appell sequence. Let $\{\beta_n, \gamma_{n+1}\}_{n\geqslant 0}$ be its recurrence coefficients in accordance with (1.4). From (1.2) and (3.33), we get

$$\mathscr{F}_{-\varepsilon}(B_n u_0) = \lambda_n B_{n+1} u_0, \quad n\geqslant 0, \tag{4.34}$$

with

$$\lambda_n := \lambda_n(\varepsilon) = \frac{(n+1)(2(n+1)+\varepsilon)}{\gamma_{n+1}}, \quad n\geqslant 0, \tag{4.35}$$

Remark that $\lambda_n \neq 0, n\geqslant 0$, since $\varepsilon \neq -2(n+1),\ n\geqslant 0$.

For $n = 0$ in (4.34) we get

$$\mathscr{F}_{-\varepsilon} u_0 = \lambda_0 B_1 u_0, \tag{4.36}$$

which is equivalent to

$$2(xu_0')' - \varepsilon u_0' = \lambda_0 B_1 u_0. \tag{4.37}$$

For $n = 1$ in Eq. (4.34), by taking into account (3.29) and (4.36), we get

$$4xu_0' = A(x)u_0, \tag{4.38}$$

where

$$A(x) = \lambda_1 B_2(x) - \lambda_0 B_1^2(x) - 2 + \varepsilon. \tag{4.39}$$

Differentiating both sides of (4.38) and using (4.37), we obtain

$$(A(x) - 2\varepsilon)u_0' = (2\lambda_0 B_1(x) - A'(x))u_0.$$

Further, by eliminating u_0' between (4.38) and the last equation and by taking into account the regularity of u_0, we finally get

$$(A(x) - 2\varepsilon)A(x) = 4x(2\lambda_0 B_1(x) - A'(x)). \tag{4.40}$$

On the basis of (4.39) and (4.40), it is easily seen that $\lambda_1 = \lambda_0$, which implies

$$\begin{cases} \lambda_0(\beta_0 - \beta_1)^2 = 8, \\ 4\beta_0 + \lambda_0\gamma_1(\beta_0 - \beta_1) = 0, \\ (\lambda_0\gamma_1 + 2 + \varepsilon)(\lambda_0\gamma_1 + 2 - \varepsilon) + 4\lambda_0\beta_0(\beta_0 - \beta_1) = 0. \end{cases}$$

But $\lambda_0 \gamma_1 = 2 + \varepsilon$ in view of (4.35) with $n = 0$. Whence

$$\beta_1 = \left(1 + \frac{4}{2+\varepsilon}\right)\beta_0, \quad \beta_0 = \sqrt{\frac{2}{\lambda_0}}\left(1 + \frac{\varepsilon}{2}\right),$$

and $A(x) = -2\sqrt{2\lambda_0}x + 2\varepsilon$ where the last equalities are obtained up to a reflection.

Following (4.38), we deduce that u_0 fulfils the functional differential equation

$$(\Phi u_0)' + \Psi u_0 = 0, \tag{4.41}$$

with $\Phi(x) = x$ and $\Psi(x) = \sqrt{(\lambda_0/2)}x - (1 + (\varepsilon/2))$.

Essentially (4.41) corresponds to the Laguerre functional Eq. (1.5) with $\alpha = \varepsilon/2$ and up to the affine transformation $\sqrt{(\lambda_0/2)}x$.

Remark 7. With the following definition: a MOPS $\{B_n\}_{n\geqslant 0}$ is called a $\mathscr{F}_\varepsilon$-classical sequence when $\{B_n^{[1]}(\cdot;\mathscr{F}_\varepsilon)\}_{n\geqslant 0}$ is also orthogonal (Hahn property with respect to $\mathscr{F}_\varepsilon$), the monic Laguerre sequence with parameter $\varepsilon/2$ is a $\mathscr{F}_\varepsilon$-classical sequence since $B_n^{[1]}(x;\mathscr{F}_\varepsilon) = B_n(x)$, $n \geqslant 0$, and the Laguerre form u_0 fulfilling (4.36) is a $\mathscr{F}_\varepsilon$-classical form. It is well known that the monic Hermite sequence possesses the same properties with respect to the operator D, [1].

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