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ORIGINAL PAPER

The symmetric D_ω-semi-classical orthogonal polynomials of class one

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Abstract We give the system of Laguerre–Freud equations associated with the D_ω-semi-classical functionals of class one, where D_ω is the divided difference operator. This system is solved in the symmetric case. There are essentially two canonical cases. The corresponding integral representations are given.

Keywords Discrete semi-classical orthogonal polynomials · Difference operator

Mathematics Subject Classifications (2000) 42C05 · 33C45

1 Introduction

There are many papers whose interest is semi-classical orthogonal polynomials related either to the differential operator D or to the divided difference operator D_ω [1, 2, 6, 8, 10, 12, 14]. In [6], M. Bachène established the system satisfied by the coefficients of the recurrence relation of D-semi-classical orthogonal

In memory of Professor Luigi Gatteschi.

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sequences of class one; in [7], S. Belmehdi found again the same system, but those authors did not solve it. By carrying out the complete description of the symmetric D-Laguerre–Hahn forms of class one, in [2], the authors, in particular, obtained the canonical cases of all symmetric D-semi-classical forms of class one. Analogously in [8], the authors established the system satisfied by the coefficients of the recurrence relation of D_ω-semi-classical orthogonal sequences of class one, but their method is short of luminousness.

So, the aim of this paper is threefold. First, to establish the Laguerre–Freud equations corresponding to D_ω-semi-classical orthogonal sequences of class one in a simpler manner and for the sake of completeness. Secondly to solve the system in the symmetric case. Thus we exhaustively describe the solutions which arise. Thirdly to identify the integral representation of each symmetric D_ω-classical-form of class one by taking into account its difference equation. Finally we investigate the limiting case: $\omega \to 0$. We recover again the three canonical cases put forth previously [2, 5].

The first section contains material of preliminary and some results regarding the class of D_ω-semi-classical forms. In the second section, the system of Laguerre–Freud equations is built. In the third section, we give some properties of symmetric D_ω-semi-classical forms of class s. The fourth section is devoted to the resolution of the system in the symmetric case. The canonical cases are described in the fifth section. The sixth section deals with integral representations of the so-called canonical forms found in the previous section. The last section is devoted to the limiting case $\omega = 0$.

2 Preliminaries

Let $\mathcal{P}$ be the vector space of polynomials with coefficients in $\mathbb{C}$ and let $\mathcal{P}'$ be its dual. We denote by $\langle u, f\rangle$ the effect of $u \in \mathcal{P}'$ on $f \in \mathcal{P}$. In particular, we denote by $(u)_n := \langle u, x^n\rangle$, $n \geq 0$ the moments of u. For any form u, any polynomial g, any $b \in \mathbb{C}$, $a \in \mathbb{C} - \{0\}$, let gu, $\tau_b u$, $h_a u$ be the forms defined by duality

$$\langle gu, f\rangle := \langle u, gf\rangle,\ \langle \tau_b u, f\rangle := \langle u, \tau_{-b} f\rangle,\ \langle h_a u, f\rangle := \langle u, h_a f\rangle,$$

where $(\tau_{-b} f)(x) = f(x+b)$, $(h_a f)(x) = f(ax)$.

We define $(x-c)^{-1}u$ by $\langle (x-c)^{-1}u, f\rangle := \langle u, \theta_c f\rangle$ where $(\theta_c f)(x) = \frac{f(x)-f(c)}{x-c}$, $c \in \mathbb{C}$.

We call polynomial sequence (PS), the sequence of polynomials $\{P_n\}_{n\geq 0}$ when $\deg P_n = n$, $n \geq 0$. Then any polynomial P_n can be supposed monic and the sequence becomes a monic polynomial sequence (MPS). Let $\{u_n\}_{n\geq 0}$ be its dual sequence defined by $\langle u_n, P_m\rangle = \delta_{n,m}$, $n, m \geq 0$.

The MPS $\{P_n\}_{n\geq 0}$ is orthogonal with respect to $u \in \mathcal{P}'$ when the following conditions hold

$$\langle u, P_m P_n\rangle = r_n \delta_{n,m},\ n, m \geq 0,\ r_n \neq 0,\ n \geq 0.$$

Then the monic orthogonal sequence (MOPS) $\{P_n\}_{n\geq 0}$ fulfils the standard recurrence relation

$$P_0(x) = 1 \quad , \quad P_1(x) = x - \beta_0,$$
$$P_{n+2}(x) = (x - \beta_{n+1})P_{n+1}(x) - \gamma_{n+1}P_n(x),\ n \geq 0. \tag{2.1}$$

In this case, the form u is said regular and it is necessary proportional to u_0. Let us introduce the Hahn's operator

$$(D_\omega f)(x) := \frac{f(x+\omega) - f(x)}{\omega},\ f \in \mathcal{P}\ ,\ \omega \neq 0.$$

We have $D_\omega = \dfrac{1}{\omega}(\tau_{-\omega} - I_{\mathcal{P}})$ where $I_{\mathcal{P}}$ is the identity operator in $\mathcal{P}$. The transposed ${}^tD_\omega$ of D_ω is ${}^tD_\omega = \dfrac{1}{\omega}(\tau_\omega - I_{\mathcal{P}'}) = -D_{-\omega}$. Thus, we have

$$\langle D_{-\omega}u, f\rangle = -\langle u, D_\omega f\rangle\ ,\ u \in \mathcal{P}'\ ,\ \omega \in \mathbb{C} - \{0\}.$$

When $\omega \to 0$, we meet again the derivative D.

Lemma 2.1 [1, 11] *Let $a \in \mathbb{C} - \{0\}$, $c \in \mathbb{C}$, $f \in \mathcal{P}$ and $u \in \mathcal{P}'$ we have*

$$h_a \circ D_{-\omega}u = aD_{-a\omega} \circ h_a u. \tag{2.2}$$
$$h_a\big(fu\big) = \big(h_{a^{-1}}f\big)\big(h_a u\big). \tag{2.3}$$
$$D_{-\omega}(fu) = (\tau_\omega f)(D_{-\omega}u) + (D_{-\omega}f)u. \tag{2.4}$$
$$(x-c)^{-1}\big((x-c)u\big) = u - (u)_0\delta_c. \tag{2.5}$$

Definition A MOPS $\{P_n\}_{n\geq 0}$ is called $D_{-\omega}$-semi-classical sequence if u_0 fulfils an equation

$$D_\omega(\phi u_0) + \psi u_0 = 0, \tag{2.6}$$

where ϕ, ψ are polynomials, ϕ monic and $\deg\psi \geq 1$. In this case, the form u_0 is called D_ω-semi-classical.

Let us introduce the integer $s(\phi, \psi) = \max\big(\deg\phi - 2, \deg\psi - 1\big)$. Then $s = \min s(\phi, \psi)$ where the minimum is taken over all the pairs (ϕ, ψ) occurring in (2.6) is called the class of u_0. By extension, the integer s is also the class of $\{P_n\}_{n\geq 0}$ [11].

Remark The case $s = 0$ has been handled in [1].

Lemma 2.2 [1] *Let $\{P_n\}_{n\geq 0}$ be a $D_{-\omega}$-semi-classical sequence. Then the sequence $\{\tilde{P}_n\}_{n\geq 0}$ where $\tilde{P}_n = a^{-n}(h_a \circ \tau_{-b}P_n)(x)$ is orthogonal with respect to $\tilde{u}_0 = (h_{a^{-1}} \circ \tau_{-b})u_0$. Moreover $\tilde{u}_0$ fulfils the equation*

$$D_{\omega a^{-1}}\big(\tilde{\phi}\tilde{u}_0\big) + \tilde{\psi}\tilde{u}_0 = 0,$$

where

$$\tilde{\phi}(x) = a^{-\deg\phi}\phi(ax+b) \quad , \quad \tilde{\psi}(x) = a^{1-\deg\phi}\psi(ax+b).$$

Lemma 2.3 *The functional equation* (2.6) *is equivalent to*

$$D_{-}\omega\big((\phi - \omega\psi)u_0\big) + \psi u_0 = 0. \tag{2.7}$$

Proof Indeed, from the definition of the operator D_ω the (2.6) is equivalent to

$$\frac{1}{\omega}\big(\tau_\omega(\phi u_0) - \phi u\big) + \psi u_0 = 0,$$

Applying the operator $\tau_{-\omega}$, the previous equation becomes

$$\frac{1}{-\omega}\Big(\tau_{-\omega}\big((\phi - \omega\psi)u_0\big) - \big(\phi - \omega\psi\big)u_0\Big) + \psi u_0 = 0,$$

by the definition of the operator $D_{-\omega}$ we get the desired result. □

The question is whether the integer $s(\phi, \psi)$ is the class of u_0.

Proposition 2.4 (compare with [8, 10]) *The form u_0 fulfiling* (2.6) *is of class $s = s(\phi, \psi)$ if and only if*

$$\prod_{c\in\ Z(\phi)} \Big(|\ \psi(c-\omega) + (\theta_c\phi)(c-\omega)\ | + |\ \langle u_0, \theta_{c-\omega}\big(\psi + \theta_c\phi\big)\rangle\ |\Big) > 0,$$

where $Z(\phi) := \{z \in \mathbb{C},\ \phi(z) = 0\}$. When there exists $c \in Z(\phi)$ such that

$$\psi(c-\omega) + (\theta_c\phi)(c-\omega) = 0,\ \ \langle u_0, \theta_{c-\omega}\big(\psi + \theta_c\phi\big)\rangle = 0,$$

equation (2.6) *becomes*

$$D_\omega((\theta_c\phi)u_0) + \big\{\theta_{c-\omega}(\psi + \theta_c\phi)\big\}u_0 = 0. \tag{2.8}$$

Proof The condition is necessary. Let c be a root of ϕ, we can write

$$\phi(x) = (x-c)(\theta_c\phi)(x).$$

Then on account of (2.4) Eq. (2.6) becomes

$$(x - c + \omega)D_\omega\big((\theta_c\phi)u_0\big) + (\psi + \theta_c\phi)u_0 = 0. \tag{2.9}$$

Carrying out the Euclidean division of $\psi + \theta_c\phi$ by $x - c + \omega$, we obtain

$$\psi(x) + (\theta_c\phi)(x) = (x - c + \omega)\psi_{c-\omega}(x) + r_{c-\omega},$$

where

$$\psi_{c-\omega}(x) = \big(\theta_{c-\omega}\big(\psi + \theta_c\phi\big)\big)(x)\ \ ,\ \ r_{c-\omega} = \psi(c-\omega) + (\theta_c\phi)(c-\omega).$$

It follows for (2.9)

$$(x - c + \omega)\big\{D_\omega\big((\theta_c\phi)u_0\big) + \psi_{c-\omega}u_0\big\} + r_{c-\omega}u_0 = 0,$$

therefore by virtue of (2.5)

$$D_\omega\big((\theta_c\phi)u_0\big) + \psi_{c-\omega}u_0 = \langle u_0, \theta_{c-\omega}\big(\psi + \theta_c\phi\big)\rangle\delta_{c-\omega}$$
$$-\big\{\psi(c-\omega) + (\theta_c\phi)(c-\omega)\big\}(x-c+\omega)^{-1}u_0. \quad (2.10)$$

Now suppose $\langle u_0, \theta_{c-\omega}\big(\psi + \theta_c\phi\big)\rangle = 0$ and $\psi(c-\omega) + (\theta_c\phi)(c-\omega) = 0$.

Then from (2.10) u_0 would be of class $\tilde{s} \leq \max\big(\deg(\theta_c\phi) - 2, \deg(\psi_{c-\omega}) - 1\big) = s - 1$, which is contradictory.

The condition is sufficient. Let us suppose u_0 to be of class $\tilde{s} \leq s$ with

$$D_\omega(\tilde{\phi}u_0) + \tilde{\psi}u_0 = 0.$$

Then there exists a polynomial Λ such that

$$\phi = \tilde{\phi}\Lambda \quad \textit{and} \quad \psi = \big(\tau_{-\omega}\Lambda\big)\tilde{\psi} - \big(D_\omega\Lambda\big)\tilde{\phi}$$

Indeed for obtaining ψ, we can write

$$\big(\tau_{-\omega}\Lambda\big)D_\omega\big(\tilde{\phi}u_0\big) + \big(\tau_{-\omega}\Lambda\big)\tilde{\psi}u_0 = 0,$$

and in accordance with (2.4), we have

$$D_\omega\big(\Lambda\tilde{\phi}u_0\big) + \big\{(\tau_{-\omega}\Lambda)\tilde{\psi} - \big(D_\omega\Lambda\big)\tilde{\phi}\big\}u_0 = 0,$$

hence ψ.

