ON A NONORTHOGONAL POLYNOMIAL SEQUENCE ASSOCIATED WITH BESSEL OPERATOR

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Abstract. By means of the Bessel operator a polynomial sequence is constructed to which several properties are given. Among them, its explicit expression, the connection with the Euler numbers, its integral representation via the Kontorovich-Lebedev transform. Despite its non-orthogonality, it is possible to associate to the canonical element of its dual sequence a positive-definite measure as long as certain stronger constraints are imposed.

1. Introduction and preliminaries

The modified Bessel function of third kind (also known as the *Macdonald function*, specially in russian literature) $K_{i\tau}(x)$ of the argument $x>0$ and the pure imaginary subscript $i\tau$ is an eigenfunction of the (Bessel) operator

$$\mathscr{A} = x^2 - x\frac{d}{dx}x\frac{d}{dx} \;=\; -x^2\frac{d^2}{dx^2} - x\frac{d}{dx} + x^2 \tag{1.1}$$

for the associated eigenvalues τ^2, *i.e.*,

$$\mathscr{A} K_{i\tau}(x) = \tau^2 K_{i\tau}(x) \tag{1.2}$$

and naturally providing the identity on the integral powers of the Bessel operator $\mathscr{A}$

$$\mathscr{A}^n K_{i\tau}(x) = \tau^{2n} K_{i\tau}(x)\ ,\ n\in\mathbb{N}, \tag{1.3}$$

inductively defined. Throughout this paper, $\mathbb{N}$ will denote the set of all positive integers, $\mathbb{N}_0=\mathbb{N}\cup\{0\}$, whereas $\mathbb{R}$ and $\mathbb{C}$ will denote, respectively, the field of the real and complex numbers. By $\mathbb{R}^+$ and $\mathbb{R}^+_0$ we respectively mean the set of all positive and nonnegative real numbers. Further notations are introduced as needed throughout the text.

Defined also by the cosine Fourier transform

$$K_{i\tau}(x) = \int_0^\infty \mathrm{e}^{-x\cosh(u)}\cos(\tau u)du\ ,\ x\in\mathbb{R}^+,\ \tau\in\mathbb{R}$$

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the function $K_{i\tau}$ is real valued and represents a kernel of the following operator of the Kontorovich-Lebedev transformation [19], given by the formula

$$\mathscr{K}_{i\tau}[f] = \int_0^\infty K_{i\tau}(x) f(x) dx \ , \tag{1.4}$$

is an isometric isomorphism (see [17]) between the two Hilbert spaces $L_2(\mathbb{R}^+; xdx)$ and $L_2(\mathbb{R}^+; \tau \sinh \pi\tau d\tau)$. Moreover the inverse formula holds

$$f(x) = \frac{2}{\pi^2 x} \int_0^\infty \tau \sinh \pi\tau \quad K_{i\tau}(x) \mathscr{K}_{i\tau}[f] \ dy \tag{1.5}$$

where the convergence of the latter integral is understood with respect to the norm in $L_2(\mathbb{R}^+; xdx)$.

The Kontorovich-Lebedev transform has been used in many applications including, for instance, fluid mechanics, quantum and nano-optics and plasmonics. This transform, where integration occurs over the index of the function rather than over the argument, proves to be useful in solving the resulting differential equations when modeling optical or electronic response of such problems.

Recently [16], the action of any power of the operator $\mathscr{A}$ over e^{-x} was investigated. Precisely, the sequence $\{\mathrm{e}^{-x}\mathscr{A}^n \mathrm{e}^x\}_{n\geqslant 0}$ was shown to be a polynomial sequence (PS), spanning the vector space $\mathscr{P}$ – the set of all polynomials with coefficients in $\mathbb{C}$ – since each of its elements has exactly degree n. However, this sequence is orthogonal with respect to the singular measure δ (the Dirac measure). In the present work, we propose instead to investigate the functions

$$\mathrm{e}^{-x} x^{-\alpha} \mathscr{A}^n \mathrm{e}^x x^\alpha \ , \quad n \in \mathbb{N}_0,$$

which end up to be a polynomials of exactly degree n as long as $\Re(\alpha) > -1/2$, as it will be revealed in Section 2. In this case the sequence $\{p_n(\cdot;\alpha)\}_{n\geqslant 0}$ defined by

$$p_n(x;\alpha) = (-1)^n \mathrm{e}^x \ x^{-\alpha} \mathscr{A}^n \ \mathrm{e}^{-x} x^\alpha \ , \quad n \in \mathbb{N}_0, \tag{1.6}$$

is indeed a PS. The properties of $\{p_n(\cdot;\alpha)\}_{n\geqslant 0}$ will be thoroughly revealed Section 2, where a generating function, the connection with the generalized Euler polynomials or the Bernoulli polynomials (when the parameter α ranges in $\mathbb{N}$) will be expounded. On the grounds of these developments lies the Kontorovich-Lebedev transform. The differential-difference equations over the polynomials $p_n(x;\alpha)$ are crucial for the subsequent developments related to the corresponding dual sequence of monic polynomial sequence (MPS) $\{P_n(x;\alpha) := a_n^{-1} p_n(x;\alpha)\}_{n\geqslant 0}$ where $a_n \neq 0$ are the leading coefficients of the polynomials $p_n(x;\alpha)$, $n \in \mathbb{N}_0$.

The research about the MPS $\{P_n(x;\alpha)\}_{n\geqslant 0}$ will then proceed toward the analysis of the corresponding dual sequence, $\{u_n(\alpha)\}_{n\geqslant 0}$, whose elements, called as *forms* (or *linear functionals*), belong to the dual space $\mathscr{P}'$ of $\mathscr{P}$ and are defined according to

$$\langle u_n(\alpha), P_k(\cdot;\alpha)\rangle := \delta_{n,k}, \ n,k \geqslant 0,$$

where $\delta_{n,k}$ represents the *Kronecker delta* function. Here, by $\langle u, f\rangle$ we mean the action of $u \in \mathscr{P}'$ over $f \in \mathscr{P}$, but a special notation is given to the action over the elements of the canonical sequence $\{x^n\}_{n\geqslant 0}$ – the *moments of* $u \in \mathscr{P}'$: $(u)_n := \langle u, x^n \rangle, n \geqslant 0$. Any element u of $\mathscr{P}'$ can be written in a series of any dual sequence $\{\mathbf{v}_n\}_{n\geqslant 0}$ of a MPS $\{B_n\}_{n\geqslant 0}$ [9]:

$$u = \sum_{n\geqslant 0} \langle u, B_n \rangle \, \mathbf{v}_n \, . \tag{1.7}$$

To infer differential equations for the elements of the dual sequence it is important to recall that a linear operator $T : \mathscr{P} \to \mathscr{P}$ has a transpose ${}^tT : \mathscr{P}' \to \mathscr{P}'$ defined by

$$\langle {}^tT(u), f \rangle = \langle u, T(f) \rangle, \quad u \in \mathscr{P}', \; f \in \mathscr{P}. \tag{1.8}$$

For example, for any form u and any polynomial g, let $Du = u'$ and gu be the forms defined as usual by $\langle u', f \rangle := -\langle u, f' \rangle$, $\langle gu, f \rangle := \langle u, gf \rangle$, where D is the differential operator [9]. Thus, D on forms is minus the transpose of the differential operator D on polynomials.

