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The generalized Bochner condition about classical orthogonal polynomials revisited ☆

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Abstract

We bring a new proof for showing that an orthogonal polynomial sequence is classical if and only if any of its polynomial fulfils a certain differential equation of order $2k$, for some $k \geqslant 1$. So, we build those differential equations explicitly. If $k = 1$, we get the Bochner's characterization of classical polynomials. With help of the formal computations made in *Mathematica*, we explicitly give those differential equations for $k = 1, 2$ and 3 for each family of the classical polynomials. Higher order differential equations can be obtained similarly.

Keywords: Classical orthogonal polynomials; Classical forms; Bochner's differential equation

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1. Introduction and preliminary results

Since the work in [5], one has known that the linear two-order Bochner equation can be generalized into a linear $2k$-order equation ($k \geqslant 1$) for characterizing classical orthogonal polynomials. But in that paper, the authors did not give an explicit and precise expression of the generalized equation. On the other hand, here, given a classical sequence, we build the necessary equation fulfilled by any classical polynomial (see Theorem 2.1). The structure of the coefficients of the equation is carefully revealed, which allows to use the technology of formal calculus (see Section 4). Concerning the reciprocal result, we bring a new proof more illuminating than the one already known (see Section 3).

The field of complex numbers is denoted as $\mathbb{C}$. The vector space of polynomials with coefficients in $\mathbb{C}$ is represented as $\mathcal{P}$ and its dual space is represented as $\mathcal{P}'$. We will simply call *polynomial* to every element of $\mathcal{P}$ and *form* to the elements in $\mathcal{P}'$. We denote by $\langle u, f\rangle$ the action of $u \in \mathcal{P}'$ on $f \in \mathcal{P}$. In particular, we denote the moments of u (with respect to the sequence $\{x^n\}_{n\geqslant 0}$) by $(u)_n := \langle u, x^n\rangle$, $n \geqslant 0$. For any form u and any polynomial h, we let $Du = u'$ and hu be the forms defined by duality as

$$\langle u', f\rangle := -\langle u, f'\rangle, \quad \langle hu, f\rangle := \langle u, hf\rangle, \quad f \in \mathcal{P}. \tag{1.1}$$

Throughout the text, the kth derivative of $p \in \mathcal{P}$ is denoted either as $D^k p$ or $(p)^{(k)}$.

We recall the definition of right-multiplication of a form by a polynomial [9]:

$$(up)(x) := \left\langle u, \frac{xp(x) - \xi p(\xi)}{x - \xi}\right\rangle, \quad u \in \mathcal{P}',\ p \in \mathcal{P}.$$

By duality, we obtain the Cauchy's product of two forms [9]:

$$\langle uv, p\rangle := \langle u, vp\rangle, \quad u, v \in \mathcal{P}',\ p \in \mathcal{P}.$$

In the sequel we shall need the following formulas, which can be easily obtained from the definitions.

For any $f \in \mathcal{P}$ and $u, v \in \mathcal{P}'$, we have:

$$D^k(fu) = \sum_{\nu=0}^{k}\left[\binom{k}{\nu}\left(D^\nu f\right)\left(D^{k-\nu}u\right)\right], \quad k \geqslant 1, \tag{1.2}$$

$$D^n u = D^n v \quad \Rightarrow \quad u = v, \quad n \geqslant 0. \tag{1.3}$$

For other properties see, among others, [9–12] or [7,8].

We will only consider sequences of polynomials $\{P_n\}_{n\geqslant 0}$ such that $\deg P_n \leqslant n$, $n \geqslant 0$. If the set $\{P_n\}_{n\geqslant 0}$ spans $\mathcal{P}$, which occurs when $\deg P_n = n$, $n \geqslant 0$, then it will be called a *polynomial sequence* (PS). Along the text it will only be considered *monic polynomial sequences* (MPS). It is always possible to associate to $\{P_n\}_{n\geqslant 0}$ a unique sequence $\{u_n\}_{n\geqslant 0}$, $u_n \in \mathcal{P}'$, $n \geqslant 0$, called the *dual sequence* of $\{P_n\}_{n\geqslant 0}$, and such that $\langle u_n, P_m\rangle := \delta_{n,m}$, $n, m \geqslant 0$, where $\delta_{n,m}$ represents the Kronecker's symbol.