Now suppose $\tilde{s} < s$, consequently $\deg(\Lambda) \geq 1$ and there exists c such that $\Lambda(c) = 0$, thus $\Lambda(x) = (x-c)(\theta_c\Lambda)(x)$. This implies

$$\psi(x) + (\theta_c\phi)(x) = (x-c+\omega)\big\{\big(\tau_{-\omega}(\theta_c\Lambda)\big)(x)\tilde{\psi}(x) - \big(D_\omega(\theta_c\Lambda)\big)(x)\tilde{\phi}(x)\big\},$$

therefore $r_{c-\omega} = 0$ and

$$\langle u_0, \psi_{c-\omega}\rangle = \langle D_{-\omega}(\tilde{\phi}u_0) + \tau_\omega(\tilde{\psi}u_0), \theta_c\Lambda\rangle = \langle D_{-\omega}\big((\tilde{\phi} - \omega\tilde{\psi})u_0\big) + \tilde{\psi}u_0, \theta_c\Lambda\rangle = 0,$$

by virtue of Lemma 2.3. It is contradictory with the assumption. So $\tilde{s} = s$. □

Remark Notice that there are misprints in the Proposition 2 given in [10], $\sigma(x) = \phi(x) - \omega\psi(x)$ must be read as $\sigma(x) = \psi(x) - (\theta_c\phi)(x)$.

3 The Laguerre–Freud equations

In the sequel we assume that $\{P_n\}$ is $D_{-\omega}$-semi-classical sequence of class one, we have

$$P_0(x) = 1 \quad , \quad P_1(x) = x - \beta_0,$$
$$P_{n+2}(x) = (x - \beta_{n+1})P_{n+1}(x) - \gamma_{n+1}P_n(x) \quad , \quad n \geq 0. \quad (3.1)$$
$$D_\omega(\phi u_0) + \psi u_0 = 0, \quad (3.2)$$

with

$$\phi(x) = b_3x^3 + b_2x^2 + b_1x + b_0 \ , \ \ \psi(x) = a_2x^2 + a_1x + a_0. \tag{3.3}$$

Let

$$I_{n,k}(\omega) = \langle u_0, x^k P_n(x) P_n(x-\omega)\rangle, \ \ 0 \le k \le 3, \tag{3.4}$$

$$J_{n,k}(\omega) = \langle u_0, x^k P_{n+1}(x) P_n(x-\omega)\rangle, \ \ 0 \le k \le 3, \tag{3.5}$$

$$K_{n,k}(\omega) = \langle u_0, x^k P_n(x) P_{n+1}(x-\omega)\rangle, \ \ 0 \le k \le 3. \tag{3.6}$$

Lemma 3.1 *For $n \ge 0$, we have the following results:*

$$\begin{aligned} & b_3\big(I_{n,3}(\omega) - I_{n,3}(-\omega)\big) + b_2\big(I_{n,2}(\omega) - I_{n,2}(-\omega)\big) \\ & \quad + b_1\big(I_{n,1}(\omega) - I_{n,1}(-\omega)\big) + b_0\big(I_{n,0}(\omega) - I_{n,0}(-\omega)\big) \\ & \quad + \omega a_2 I_{n,2}(-\omega) + \omega a_1 I_{n,1}(-\omega) + \omega a_0 I_{n,0}(-\omega) = 0. \end{aligned} \tag{3.7}$$

$$\begin{aligned} & b_3\big(K_{n,3}(\omega) - K_{n,3}(-\omega) + J_{n,3}(\omega) - J_{n,3}(-\omega)\big) + b_2\big(K_{n,2}(\omega) - K_{n,2}(-\omega)\big) \\ & \quad + b_2\big(J_{n,2}(\omega) - J_{n,2}(-\omega)\big) + b_1\big(K_{n,1}(\omega) - K_{n,1}(-\omega) \\ & \quad + J_{n,1}(\omega) - J_{n,1}(-\omega)\big) + b_0\big(K_{n,0}(\omega) - K_{n,0}(-\omega) \\ & \quad + J_{n,0}(\omega) - J_{n,0}(-\omega)\big) + \omega a_2\big(J_{n,2}(-\omega) + K_{n,2}(-\omega)\big) \\ & \quad + \omega a_1\big(J_{n,1}(-\omega) + K_{n,1}(-\omega)\big) + \omega a_0\big(J_{n,0}(-\omega) + K_{n,0}(-\omega)\big) = 0. \end{aligned} \tag{3.8}$$

Proof By (3.2), we get $\langle D_\omega(\phi u_0) + \psi u_0, P_n(x)P_n(x+\omega)\rangle = 0, \ n \ge 0$ it is equivalent to

$$\langle \phi u_0, P_n(x)P_n(x-\omega)\rangle + \langle (\omega\psi - \phi)u_0, P_n(x)P_n(x+\omega)\rangle = 0, \ n \ge 0,$$

then we can deduce (3.7).

We have $\langle D_\omega(\phi u_0) + \psi u_0, P_n(x)P_{n+1}(x+\omega) + P_n(x+\omega)P_{n+1}(x)\rangle = 0, \ n \ge 0,$ then

$$\begin{aligned} & \langle \phi u_0, P_n(x-\omega)P_{n+1}(x)\rangle + \langle (\omega\psi - \phi)u_0, P_n(x)P_{n+1}(x+\omega)\rangle \\ & \quad + \langle \phi u_0, P_n(x)P_{n+1}(x-\omega)\rangle + \langle (\omega\psi - \phi)u_0, P_{n+1}(x)P_n(x+\omega)\rangle = 0, \ n \ge 0, \end{aligned}$$

it follows (3.8). □

In order to determine $\{I_{n,k}(\omega)\}_{n\geq 0}$, $\{J_{n,k}(\omega)\}_{n\geq 0}$, $\{K_{n,k}(\omega)\}_{n\geq 0}$, $0 \leq k \leq 3$, we need the following results:

Lemma 3.2 *We have the following formulas:*

$$(u_0)_1 = \beta_0.$$

$$(u_0)_2 = \gamma_1 + \beta_0^2.$$

$$(u_0)_3 = \beta_0^3 + (2\beta_0 + \beta_1)\gamma_1.$$

Lemma 3.3 *Let* $\{a_n\}_{n\geq 0}$ *with* $a_n \neq 0$, $n \geq 0$, $\{b_n\}_{n\geq 0}$ *two sequences and* $\{x_n\}_{n\geq 0}$ *the sequence satisfying the recurrence relation:*

$$x_{n+1} = a_n x_n + b_n \ , \ \ n \geq 0 \ , \ \ x_0 = a \in \mathbb{C} - \{0\}.$$

We have

$$x_{n+1} = \prod_{k=0}^{n} a_k \left\{ a + \sum_{k=0}^{n} \left(\prod_{\mu=0}^{k} a_\mu \right)^{-1} b_k \right\}, \ \ n \geq 0.$$

Lemma 3.4 *Let* $\{a_n\}_{n\geq 0}$ *be a sequence we have the following formulas:*

$$\sum_{\nu=0}^{n} \sum_{\mu=0}^{\nu} a_\mu = (n+1) \sum_{\nu=0}^{n} a_\nu - \sum_{\nu=0}^{n} \nu a_\nu, \ n \geq 0. \tag{3.9}$$

$$\sum_{\nu=0}^{n} \nu \sum_{\mu=0}^{\nu} a_\mu = \frac{1}{2} n(n+1) \sum_{\nu=0}^{n} a_\nu - \frac{1}{2} \sum_{\nu=0}^{n} \nu(\nu-1) a_\nu, \ n \geq 0. \tag{3.10}$$

Lemma 3.5 *We have*

$$I_{n,0}(\omega) = \langle u_0, P_n^2 \rangle, \ n \geq 0. \tag{3.11}$$

$$I_{n,1}(\omega) = (\beta_n - n\omega) \langle u_0, P_n^2 \rangle, \ n \geq 0. \tag{3.12}$$

$$I_{0,2}(\omega) = \gamma_1 + \beta_0^2. \tag{3.13}$$

$$I_{n,2}(\omega) = \left\{ \gamma_n + \gamma_{n+1} + \beta_n^2 - n\omega\beta_n - \omega \sum_{\nu=0}^{n-1} \beta_\nu + \frac{1}{2} n(n-1)\omega^2 \right\} \langle u_0, P_n^2 \rangle,$$

$$n \geq 1. \tag{3.14}$$

$$I_{0,3}(\omega) = \big(2\beta_0 + \beta_1\big)\gamma_1 + \beta_0^3. \tag{3.15}$$

$$I_{n,3}(\omega) = \Bigg\{-2\omega\sum_{\nu=0}^{n-1}\gamma_{\nu+1} + \big(\beta_{n-1}+2\beta_n-(n-2)\omega\big)\gamma_n + \big(2\beta_n+\beta_{n+1}-n\omega\big)\gamma_{n+1} - \omega\sum_{\nu=0}^{n-1}\beta_\nu^2 + (n-1)\omega^2\sum_{\nu=0}^{n-1}\beta_\nu - \omega\beta_n\sum_{\nu=0}^{n}\beta_\nu + \beta_n^3 - (n-1)\omega\beta_n^2 + \frac{1}{2}n(n-1)\omega^2\beta_n - \frac{1}{6}n(n-1)(n-2)\omega^3\Bigg\}\langle u_0, P_n^2\rangle,\ n\geq 1. \tag{3.16}$$

$$J_{n,0}(\omega) = 0,\ n\geq 0. \tag{3.17}$$

$$J_{n,1}(\omega) = \gamma_{n+1}\langle u_0, P_n^2\rangle,\ n\geq 0. \tag{3.18}$$

$$J_{n,2}(\omega) = (\beta_n+\beta_{n+1}-n\omega)\,\gamma_{n+1}\langle u_0, P_n^2\rangle,\ n\geq 0. \tag{3.19}$$

$$J_{0,3}(\omega) = \gamma_1\left\{\gamma_1+\gamma_2+\beta_0^2+\beta_0\beta_1+\beta_1^2\right\}. \tag{3.20}$$

$$J_{n,3}(\omega) = \Bigg\{\gamma_n+\gamma_{n+1}+\gamma_{n+2}+\beta_n(\beta_n-n\omega)+\beta_{n+1}(\beta_n+\beta_{n+1}-n\omega) - \omega\sum_{\nu=0}^{n-1}\beta_\nu + \frac{1}{2}n(n-1)\omega^2\Bigg\}\gamma_{n+1}\langle u_0, P_n^2\rangle,\ n\geq 1. \tag{3.21}$$

$$K_{n,0}(\omega) = -(n+1)\omega\langle u_0, P_n^2\rangle,\ n\geq 0. \tag{3.22}$$

$$K_{n,1}(\omega) = \left\{\gamma_{n+1} - \omega\sum_{\nu=0}^{n}\beta_\nu + \frac{1}{2}n(n+1)\omega^2\right\}\langle u_0, P_n^2\rangle,\ n\geq 0. \tag{3.23}$$

$$K_{0,2}(\omega) = (\beta_0+\beta_1-\omega)\gamma_1 - \omega\beta_0^2, \tag{3.24}$$

$$K_{n,2}(\omega) = \Bigg\{-2\omega\sum_{\nu=0}^{n-1}\gamma_{\nu+1} + \big(\beta_n+\beta_{n+1}-(n+1)\omega\big)\gamma_{n+1} - \omega\sum_{\nu=0}^{n}\beta_\nu^2 + n\omega^2\sum_{\nu=0}^{n}\beta_\nu - \frac{1}{6}n(n^2-1)\omega^3\Bigg\}\langle u_0, P_n^2\rangle,\ n\geq 1. \tag{3.25}$$

$$K_{0,3}(\omega) - K_{0,3}(-\omega) = -2\omega\big(\beta_0^3 + (2\beta_0+\beta_1)\gamma_1\big), \tag{3.26}$$

$$K_{n,3}(\omega) - K_{n,3}(-\omega) = -2\omega\Bigg\{\gamma_{n+1}\left((n+1)(\beta_n+\beta_{n+1}) + \sum_{\nu=0}^{n}\beta_\nu\right) + \sum_{\nu=0}^{n}\beta_\nu^3 + 3\sum_{\nu=0}^{n-1}\gamma_{\nu+1}(\beta_\nu+\beta_{\nu+1}) + \frac{1}{2}n(n-1)\omega^2\sum_{\nu=0}^{n}\beta_\nu\Bigg\}\langle u_0, P_n^2\rangle,\ n\geq 1. \tag{3.27}$$