In Section 3 we will come to the conclusion that it is possible to associate to $u_0(\alpha)$ a definite-positive measure as long as $\Re(\alpha) > 0$. This implies the regularity of any form proportional to $u_0(\alpha)$, a concept that is recalled in detail on page 12. Concomitantly, the existence of a unique MPS regularly orthogonal (hereafter MOPS) with respect to $u_0(\alpha)$ is ensured, that is, there exists a unique $\{Q_n(x;\alpha)\}_{n\geqslant 0}$ such that $\langle u_0(\alpha), Q_n(x;\alpha) Q_m(x;\alpha) \rangle = k_n(\alpha) \delta_{n,m}$ with $k_n \neq 0$ for any $n, m \in \mathbb{N}_0$.

2. Algebraic and differential properties

The functions defined by (1.6) are actually polynomials of exactly degree n, as previously claimed and below proved, and therefore $\{p_n(x;\alpha)\}_{n\geqslant 0}$ forms a PS. Once this is guaranteed we will proceed to obtain an explicit expression for these polynomials $p_n(x;\alpha)$ in §2.1, an integral representation for such polynomials by means of the Kontorovich-Lebedev transform in §2.2 and finally, by inverting this integral transform, to derive a relation between the generalized Euler polynomials and $p_n(x;\alpha)$ in §2.3.

Lemma 2.1. *For any complex number* α *such that* $\Re(\alpha) > -1/2$ *the functions* $p_n(x;\alpha)$ *defined in* (1.6) *are actually polynomials of exactly degree* n *fulfilling*

$$p_{n+1}(x;\alpha) = x^2 p_n''(x;\alpha) - x(2x - 1 - 2\alpha) p_n'(x;\alpha) - \Big((2\alpha+1)x - \alpha^2 \Big) p_n(x;\alpha) \; , \; n \in \mathbb{N}_0, \tag{2.1}$$

and also

$$(2\alpha+1) \; x \; p_n(x;\alpha+1) = -p_{n+1}(x;\alpha) + \alpha^2 p_n(x;\alpha) \; , \; n \in \mathbb{N}_0. \tag{2.2}$$

Moreover, $p_n(0;\alpha) = \alpha^{2n}$ *and* $p_n(x;\alpha) = (-2)^n (\alpha + 1/2)_n x^n + a_{n-1}(x)$ *with* $\deg a_{n-1} \leqslant n-1$ *for any* $n \in \mathbb{N}_0$ *(under the convention* $a_{-1}(x) = 0$*) and* $(y)_n$ *representing the Pochhammer symbol:* $(y)_n = \prod_{\tau=0}^{n-1} (y + \tau)$ *for* $n \in \mathbb{N}$ *and* $(y)_0 = 1$.

Proof. According to the definition (1.6), we may write

$$p_{n+k}(x;\alpha) = (-1)^k \mathrm{e}^x x^{-\alpha} \mathscr{A}^k \Big(\mathrm{e}^{-x} x^{\alpha} \Big(\mathrm{e}^x x^{-\alpha} \mathscr{A}^n \mathrm{e}^{-x} x^{\alpha} \Big) \Big) \; , \; n, k \in \mathbb{N},$$

providing

$$p_{n+k}(x;\alpha) = (-1)^k \mathrm{e}^x x^{-\alpha} \mathscr{A}^k\left(\mathrm{e}^{-x}x^{\alpha}p_n(x;\alpha)\right)\ ,\ n,k\in\mathbb{N}. \tag{2.3}$$

On behalf of the property,

$$\mathrm{e}^x x^{-\alpha}\mathscr{A}\left(\mathrm{e}^{-x}x^{\alpha}f(x)\right) = -x^2\frac{d^2}{dx^2}f(x) + x(2x-1-2\alpha)\frac{d}{dx}f(x) + \Big((2\alpha+1)x-\alpha^2\Big)f(x), \tag{2.4}$$

that holds for any analytic function f, the particular choice of $k=1$ in (2.3) furnishes (2.1), whereas the choice of $n=1$ with k varying within $\mathbb{N}$ leads to the equality (2.2).

Insofar as computing (1.6) for $n=0$ and $n=1$ we respectively extract that $p_0(x;\alpha)=1$ and $p_1(x;\alpha)=-(2\alpha+1)x+\alpha^2$, then on the basis of (2.1), by finite induction we derive that $p_n(x;\alpha)$ is a polynomial that has exactly degree n as long as $\Re(\alpha)>-1/2$.

Finally, the particular choice of $x=0$ in (2.2) provides $p_{n+1}(0;\alpha)=\alpha^{2n+2}p_0(0;\alpha)$, while from (2.1) we deduce that the leading coefficient $c_{n,n}(\alpha)$ of $p_n(x;\alpha)$ satisfies $c_{n+1,n+1}(\alpha)=-(2n+2\alpha+1)c_{n,n}(\alpha)$ for any $n\in\mathbb{N}$, whence the result. □

Directly from (2.3), we obtain

$$\mathscr{A}^k\left(\mathrm{e}^{-x}x^{\alpha}p_n(x;\alpha)\right) = \mathscr{A}^n\left(\mathrm{e}^{-x}x^{\alpha}p_k(x;\alpha)\right)\ ,\ n,k\in\mathbb{N}.$$

From the insertion of (2.2) into (2.1) we derive

$$(2\alpha+1)p_n(x;\alpha+1) = -x\ p_n''(x;\alpha) + (2x-1-2\alpha)p_n'(x;\alpha) + (1+2\alpha)p_n(x;\alpha),\ n\geqslant 0. \tag{2.5}$$

The polynomial sequence treated in [16] corresponds to the special case $\{p_n(x;0)\}_{n\geqslant 0}$. The introduction of this slight modification on the sequence $\{p_n(x;0)\}_{n\geqslant 0}$ – more precisely, the inclusion of $x^{-\alpha}$ on the left and of x^{α} on the right hand side – has the merit and pertinency of guaranteeing that we will be able to deal with regular forms as long as $\Re(\alpha)>0$, as it will be analyzed on Section 3. Notwithstanding this advantage, the analysis of $\{p_n(x;\alpha)\}_{n\geqslant 0}$ became significantly more hard to deal with.

Before proceeding into this, we list a few elements of $\{p_n(x;\alpha)\}_{n\geqslant 0}$

$$\begin{aligned}
p_1(x;\alpha) &= x(-2\alpha-1)+\alpha^2\\
p_2(x;\alpha) &= x^2(4\alpha(\alpha+2)+3)+x(-2\alpha(\alpha(2\alpha+3)+2)-1)+\alpha^4\\
p_3(x;\alpha) &= -x^3(2\alpha+1)(2\alpha+3)(2\alpha+5)+x^2(2\alpha+1)(2\alpha+3)(3\alpha(\alpha+2)+5)\\
&\quad -x(2\alpha+1)\left(\alpha^2+\alpha+1\right)(3\alpha(\alpha+1)+1)+\alpha^6
\end{aligned}$$

The polynomials $p_n(x;\alpha)$ in the variable x are also polynomials in the variable α, but of degree $2n$ for each $n\in\mathbb{N}$. This can be deduced from (2.2).