Lemma 1.1. [9] *For any $u \in \mathcal{P}'$ and any integer $m \geqslant 1$, the following statements are equivalent*:

(1) $\langle u, P_{m-1}\rangle \neq 0$, $\langle u, P_n\rangle = 0$, $n \geqslant m$.
(2) $\exists \lambda_\nu \in \mathbb{C}$, $0 \leqslant \nu \leqslant m-1$, $\lambda_{m-1} \neq 0$ *such that* $u = \sum_{\nu=0}^{m-1} \lambda_\nu u_\nu$.

Furthermore, $\lambda_\nu = \langle u, P_\nu\rangle$, $0 \leqslant \nu \leqslant m-1$.

We call the *normalized derivative sequence* of $\{P_n\}_{n\geqslant 0}$ to the sequence $\{P_n^{[1]}\}_{n\geqslant 0}$ defined as follows:

$$P_n^{[1]}(x) := \frac{P'_{n+1}(x)}{n+1}, \quad n \geqslant 0.$$

The normalized derivative sequences of higher orders, say $k \geqslant 1$, are recursively defined by

$$P_n^{[k+1]}(x) = \frac{(P_{n+1}^{[k]}(x))'}{n+1}, \quad n \geqslant 0. \tag{1.4}$$

As a consequence of Lemma 1.1, we have that the dual sequence associated to $\{P_n^{[k]}\}_{n\geqslant 0}$, say $\{u_n^{[k]}\}_{n\geqslant 0}$, fulfils the recurrence relation:

$$Du_n^{[k]} = -(n+1)u_{n+1}^{[k-1]}, \quad \text{with } u_n^{[0]} = u_n,\ n \geqslant 0. \tag{1.5}$$

Writing $v_n = u_n^{[k]}$, the kth derivative of v_n is given by

$$(v_n)^{(k)} = (-1)^k \prod_{\mu=1}^{k} (n+\mu) u_{n+k}, \quad n \geqslant 0,\ k \geqslant 1, \tag{1.6}$$

which can be easily obtained by finite induction (see [7,8]).

A form u is called *regular* if we can associate with it a sequence $\{P_n\}_{n\geqslant 0}$ such that

$$\langle u, P_n P_m\rangle = r_n \delta_{n,m} \quad \text{with } r_n \neq 0,\ n \geqslant 0.$$

In this case, $\{P_n\}_{n\geqslant 0}$ is called an *orthogonal polynomial sequence* (OPS) with respect to u. From the definition, it results that every OPS is a PS. Therefore, if each element is taken monic, the sequence is said to be a *monic orthogonal polynomial sequence* (MOPS).

Theorem 1.2. [2,9,10] *Let $\{P_n\}_{n\geqslant 0}$ be a PS and $\{u_n\}_{\geqslant 0}$ its dual sequence. Then the following statements are equivalent*:

(1) *The sequence $\{P_n\}_{n\geqslant 0}$ is orthogonal with respect to u_0.*

(2) $$\begin{cases} P_0(x) = 1, \quad P_1(x) = x - \beta_0, \\ P_{n+2}(x) = (x - \beta_{n+1})P_{n+1}(x) - \gamma_{n+1}P_n(x), \quad \gamma_{n+1} \neq 0,\ n \geqslant 0, \end{cases}$$

with $\beta_{n+1} = \frac{\langle u, xP_{n+1}^2\rangle}{\langle u, P_{n+1}^2\rangle}$ *and* $\gamma_{n+1} = \frac{\langle u, P_{n+1}^2\rangle}{\langle u, P_n^2\rangle}$.