Proof From the orthogonality of $\{P_n\}_{n\geq 0}$, we can deduce (3.11), (3.17) and (3.18). We have $I_{0,1}(\omega) = (u_0)_1$ by the Lemma 3.2, we get

$$I_{0,1}(\omega) = \beta_0. \tag{3.28}$$

For $n \geq 0$, by (3.1) we have

$$I_{n+1,1}(\omega) = \langle u_0, \{P_{n+2}(x) + \beta_{n+1}P_{n+1}(x) + \gamma_{n+1}P_n(x)\}P_{n+1}(x-\omega)\rangle\,,$$

taking the orthogonality of $\{P_n\}_{n\geq 0}$ into account, we can deduce that

$$I_{n+1,1}(\omega) = \beta_{n+1}\left\langle u_0, P_{n+1}^2\right\rangle + \gamma_{n+1}K_{n,0}(\omega),\ n \geq 0. \tag{3.29}$$

By (3.1) and the orthogonality of $\{P_n\}_{n\geq 0}$, we have

$$K_{n,0}(\omega) = I_{n,1}(\omega) - (\omega + \beta_n)\left\langle u_0, P_n^2\right\rangle,\ n \geq 0. \tag{3.30}$$

By virtue of the last relation, (3.29) can be written

$$I_{n+1,1}(\omega) = \gamma_{n+1}I_{n,1}(\omega) + \left(\beta_{n+1} - \beta_n - \omega\right)\left\langle u_0, P_{n+1}^2\right\rangle,\ n \geq 0,$$

consequently from the Lemma 3.3 and (3.28), we can deduce (3.12).
We remark that by (3.12), (3.30) give (3.22).
We have $I_{0,2}(\omega) = \left\langle u_0, x^2\right\rangle = (u_0)_2$, from the Lemma 3.3, we get (3.13).
For $n \geq 0$, by (3.1), we can write

$$I_{n+1,2}(\omega) = \langle u_0, x\{P_{n+2}(x) + \beta_{n+1}P_{n+1}(x) + \gamma_{n+1}P_n(x)\}P_{n+1}(x-\omega)\rangle\,,$$

by the orthogonality of $\{P_n\}_{n\geq 0}$, we obtain

$$I_{n+1,2}(\omega) = \left\langle u_0, P_{n+2}^2\right\rangle + \beta_{n+1}I_{n+1,1}(\omega) + \gamma_{n+1}K_{n,1}(\omega),\ n \geq 0. \tag{3.31}$$

We can write $K_{0,1}(\omega) = (u_0)_2 - (\omega + \beta_0)(u_0)_1$, by the Lemma 3.3, we obtain

$$K_{0,1}(\omega) = \gamma_1 - \omega\beta_0. \tag{3.32}$$

Making $n = 0$ in (3.31), by (3.12) and (3.32), it follows that

$$I_{1,2}(\omega) = \gamma_1\left\{\gamma_1 + \gamma_2 + \beta_1^2 - \omega(\beta_0 + \beta_1)\right\}. \tag{3.33}$$

When $n \geq 1$, by (3.1) and the orthogonality of $\{P_n\}_{n\geq 0}$, we get

$$K_{n,1}(\omega) = I_{n,2}(\omega) - (\omega + \beta_n)I_{n,1}(\omega) - \gamma_n\left\langle u_0, P_n^2\right\rangle,$$

by (3.12), we can deduce that

$$K_{n,1}(\omega) = I_{n,2}(\omega) + \left\{(n\omega - \beta_n)(\omega + \beta_n) - \gamma_n\right\}\left\langle u_0, P_n^2\right\rangle,\ n \geq 1. \tag{3.34}$$

By virtue of (3.34), (3.31) becomes

$$I_{n+1,2}(\omega) = \gamma_{n+1}I_{n,2}(\omega) + \left\{\beta_{n+1}^2 - (n+1)\omega\beta_{n+1} - \beta_n^2 + (n-1)\omega\beta_n + n\omega^2 + \gamma_{n+2} - \gamma_n\right\}\left\langle u_0, P_{n+1}^2\right\rangle,\ n \geq 1,$$

by (3.33) and the Lemma 3.3, we can deduce (3.14).
We can remark that by the relation (3.14), (3.34) and (3.32) give (3.23).
We have $I_{0,3}(\omega) = \left\langle u_0, x^3\right\rangle = (u_0)_3$, from the Lemma 3.3, we get (3.15).

When $n \geq 0$, from (3.1) we have

$$I_{n+1,3}(\omega) = \left\langle u_0, x^2\{P_{n+2}(x) + \beta_{n+1}P_{n+1}(x) + \gamma_{n+1}P_n(x)\}P_{n+1}(x-\omega)\right\rangle,$$

taking the orthogonality of $\{P_n\}_{n\geq 0}$ into account, we can deduce that

$$I_{n+1,3}(\omega) = J_{n+1,2}(\omega) + \beta_{n+1}I_{n+1,2}(\omega) + \gamma_{n+1}K_{n,2}(\omega),\ n \geq 0. \tag{3.35}$$

By (3.1), $J_{n,2}(\omega) = \beta_{n+1}\left\langle u_0,\, P_{n+1}^2\right\rangle + \gamma_{n+1}I_{n,1}(\omega),\ n \geq 0$, by (3.12) we get (3.19). On the other hand $K_{0,2}(\omega) = (u_0)_3 - (\omega + \beta_0)(u_0)_2$, by the Lemma 3.3 we get (3.24).

Making $n = 0$ in (3.35) and taking (3.14), (3.19), (3.24) into account, we obtain

$$I_{1,3}(\omega) = \gamma_1\left\{(\beta_0 + 2\beta_1 - \omega)\gamma_1 + (2\beta_1 + \beta_2 - \omega)\gamma_2 - \omega\beta_0\beta_1 - \omega\beta_0^2 - \omega\beta_1^2 + \beta_1^3\right\}. \tag{3.36}$$

From (3.1) and the orthogonality of $\{P_n\}_{n\geq 0}$, it follows that

$$K_{n,2}(\omega) = I_{n,3}(\omega) - (\omega + \beta_n)I_{n,2}(\omega) - \gamma_n J_{n-1,2}(\omega),\ n \geq 1. \tag{3.37}$$

On account of (3.14), (3.19), (3.35), (3.37) can be written

$$\begin{aligned} I_{n+1,3}(\omega) = \gamma_{n+1}I_{n,3}(\omega) &+ \left\langle u_0,\, P_{n+1}^2\right\rangle \Big\{\gamma_{n+2}\big(2\beta_{n+1} + \beta_{n+2} - (n+1)\omega\big) \\ &- \gamma_{n+1}\big(2\beta_n + \beta_{n+1} - n\omega\big) \\ &+ \gamma_{n+1}\big(\beta_n + 2\beta_{n+1} - (n+1)\omega\big) \\ &- \gamma_n\big(\beta_{n-1} + 2\beta_n - n\omega\big) - 2\omega\gamma_n + \beta_{n+1}^3 - \beta_n^3 \\ &- (n+1)\omega\beta_{n+1}^2 + n\omega\beta_n^2 - \omega\beta_n^2 + n\omega^2\beta_n \\ &+ \frac{1}{2}n(n+1)\omega^2\beta_{n+1} - \frac{1}{2}n(n-1)\omega^2\beta_n \\ &- \omega\beta_{n+1}\sum_{\nu=0}^{n}\beta_\nu + \omega\beta_n\sum_{\nu=0}^{n-1}\beta_\nu + \omega^2\sum_{\nu=0}^{n-1}\beta_\nu \\ &- \frac{1}{2}n(n-1)\omega^3\Big\},\ n \geq 1, \end{aligned}$$

taking the Lemma 3.3 into account, we can deduce that

$$
\begin{aligned}
I_{n+1,3}(\omega) = \langle u_0, P_{n+1}^2 \rangle \\
\times \Bigg\{ & \gamma_{n+1}\big(\beta_n + 2\beta_{n+1} - (n+1)\omega\big) + \gamma_{n+2}\big(2\beta_{n+1} + \beta_{n+2} - (n+1)\omega\big) \\
& - 2\omega \sum_{\nu=0}^{n} \gamma_\nu + \beta_{n+1}^3 - (n+1)\omega\beta_{n+1}^2 - \omega \sum_{\nu=0}^{n} \beta_\nu^2 \\
& + \frac{1}{2} n(n+1)\omega^2 \beta_{n+1} + \omega^2 \sum_{\nu=0}^{n} \nu\beta_\nu - \omega\beta_{n+1} \sum_{\nu=0}^{n} \beta_\nu \\
& + \omega^2 \sum_{\nu=0}^{n-1} \sum_{\mu=0}^{\nu} \beta_\mu - \frac{1}{6} n(n^2-1)\omega^3 \Bigg\}, \; n \geq 1,
\end{aligned}
$$

by virtue of (3.9) and (3.36), we get (3.16).

By (3.14), (3.18) and (3.16) the relation (3.37) give (3.25).

Let $n \geq 0$, by (3.1) and the orthogonality of $\{P_n\}_{n\geq 0}$, we get

$$
J_{n,3}(\omega) = \langle u_0, P_{n+2}^2 \rangle + \beta_{n+1} J_{n,2}(\omega) + \gamma_{n+1} I_{n,2}(\omega)
$$

by virtue of (3.19), (3.13) and (3.14), we obtain (3.20) and (3.21).

We have $K_{0,3}(\omega) - K_{0,3}(-\omega) = -2\omega(u_0)_3$, from the Lemma 3.3 , we obtain (3.26).

When $n \geq 0$ by (3.1) and the orthogonality of $\{P_n\}_{n\geq 0}$ we can write

$$
K_{n+1,3}(\omega) = I_{n+2,2}(\omega) + \beta_{n+1} K_{n+1,2}(\omega) + \gamma_{n+1} \langle u_0, x^2 P_n(x) P_{n+2}(x-\omega) \rangle, \; n \geq 0, \tag{3.38}
$$

but, by (3.1) and the orthogonality of $\{P_n\}_{n\geq 0}$, we get

$$
\langle u_0, x^2 P_n(x) P_{n+2}(x-\omega) \rangle = K_{n,3}(\omega) - (\omega + \beta_{n+1}) K_{n,2}(\omega) - \gamma_{n+1} I_{n,2}(\omega), \; n \geq 0,
$$

then (3.38) becomes

$$
\begin{aligned}
K_{n+1,3}(\omega) = \gamma_{n+1} K_{n,3}(\omega) + I_{n+2,2}(\omega) - \gamma_{n+1}^2 I_{n,2}(\omega) + \beta_{n+1} K_{n+1,2}(\omega) \\
-(\omega + \beta_{n+1})\gamma_{n+1} K_{n,2}(\omega), \; n \geq 0,
\end{aligned} \tag{3.39}
$$

on account of (3.13), (3.14), (3.24–3.26) and (3.39) it follows that

$$
K_{1,3}(\omega) - K_{1,3}(-\omega) = -2\omega\gamma_1\{\beta_0^3 + \beta_1^3 + 3\gamma_1(\beta_0 + \beta_1) + \gamma_2(\beta_0 + 3\beta_1 + 2\beta_2)\},
$$

and

$$K_{n+1,3}(\omega) - K_{n+1,3}(-\omega) = \gamma_{n+1}(K_{n,3}(\omega) - K_{n,3}(-\omega))$$
$$+ 2\omega\left\langle u_0, P_{n+1}^2\right\rangle\left\{-\gamma_{n+2}\left\{(n+3)\beta_{n+1} + (n+2)\beta_{n+2} + \sum_{\nu=0}^{n}\beta_\nu\right\}\right.$$
$$+ \gamma_{n+1}\left\{(n+2)\beta_n + (n+1)\beta_{n+1} + \sum_{\nu=0}^{n-1}\beta_\nu\right\}$$
$$- 3\gamma_{n+1}(\beta_n + \beta_{n+1}) - \beta_{n+1}^3 - \frac{1}{2}\omega^2 n(n+1)\beta_{n+1}$$
$$\left. - n\omega^2\sum_{\nu=0}^{n}\beta_\nu\right\},\ n \geq 1,$$

on account of the Lemma 3.3 and (3.26), we obtain

$$K_{n+1,3}(\omega) - K_{n+1,3}(-\omega)$$
$$= 2\omega\left\langle u_0, P_{n+1}^2\right\rangle \times \left\{-\gamma_{n+2}\left\{(n+3)\beta_{n+1} + (n+2)\beta_{n+2} + \sum_{\nu=0}^{n+1}\beta_\nu\right\}\right.$$
$$-3\sum_{\nu=1}^{n+1}\gamma_\nu(\beta_{\nu-1} + \beta_\nu) - \frac{1}{2}\omega^2\sum_{\nu=0}^{n+1}\nu(\nu-1)\beta_\nu$$
$$\left. -\omega^2\sum_{\nu=0}^{n}\nu\sum_{\mu=0}^{\nu}\beta_\mu - \sum_{\nu=0}^{n+1}\beta_\nu^3\right\},\ n \geq 0,$$

by (3.10), we obtain (3.27). □

Proposition 3.6 *We have the following system*

$$a_2\gamma_1 = -\psi(\beta_0). \tag{3.40}$$

$$\big(a_2 - 2nb_3\big)(\gamma_n + \gamma_{n+1}) - 4b_3\sum_{\nu=0}^{n-2}\gamma_{\nu+1} = -\psi(\beta_n) + \sum_{\nu=0}^{n-1}\big(\theta_{\beta_n}(2\phi - \omega\psi)\big)(\beta_\nu)$$
$$+ \frac{1}{3}n(n-1)(n-2)\omega^2 b_3 - \frac{1}{2}n(n-1)\omega^2 a_2,\ \ n \geq 1, \tag{3.41}$$

with $\sum_{\nu=0}^{-1} = 0$.