2.1. **Connection coefficients with the canonical sequence.** On the grounds of (2.1), by performing straightforward computations we deduce that the coefficients $c_{n,\nu}(\alpha)$ defined through

$$p_n(x;\alpha) = \sum_{\nu=0}^{n} c_{n,\nu}(\alpha)x^{\nu}$$

fulfill the recurrence relation

$$c_{n+1,\nu}(\alpha) = \big(\nu+\alpha\big)^2 c_{n,\nu}(\alpha) - \big(2\nu+2\alpha-1\big)c_{n,\nu-1}(\alpha)\ ,\ 0\leqslant \nu \leqslant n,\ ,\ n\geqslant 0, \tag{2.6}$$

under the convention of $c_{n,-1}(\alpha)=c_{n,n+1}(\alpha)=0$, while (2.2) yields

$$(2\alpha+1)c_{n,\nu-1}(\alpha+1) = -c_{n+1,\nu}(\alpha)+\alpha^2 c_{n,\nu}(\alpha)\ ,\ 1\leqslant \nu\leqslant n+1\ ,n\geqslant 0. \tag{2.7}$$

Lemma 2.2. *The sequence $p_n(x;\alpha)$ is then given by*

$$p_n(x;\alpha)=\sum_{\nu=0}^{n}\left\{\frac{2^{\nu+1}(\alpha+1/2)_\nu}{\nu!}\sum_{\mu=0}^{\nu}\binom{\nu}{\mu}\frac{(-1)^\mu\ \Gamma(2\alpha+\mu)}{\Gamma(2\alpha+\mu+\nu+1)}(\alpha+\mu)^{2n+1}\right\}x^\nu\ ,\ n\geqslant 0. \tag{2.8}$$

Proof. Let us set $c_{n,\nu}(\alpha)=(-2)^\nu(\alpha+1/2)_\nu\ \widetilde{c}_{n,\nu}(\alpha)$ in order to deduce an explicit expression for $\widetilde{c}_{n,\nu}(\alpha)$. The "triangular" relation (2.6) ensures another "triangular" relation for the new set of coefficients $\{\widetilde{c}_{n,\nu}(\alpha)\}_{0\leqslant\nu\leqslant n}$

$$\begin{cases}\widetilde{c}_{n+1,\nu}(\alpha)=\widetilde{c}_{n,\nu-1}(\alpha)+\big(\nu+\alpha\big)^2\widetilde{c}_{n,\nu}(\alpha)\ ,\ 0\leqslant\nu\leqslant n,\ ,\ n\geqslant 0,\\ \widetilde{c}_{n,0}(\alpha)=\alpha^{2n}\quad ,\quad \widetilde{c}_{n,\nu}(\alpha)=0\ ,\ \nu\geqslant n+1\ ,\ n\geqslant 0.\end{cases} \tag{2.9}$$

The particular choice of $n=0,1$ or 2 in (2.9) furnishes the identities

$$\widetilde{c}_{1,0}(\alpha)=\alpha^2,\ \widetilde{c}_{1,1}(\alpha)=1,\ \widetilde{c}_{2,0}(\alpha)=\alpha^4,\ \widetilde{c}_{2,1}(\alpha)=1+2\alpha(\alpha+1),\ \widetilde{c}_{2,2}(\alpha)=1\ ,$$

which show the validity of the identity

$$\widetilde{c}_{n,\nu}(\alpha)=\frac{2}{\nu!}\sum_{\mu=0}^{\nu}\binom{\nu}{\mu}\frac{(-1)^{\mu+\nu}\ \Gamma(2\alpha+\mu)}{\Gamma(2\alpha+\mu+\nu+1)}(\alpha+\mu)^{2n+1}\ ,\ 0\leqslant\nu\leqslant n \tag{2.10}$$

for at least $n=0,1$ and 2. By finite induction, we will show that (2.10) actually holds for any $n\in\mathbb{N}$. Indeed, according to (2.9), it follows

$$\begin{aligned}\widetilde{c}_{n+1,\nu}(\alpha) &= \frac{2}{(\nu-1)!}\sum_{\mu=0}^{\nu-1}\binom{\nu-1}{\mu}\frac{(-1)^{\mu+\nu-1}\ \Gamma(2\alpha+\mu)}{\Gamma(2\alpha+\mu+\nu)}(\alpha+\mu)^{2n+1}\\ &\quad+\frac{2(\nu+\alpha)^2}{\nu!}\sum_{\mu=0}^{\nu}\binom{\nu}{\mu}\frac{(-1)^{\mu+\nu}\ \Gamma(2\alpha+\mu)}{\Gamma(2\alpha+\mu+\nu+1)}(\alpha+\mu)^{2n+1}\\ &=\frac{2}{\nu!}\sum_{\mu=0}^{\nu}\binom{\nu}{\mu}\frac{(-1)^{\mu+\nu}\ \Gamma(2\alpha+\mu)}{\Gamma(2\alpha+\mu+\nu+1)}(\alpha+\mu)^{2n+1}\Big(-(\nu-\mu)(\nu+\mu+2\alpha)+(\nu+\alpha)^2\Big)\\ &=\frac{2}{\nu!}\sum_{\mu=0}^{\nu}\binom{\nu}{\mu}\frac{(-1)^{\mu+\nu}\ \Gamma(2\alpha+\mu)}{\Gamma(2\alpha+\mu+\nu+1)}(\alpha+\mu)^{2n+3}\end{aligned}$$

which corresponds to (2.10) when n is replaced by $n+1$. □

The particular choice of $\alpha=0$ gives a simpler expression for $p_n(x;0)$ than the one obtained in [16]. Moreover, in this case, the coefficients $\widetilde{c}_{n,\nu}(0)$ coincide with the 0-*modified Striling numbers of second kind*, a particular case of the A-modified Stirling numbers of second kind treated in [7] – they correspond to the *Jacobi-Stirling numbers* expounded in [4] and explored from a purely combinatorial point of view in [5]. Indeed, the set of numbers $\{\widetilde{c}_{n,\nu}(\alpha)\}$ bear some resemblance with the *Jacobi-Stirling numbers*, triggering the problem of giving them some combinatorial significance. To avoid dispersion, we defer this study for a future work.

2.2. **Generating function.** From this point forth we consider $\Re(\alpha)>0$. Using relation (2.16.6.4) of [13] and the reciprocal formula (1.5) we obtain the representation

$$\mathrm{e}^{-x}x^{\alpha}=\frac{2^{1-\alpha}}{\pi^{3/2}\Gamma(\alpha+1/2)}\int_0^{\infty}\tau\sinh(\pi\tau)|\Gamma(\alpha+i\tau)|^2K_{i\tau}(x)d\tau. \tag{2.11}$$

The absolute and uniform convergence of

$$\int_0^{\infty}\frac{\partial^m K_{i\tau}(x)}{\partial x^m}\ \tau^{2n+1}\sinh(\pi\tau)|\Gamma(\alpha+i\tau)|^2d\tau\ ,\quad m,n\in\mathbb{N}_0 \tag{2.12}$$

with respect to $x\geqslant x_0>0$, easily verified by taking into account inequality

$$\left|\frac{\partial^m K_{i\tau}(x)}{\partial x^m}\right|\leqslant \mathrm{e}^{-\delta\tau}K_m(x\cos\delta),\quad x>0,\ \tau>0,\ m\in\mathbb{N}_0 \tag{2.13}$$

with $\delta\in(0,\pi/2)$, permits to interchange the order between the integral (2.11) and the operator $\mathscr{A}^n$. Hence, appealing to (1.3) combined with (2.11), we derive

$$\mathscr{A}^n\Big(\mathrm{e}^{-x}x^{\alpha}\Big)=\frac{2^{1-\alpha}}{\pi^{3/2}\Gamma(\alpha+1/2)}\int_0^{\infty}\tau^{2n+1}\sinh(\pi\tau)|\Gamma(\alpha+i\tau)|^2K_{i\tau}(x)d\tau\ ,\ n\in\mathbb{N}_0. \tag{2.14}$$

Therefore, recalling (1.6), the expression for polynomials $p_n(x;\alpha)$, we obtain the integral representation

$$p_n(x;\alpha)=(-1)^n\ \frac{2^{1-\alpha}\ \mathrm{e}^{x}x^{-\alpha}}{\pi^{3/2}\Gamma(\alpha+1/2)}\int_0^{\infty}\tau^{2n+1}\sinh(\pi\tau)|\Gamma(\alpha+i\tau)|^2K_{i\tau}(x)d\tau\ ,\ n\in\mathbb{N}_0, \tag{2.15}$$

where the latter integral is absolutely convergent for all $x>0$ (see (2.12)).