(3) $xu_n = u_{n-1} + \beta_n u_n + \chi_{n,n} u_{n+1}$, *where* $\beta_n = \langle u_n, xP_n\rangle$ *and* $\chi_{n,n} = \langle u_n, xP_{n+1}\rangle \neq 0$, $n \geqslant 0$, $u_{-1} = 0$.

(4) $u_n = (\langle u_0, P_n^2\rangle)^{-1} P_n u_0$, $n \geqslant 0$.

From the statement (2) of the previous theorem we conclude that

$$\langle u_0, P_{n+1}^2\rangle = \prod_{\nu=0}^{n} \gamma_{\nu+1}, \quad n \geqslant 0. \tag{1.7}$$

Lemma 1.3. [7–10] *For any regular form u and any polynomial A such that $Au = 0$, we necessarily have $A = 0$.*

The pair (Φ, Ψ) of two polynomials, such that Φ is monic, $\deg(\Phi) = t$ and $\deg(\Psi) = p \geqslant 1$, is said to be *admissible* when the equation that it generates,

$$D(\Phi u) + \Psi = 0, \tag{1.8}$$

possesses at least one solution fulfilling $(u)_0 = 1$, see [10].

Let $s = \max(\deg(\Phi) - 2, \deg(\Psi) - 1) = (t - 2, p - 1)$.

The form u is called *semi-classical* if it is regular and satisfies (1.8), where the pair (Φ, Ψ) is admissible. The sequence $\{P_n\}_{n\geqslant 0}$ orthogonal with respect to u ($u = \lambda u_0$) is also called *semi-classical*.

When $s = 0$, u is a *classical* form and we get the *classical orthogonal polynomials* (Hermite, Laguerre, Bessel and Jacobi), see [10] or [11]. In this case, let us recall the following result:

Theorem 1.4. *For any orthogonal sequence $\{P_n\}_{n\geqslant 0}$, the following statements are equivalent*:

(1) *The sequence $\{P_n\}_{n\geqslant 0}$ is classical.*
(2) *The sequence $\{P_n^{[1]}\}_{n\geqslant 0}$ is orthogonal* (*Hahn's property*) [3,4].
(3) *There exists a $k \geqslant 1$ such that $\{P_n^{[k]}\}_{n\geqslant 0}$ is orthogonal* (*Hahn's theorem*) [4,12].
(4) *There exists two polynomials, Φ monic, $\deg \Phi \leqslant 2$, Ψ, $\deg \Psi = 1$ and a sequence $\{\lambda_n\}_{n\geqslant 0}$, $\lambda_n \neq 0$, $n \geqslant 0$ such that*

$$\Phi P''_{n+1} - \Psi P'_{n+1} + \lambda_n P_{n+1} = 0, \quad n \geqslant 0 \quad (\textit{Bochner's property}) \text{ [1]}.$$

(5) *There exists two polynomials, Φ monic, $\deg \Phi \leqslant 2$, and Ψ, $\deg \Psi = 1$ such that*

$$D(\Phi u_0) + \Psi u_0 = 0.$$

Corollary 1.5. [7–10] *If the sequence $\{P_n\}_{n\geqslant 0}$ is classical, then so is $\{P_n^{[k]}\}_{n\geqslant 0}$, whenever $k \geqslant 1$, and any polynomial $P_{n+1}^{[k]}$ fulfils the following differential equation*:

$$\Phi \big(P_{n+1}^{[k]}\big)'' - (\Psi - k\Phi')\big(P_{n+1}^{[k]}\big)' + \lambda_n^{[k]}\big(P_{n+1}^{[k]}\big) = 0. \tag{1.9}$$

Our main aim is to generalize the Bochner's property, mentioned above.

First we give a differential equation of order $2k$ fulfilled by any classical polynomial sequence. In the third section we show that if an OPS fulfills a certain differential equation of order $2k$, it is necessarily a classical sequence. This allows us to characterize the classical sequences. In the final section, we present some of the results obtained in the *Mathematica*.

2. Generalized Bochner's differential equation

For the sake of simplicity, let us denote $Q_n := P_n^{[k]}$ and $\{v_n\}_{n\geqslant 0}$ the dual sequence of $\{Q_n\}_{n\geqslant 0}$ ($v_n = u_n^{[k]}$).