$$\left\{2(a_2 - b_3)\beta_1 - 2(2b_3 - a_2)\beta_0 - 2b_2 + \omega a_2 + 2a_1\right\}\gamma_1 = (2\phi - \omega\psi)(\beta_0). \tag{3.42}$$

$$\begin{aligned}\Xi_n\gamma_{n+1} &- 2(2b_2 - \omega a_2)\sum_{\nu=0}^{n-1}\gamma_{\nu+1} - 6b_3\sum_{\nu=0}^{n-1}\gamma_{\nu+1}(\beta_\nu + \beta_{\nu+1})\\ &= \sum_{\nu=0}^{n}(2\phi - \omega\psi)(\beta_\nu) + \omega^2 n\big((n-1)b_3 - a_2\big)\sum_{\nu=0}^{n}\beta_\nu\\ &\quad - \frac{1}{2}n(n+1)\omega^2 a_1 + \frac{1}{6}n(n^2-1)\omega^2(2b_2 - \omega a_2),\ \ n \geq 1,\end{aligned} \tag{3.43}$$

where

$$\begin{aligned}\Xi_n &= 2\big(a_2 - (2n+1)b_3\big)\beta_{n+1} + 2\big(a_2 - 2nb_3\big)\beta_n - 4b_3\sum_{\nu=0}^{n}\beta_\nu\\ &\quad -(2n+1)(2b_2 - \omega a_2) + 2a_1,\ n \geq 1.\end{aligned} \tag{3.44}$$

Proof By virtue of the previous lemma, we can deduce that

$$I_{n,0}(\omega) - I_{n,0}(-\omega) = 0,\ n \geq 0. \tag{3.45}$$

$$I_{n,1}(\omega) - I_{n,1}(-\omega) = -2n\omega\left\langle u_0,\, P_n^2\right\rangle,\ n \geq 0. \tag{3.46}$$

$$I_{0,2}(\omega) - I_{0,2}(-\omega) = 0. \tag{3.47}$$

$$I_{n,2}(\omega) - I_{n,2}(-\omega) = -2\omega\left(n\beta_n + \sum_{\nu=0}^{n-1}\beta_\nu\right)\left\langle u_0,\, P_n^2\right\rangle,\ n \geq 1. \tag{3.48}$$

$$I_{0,3}(\omega) - I_{0,3}(-\omega) = 0, \tag{3.49}$$

$$\begin{aligned}I_{n,3}(\omega) - I_{n,3}(-\omega) &= -2\omega\left\{2\sum_{\nu=0}^{n-1}\gamma_{\nu+1} + (n-2)\gamma_n + n\gamma_{n+1} + \sum_{\nu=0}^{n-1}\beta_\nu^2 + \beta_n\sum_{\nu=0}^{n}\beta_\nu\right.\\ &\qquad\left. + (n-1)\beta_n^2 + \frac{1}{6}n(n-1)(n-2)\omega^2\right\}\\ &\qquad \times\left\langle u_0,\, P_n^2\right\rangle,\ n \geq 1.\end{aligned} \tag{3.50}$$

$$J_{n,0}(\omega) - J_{n,0}(-\omega) = 0,\ n \geq 0. \tag{3.51}$$

$$J_{n,1}(\omega) - J_{n,1}(-\omega) = 0,\ n \geq 0. \tag{3.52}$$

$$J_{n,2}(\omega) - J_{n,2}(-\omega) = -2n\omega\gamma_{n+1}\left\langle u_0,\, P_n^2\right\rangle,\ n \geq 0, \tag{3.53}$$

$$J_{0,3}(\omega) - J_{0,3}(-\omega) = 0, \tag{3.54}$$

$$J_{n,3}(\omega) - J_{n,3}(-\omega) = -2\omega \left\{ n\beta_n + n\beta_{n+1} + \sum_{\nu=0}^{n-1} \beta_\nu \right\} \gamma_{n+1} \left\langle u_0,\, P_n^2 \right\rangle, n \geq 1. \tag{3.55}$$

$$K_{n,0}(\omega) - K_{n,0}(-\omega) = -2(n+1)\omega \left\langle u_0,\, P_n^2 \right\rangle,\ n \geq 0. \tag{3.56}$$

$$K_{n,1}(\omega) - K_{n,1}(-\omega) = -2\omega \left\langle u_0,\, P_n^2 \right\rangle \sum_{\nu=0}^{n} \beta_\nu,\ n \geq 0. \tag{3.57}$$

$$K_{0,2}(\omega) - K_{0,2}(-\omega) = -2\omega(\beta_0^2 + \gamma_1). \tag{3.58}$$

$$K_{n,2}(\omega) - K_{n,2}(-\omega) = -2\omega \left\langle u_0,\, P_n^2 \right\rangle \left\{ 2\sum_{\nu=0}^{n-1} \gamma_{\nu+1} + (n+1)\gamma_{n+1} + \sum_{\nu=0}^{n} \beta_\nu^2 + \frac{1}{6} n\left(n^2 - 1\right)\omega^2 \right\},\ n \geq 1. \tag{3.59}$$

Making $n = 0$ in (3.7) and taking the relations (3.11)–(3.13), (3.45)–(3.47), (3.49) into account, we can deduce (3.40).

Let $n \geq 1$, by virtue of the relations (3.11), (3.12), (3.14), (3.45), (3.46), (3.48) and (3.50), the equation (3.7) becomes

$$\begin{aligned}
&-2b_3 \left\{ 2\sum_{\nu=0}^{n-1} \gamma_{\nu+1} + (n-2)\gamma_n + n\gamma_{n+1} + \sum_{\nu=0}^{n-1} \beta_\nu^2 + \beta_n \sum_{\nu=0}^{n} \beta_\nu + (n-1)\beta_n^2 \right. \\
&\qquad \left. + \frac{1}{6} n(n-1)(n-2)\omega^2 \right\} \\
&-2b_2 \left(n\beta_n + \sum_{\nu=0}^{n-1} \beta_\nu \right) - 2nb_1 + a_2 \left\{ \gamma_n + \gamma_{n+1} + \beta_n^2 + n\omega\beta_n + \omega \sum_{\nu=0}^{n-1} \beta_\nu \right. \\
&\qquad \left. + \frac{1}{2} n(n-1)\omega^2 \right\} + a_1(\beta_n + n\omega) + a_0 = 0,
\end{aligned}$$

it is equivalent to

$$\begin{aligned}
&\left(a_2 - 2nb_3\right)(\gamma_n + \gamma_{n+1}) - 4b_3 \sum_{\nu=0}^{n-2} \gamma_{\nu+1} = -a_2\beta_n^2 - a_1\beta_n - a_0 \\
&\quad + \sum_{\nu=0}^{n-1} \left\{ 2b_3\left(\beta_\nu^2 + \beta_\nu\beta_n + \beta_n^2\right) + (2b_2 - \omega a_2)(\beta_\nu + \beta_n) + 2b_1 - \omega a_1 \right\} \\
&\quad \times \frac{1}{3} n(n-1)(n-2)\omega^2 b_3 - \frac{1}{2} n(n-1) a_2 \omega^2,
\end{aligned}$$

but

$$\left(\theta_{\beta_n}(2\phi-\omega\psi)\right)(\beta_\nu)=2b_3\left(\beta_\nu^2+\beta_\nu\beta_n+\beta_n^2\right)+(2b_2-\omega a_2)(\beta_\nu+\beta_n)$$
$$+2b_1-\omega a_1,$$

then we can deduce (3.41).

Let $n=0$ in (3.8), by virtue of (3.17)–(3.19), (3.22)–(3.24), (3.26), (3.51)–(3.58), we get (3.42).

When $n\geq 1$, on account of (3.17)–(3.19), (3.22), (3.23), (3.25), (3.27), (3.51)–(3.53), (3.55)–(3.57) and (3.59), (3.8) becomes

$$-2b_3\left\{(2n+1)(\beta_n+\beta_{n+1})\gamma_{n+1}+2\gamma_{n+1}\sum_{\nu=0}^{n}\beta_\nu-\gamma_{n+1}\beta_n+\sum_{\nu=0}^{n}\beta_\nu^3\right.$$
$$\left.+3\sum_{\nu=0}^{n-1}\gamma_{\nu+1}(\beta_\nu+\beta_{\nu+1}+\frac{1}{2}n(n-1)\omega^2\sum_{\nu=0}^{n}\beta_\nu\right\}$$
$$-2b_2\left\{2\sum_{\nu=0}^{n-1}\gamma_{\nu+1}+(2n+1)\gamma_{n+1}+\sum_{\nu=0}^{n}\beta_\nu^2+\frac{1}{6}n(n^2-1)\omega^2\right\}-2b_1\sum_{\nu=0}^{n}\beta_\nu$$
$$-2b_0(n+1)+a_2\left\{\left(2\beta_n+2\beta_{n+1}+(2n+1)\omega\right)\gamma_{n+1}+2\omega\sum_{\nu=0}^{n-1}\gamma_{\nu+1}+n\omega^2\sum_{\nu=0}^{n}\beta_\nu^2\right.$$
$$\left.+\frac{1}{6}n(n^2-1)\omega^3\right\}$$
$$a_1\left\{2\gamma_{n+1}+\omega\sum_{\nu=0}^{n}\beta_\nu+\frac{1}{2}n(n+1)\omega^2\right\}+\omega a_0(n+1)=0,$$

this leads to

$$\left\{2\left(a_2-(2n+1)b_3\right)\beta_{n+1}+2\left(a_2-2nb_3\right)\beta_n-4b_3\sum_{\nu=0}^{n}\beta_\nu\right.$$
$$\left.-(2n+1)(2b_2-\omega a_2)+2a_1\right\}\gamma_{n+1}-6b_3\sum_{\nu=0}^{n-1}\gamma_{\nu+1}(\beta_\nu+\beta_{\nu+1})$$
$$-2(2b_2-\omega a_2)\sum_{\nu=0}^{n-1}\gamma_{\nu+1}=\sum_{\nu=0}^{n}\left\{2b_3\beta_\nu^3+(2b_2-\omega a_2)\beta_\nu^2+(2b_1-\omega a_1)\beta_\nu\right.$$
$$\left.+2b_0-\omega a_0\right\}+n\left((n-1)b_3-a_2\right)\omega^2\sum_{\nu=0}^{n}\beta_\nu$$
$$-\frac{1}{2}n(n+1)a_1\omega^2+\frac{1}{6}n(n^2-1)(2b_2-\omega a_2),$$

then we can deduce (3.43). □

4 Some properties of the symmetric D_ω-semi-classical forms of class s

In the sequel we assume that u_0 is a symmetric D_ω-semi-classical form satisfying

$$D_\omega(\phi u_0) + \psi u_0 = 0. \tag{4.1}$$

Lemma 4.1 *The following functional equations hold:*

$$D_\omega\Big(\big(\phi(-x) - \omega\psi(-x)\big)u_0\Big) - \psi(-x)u_0 = 0, \tag{4.2}$$

$$D_\omega\Big(\big(\phi(x) + \phi(-x) - \omega\psi(-x)\big)u_0\Big) + \big(\psi(x) - \psi(-x)\big)u_0 = 0, \tag{4.3}$$

$$D_\omega\Big(\big(\phi(x) - \phi(-x) + \omega\psi(-x)\big)u_0\Big) + \big(\psi(x) + \psi(-x)\big)u_0 = 0. \tag{4.4}$$

Proof On account of the Lemma 2.3, the equation (4.1) is equivalent to

$$D_{-\omega}\big(\phi - \omega\psi)u_0\big) + \psi u_0 = 0.$$

Applying the operator h_{-1} to the previous functional equation we get

$$h_{-1} \circ D_{-\omega}\Big(\big(\phi(x) - \omega\psi(x)\big)u_0\Big) + h_{-1}\big(\psi(x)u_0\big) = 0,$$

taking (2.2) and (2.3) into account we obtain

$$D_\omega\Big(\big(\phi(-x) - \omega\psi(-x)\big)h_{-1}u_0\Big) - \psi(-x)h_{-1}u_0 = 0,$$

but u is a symmetric form then $h_{-1}u_0 = u_0$, the previous equation give (4.2).