By setting

$$F_{\alpha}(u,x)=\frac{2^{1-\alpha}}{\pi^{3/2}\Gamma(\alpha+1/2)}\ \mathrm{e}^{x}x^{-\alpha}\int_0^{\infty}\cos(\tau u)\ \tau\ \sinh(\pi\tau)\ |\Gamma(\alpha+i\tau)|^2K_{i\tau}(x)d\tau \tag{2.16}$$

and by taking into account that $\frac{\partial^{2n}F_{\alpha}(u,x)}{\partial u^{2n}}, n\in\mathbb{N}_0$, is uniformly convergent by u in $\mathbb{R}$ via the absolutely convergent integral (2.15), we obtain

$$\lim_{u\to 0}\frac{\partial^{2n}F_{\alpha}(u,x)}{\partial u^{2n}}=\frac{2^{1-\alpha}\ (-1)^n\ \mathrm{e}^{x}x^{-\alpha}}{\pi^{3/2}\Gamma(\alpha+1/2)}\int_0^{\infty}\tau^{2n+1}\sinh(\pi\tau)|\Gamma(\alpha+i\tau)|^2K_{i\tau}(x)d\tau=p_n(x;\alpha)\ ,\ n\in\mathbb{N}_0, \tag{2.17}$$

whereas

$$\lim_{u\to 0}\frac{\partial^{2n+1}F_{\alpha}(u,x)}{\partial u^{2n+1}}=0\ ,\ n\in\mathbb{N}_0. \tag{2.18}$$

As a consequence, the series representation is allowed

$$F_{\alpha}(t,x)=\sum_{n\geqslant 0}\frac{p_n(x;\alpha)}{(2n)!}\ t^{2n}\ . \tag{2.19}$$

We stress that

$$\frac{\partial^2 F_\alpha(u,x)}{\partial u^2} = -\mathrm{e}^x x^{-\alpha} \mathscr{A}\, \mathrm{e}^{-x} x^\alpha F_\alpha(u,x),$$

which, owing to (2.4), corresponds to a second order partial differential equation. Besides, the insertion of (2.19) into this latter relation yields (2.1). Moreover, we have

$$\frac{\partial^{2n} F_\alpha(u,x)}{\partial u^{2n}} = (-1)^n \mathrm{e}^x x^{-\alpha} \mathscr{A}^n \mathrm{e}^{-x} x^\alpha F_\alpha(u,x)\ ,\ n \in \mathbb{N}_0.$$

2.3. **Connection with the generalized Euler numbers.** We integrate through (2.15) multiplied by $\mathrm{e}^{-2x} x^{\alpha+\varepsilon-1}$ with respect to x over $\mathbb{R}^+$, changing the order of integration in its right-hand side due to Fubini's theorem (use inequality (2.13)). Now, we invoke that

$$\int_0^\infty x^{\varepsilon-1} \mathrm{e}^{-x} K_{i\tau}(x) dx = 2^{-\varepsilon} \sqrt{\pi} \frac{|\Gamma(\varepsilon+i\tau)|^2}{\Gamma(\varepsilon+1/2)}$$

to obtain

$$\begin{aligned} &\lim_{\varepsilon\to 0} \int_0^\infty \mathrm{e}^{-2x} x^{\alpha+\varepsilon-1} p_n(x;\alpha) dx \\ &= \lim_{\varepsilon\to 0} \frac{2^{1-\alpha-\varepsilon}}{\pi\Gamma(\alpha+1/2)\Gamma(\varepsilon+1/2)} \int_0^\infty \tau^{2n+1} \sinh(\pi\tau) |\Gamma(\alpha+i\tau)|^2 |\Gamma(\varepsilon+i\tau)|^2 d\tau\ ,\ n \in \mathbb{N}_0, \end{aligned}$$

which yields

$$\int_0^\infty \mathrm{e}^{-2x} x^{\alpha-1} p_n(x;\alpha) dx = \frac{(-1)^n 2^{1-\alpha}}{\sqrt{\pi}\Gamma(\alpha+1/2)} \int_0^\infty \tau^{2n} |\Gamma(\alpha+i\tau)|^2 d\tau\ ,\ n \in \mathbb{N}_0. \tag{2.20}$$

We recall the *Generalized Euler polynomials* $E_n^{2\alpha}(x)$ [1, Vol. III][8, 12, 15] defined through

$$\left(\frac{2}{\mathrm{e}^t+1}\right)^{2\alpha} \mathrm{e}^{xt} = \sum_{n\geqslant 0} E_n^{2\alpha}(x) \frac{t^n}{n!}\ . \tag{2.21}$$

The particular choice of $x = \alpha$ brings the identity

$$(\cosh(t/2))^{-2\alpha} = \sum_{n\geqslant 0} E_n^{2\alpha}(\alpha) \frac{t^n}{n!}\ . \tag{2.22}$$

On the other hand, [19, (1.104)]

$$(\cosh(u/2))^{-2\alpha} = \frac{2^{2\alpha-1}}{\pi\ \Gamma(2\alpha)} \int_0^\infty \cos(\tau u) |\Gamma(\alpha+i\tau)|^2 d\tau$$

providing similarly to (2.17) the equality

$$\lim_{u\to 0} \frac{\partial^{2n}}{\partial u^{2n}} (\cosh(u/2))^{-2\alpha} = \frac{(-1)^n\ 2^{2\alpha-1}}{\pi\ \Gamma(2\alpha)} \int_0^\infty \tau^{2n} |\Gamma(\alpha+i\tau)|^2 d\tau\ , \quad n \in \mathbb{N}_0.$$

Hence, (2.20) together with (2.22) imply

$$E_{2n}^{2\alpha}(\alpha) = \frac{2^{\alpha-1}}{\Gamma(\alpha)} (-1)^n \int_0^\infty \mathrm{e}^{-2x} x^{\alpha-1}\ p_n(x;\alpha) dx\ , \quad n \in \mathbb{N}_0, \tag{2.23}$$

which, according to (1.6), amounts to the same as

$$E_{2n}^{2\alpha}(\alpha)=\frac{2^{\alpha-1}}{\Gamma(\alpha)}\int_0^\infty \frac{\mathrm{e}^{-x}}{x}\mathscr{A}^n e^{-x}x^\alpha dx\ ,\quad n\in\mathbb{N}_0.$$

Aware of the many formulas that can be found in the literature, we give here a formula that, as far as we are concerned, is new in the theory.

Lemma 2.3. *For any complex parameter α such that $\Re(\alpha)>0$, the Generalized Euler polynomials of parameter 2α, $E_n^{2\alpha}(x)$ are given by*

$$E_n^{2\alpha}(x)=x^n+\sum_{k=0}^{n-1}\left\{\binom{n}{k}\sum_{\tau=1}^{n-k}(-2)^{-\tau}(2\alpha)_\tau\ \mathbf{S}(n-k,\tau)\right\}x^k \tag{2.24}$$

where $\mathbf{S}(k,\tau)$ represent the Stirling numbers of second kind.