Theorem 2.1. *Let $\{P_n\}_{n\geqslant 0}$ be an OPS. Suppose there is an integer $k \geqslant 1$ such that $\{Q_n\}_{n\geqslant 0}$ is an OPS. Then any polynomial P_{n+k} fulfills the following differential equation of order $2k$:*

$$\sum_{\nu=0}^{k} \Lambda_\nu(k;x)(P_{n+k})^{(k+\nu)}(x) = \Xi_n(k)P_{n+k}(x), \quad n \geqslant 0, \tag{2.10}$$

where

$$\Lambda_\nu(k;x) = \frac{1}{\nu!}\sum_{\mu=0}^{\nu} \lambda_\mu^k \Omega_{\nu-\mu}^k(\nu;x)P_{k+\mu}(x), \quad 0 \leqslant \nu \leqslant k, \tag{2.11}$$

$$\Xi_n(k) = \lambda_n^k \prod_{\mu=1}^{k}(n+\mu), \quad n \geqslant 0, \tag{2.12}$$

$$\lambda_n^k = (-1)^k \frac{\langle v_0, Q_n^2\rangle}{\langle u_0, P_{n+k}^2\rangle}\prod_{\mu=1}^{k}(n+\mu), \quad n \geqslant 0, \tag{2.13}$$

and

$$\begin{cases} \Omega_0^k(0;\cdot) = 1, \\ \Omega_0^k(\mu+1;\cdot) = 1, \quad \mu \geqslant 0, \\ \Omega_{\mu+1-\xi}^k(\mu+1;\cdot) = -\sum_{\nu=\xi}^{\mu} \frac{1}{\nu!}(Q_{\mu+1})^{(\nu)}\Omega_{\nu-\xi}^k(\nu;\cdot), \quad 0 \leqslant \xi \leqslant \mu. \end{cases} \tag{2.14}$$

Proof. As $\{P_n\}_{n\geqslant 0}$ and $\{Q_n\}_{n\geqslant 0}$ are OPS, from Theorem 1.2(4) we have

$$u_n = \left(\langle u_0, P_n^2\rangle\right)^{-1} P_n u_0, \quad n \geqslant 0, \tag{2.15}$$

$$v_n = \left(\langle v_0, Q_n^2\rangle\right)^{-1} Q_n v_0, \quad n \geqslant 0. \tag{2.16}$$

Recalling (1.6), the relation (2.16) becomes

$$(Q_n v_0)^{(k)} = \lambda_n^k P_{n+k} u_0, \quad n \geqslant 0, \tag{2.17}$$

with

$$\lambda_n^k = (-1)^k \frac{\langle v_0, Q_n^2\rangle}{\langle u_0, P_{n+k}^2\rangle}\prod_{\mu=1}^{k}(n+\mu), \quad n \geqslant 0. \tag{2.18}$$

Using the Leibniz relation (1.2), we have

$$(Q_n v_0)^{(k)} = \sum_{\nu=0}^{k}\binom{k}{\nu}(Q_n)^{(\nu)}(v_0)^{(k-\nu)}, \quad n \geqslant 0, \tag{2.19}$$

which allows us to determine $(v_0)^{(k-\nu)}$, $0 \leqslant \nu \leqslant k$, if we use (2.19) in (2.17):

$$\sum_{\nu=0}^{k}\binom{k}{\nu}(Q_n)^{(\nu)}(v_0)^{(k-\nu)} = \lambda_n^k P_{n+k}u_0, n \geqslant 0. \tag{2.20}$$

Since $(Q_n)^{(\nu)} = 0$ whenever $\nu \geqslant n+1$, we have

$$\sum_{\nu=0}^{n}\binom{k}{\nu}(Q_n)^{(\nu)}(v_0)^{(k-\nu)} = \lambda_n^k P_{n+k}u_0, \quad 0 \leqslant n \leqslant k. \tag{2.21}$$