The equation (4.3) is obtained by adding the equations (4.1) and (4.2).

Subtracting both sides of the equations (4.1) and (4.2) we obtain (4.4). □

Proposition 4.2 *Let $\widetilde{s}$ be the class of u_0 we have the following results:*

a) *If $\widetilde{s}$ is odd then $2\phi - \omega\psi$ is odd and ψ is even.*
b) *If $\widetilde{s}$ is even then $2\phi - \omega\psi$ is even and ψ is odd.*

Proof We can write

$$\phi(x) = \phi^e(x^2) + x\phi^o(x^2) \quad , \quad \psi(x) = \psi^e(x^2) + x\psi^o(x^2). \tag{4.5}$$

Let

$$\deg(\phi^e) = t_1,\ \deg(\phi^o) = t_2,\ \deg(\psi^e) = p_1,\ \deg(\psi^o) = p_2, \tag{4.6}$$

by the definition of $\widetilde{s}$ we can deduce that

$$\max(2t_1, 2t_2 + 1) \leq \widetilde{s} + 2, \quad \max(2p_1, 2p_2 + 1) \leq \widetilde{s} + 1. \tag{4.7}$$

Taking (4.5) into account, the equations (4.3) and (4.4) become respectively

$$D_\omega\big(\phi_0(x)u_0\big) + \psi_0(x)u_0 = 0, \tag{4.8}$$

$$D_\omega\big(\phi_1(x)u_0\big) + \psi_1(x)u_0 = 0, \tag{4.9}$$

where

$$\phi_0(x) = 2\phi^e(x^2) - \omega\psi^e(x^2) + \omega x\psi^o(x^2), \qquad \psi_0(x) = 2x\psi^o(x^2), \quad (4.10)$$

$$\phi_1(x) = 2x\phi^o(x^2) + \omega\psi^e(x^2) - \omega x\psi^o(x^2), \quad \psi_1(x) = 2\psi^e(x^2). \quad (4.11)$$

a) In this case we can write $\widetilde{s} = 2s + 1$, from (4.7) we get

$$t_1 \leq s+1, \ \ t_2 \leq s+1, \ \ p_1 \leq s+1, \ \ p_2 \leq s. \quad (4.12)$$

On account of (4.12), it follows

$$\deg(\phi_0) - 2 \leq 2s < \widetilde{s}, \quad \deg(\psi_0) - 1 \leq 2s < \widetilde{s}. \quad (4.13)$$

By virtue of the definition of $\widetilde{s}$, from (4.8) we can deduce that

$$\phi_0(x) = 0 \quad , \quad \psi_0(x) = 0. \quad (4.14)$$

then we get

$$2x\psi^o(x^2) = 0 \quad , \quad 2\phi^e(x^2) - \omega\psi^e(x^2) - \omega x\psi^o(x^2) = 0,$$

we can deduce that

$$\psi^o(x) = 0 \quad , \quad 2\phi^e(x) - \omega\psi^e(x) = 0. \quad (4.15)$$

By the previous relations we can deduce that $2\phi - \omega\psi$ is odd and ψ is even. Hence the desired results.

b) We have $\widetilde{s} = 2s$ from (4.7), it follows

$$t_1 \leq s+1, \ \ t_2 \leq s, \ \ p_1 \leq s, \ \ p_2 \leq s. \quad (4.16)$$

By virtue of the last relations, we obtain

$$\deg(\phi_1) - 2 \leq 2s < \widetilde{s}, \quad \deg(\psi_1) - 1 \leq 2s < \widetilde{s}. \quad (4.17)$$

Taking the definition of $\widetilde{s}$ and (4.17) and (4.9) into account, we get

$$\phi_1(x) = 0 \quad , \quad \psi_1(x) = 0,$$

which means that

$$2x\psi^e(x^2) = 0 \quad , \quad 2x\phi^o(x^2) + \omega\psi^e(x^2) - \omega x\psi^o(x^2) = 0,$$

then

$$\psi^e(x) = 0 \quad , \quad 2\phi^o(x) - \omega\psi^o(x) = 0. \quad (4.18)$$

On account of the previous relations, we can deduce the desired results.

□

5 The symmetric case when s = 1

In the sequel we assume that $\{W_n\}_{n\geq 0}$ is a symmetric $D_{-\omega}$-semi-classical orthogonal sequence of class one and $\{w_n\}_{n\geq 0}$ its dual sequence.

Then we have

$$W_0(x) = 1 \quad , \quad W_1(x) = x,$$
$$W_{n+2}(x) = xW_{n+1}(x) - \gamma_{n+1}W_n(x) \quad , \quad n \geq 0. \tag{5.1}$$

By virtue of the Proposition 4.2, it follows that

$$D_\omega(\phi w_0) + \psi w_0 = 0, \tag{5.2}$$

with

$$\phi(x) = b_3x^3 + \frac{1}{2}\omega a_2x^2 + b_1x + \frac{1}{2}\omega a_0 \ , \ \ \psi(x) = a_2x^2 + a_0. \tag{5.3}$$

In this case the system (3.40)–(3.43) becomes

$$a_2\gamma_1 = -a_0. \tag{5.4}$$

$$\big(a_2 - 2nb_3\big)(\gamma_n + \gamma_{n+1}) - 4b_3\sum_{\nu=0}^{n-2}\gamma_{\nu+1} = -a_0 + 2nb_1 + \frac{1}{3}n(n-1)(n-2)\omega^2b_3$$
$$-\frac{1}{2}n(n-1)\omega^2a_2 \ , n \geq 1. \tag{5.5}$$

Let

$$T_n = \sum_{\nu=0}^{n}\gamma_{\nu+1}, \ n \geq 0. \tag{5.6}$$

Then

$$T_n - T_{n-2} = \gamma_n + \gamma_{n+1}, \ n \geq 1 \ , \ \ T_{-1} = 0, \tag{5.7}$$

$$T_n - T_{n-1} = \gamma_{n+1}, \ n \geq 0. \tag{5.8}$$

Taking the relations (5.6), (5.7) and (5.8) into account, the system (5.4)–(5.5) becomes

$$T_0 = -\frac{a_0}{a_2}, \tag{5.9}$$

$$\big(a_2 - 2nb_3\big)(T_n - T_{n-2}) - 4b_3T_{n-2} = -a_0 + 2nb_1 - \frac{1}{2}n(n-1)\omega^2a_2$$
$$+\frac{1}{3}n(n-1)(n-2)\omega^2b_3, \ \ n \geq 1. \tag{5.10}$$

We need the following result:

Lemma 5.1 *Let*

$$S_n(m) = \sum_{\nu=1}^{n}\nu^m \ , \ \ n \geq 1 \ , \ \ m \geq 0. \tag{5.11}$$

We have

$$S_n(1) = \frac{1}{2}n(n+1)\ ,\ \ n \geq 1\ \ ;\quad S_n(2) = \frac{1}{6}n(n+1)(2n+1)\ ,\ \ n \geq 1;$$

$$S_n(3) = \frac{1}{4}n^2(n+1)^2\ ,\ \ n \geq 1.$$

Proposition 5.2 *We have*

$$T_{2n} = (n+1)\frac{a_0 - 2nb_1 + \frac{1}{6}n(4n-1)\omega^2 a_2 - \frac{2}{3}n^2(n-1)\omega^2 b_3}{4nb_3 - a_2}\ ,\ \ n \geq 0. \tag{5.12}$$

$$T_{2n+1} = (n+1)\frac{a_0 - 2(n+1)b_1 + \frac{1}{6}n(4n+5)\omega^2 a_2 - \frac{1}{3}n\big(2n^2+2n-1\big)\omega^2 b_3}{2(2n+1)b_3 - a_2},$$
$$n \geq 0. \tag{5.13}$$

Proof Making $n = 1$ in (5.10), we obtain

$$(a_2 - 2b_3)T_1 = 2b_1 - a_0. \tag{5.14}$$

The equation (5.10) can be written

$$\big(2nb_3 - a_2\big)T_n - \Big(2(n-2)b_3 - a_2\Big)T_{n-2} = a_0 - 2nb_1 + \frac{1}{2}n(n-1)\omega^2 a_2$$
$$-\frac{1}{3}n(n-1)(n-2)\omega^2 b_3\ ,\ \ n \geq 1. \tag{5.15}$$

From the previous equation, we get

$$\big(4nb_3 - a_2\big)T_{2n} - (4(n-1)b_3 - a_2)\ T_{2n-2} = a_0 - 4nb_1 + n(2n-1)\omega^2 a_2$$
$$-\frac{4}{3}n(n-1)(2n-1)\omega^2 b_3\ ,\ \ n \geq 1,$$

so we obtain

$$\big(4nb_3 - a_2\big)T_{2n} + a_2 T_0 = \sum_{\nu=1}^{n}\Big\{a_0 - 4\nu b_1 + \nu(2\nu-1)\omega^2 a_2$$
$$-\frac{4}{3}\nu(\nu-1)(2\nu-1)\omega^2 b_3\Big\}$$
$$= na_0 - 4b_1 S_n(1) + \big(2S_n(2) - S_n(1)\big)\omega^2 a_2$$
$$-\frac{4}{3}\big\{2S_n(3) - 3S_n(2) + S_n(1)\big\}\omega^2 b_3\ ,\ \ n \geq 1.$$

From the relation (5.9) we have $a_2 T_0 = -a_0$, then we get

$$\big(4nb_3 - a_2\big)T_{2n} = (n+1)a_0 - 4b_1 S_n(1) + \big(2S_n(2) - S_n(1)\big)\omega^2 a_2 - \frac{4}{3}\left\{2S_n(3) - 3S_n(2) + S_n(1)\right\}\omega^2 b_3, \ \ n \ge 1.$$

Taking the Lemma 5.1 into account, we obtain

$$T_{2n} = (n+1)\frac{a_0 - 2nb_1 + \frac{1}{6}n(4n-1)\omega^2 a_2 - \frac{2}{3}n^2(n-1)\omega^2 b_3}{4nb_3 - a_2}, \ \ n \ge 1,$$

by (5.9), we can deduce that the last relation is valid for $n = 0$. Hence (5.12). Likewise, making $n \to 2n+1$ in (5.15), we get

$$\big(2(2n+1)b_3 - a_2\big)T_{2n+1} - \big(2(2n-1)b_3 - a_2\big)T_{2n-1} = a_0 - 2(2n+1)b_1 + n(2n+1)\omega^2 a_2 - \frac{2}{3}n(4n^2-1)\omega^2 b_3\ , \ \ n \ge 1,$$

it follows that

$$\begin{aligned}
&(2(2n+1)b_3 - a_2)T_{2n+1} - \big(2b_3 - a_2\big)T_1 \\
&= \sum_{\nu=1}^{n}\left\{a_0 - 2(2\nu+1)b_1 + \nu(2\nu+1)\omega^2 a_2 - \frac{2}{3}\nu(4\nu^2-1)\omega^2 b_3\right\} \\
&= na_0 - \big(4S_n(1) + 2n\big)b_1 + \big(2S_n(2) + S_n(1)\big)\omega^2 a_2 \\
&\quad - \frac{2}{3}\big(4S_n(3) - S_n(1)\big)\omega^2 b_3\ , \ \ n \ge 1.
\end{aligned}$$

Using (5.14), the last equation can be written

$$\begin{aligned}
\big(2(2n+1)b_3 - a_2\big)T_{2n+1} = (n+1)a_0 &- \big(4S_n(1) + 2n + 2\big)b_1 \\
&+ \big(2S_n(2) + S_n(1)\big)\omega^2 a_2 \\
&- \frac{2}{3}\big(4S_n(3) - S_n(1)\big)\omega^2 b_3\ , \ \ n \ge 1.
\end{aligned}$$