Proof. The nth order derivative of the left-hand side of (2.21) evaluated at the point $t=0$ furnishes an expression for $E_n^{2\alpha}(x)$. The Leibniz rule for the derivative of the product of two functions permits to write

$$\frac{\partial^n}{\partial t^n}\left(\frac{2}{\mathrm{e}^t+1}\right)^{2\alpha}\mathrm{e}^{xt}\Bigg|_{t=0}=\sum_{\nu=0}^{n}\binom{n}{\nu}x^\nu\ \frac{d^{n-\nu}}{dt^{n-\nu}}\left(\frac{2}{\mathrm{e}^t+1}\right)^{2\alpha}\Bigg|_{t=0}\ ,\quad n\in\mathbb{N}_0\ .$$

According to the Faà di Bruno's formula for the kth order of derivative of the composition of functions (we refer to [3, Chapter III] for notation and a compendium of results for the Bell polynomials and the Faà di Bruno's formula), for any $k\geqslant 1$, we have

$$\frac{d^k}{dt^k}\left(\frac{2}{\mathrm{e}^t+1}\right)^{2\alpha}\Bigg|_{t=0}=\sum_{\mu=1}^{k}(\alpha)_\mu(-1)^\mu B_{k,\mu}(\frac{\mathrm{e}^t}{2},\frac{\mathrm{e}^t}{2},\frac{\mathrm{e}^t}{2},\frac{\mathrm{e}^t}{2},\dots)\Bigg|_{t=0} \tag{2.25}$$

where $B_{n,k}(x_1,x_2,\dots,x_{n-k+1})$, $1\leqslant k\leqslant n$, represents the *Bell polynomials* evaluated at the n-tuple $(x_1,x_2,\dots,x_n)$. On the basis of the properties of the Bell polynomials it follows (see [3, p.135])

$$B_{k,\mu}(\frac{\mathrm{e}^t}{2},\frac{\mathrm{e}^t}{2},\frac{\mathrm{e}^t}{2},\frac{\mathrm{e}^t}{2},\dots)=\left(\frac{\mathrm{e}^t}{2}\right)^\mu B_{k,\mu}(1,1,1,1,1,\dots)=\left(\frac{\mathrm{e}^t}{2}\right)^\mu\mathbf{S}(k,\mu)$$

with $\mathbf{S}(k,\mu)$ representing the Stirling numbers of second kind [3, Chapter V]. The insertion of this information into (2.25), ensures

$$\frac{d^k}{dt^k}\left(\frac{2}{\mathrm{e}^t+1}\right)^{2\alpha}\Bigg|_{t=0}=\sum_{\mu=1}^{k}(\alpha)_\mu(-2)^{-\mu}\mathbf{S}(k,\mu)$$

whence the result. □

With x taken equal to α in (2.24), it follows that $E_{2n+1}^{2\alpha}(\alpha)=0$, while

$$E_{2n}^{2\alpha}(\alpha)=\alpha^n+\sum_{k=1}^{n}\binom{n}{k}\alpha^{n-k}\sum_{\tau=1}^{k}(-2)^{-\tau}(2\alpha)_\tau\mathbf{S}(k,\tau)\ ,\quad n\in\mathbb{N}_0,$$

are polynomials of degree n in α. We list a few examples of the (polynomial) sequence $\{E_{2n}^{2\alpha}(\alpha)\}_{n\geqslant 0}$:

$$E_2^{2\alpha}(\alpha)=-\tfrac{\alpha}{2} \qquad E_4^{2\alpha}(\alpha)=\tfrac{3\alpha^2}{4}+\tfrac{\alpha}{4}$$
$$E_6^{2\alpha}(\alpha)=-\tfrac{15\alpha^3}{8}-\tfrac{15\alpha^2}{8}-\tfrac{\alpha}{2} \qquad E_8^{2\alpha}(\alpha)=\tfrac{105\alpha^4}{16}+\tfrac{105\alpha^3}{8}+\tfrac{147\alpha^2}{16}+\tfrac{17\alpha}{8}$$

2.4. **Connection with the Bernoulli polynomials when α is a positive integer.** Consider the (modified) falling factorial of a complex number x,

$$[x]_n := \left(\prod_{\sigma=0}^{m-1}(x-\sigma^2)\right) \ , \ n\in\mathbb{N}, \quad \text{and} \quad [x]_0 := 1.$$

This is actually a polynomial of degree n whose coefficients in the canonical basis are the so-called 0-*modified Stirling numbers of first kind*, denoted by $\widehat{s}_0(n,\nu)$, and treated in [7]. Regarding their importance for the subsequent developments, we recall

$$[x]_n=\sum_{\nu=0}^{n}\widehat{s}_0(n,\nu)x^{\nu} \ , \quad n\in\mathbb{N}_0. \tag{2.26}$$

This gives grounds for writing $|\Gamma(\alpha+i\tau)|^2$, whenever $\alpha=m\in\mathbb{N}$, as follows:

$$|\Gamma(m+i\tau)|^2=\left(\prod_{\sigma=0}^{m-1}(\tau^2+\sigma^2)\right)\frac{\pi}{\tau\sinh(\pi\tau)}=\frac{(-1)^m\pi}{\tau\sinh(\pi\tau)}[-\tau^2]_m=\frac{\pi}{\tau\sinh(\pi\tau)}\sum_{\sigma=0}^{m}(-1)^{m+\sigma}\widehat{s}_0(m,\sigma)\tau^{2\sigma}.$$

As a consequence, the generating function $F_m(u,x)$ given by (2.16) becomes

$$F_m(u,x)=C_m\ \pi\ x^{-m}\mathrm{e}^x\sum_{\sigma=0}^{m}(-1)^{m+\sigma}\widehat{s}_0(m,\sigma)\int_0^\infty\tau^{2\sigma}\cos(\tau u)K_{i\tau}(x)\ d\tau$$

which amounts to the same as

$$F_m(t,x)=C_m\ (-1)^m\ x^{-m}\sum_{\sigma=0}^{m}\frac{\pi^2}{2}\ \widehat{s}_0(m,\sigma)\ \frac{\partial^{2\sigma}}{\partial t^{2\sigma}}\left(\mathrm{e}^{-2x\sinh^2(u/2)}\right) \tag{2.27}$$

because of the equality [16, (1.12)]

$$\int_0^\infty\tau^{2n}\cos(\tau u)K_{i\tau}(x)\ d\tau=\frac{(-1)^{\sigma}\ \pi}{2}\ \frac{\partial^{2n}}{\partial t^{2n}}\mathrm{e}^{-x\cosh(u)}\ ,\ n\in\mathbb{N}_0.$$

Concomitantly, the polynomials $p_n(x;m)$ for $m\in\mathbb{N}$ are related to $p_n(x;0)$ through

$$x^m p_n(x;m)=C_m\ (-1)^m\frac{\pi^2}{2}\ \sum_{\sigma=0}^{m}\widehat{s}_0(m,\sigma)p_{n+\sigma}(x;0)\ ,\ n\geqslant 0. \tag{2.28}$$

with $C_m = \dfrac{2}{\pi} \cdot \dfrac{1}{2^m (1/2)_m}$. Recalling [16, (3.14)], it follows a connection with the Bernoulli polynomials evaluated at $\frac{1-i\tau}{2}$

$$x^m p_n(x;m) = C_m \ (-1)^m \frac{\pi^2}{2} \sum_{\sigma=0}^{m} \widehat{s}_0(m,\sigma) \frac{-2^{2(n+1)} \mathrm{e}^x}{(2n+1)\pi i} \int_0^\infty \tau \frac{K_{i\tau}(x)}{x} B_{2n+1}\left(\frac{1-i\tau}{2}\right) d\tau \ , \ n \geqslant 0. \tag{2.29}$$

Such equality has the merit of deriving analog properties for $p_n(x;m)$, when $m \in \mathbb{N}$, to those obtained in [16]. Namely, the connection with the Bernoulli or Euler numbers as well as the Bernoulli polynomials. For further considerations, we refer to [16, (3.3), (3.13)-(3.14) and (3.17)]. As a consequence of the aforementioned, we have the following result, which, as far as we are concerned, is new in the theory.