Taking $n=0$ and $n=1$ in (2.21) we respectively obtain

$$(v_0)^{(k)} = \lambda_0^k P_k u_0 \tag{2.22}$$

and

$$k(v_0)^{(k-1)} = \big(\lambda_1^k P_{k+1} - \lambda_0^k Q_1 P_k\big)u_0. \tag{2.23}$$

Let us now suppose that

$$\begin{cases} \Omega_0^k(\nu;x) = 1, \\ \frac{k!}{(k-\nu)!}(v_0)^{(k-\nu)} = \big(\sum_{\zeta=0}^{\nu} \lambda_\zeta^k \Omega_{\nu-\zeta}^k(\nu;x) P_{k+\zeta}(x)\big)u_0, \end{cases} \qquad 0 \leqslant \nu \leqslant \mu < k. \tag{2.24}$$

Following (2.22) and (2.23), we have

$$\Omega_0^k(0;x) = 1,$$
$$\Omega_1^k(1;x) = -Q_1(x), \qquad \Omega_0^k(1;x) = 1. \tag{2.25}$$

For $n = \mu + 1$, (2.21) becomes like

$$\frac{k!}{(k-\mu-1)!}(v_0)^{(k-\mu-1)} = \lambda_{\mu+1}^k P_{\mu+1+k} u_0 - \sum_{\nu=0}^{\mu} \binom{k}{\nu} (Q_{\mu+1})^{(\nu)} (v_0)^{(k-\nu)}.$$

Taking into account the assumption (2.24), we get

$$\frac{k!}{(k-\mu-1)!}(v_0)^{(k-\mu-1)}$$
$$= \left[\lambda_{\mu+1}^k P_{\mu+1+k}(x) - \sum_{\nu=0}^{\mu}\sum_{\zeta=0}^{\nu} \frac{1}{\nu!}\big(Q_{\mu+1}(x)\big)^{(\nu)} \lambda_\zeta^k \Omega_{\nu-\zeta}^k(\nu;x) P_{k+\zeta}(x)\right]u_0,$$

which may be expressed as

$$\frac{k!}{(k-\mu-1)!}(v_0)^{(k-\mu-1)} = \left[\lambda_{\mu+1}^k P_{\mu+1+k} - \sum_{\zeta=0}^{\mu} \lambda_\zeta^k P_{k+\zeta} \sum_{\nu=\zeta}^{\mu} \frac{1}{\nu!} (Q_{\mu+1})^{(\nu)} \Omega_{\nu-\zeta}^k(\nu;\cdot)\right]u_0.$$

This last relation is read as

$$\frac{k!}{(k-\mu-1)!}(v_0)^{(k-\mu-1)} = \sum_{\zeta=0}^{\mu+1} \lambda_\zeta^k \Omega_{\mu+1-\zeta}^k(\mu+1;\cdot) P_{k+\zeta} u_0, \tag{2.26}$$

by virtue of (2.14). Substituting $(v_0)^{(k-\nu)}$ given by (2.24) into (2.20), we obtain

$$\sum_{\nu=0}^{k} \binom{k}{\nu} (Q_n)^{(\nu)}(x) \frac{(k-\nu)!}{k!} \left(\sum_{\zeta=0}^{\nu} \lambda_\zeta^k \Omega_{\nu-\zeta}^k(\nu;x) P_{k+\zeta}(x) u_0\right) = \lambda_n^k P_{n+k} u_0,$$

for $n \geqslant 0$, or, equivalently,

$$\sum_{\nu=0}^{k} \frac{1}{\nu!} \left(\sum_{\zeta=0}^{\nu} \lambda_\zeta^k \Omega_{\nu-\zeta}^k(\nu;x) P_{k+\zeta}(x)\right) (Q_n)^{(\nu)} u_0 = \lambda_n^k P_{n+k} u_0, \quad n \geqslant 0.$$