On account of the Lemma 5.1, we obtain

$$T_{2n+1} = (n+1)\frac{a_0 - 2(n+1)b_1 + \frac{1}{6}n(4n+5)\omega^2 a_2 - \frac{1}{3}n\big(2n^2 + 2n - 1\big)\omega^2 b_3}{2(2n+1)b_3 - a_2}, \quad n \ge 1,$$

by (5.14), we see that the last equation is valid for $n = 0$. Hence (5.13). □

Corollary 5.3 *The sequence* $\{\gamma_{n+1}\}_{n\geq 0}$ *is defined by*

$$\gamma_{2n+1} = \frac{\Big(2(n-1)b_3 - a_2\Big)\Big(a_0 - 2nb_1 - n^2(2nb_3 - a_2)\omega^2\Big)}{\big(4nb_3 - a_2\big)\big((4n-2)b_3 - a_2\big)}, \quad n \geq 0, \qquad (5.16)$$

$$\gamma_{2n+2} = \frac{(n+1)\Big(-2a_0b_3 + 2(a_2 - 2nb_3)b_1 - n(2nb_3 - a_2)^2\omega^2\Big)}{\big(4nb_3 - a_2\big)\big((4n+2)b_3 - a_2\big)}, \quad n \geq 0. \qquad (5.17)$$

Proof From (5.6) we get $\gamma_{2n+1} = T_{2n} - T_{2n-1}$, $n \geq 0$. By the Proposition 5.2, we get

$$\begin{aligned}
&\big(4nb_3 - a_2\big)\big((4n-2)b_2 - a_2\big)\gamma_{2n+1}\\
&\quad = (n+1)\big((4n-2)b_2 - a_2\big)\Big\{a_0 - 2nb_1 + \frac{2}{3}n^2(a_2 - nb_3)\omega^2\\
&\qquad\qquad + \frac{1}{6}n\big(4nb_3 - a_2\big)\omega^2\Big\}\\
&\qquad - n\big(4nb_3 - a_2\big)\Big\{a_0 - 2nb_1 + \frac{2}{3}n^2(a_2 - nb_3)\omega^2\\
&\qquad\qquad - \frac{1}{6}\big((-8n^2 + 2n + 2)b_3 + (3n+1)a_2\big)\omega^2\Big\}\\
&\quad = \big(2(n-1)b_3 - a_2\big)\Big\{a_0 - 2nb_1 + \frac{2}{3}n^2(a_2 - nb_3)\omega^2\Big\}\\
&\qquad - \frac{1}{3}n^2\big(4nb_3 - a_2\big)\big(2(n-1)b_3 - a_2\big),\ n \geq 0,
\end{aligned}$$

then we can deduce (5.16).

On account of (5.6), we have $\gamma_{2n+2} = T_{2n+1} - T_{2n}$, $n \geq 0$, by (5.12) and (5.13) we obtain

$$\begin{aligned}
&\big(4nb_3 - a_2\big)\big(((4n+2)b_3 - a_2\big)\gamma_{2n+2}\\
&\quad = (n+1)\big(4nb_3 - a_2\big)\Big\{a_0 - 2(n+1)b_1 + \frac{1}{6}n(4n+5)\omega^2 a_2\\
&\qquad\qquad - \frac{1}{3}n\big(2n^2 + 2n - 1\big)\omega^2 b_3\Big\}\\
&\qquad - (n+1)\big((4n+2)b_3 - a_2\big)\Big\{a_0 - 2nb_1 + \frac{1}{6}n(4n-1)\omega^2 a_2\\
&\qquad\qquad - \frac{2}{3}n^2(n-1)\omega^2 b_3\Big\}
\end{aligned}$$

$$= (n+1)\Big\{ -2a_0b_3 + 2\big(a_2 - 2nb_3\big)b_1 + \frac{1}{3}n\big((8n+1)b_3 - 3a_2\big)\omega^2 a_2$$
$$- \frac{1}{3}n\big(12n^2b_3 + (-4n+1)a_2\big)\omega^2 b_3\Big\},\ n \geq 0,$$

this is

$$\big(4nb_3 - a_2\big)\big(((4n+2)b_3 - a_2\big)\gamma_{2n+2}$$
$$= (n+1)\left\{-2a_0b_3 + 2\big(a_2 - 2nb_3\big)b_1 - n\big(a_2^2 - 4na_2b_3 + 4n^2b_3^2\big)\right\},\ n \geq 0.$$

Hence (5.17). □

6 The canonical cases

Before quoting the different canonical situations, let us proceed to the general transformation see the Lemma 2.2

$$\tilde{W}_n = A^{-n}W_n(Ax)\ ,\ \ n \geq 0. \tag{6.1}$$

$$\tilde{\gamma}_{n+1} = A^{-2}\gamma_{n+1},\ n \geq 0. \tag{6.2}$$

The form $\tilde{w}_0 = h_{A^{-1}}w_0$ fulfils

$$D_{\frac{\omega}{A}}\big(A^{-\deg\phi}\phi(Ax)\tilde{w}_0\big) + A^{1-\deg\phi}\psi(Ax)\tilde{w}_0 = 0. \tag{6.3}$$

Any so-called canonical case will be denoted by $\hat{\gamma}_{n+1},\ \ \hat{w}_0$.
On account of (5.3) and the Corollary 5.3, we get the general situation

$$\begin{cases} \gamma_{2n+1} = \dfrac{\big(2(n-1)b_3 - a_2\big)\big(a_0 - 2nb_1 - n^2(2nb_3 - a_2)\omega^2\big)}{\big(4nb_3 - a_2\big)\big((4n-2)b_3 - a_2\big)},\ n \geq 0, \\ \gamma_{2n+2} = \dfrac{(n+1)\Big(-2a_0b_3 + 2\big(a_2 - 2nb_3\big)b_1 - n(2nb_3 - a_2)^2\omega^2\Big)}{\big(4nb_3 - a_2\big)\big((4n+2)b_3 - a_2\big)},\ n \geq 0, \\ D_\omega\Big(\big(b_3x^3 + \frac{1}{2}\omega a_2x^2 + b_1x + \frac{1}{2}\omega a_0\big)w_0\Big) + (a_2x^2 + a_0)w_0 = 0,\ \ (w_0)_1 = 0. \end{cases} \tag{6.4}$$

Theorem 6.1 *The following canonical cases arise:*

a) *When* $\deg\hat{\phi} = 2$*, we have*

$$\begin{cases} \hat{\gamma}_{2n+1} = (n+\alpha)(n+\beta),\ n \geq 0, \\ \hat{\gamma}_{2n+2} = (n+1)(n+\alpha+\beta),\ n \geq 0, \\ D_{-i}\Big((x+i\alpha)(x+i\beta)\hat{w}_0(\alpha,\beta)\Big) + 2i\big(x^2 - \alpha\beta\big)\hat{w}_0(\alpha,\beta) = 0, \\ \big(\hat{w}_0(\alpha,\beta)\big)_1 = 0. \end{cases} \tag{6.5}$$

b) *The case when* $\deg \hat{\phi} = 3$, *we obtain the canonical case below*

$$\begin{cases} \hat{\gamma}_{2n+1} = \dfrac{(n+\tau)(n+\mu)(n+\delta)(n+\tau+\mu+\delta-1)}{(2n+\tau+\mu+\delta-1)(2n+\tau+\mu+\delta)}, \; n \geq 0. \\ \hat{\gamma}_{2n+2} = \dfrac{(n+1)(n+\tau+\mu)(n+\tau+\delta)(n+\mu+\delta)}{(2n+\tau+\mu+\delta)(2n+\tau+\mu+\delta+1)}, \; n \geq 0. \\ D_i\big((x-i\tau)(x-i\mu)(x-i\delta))\hat{w}_0(\tau,\mu,\delta)\big) \\ \quad -2\big((\tau+\mu+\delta)x^2-\tau\mu\delta\big)\hat{w}_0(\tau,\mu,\delta)=0, \; \big(\hat{w}_0(\tau,\mu,\delta)\big)_1=0. \end{cases} \tag{6.6}$$

Proof

a) In this case (6.4) becomes

$$\begin{cases} \gamma_{2n+1} = -\omega^2\big(n^2 - 2nb_1a_2^{-1}\omega^{-2} + a_0a_2^{-1}\omega^{-2}\big), \; n \geq 0, \\ \gamma_{2n+2} = -\omega^2(n+1)\big(n - 2b_1a_2^{-1}\omega^{-2}\big), \; n \geq 0, \\ D_\omega\Big((\tfrac{1}{2}\omega a_2x^2 + b_1x + \tfrac{1}{2}\omega a_0)w_0\Big) + \Big(a_2x^2 + a_0\Big)w_0 = 0, \;\; (w_0)_1 = 0. \end{cases}$$

That is

$$\begin{cases} \gamma_{2n+1} = -\omega^2\big(n^2 - 2nb_1a_2^{-1}\omega^{-2} + a_0a_2^{-1}\omega^{-2}\big), \; n \geq 0, \\ \gamma_{2n+2} = -\omega^2(n+1)(n - 2b_1a_2^{-1}\omega^{-2}), \; n \geq 0, \\ D_\omega\big((x^2 + 2b_1a_2^{-1}\omega^{-1}x + a_0a_2^{-1})w_0\big) + 2\big(\omega^{-1}x^2 + a_0a_2^{-1}\omega^{-1}\big)w_0 = 0, \\ (w_0)_1 = 0. \end{cases}$$

With the choice $A = i\omega$ and putting $\alpha+\beta = -2b_1a_2^{-1}\omega^{-2}$, $\alpha\beta = a_0a_2^{-1}\omega^{-2}$, we get (6.5).

b) In this case (6.4) can be written

$$\begin{cases} \gamma_{2n+1} = -\omega^2 \dfrac{\big(n-1-\frac{1}{2}a_2b_3^{-1}\big)\Big(n^3 - \frac{1}{2}a_2b_3^{-1}n^2 + nb_1b_3^{-1}\omega^{-2} - \frac{1}{2}a_0b_3^{-1}\omega^{-2}\Big)}{\big(2n-1-\frac{1}{2}a_2b_3^{-1}\big)\big(2n-\frac{1}{2}a_2b_3^{-1}\big)}, \\ \qquad n \geq 0. \\ \gamma_{2n+2} = \\ \qquad -\omega^2 \dfrac{(n+1)\Big(n^3 - a_2b_3^{-1}n^2 + \big(\frac{1}{4}a_2^2b_3^{-2} + b_1b_3^{-1}\omega^{-2}\big)n + \frac{1}{2}a_0b_3^{-1}\omega^{-2} - \frac{1}{2}a_2b_1b_3^{-2}\omega^{-2}\Big)}{\big(2n-\frac{1}{2}a_2b_3^{-1}\big)\big(2n+1-\frac{1}{2}a_2b_3^{-1}\big)}, \\ \qquad n \geq 0. \\ D_\omega\Big(\big(x^3 + \frac{1}{2}\omega a_2b_3^{-1}x^2 + b_1b_3^{-1}x + \frac{1}{2}\omega a_0b_3^{-1}\big)w_0\Big) + \big(a_2b_3^{-1}x^2 + a_0b_3^{-1}\big)w_0 = 0, \\ \quad (w_0)_1 = 0. \end{cases}$$

The choice $A = -i\omega$ and putting $\tau+\mu+\delta = -\frac{1}{2}a_2b_3^{-1}$, $\tau\mu+\tau\delta+\delta\mu = b_1b_3^{-1}\omega^{-2}$ and $\tau\mu\delta = -\frac{1}{2}a_0b_3^{-1}\omega^{-2}$, we get (6.6). □

Remark

1. The form $\hat{w}_0(\alpha,\beta)$ given by (6.5) is regular if and only if $\alpha \neq -n$, $n \geq 0$, $\beta \neq -n$, $n \geq 0$ and $\alpha+\beta \neq -n$, $n \geq 0$. When $\alpha > 0$ and $\beta > 0$, the form $\hat{w}_0(\alpha,\beta)$ is positive definite.
2. The form $\hat{w}_0(\tau,\mu,\delta)$ is regular if and only if $\tau \neq -n$, $n \geq 0$, $\mu \neq -n$, $n \geq 0$, $\delta \neq -n$, $n \geq 0$, $\tau+\mu \neq -n$, $n \geq 0$, $\tau+\delta \neq -n$, $n \geq 0$, $\mu+\delta \neq -n$,

$n \geq 0$, $\tau + \mu + \delta - 1 \neq -n$, $n \geq 0$. When $\tau > 0$, $\mu > 0$, $\delta > 0$, $\tau + \mu + \delta - 1 > 0$, it is positive definite.