Lemma 2.4. *The Generalized Euler numbers of parameter* $2m$ *and the Bernoulli numbers are related by*

$$E_{2n}^{2m}(m) = \frac{(-1)^{n+m} 2^{2m-2}}{(2m-1)!} \sum_{\sigma=0}^{m} \frac{1-2^{2n+2\sigma}}{n+\sigma} \widehat{s}_0(m,\sigma) B_{2n+2\sigma} \ , \ n \in \mathbb{N}_0.$$

Proof. The relation (3.3) pointed out in [16] furnishes an identity between the Bernoulli numbers and the polynomials $p_n(x;0)$ by means of an integral

$$B_{2n} = \frac{n}{1-2^{2n}} \int_0^\infty \mathrm{e}^{-2x} p_n(x;0) \frac{dx}{x} \ , \ n \in \mathbb{N}_0.$$

The combination of this latter with (2.23) together with (2.28) brings to light the desired equality. □

The forthcoming developments leave aside aspects of the polynomial sequence $\{p_n(x;\alpha)\}_{n\geqslant 0}$ itself in order to embrace outer consequences. The study will be essentially focused on the dual sequence corresponding to the monic polynomial sequence, hereafter $\{P_n(x;\alpha)\}_{n\geqslant 0}$, obtained from $\{p_n(x;\alpha)\}_{n\geqslant 0}$ by dividing each of its elements by the respective leading coefficient:

$$P_n(x;\alpha) = \frac{1}{(-2)^n \ (\alpha+1/2)_n} p_n(x;\alpha) = \frac{1}{2^n \ (\alpha+1/2)_n} \ \mathrm{e}^x \ x^{-\alpha} \mathscr{A}^n \ \mathrm{e}^{-x} x^\alpha \ , \quad n \in \mathbb{N}_0. \tag{2.30}$$

3. The monic polynomial sequence and its corresponding dual sequence

As, from now on, we will deal exclusively with the monic polynomial sequences $\{P_n(x;\alpha)\}_{n\geqslant 0}$, we recreate the needed properties based on those already obtained for the original sequence. Indeed, in accordance with (2.1), $\{P_n(x;\alpha)\}_{n\geqslant 0}$ fulfills

$$x^2 \ P_n''(x;\alpha) - x\Big(2x-1-2\alpha\Big) \ P_n'(x;\alpha) - \Big((2\alpha+1)x - \alpha^2\Big) \ P_n(x;\alpha) = -\big(2n+2\alpha+1\big) P_{n+1}(x;\alpha) \tag{3.1}$$

for all $n \in \mathbb{N}_0$, whereas the equations (2.2)-(2.5) yield

$$x\,P_n(x;\alpha+1) = P_{n+1}(x;\alpha) + \frac{\alpha^2}{(2n+2\alpha+1)}\,P_n(x;\alpha), \quad n \in \mathbb{N}_0. \tag{3.2}$$

In the next §3.1 we will ferret out properties of the corresponding dual sequence, $\{u_n(\alpha)\}_{n\geqslant 0}$, which will trigger some interesting results, namely, the positive-definite character of the canonical element $u_0(\alpha)$ as it will be shown in §3.2.

3.1. **The dual sequence.** The differential and difference equations satisfied by the MPS $\{P_n(x;\alpha)\}_{n\geqslant 0}$ enable the differential and difference equations fulfilled by the corresponding dual sequence $\{u_n(\alpha)\}_{n\geqslant 0}$.

Lemma 3.1. *The dual sequence $\{u_n(\alpha)\}_{n\geqslant 0}$ associated to $\{P_n(x;\alpha)\}_{n\geqslant 0}$ is such that*

$$\Big((x^2u_0(\alpha))' + x(2x-(1+2\alpha))u_0(\alpha)\Big)' - ((1+2\alpha)x-\alpha^2)u_0(\alpha) = 0 \tag{3.3}$$

$$\Big((x^2u_{n+1}(\alpha))' + x(2x-(1+2\alpha))u_{n+1}(\alpha)\Big)' - ((1+2\alpha)x-\alpha^2)u_{n+1}(\alpha) = (2n+2\alpha+1)u_n(\alpha) \tag{3.4}$$

for $n \in \mathbb{N}_0$. Moreover the moments of the canonical form u_0 are given by

$$(u_0(\alpha))_n = \frac{[(\alpha)_n]^2}{2^n(\alpha+1/2)_n} \quad , \quad n \in \mathbb{N}_0. \tag{3.5}$$

Proof. The action of u_0 over the relation (3.1) furnishes

$$\langle u_0(\alpha), -x^2\,P_n''(x) + x\Big(2x-1-2\alpha\Big)\,P_n'(x) + \Big((2\alpha+1)x-\alpha^2\Big)\,P_n(x)\rangle = 0, \quad n \geqslant 0,$$

which, by transposition, in accordance with (1.8), becomes

$$\Big\langle -\big(x^2\,u_0(\alpha)\big)'' - \big(x\big(2x-1-2\alpha\big)\,u_0(\alpha)\big)' + \Big((2\alpha+1)x-\alpha^2\Big)u_0(\alpha)\ ,\ P_n(x)\Big\rangle = 0, \quad n \geqslant 0,$$

whence (3.3). Likewise, the action of u_{k+1} over (3.1) provides

$$\Big\langle u_{k+1}(\alpha)\ ,\ -x^2\,P_n''(x) + x\Big(2x-1-2\alpha\Big)\,P_n'(x) + \Big((2\alpha+1)x-\alpha^2\Big)\,P_n(x)\Big\rangle = (2n+2\alpha+1)\,\delta_{n,k},\ n \geqslant 0,$$

and, again, by transposition, this latter leads to

$$\langle -\big(x^2\,u_{k+1}(\alpha)\big)'' - \big(x\big(2x-1-2\alpha\big)\,u_{k+1}(\alpha)\big)' + \big((2\alpha+1)x-\alpha^2\big)u_{k+1}(\alpha)\ ,\ P_n(x)\rangle = (2n+2\alpha+1)\,\delta_{n,k},\ n \geqslant 0,$$

therefore, according to (1.7), we derive (3.4).On the other hand, the action of both sides of (3.3) over each element of the sequence $\{x^n\}_{n\geqslant 0}$ leads to the following difference equation having the moments of $u_0(\alpha)$ as solution

$$(2n+2\alpha+1)\big(u_0(\alpha)\big)_{n+1} = -(n+\alpha)^2\big(u_0(\alpha)\big)_n \quad , \quad n \in \mathbb{N}_0.$$

providing (3.5). □

In addition, according to (1.7), we have as well

$$xu_0(\alpha) = \sum_{\nu\geqslant 0} \langle x\ u_0(\alpha), P_n(x;\alpha+1)\rangle u_n(\alpha+1).$$

Considering (1.8) and (3.2), it follows

$$\langle x\ u_0(\alpha), P_n(x;\alpha+1)\rangle = \langle u_0(\alpha), P_{n+1}(x;\alpha) + \frac{\alpha^2}{2n+2\alpha+1}P_n(x;\alpha)\rangle = \frac{\alpha^2}{2n+2\alpha+1}\delta_{n,0},$$

that holds for any $n\in\mathbb{N}_0$. Therefore the two forms $u_0(\alpha)$ and $u_0(\alpha+1)$ are related by

$$x\ u_0(\alpha) = \frac{\alpha^2}{2\alpha+1}\ u_0(\alpha+1)\ . \tag{3.6}$$

The question on whether the MPS $\{P_n(x;\alpha)\}_{n\geqslant 0}$ can be orthogonal arises in a natural way and, concomitantly, on whether the form $u_0(\alpha)$ is or is not regular.