Corollary 6.2

a) *The form $\hat{w}_0(\alpha, \beta)$ given by (6.5) is D_{-i}-semi-classical of class one if and only if $\alpha \neq 1/2$ and $\beta \neq 1/2$.*
b) *The form $\hat{w}_0(\tau, \mu, \delta)$ is D_i-semi-classical of class one if and only if $\tau \neq 1/2$, $\mu \neq 1/2$ and $\delta \neq 1/2$.*

Proof

a) We have $\hat{\phi}(x) = (x + i\alpha)(x + i\beta)$, $\hat{\psi}(x) = 2i(x^2 - \alpha\beta)$ and $\omega = -i$. Let $c = -i\alpha$, we can deduce that

$$\theta_c\hat{\phi}(x) = x + i\beta.$$

$$\psi(c - \omega) + \big(\theta_c\hat{\phi}\big)(c - \omega) = i\{-2(\alpha - 1)^2 - 2\alpha\beta + 1 + \beta - \alpha\}. \quad (6.7)$$

$$\theta_{c-\omega}\big(\hat{\psi} + \theta_c\hat{\phi}\big)(x) = 2ix + 2\alpha - 1. \quad (6.8)$$

From (6.8), it follows that

$$\Big\langle \hat{w}_0(\alpha, \beta), \theta_{c-\omega}\big(\hat{\psi} + \theta_c\hat{\phi}\big)\Big\rangle = 2i\big(\hat{w}_0(\alpha, \beta)\big)_1 + 2\alpha - 1.$$

But, $(\hat{w}_0(\alpha, \beta))_1 = 0$ since $\hat{w}_0(\alpha, \beta)$ is a symmetric form, then

$$\Big\langle \hat{w}_0(\alpha, \beta), \theta_{c-\omega}\big(\hat{\psi} + \theta_c\hat{\phi}\big)\Big\rangle = 2\alpha - 1. \quad (6.9)$$

On account of (6.7), (6.9) and the Proposition 2.4 we get the desired result.
b) We have $\hat{\phi}(x) = (x - i\tau)(x - i\mu)(x - i\delta)$, $\hat{\psi}(x) = -2((\tau + \mu + \delta)x^2 - \tau\mu\delta)$ and $\omega = i$. Let $c = i\tau$, then

$$\theta_c\hat{\phi}(x) = (x - i\mu)(x - i\delta).$$

$$\begin{aligned}\psi(c - \omega) + \big(\theta_c\hat{\phi}\big)(c - \omega) &= \big(2\tau + 2\mu + 2\delta - 1\big)(\tau + 1)^2 \\ &\quad - (\mu + \delta)(\tau + 1) + \mu\delta(2\tau - 1). \quad (6.10)\end{aligned}$$

$$\begin{aligned}\theta_{c-\omega}\big(\hat{\psi} + \theta_c\hat{\phi}\big)(x) &= -(2\tau + 2\mu + 2\delta - 1)x - i(\mu + \delta) \\ &\quad + i(2\tau + 2\mu + 2\delta - 1)(\tau + 1). \quad (6.11)\end{aligned}$$

By (6.11), we obtain

$$\Big\langle \hat{w}_0(\tau, \mu, \delta), \theta_{c-\omega}\big(\hat{\psi} + \theta_c\hat{\phi}\big)\Big\rangle = i(2\tau + 2\mu + 2\delta - 1)(\tau + 1) - i(\mu + \delta). \quad (6.12)$$

Therefore, by (6.10), (6.12) and the Proposition 2.4, we can deduce the desired result. □

Corollary 6.3

a) *When $\alpha = 1/2$ in (6.5), we obtain*

$$\begin{cases} \hat{\gamma}_{n+1} = \frac{1}{4}(n+1)(n+2\beta), \quad n \geq 0. \\ D_{-i}\Big((x+i\beta)\hat{w}_0(1/2,\beta)\Big) + 2ix\hat{w}_0(1/2,\beta) = 0. \end{cases} \tag{6.13}$$

The form $\hat{w}_0(1/2,\beta)$ is D_{-i}-classical.

b) *Let $\tau = 1/2$ in (6.6), we have*

$$\begin{cases} \hat{\gamma}_{n+1} = \frac{1}{4}\frac{(n+1)(n+2\mu)(n+2\delta)(n+2\mu+2\delta-1)}{(2n+2\mu+2\delta-1)(2n+2\mu+2\delta+1)}, \; n \geq 0. \\ D_i\big((x-i\mu)(x-i\delta)\hat{w}_0(1/2,\mu,\delta)\big) - 2(\mu+\delta)x\hat{w}_0(1/2,\mu,\delta) = 0. \end{cases} \tag{6.14}$$

The form $\hat{w}_0(1/2,\mu,\delta)$ is D_i- classical form.

Proof

a) When $\alpha = 1/2$, by virtue of Corollary 6.2, the functional equation in (6.5) can be simplified by the factor $x + \frac{i}{2}$. In accordance of Proposition 5.1 we can deduce (6.13).

b) If $\tau = 1/2$, according to Corollary 6.2, the functional equation in (6.5) can be simplified by the factor $x - \frac{i}{2}$. Therefore the Proposition 5.1 give the desired result. □

Remark

1. The form $\hat{w}_0(1/2,\beta)$ is the symmetric Meixner–Pollaczek polynomials [5], see also [1, pp. 15]
2. The form $\hat{w}_0(1/2,\alpha/2+1/2,\beta/2+1/2)$ is given in [1, pp. 17, (3.22)].
3. The form $\hat{w}_0(1/2,\delta/2+1/2,\alpha+1/2-\delta/2)$ is studied in [1, pp. 17, (3.21)].
4. The form $\hat{w}_0(1/2,\alpha/2,\alpha/2)$ is given in [1, pp. 16, (4.10) for $\mu = 0$].

7 Integral representations

The scope of this section is to determine an integral representation for any canonical case.

When $\omega \to i\omega$, $\omega \in \mathbb{R}$, we are looking for a weight function U such that

$$\langle \hat{w}_0, f \rangle = \int_{-\infty}^{+\infty} U(x) f(x) dx, \quad f \in \mathcal{P}, \tag{7.1}$$

where we suppose that U is regular as far as it is necessary.

In the case where $\hat{w}_0$ is a symmetric form we have $\hat{w}_0 = h_{-1}\hat{w}_0$, consequently

$$\langle \hat{w}_0, f\rangle = \langle h_{-1}\hat{w}_0, f\rangle = \int_{-\infty}^{+\infty} U(-x) f(x) dx, \quad f \in \mathcal{P},$$

on account of (7.1) we can deduce that

$$\langle \hat{w}_0, f\rangle = \int_{-\infty}^{+\infty} \frac{1}{2}\Big(U(x) + U(-x)\Big) f(x) dx, \ \ f \in \mathcal{P}. \tag{7.2}$$

On account of (7.1), we get [1]

$$\int_{-\infty}^{+\infty} \Big\{ \big(D_{i\omega}(\phi U)\big)(x) + \psi(x) U(x) \Big\} f(x) dx = 0, \ \ f \in \mathcal{P},$$

with the additional condition

$$\int_{-\infty+i\omega}^{+\infty+i\omega} U(x+i\omega)\phi(x+i\omega) f(x) dx = \int_{-\infty}^{+\infty} U(x+i\omega)\phi(x+i\omega) f(x) dx, \quad f \in \mathcal{P}. \tag{7.3}$$

Therefore

$$\big(D_{i\omega}(\phi U)\big)(x) + \psi(x) U(x) = \lambda g(x), \tag{7.4}$$

where $\lambda \in \mathbb{C}$ and g is a locally integrable function with rapid decay representing the null form. For instance

$$g(x) = \begin{cases} 0 \ , & x \leq 0 \ , \\ \exp(-x^{1/4}) \sin(x^{1/4}) \ , & x > 0 \ , \end{cases}$$

was given by Stieltjes [13]. When $\lambda = 0$, the equation (7.4), becomes

$$\phi(x+i\omega) U(x+i\omega) = \Big(\phi(x) - i\omega\psi(x)\Big) U(x),$$

so, that, if $\omega = 1$, we have

$$U(x+i) = \frac{\phi(x) - i\psi(x)}{\phi(x+i)} U(x) \ , \ \ x \in \mathbb{R}, \tag{7.5}$$

and if $\omega = -1$, with $x \to x+i$, we have

$$U(x+i) = \frac{\phi(x)}{\phi(x+i) + i\psi(x+i)} U(x), \ \ x \in \mathbb{R}. \tag{7.6}$$

Now we are able to give the integral representations for any canonical form.

Theorem 7.1

a) *The form $\hat{w}_0(\alpha, \beta)$ given in (6.5) possesses the following integral representation:*

$$\left\langle \hat{w}_0(\alpha,\beta),\ f\right\rangle = K_1 \int_{-\infty}^{+\infty} \frac{\mid \Gamma(ix)\mid^2 \Gamma(\alpha+ix)\mid^2\mid \Gamma(\beta+ix)\mid^2}{\mid \Gamma(2ix)\mid^2} \times f(x)dx,\ f\in\mathcal{P},\ \alpha,\beta>0, \tag{7.7}$$

with

$$K_1^{-1} = \pi\Gamma(\alpha)\Gamma(\beta)\Gamma(\alpha+\beta). \tag{7.8}$$

b) *When $\tau > 0$, $\mu > 0$, $\delta > 0$ and $\tau+\mu+\delta-1>0$, the form $\hat{w}_0(\tau,\mu,\delta)$ defined in (6.6) have the following integral representation:*

$$\left\langle \hat{w}_0(\tau,\mu,\delta),\ f\right\rangle = K_2 \int_{-\infty}^{+\infty} \frac{\mid \Gamma(ix)\mid^2\mid \Gamma(\tau+ix)\mid^2\mid \Gamma(\mu+ix)\mid^2\mid \Gamma(\delta+ix)\mid^2}{\mid \Gamma(2ix)\mid^2} \times f(x)dx,\ f\in\mathcal{P}, \tag{7.9}$$

with

$$K_2^{-1} = \pi\frac{\Gamma(\tau)\Gamma(\mu)\Gamma(\delta)\Gamma(\tau+\mu)\Gamma(\tau+\delta)\Gamma(\mu+\delta)}{\Gamma(\tau+\mu+\delta)}. \tag{7.10}$$

Proof We need the following formulas [3, 4, 9]

$$\int_0^{+\infty} \frac{\mid \Gamma(a+ix)\mid^2\mid \Gamma(b+ix)\mid^2\mid \Gamma(c+ix)\mid^2}{\mid \Gamma(2ix)\mid^2}dx = 2\pi\Gamma(a+b)\Gamma(a+c)\Gamma(b+c),\ a,b,c\geq 0. \tag{7.11}$$

$$\int_0^{+\infty} \frac{\mid \Gamma(a+ix)\mid^2\mid \Gamma(b+ix)\mid^2\mid \Gamma(c+ix)\mid^2\mid \Gamma(d+ix)\mid^2}{\mid \Gamma(2ix)\mid^2}dx = \frac{2\pi\Gamma(a+b)\Gamma(a+c)\Gamma(a+d)\Gamma(b+c)\Gamma(b+d)\Gamma(c+d)}{\Gamma(a+b+c+d)},\ a,b,c,d\geq 0. \tag{7.12}$$

a) In the case (6.5) we have $\phi(x) = (x+i\alpha)(x+i\beta)$, $\psi(x) = 2i(x^2-\alpha\beta)$ and $\omega=-1$. Supposing $\alpha>0$, $\beta>0$. Then the equation (7.6) becomes

$$U(x+i) = -\frac{(x+i\alpha)(x+i\beta)}{(x-i(\alpha-1))(x-i(\beta-1))}U(x),\ x\in\mathbb{R},$$

hence

$$U(x) = e^{\pi x} \frac{\Gamma(\alpha - ix)\Gamma(\beta - ix)}{\Gamma(1-\alpha - ix)\Gamma(1-\beta - ix)} A(x), \quad x \in \mathbb{R},$$

then $A(x+i) = A(x)$, $x \in \mathbb{R}$.
Taking account of

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)} \ , \tag{7.13}$$

we have

$$U(x) = \pi^{-2} e^{\pi x} \mid \Gamma(\alpha + ix) \mid^2 \mid \Gamma(\beta + ix) \mid^2 \sin(\pi(\alpha + ix)) \sin(\pi(\beta + ix)) A(x).$$