As a matter of fact, we recall that a form v is said to be *regular* if we can associate with it a PS $\{Q_n\}_{n\geqslant 0}$ such that $\langle v, Q_nQ_m\rangle = k_n\delta_{n,m}$ with $k_n\neq 0$ for all $n,m\in\mathbb{N}_0$ [9, 10]. The PS $\{Q_n\}_{n\geqslant 0}$ is then said to be orthogonal with respect to v and we can assume the system (of orthogonal polynomials) to be monic. Therefore, there exists a dual sequence $\{v_n\}_{n\geqslant 0}$ and the original form is proportional to v_0. Furthermore, it holds

$$v_{n+1} = \left(\langle v_0, Q_{n+1}^2(\cdot)\rangle\right)^{-1}\ Q_{n+1}(x)v_0\ ,\ n\in\mathbb{N}_0. \tag{3.7}$$

Moreover, when $v\in\mathscr{P}'$ is regular, let Φ be a polynomial such that $\Phi v=0$, then $\Phi=0$ [9, 10].

In this case we call this unique MPS $\{Q_n(x)\}_{n\geqslant 0}$ as *monic orthogonal polynomial sequence* - hereafter MOPS - and it can be characterized by the popular second order recurrence relation

$$\begin{cases} Q_0(x;\alpha)=1 \quad ; \quad Q_1(x;\alpha) = x-\beta_0(\alpha) \\ Q_{n+2}(x;\alpha) = (x-\beta_{n+1}(\alpha))Q_{n+1}(x;\alpha) - \gamma_{n+1}(\alpha)\,Q_n(x;\alpha) \quad , \quad n\in\mathbb{N}_0. \end{cases} \tag{3.8}$$

Here, we consider the dependence on the complex parameter α to avoid repetition of similar formulas. We will systematically refer to the pair $(\beta_n(\alpha),\gamma_{n+1}(\alpha))_{n\geqslant 0}$ as the recurrence coefficients of $\{Q_n(x;\alpha)\}_{n\geqslant 0}$, necessarily fulfilling $\gamma_{n+1}(\alpha)\neq 0$, $n\geqslant 0$. See [9, 10] for notation and a compendium of results about algebraic properties of orthogonal polynomial sequences along with regular forms.

Somehow expected, we have the following result.

Lemma 3.2. *The MPS $\{P_n(x;\alpha)\}_{n\geqslant 0}$ cannot be regularly orthogonal.*

Proof. Under the assumption that $\{P_n(x;\alpha)\}_{n\geqslant 0}$ defined through (2.30) is orthogonal, we may insert relation (3.7), with v_n replaced by $u_n(\alpha)$ and Q_n by $P_n(\cdot;\alpha)$, in (3.4) which provides

$$\begin{aligned}\Big((x^2P_{n+1}(x;\alpha)u_0(\alpha))' + x(2x-(1+2\alpha))P_{n+1}(x;\alpha)u_0(\alpha)\Big)' \\ -((1+2\alpha)x-\alpha^2)P_{n+1}(x;\alpha)u_0(\alpha) = \lambda_n(\alpha)P_n(x;\alpha)u_0(\alpha), \quad n\in\mathbb{N}_0,\end{aligned} \tag{3.9}$$

where $\lambda_n(\alpha) = (2n+2\alpha+1) < u_0(\alpha), P_{n+1}^2(x;\alpha) > \big(< u_0(\alpha), P_n^2(x;\alpha) >\big)^{-1}$, for all $n \in \mathbb{N}_0$. Taking into account (3.3), then (3.9) implies the differential equation

$$2P'_{n+1}(x;\alpha)(x^2u_0(\alpha))' + \Big(x^2P''_{n+1}(x;\alpha) + x(2x-(1+2\alpha)P'_{n+1}(x;\alpha)\Big)u_0(\alpha) = \lambda_n(\alpha)P_n(x;\alpha)u_0(\alpha),\ n \geqslant 0.$$

When the equation obtained by the particular choice of $n=0$, *i.e.*

$$2(x^2u_0(\alpha))' + x(2x-(1+2\alpha)u_0(\alpha) = \lambda_0(\alpha)u_0(\alpha),$$

is inserted into the original equation, we obtain

$$\Big(x^2P''_{n+1}(x;\alpha)\Big)u_0(\alpha) = \Big(\lambda_n(\alpha)P_n(x;\alpha) - \lambda_0(\alpha)P'_{n+1}(x;\alpha)\Big)u_0(\alpha).$$

The regularity of $u_0(\alpha)$ would now imply the condition

$$x^2P''_{n+1}(x;\alpha) = \lambda_n(\alpha)P_n(x;\alpha) - \lambda_0(\alpha)P'_{n+1}(x;\alpha)\ ,\ n \geqslant 0,$$

contradicting $\deg P_n(x;\alpha) = n$, and therefore crumbling the possibility of $\{P_n(x;\alpha)\}$ to be orthogonal. □

Despite the non-(regular)orthogonality of $\{P_n(x;\alpha)\}_{n\geqslant 0}$ with respect to the form $u_0(\alpha)$, we cannot exclude the existence of an orthogonal polynomial sequence, say $\{Q_n(x;\alpha)\}_{n\geqslant 0}$, with respect to $u_0(\alpha)$, which amounts to the same as ensuring the regularity of $u_0(\alpha)$. This question is handled in the next section.

3.2. **About the regularity of u_0.** We begin by rewriting (3.3) as follows

$$\Big((\phi u_0(\alpha))' + \psi u_0(\alpha)\Big)' + \chi u_0(\alpha) = 0 \tag{3.10}$$

with

$$\phi(x) := \phi(x;\alpha) = x^2,\ \psi(x) := \psi(x;\alpha) = x(2x-2\alpha-1),\ \chi(x) := \chi(x;\alpha) = -(2\alpha+1)x+\alpha^2. \tag{3.11}$$

Actually, while seeking an integral representation for $u_0(\alpha)$, we realize that it is indeed regular.

Lemma 3.3. *For any positive real value of the parameter α, the form $u_0(\alpha)$ is positive definite (therefore regular) admitting the integral representation*

$$\langle u_0, f(x)\rangle = \frac{2^\alpha\Gamma(\alpha+1/2)}{\sqrt{\pi}\ \Gamma(\alpha)^2}\int_0^{+\infty} f(x)\ e^{-x}x^{\alpha-1}K_0(x)\ dx \quad ,\quad \forall f \in \mathscr{P}. \tag{3.12}$$

Proof. We seek a function $U(x) := U(x;\alpha)$ such that (3.12) holds in a certain domain C. Since $\langle u_0, 1\rangle = 1 \neq 0$, we must have

$$\int_C U(x)dx = 1 \neq 0 \tag{3.13}$$

By virtue of (3.10), we have, for any $f \in \mathscr{P}$

$$\begin{aligned} 0 &= \langle \Big((\phi(x)u_0)' + \psi(x)u_0 \Big)' + \chi(x)u_0, f(x) \rangle = \langle u_0, \phi(x) f''(x) + \psi(x) f'(x) + \chi(x) f(x) \rangle \\ &= \int_C \big((\phi(x)U(x))'' + (\psi(x)U(x))' + \chi(x)U(x) \big) f(x) dx \\ &\quad - \Big(\phi(x)U(x) f'(x) - (\phi(x)U(x))' f(x) - \psi(x)U(x) f(x) \Big) \Big|_C \end{aligned}$$

therefore, $U(x)$ is a function simultaneously fulfilling

$$\int_C \big((\phi(x)U(x))'' + (\psi(x)U(x))' + \chi(x)U(x) \big) f(x) dx = 0 \quad , \quad \forall f \in \mathscr{P} \tag{3.14}$$

$$\Big(\phi(x)U(x) f'(x) - (\phi(x)U(x))' f(x) - \psi(x)U(x) f(x) \Big) \Big|_C = 0 \quad , \quad \forall f \in \mathscr{P} \ . \tag{3.15}$$