Choosing

$$A(x) = K_1 \frac{\pi^2}{\sin(\pi(\alpha + ix)) \sin(\pi(\beta + ix))},$$

we obtain

$$U(x) = K_1 e^{\pi x} \mid \Gamma(\alpha + ix) \mid^2 \mid \Gamma(\beta + ix) \mid^2, \quad x \in \mathbb{R}. \tag{7.14}$$

It follows that

$$\frac{1}{2}\big(U(x) + U(-x)\big) = K_1 \cosh(\pi x) \mid \Gamma(\alpha + ix) \mid^2 \mid \Gamma(\beta + ix) \mid^2, \ x \in \mathbb{R}.$$

But

$$\cosh(\pi x) = \frac{\mid \Gamma(ix) \mid^2}{4 \mid \Gamma(2ix) \mid^2}, \ x \in \mathbb{R}. \tag{7.15}$$

Therefore

$$\frac{1}{2}\big(U(x) + U(-x)\big) = \frac{K_1}{4} \frac{\mid \Gamma(ix) \mid^2 \mid \Gamma(\alpha + ix) \mid^2 \mid \Gamma(\beta + ix) \mid^2}{\mid \Gamma(2ix) \mid^2}, \ x \in \mathbb{R}. \tag{7.16}$$

where

$$K_1^{-1} = \frac{1}{2} \int_0^{+\infty} \frac{\mid \Gamma(ix) \mid^2 \Gamma(\alpha + ix) \mid^2 \mid \Gamma(\beta + ix) \mid^2}{\mid \Gamma(2ix) \mid^2} dx. \tag{7.17}$$

Finally, on account of (7.2), (7.16), (7.17) and (7.11), we get (7.7) and (7.8).

b) For the case (6.6), we have $\phi(x) = (x - i\tau)(x - i\mu)(x - i\delta)$, $\psi(x) = -2(\tau + \mu + \delta)x^2 + 2\tau\mu\delta$ and $\omega = 1$. Then (7.5) can be written

$$U(x+i) = \frac{(x + i\tau)(x + i\mu)(x + i\delta)}{(x + i(1-\tau))(x + i(1-\mu))(x + i(1-\delta))} U(x), \ x \in \mathbb{R}.$$

This leads to

$$U(x) = \frac{\Gamma(\tau - ix)\Gamma(\mu - ix)\Gamma(\delta - ix)}{\Gamma(1-\tau - ix)\Gamma(1-\mu - ix)\Gamma(1-\delta - ix)} A(x), \quad x \in \mathbb{R},$$

with $A(x+i) = A(x), \quad x \in \mathbb{R}$

Taking (7.13) into account, we obtain

$$U(x) = \pi^{-3} \mid \Gamma(\tau + ix) \mid^2 \mid \Gamma(\mu + ix) \mid^2 \mid \Gamma(\delta + ix) \mid^2$$
$$\times \sin(\pi(\tau + ix)) \sin(\pi(\mu + ix)) \sin(\pi(\delta + ix)) A(x), \quad x \in \mathbb{R},$$

choosing

$$A(x) = \frac{K_2 \pi^3 e^{\pi x}}{\sin(\pi(\tau + ix)) \sin(\pi(\mu + ix)) \sin(\pi(\delta + ix))}, \quad x \in \mathbb{R},$$

we get

$$U(x) = K_2 e^{\pi x} \mid \Gamma(\tau + ix) \mid^2 \mid \Gamma(\mu + ix) \mid^2 \mid \Gamma(\delta + ix) \mid^2, \quad x \in \mathbb{R}. \tag{7.18}$$

Consequently

$$\frac{1}{2}\big(U(x) + U(-x)\big) = K_2 \cosh(\pi x) \mid \Gamma(\tau + ix) \mid^2 \mid \Gamma(\mu + ix) \mid^2 \mid \Gamma(\delta + ix) \mid^2,$$
$$x \in \mathbb{R},$$

by (7.15), we obtain

$$\frac{1}{2}\big(U(x) + U(-x)\big) = \frac{K_2}{4} \frac{\mid \Gamma(ix) \mid^2 \mid \Gamma(\tau + ix) \mid^2 \mid \Gamma(\mu + ix) \mid^2 \mid \Gamma(\delta + ix) \mid^2}{\mid \Gamma(2ix) \mid^2},$$
$$x \in \mathbb{R}. \tag{7.19}$$

with

$$K_2^{-1} = \frac{1}{2} \int_0^{+\infty} \frac{\mid \Gamma(ix) \mid^2 \mid \Gamma(\tau + ix) \mid^2 \mid \Gamma(\mu + ix) \mid^2 \mid \Gamma(\delta + ix) \mid^2}{\mid \Gamma(2ix) \mid^2} dx. \tag{7.20}$$

Then by (7.2), (7.19), (7.20) and (7.12) we can deduce (7.9) and (7.10). □

Remark In any case (7.14) and (7.18), the condition (7.3) is fulfilled by virtue of the standard asymptotic formula

$$\mid \Gamma(a + ix) \mid = \sqrt{2\pi} e^{-\pi|x|/2} \mid x \mid^{a-1/2} (1 + r(a, x)),$$

where $r(a, x) \to 0$, as $\mid x \mid \to +\infty$, uniformly for bounded $\mid a \mid$.

8 The limiting cases: $\omega \to 0$

When $\omega \to 0$ in (6.4), we obtain the following general situation:

$$\begin{cases} \gamma_{2n+1} = \dfrac{\big(2(n-1)b_3 - a_2\big)\big(a_0 - 2nb_1\big)}{\big(4nb_3 - a_2\big)\big((4n-2)b_3 - a_2\big)}, \quad n \geq 0, \\ \gamma_{2n+2} = \dfrac{(n+1)\Big(-2a_0 b_3 + 2\big(a_2 - 2nb_3\big)b_1\Big)}{\big(4nb_3 - a_2\big)\big((4n+2)b_3 - a_2\big)}, \quad n \geq 0, \\ \Big(\big(b_3 x^3 + b_1 x\big) w_0\Big)' + (a_2 x^2 + a_0) w_0 = 0, \quad (w_0)_1 = 0. \end{cases} \tag{8.1}$$

A$_1$. **deg ϕ = 1**. Then (8.1) becomes

$$\begin{cases} \gamma_{2n+1} = a_2^{-1} b_1 \big(2n - a_0 b_1^{-1}\big),\ n \geq 0. \\ \gamma_{2n+2} = 2 a_2^{-1} b_1 (n+1),\ n \geq 0. \\ (x w_0)' + \big(a_2 b_1^{-1} x^2 + a_0 b_1^{-1}\big) w_0 = 0,\ \ (w_0)_1 = 0. \end{cases} \tag{8.2}$$

Choosing $a_2 b_1^{-1} = 2$ and $a_0 b_1^{-1} = -2\mu - 1$ we recover the generalized Hermite form [2, 5].

$$\begin{cases} \hat{\gamma}_{2n+1} = \frac{1}{2}(2n + 2\mu + 1)\ ,\ \ n \geq 0. \\ \hat{\gamma}_{2n+2} = (n+1)\ ,\ \ n \geq 0. \\ (x \hat{w}_0)' + \big(2x^2 - 2\mu - 1\big) \hat{w}_0 = 0,\ \ (\hat{w}_0)_1 = 0. \end{cases} \tag{8.3}$$

The form $\hat{w}_0$ is regular if and only if $2\mu + 1 \neq -2n\ ,\ \ n \geq 0$. The form $\hat{w}_0$ is semi-classical of class $s = 1$ if $\mu \neq 0$ and $s = 0$ if $\mu = 0$.

A$_2$. **deg ϕ = 3**. For (8.1) we get

$$\begin{cases} \gamma_{2n+1} = b_3^{-1} \dfrac{(2n - 2 - a_2 b_3^{-1})(a_0 - 2n b_1)}{(4n - a_2 b_3^{-1})\big(4n - 2 - a_2 b_3^{-1}\big)},\ n \geq 0. \\ \gamma_{2n+2} = b_3^{-1} \dfrac{(n+1)\big(-2a_0 + 2\big(a_2 b_3^{-1} - 2n\big) b_1\big)}{(4n - a_2 b_3^{-1})\big(4n + 2 - a_2 b_3^{-1}\big)},\ n \geq 0. \\ \Big(\big(x^3 + b_1 b_3^{-1} x\big) w_0\Big)' + \big(a_2 b_3^{-1} x^2 + a_0 b_3^{-1}\big) w_0 = 0,\ \ (w_0)_1 = 0. \end{cases} \tag{8.4}$$

Two cases arise

A$_{21}$. $b_1 = 0$. Then we obtain for (8.4):

$$\begin{cases} \gamma_{2n+1} = a_0 b_3^{-1} \dfrac{2n - 2 - a_2 b_3^{-1}}{\big(4n - a_2 b_3^{-1}\big)\big(4n - 2 - a_2 b_3^{-1}\big)},\ n \geq 0. \\ \gamma_{2n+2} = -2 a_0 b_3^{-1} \dfrac{n+1}{\big(4n - a_2 b_3^{-1}\big)\big(4n + 2 - a_2 b_3^{-1}\big)},\ n \geq 0. \\ \big((x^3 w_0\big)' + \big(a_2 b_3^{-1} x^2 + a_0 b_3^{-1}\big) w_0 = 0,\ \ (w_0)_1 = 0. \end{cases}$$

Choosing $a_2 b_3^{-1} = -2(\nu + 1)$ and $a_0 b_3^{-1} = -\frac{1}{2}$, we rediscover the following situation [2]

$$\begin{cases} \hat{\gamma}_{2n+1} = -\dfrac{1}{4} \dfrac{n + \nu}{(2n + \nu)(2n + \nu + 1)},\ n \geq 0. \\ \hat{\gamma}_{2n+2} = \dfrac{1}{4} \dfrac{n+1}{(2n + \nu + 1)(2n + \nu + 2)},\ n \geq 0. \\ \big((x^3 \hat{w}_0\big)' - \big(2(\nu+1) x^2 + \frac{1}{2}\big) \hat{w}_0 = 0,\ \ (\hat{w}_0)_1 = 0. \end{cases} \tag{8.5}$$

The form $\hat{w}_0$ is regular if and only if $\nu \neq -n\ ,\ \ n \geq 0$ and semi-classical of class one.

A_{22}. $b_1 \neq 0$. Then, (8.4) can be written as follow

$$\begin{cases} \gamma_{2n+1} = b_1 b_3^{-1} \dfrac{\left(2n-2-a_2 b_3^{-1}\right)\left(a_0 b_1^{-1} - 2n\right)}{\left(4n - a_2 b_3^{-1}\right)\left(4n-2-a_2 b_3^{-1}\right)}, \quad n \geq 0. \\ \gamma_{2n+2} = b_1 b_3^{-1} \dfrac{(n+1)\left(+4n + 2a_2 b_3^{-1} - 2a_0 b_1^{-1}\right)}{\left(4n - a_2 b_3^{-1}\right)\left(4n+2-a_2 b_3^{-1}\right)}, \quad n \geq 0. \\ \left(\left(x^3 + b_1 b_3^{-1} x\right) w_0\right)' + \left(a_2 b_3^{-1} x^2 + a_0 b_3^{-1}\right) w_0 = 0, \quad (w_0)_1 = 0. \end{cases}$$

With the choice $b_1 b_3^{-1} = -1$, $a_0 b_3^{-1} = 2\beta + 2$ and $a_2 b_3^{-1} = -2\alpha - 2\beta - 4$,
we obtain

$$\begin{cases} \hat{\gamma}_{2n+1} = \dfrac{(n+\beta+1)(n+\alpha+\beta+1)}{(2n+\alpha+\beta+1)(2n+\alpha+\beta+2)}, \quad n \geq 0. \\ \hat{\gamma}_{2n+2} = \dfrac{(n+1)(n+\alpha+1)}{(2n+\alpha+\beta+2)(2n+\alpha+\beta+3)}, \quad n \geq 0. \\ \left(x(x^2-1)\hat{w}_0\right)' - 2((\alpha+\beta+2)x^2 - \beta - 1)\hat{w}_0 = 0, \quad (\hat{w}_0)_1 = 0. \end{cases} \tag{8.6}$$

It is the definition of the form studied in [2, 5]. The form $\hat{w}_0$ is regular if and only if $\alpha \neq -n-1$, $n \geq 0$, $\beta \neq -n-1$, $n \geq 0$, $\alpha + \beta \neq -n-1$, $n \geq 0$ and it is semi-classical of class one if $\beta \neq -1/2$.
When $\beta = -1/2$, $\alpha \to \alpha + 1$, we obtain the classical Gegenbauer form.

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