The first equation implies

$$(\phi(x)U(x))'' + (\psi(x)U(x))' + \chi(x)U(x) = \lambda g(x)$$

where λ is a complex number and $g(x) \neq 0$ is a function representing the null form, that is, a function such that

$$\int_C g(x) f(x) dx = 0 \quad , \quad \forall f \in \mathscr{P}.$$

We begin by choosing $\lambda = 0$ and we search a regular solution of the differential equation

$$(\phi(x)U(x))'' + (\psi(x)U(x))' + \chi(x)U(x) = 0$$

Upon the change of variable $U(x) = \mathrm{e}^{-x} x^{\alpha-1} y(x)$ we have

$$x^2 y''(x) + x y'(x) - x^2 y(x) = 0 \quad \Leftrightarrow \quad \mathscr{A}\Big(y(x) \Big) = 0$$

whose general solution is: $y(x) = c_1 I_0(x) + c_2 K_0(x)$, $x \geqslant 0$, for some arbitrary constants c_1, c_2 and $y(x) = 0$ when $x < 0$ [1]. As a consequence $U(x) = \mathrm{e}^{-x} x^{\alpha-1} \Big\{ c_1 I_0(x) + c_2 K_0(x) \Big\}$, $x \geqslant 0$. Insofar as $U(x)$ must be a rapidly decreasing sequence (that is, such that $\lim_{x\to+\infty} f(x)U(x) = 0$ for any polynomial f), we set $c_1 = 0$ and $c_2 \neq 0$ and we write $U(x) = c_2 \mathrm{e}^{-x} x^{\alpha-1} K_0(x)$, $x \in C =]0, +\infty[$ with c_2 set in order to realize (3.13), which amounts to the same as $c_2 = \frac{2^\alpha \Gamma(\alpha+1/2)}{\sqrt{\pi}\ \Gamma(\alpha)^2}$. In this case $U(x)$ also fulfills the condition (3.13) as long as $\alpha > 0$ because $\int_0^{+\infty} \mathrm{e}^{-x} x^{\alpha-1} K_0(x) dx$ is a stricktly positive convergent integral. Besides, since $K_\nu(x)$ has the asymptotic behaviour with respect to x [1, Vol. II][19]

$$K_\nu(x) = \Big(\frac{\pi}{2x} \Big)^{1/2} \mathrm{e}^{-x} [1 + O(1/x)], \quad x \to +\infty$$

$$K_\nu(x) = O(x^{-\Re(\nu)}) \ , K_0(x) = O(-\log x) \ , \quad x \to 0,$$

then we immediately deduce that (3.15) (3.15) is fulfilled by any element of the PS $\{x^n\}_{n\geqslant 0}$ (that spans $\mathscr{P}$) and hence it is fulfilled by any element f of $\mathscr{P}$. Moreover, for every polynomial $g(x)$ that is not identically zero and is non-negative for all real x we have $\langle u_0, g(x) \rangle = \dfrac{2^\alpha \Gamma(\alpha+1/2)}{\sqrt{\pi}\ \Gamma(\alpha)^2} \displaystyle\int_0^{+\infty} g(x)\ \mathrm{e}^{-x} x^{\alpha-1} K_0(x)\ dx > 0$ and therefore u_0 is **positive-definite** form. This implies that u_0 has real moments and a corresponding MOPS exists (*i.e.*, u_0 is a regular form). □

The regularity of $u_0(\alpha)$ raises the problem of characterizing a corresponding orthogonal polynomial sequence, say $\{Q_n(x;\alpha)\}_{n\geqslant 0}$, whenever α is a positive real number. From this point forth we consider $\alpha \in \mathbb{R}^+$.

Entailed in the question of characterizing the arisen MOPS $\{Q_n(x;\alpha)\}_{n\geqslant 0}$ (with respect to $u_0(\alpha)$), comes out the problem of determining the associated recurrence coefficients or even a differential equation fulfilled by $\{Q_n(x;\alpha)\}_{n\geqslant 0}$ or an analogue of Rodrigues' formula or a generating function. However, this has revealed to be a tricky problem to solve, reminding other open problems such as the one posed by Prudnikov [14]. Indeed, the difficult part of this problem is connected to the fact that the regular form $u_0(\alpha)$ is solution of a differential equation of order higher than one. As far as we can tell the gap in the theory concerned to this problem, avoids us to attain further results. Notwithstanding this, based on the work [11], it is possible to reach a finite-type relation (of order two actually) between the two MOPSs $\{Q_n(x;\alpha)\}_{n\geqslant 0}$ and $\{Q_n(x;\alpha+1)\}_{n\geqslant 0}$. Again, this adds very few to the answer of the problem.

The recurrence coefficients $(\beta_n(\alpha),\gamma_{n+1}(\alpha))_{n\geqslant 0}$ associated to the second order recurrence relation (3.8) fulfilled by $\{Q_n(\cdot;\alpha)\}_{n\geqslant 0}$ may be successively computed either by means of the weight function or using the Hankel determinant of the moments of $u_0(\alpha)$ [2]. Making use of the first process, for each positive integer n we have

$$\beta_n(\alpha) = \frac{\langle u_0(\alpha), xQ_n^2(x;\alpha)\rangle}{\langle u_0(\alpha), Q_n^2(x;\alpha)\rangle} = \frac{\displaystyle\int_0^{+\infty} Q_n^2(x;\alpha)\ \mathrm{e}^{-x}x^{\alpha}K_0(x)\ dx}{\displaystyle\int_0^{+\infty} Q_n^2(x;\alpha)\ \mathrm{e}^{-x}x^{\alpha-1}K_0(x)\ dx} \tag{3.16}$$

$$\gamma_{n+1}(\alpha) = \frac{\langle u_0(\alpha), Q_{n+1}^2(x;\alpha)\rangle}{\langle u_0(\alpha), Q_n^2(x;\alpha)\rangle} = \frac{\displaystyle\int_0^{+\infty} Q_{n+1}^2(x;\alpha)\ \mathrm{e}^{-x}x^{\alpha-1}K_0(x)\ dx}{\displaystyle\int_0^{+\infty} Q_n^2(x;\alpha)\ \mathrm{e}^{-x}x^{\alpha-1}K_0(x)\ dx}. \tag{3.17}$$

According to this latter, we list the first elements

$$\begin{array}{llll} \beta_0(\alpha) = \frac{\alpha^2}{2\alpha+1} & , & \beta_1(\alpha) = \frac{\alpha(2\alpha(\alpha+4)+7)(\alpha(2\alpha+5)+4)+4}{(2\alpha+1)(2\alpha+5)(2\alpha(\alpha+2)+1)} & \\ \gamma_1(\alpha) = \frac{\alpha^2(2\alpha(\alpha+2)+1)}{(2\alpha+1)^2(2\alpha+3)} & , & \gamma_2(\alpha) = \frac{4(\alpha+1)^2(2\alpha+1)(\alpha(\alpha(2\alpha(\alpha(2\alpha(\alpha+12)+113)+262)+613)+325)+51)}{(2\alpha+3)(2\alpha+5)^2(2\alpha+7)(2\alpha(\alpha+2)+1)^2} & . \end{array}$$

The determination of recurrence coefficients of higher order has indeed revealed to be a ticklish problem.

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