Since u_0 is regular, on attempt of Lemma 1.3, the previous relation becomes

$$\sum_{\nu=0}^{k} \frac{1}{\nu!} \left(\sum_{\zeta=0}^{\nu} \lambda_\zeta^k \Omega_{\nu-\zeta}^k(\nu; x) P_{k+\zeta}(x) \right) (Q_n)^{(\nu)} = \lambda_n^k P_{n+k}, \quad n \geqslant 0. \tag{2.27}$$

Since

$$(Q_n)^{(\nu)}(x) = \left(\prod_{\mu=1}^{k} (n+\mu) \right)^{-1} (P_{n+k})^{(k+\nu)}(x), \quad \nu \geqslant 0, \tag{2.28}$$

we easily obtain (2.10)–(2.12). □

Remark 2.2. The polynomials Λ_i, $i = 0, 1, 2$, in (2.11) are given by

$$\Lambda_0(k; x) = \lambda_0^k P_k(x),$$
$$\Lambda_1(k; x) = E_k(x) P_{k+1}(x) + F_k(x) P_k(x),$$
$$\Lambda_2(k; x) = G_k(x) P_{k+1}(x) + H_k(x) P_k(x),$$

where

$$E_k(x) = \lambda_1^k,$$
$$F_k(x) = -\lambda_0^k Q_1(x),$$
$$G_k(x) = \frac{1}{2}\left\{ -\lambda_1^k Q_2'(x) + \lambda_2^k (x - \beta_{k+1}) \right\},$$
$$H_k(x) = \frac{1}{2}\left\{ \lambda_0^k \left(-Q_2(x) + Q_2'(x) Q_1(x) \right) - \lambda_2^k \gamma_{k+1} \right\}.$$

Naturally, for $k \geqslant 1$, $\deg(E_k) = 0$, $\deg(F_k) = 1$, $\deg(G_k) \leqslant 1$ and $\deg(H_k) = 2$.

It appears to be important to know more about the degree of the Λ-polynomials given in (2.11). As this depends on the degree of Ω-polynomials presented in (2.14), we are obliged to analyze first these elements.

Lemma 2.3. *The polynomials $\Omega_\mu^k(\nu, \cdot)$ have degree μ, $0 \leqslant \mu \leqslant \nu$; precisely*

$$\Omega_\mu^k(\nu; x) = (-1)^\mu \binom{\nu}{\nu-\mu} x^\mu + \cdots, \quad 0 \leqslant \mu \leqslant \nu. \tag{2.29}$$

Consequently, we have the following results:

- *for Hermite and Laguerre cases,*

$$\deg \Lambda_0(k; x) = k,$$
$$\deg \Lambda_\nu(k; x) \leqslant \nu + k - 1, \quad \nu \geqslant 1; \tag{2.30}$$

- *for Bessel and Jacobi cases,*

$$\deg \Lambda_\nu(k; x) = k + \nu, \quad 0 \leqslant \nu \leqslant k,$$
$$\deg \Lambda_\nu(k; x) \leqslant \nu + k - 1, \quad \nu \geqslant k + 1. \tag{2.31}$$

Proof. Writing $\Omega_\mu^k(\nu;x)=\omega_\mu^k(\nu)x^\mu+\cdots$, from (2.14) and (2.28), we easily obtain

$$\omega^k_{\mu+1-\xi}(\mu+1)=-\sum_{\nu=\xi}^{\mu}\binom{\mu+1}{\nu}\omega^k_{\nu-\xi}(\nu),\quad 0\leqslant\xi\leqslant\mu. \tag{2.32}$$

Now, taking $\xi=\mu$, we have

$$\omega_1^k(\mu+1)=-(\mu+1)\omega_0^k(\mu)=-\binom{\mu+1}{\mu},\quad \mu\geqslant 0,$$

since $\omega_0^k(\mu)=1$, $\mu\geqslant 0$, according to the definition.

When $\xi=\mu-1$, for $\mu\geqslant 1$, we obtain from (2.32)

$$\omega_2^k(\mu+1)=\binom{\mu+1}{\mu-1}.$$

Let us take $\xi=\mu-\tau$, $0\leqslant\tau\leqslant\mu$. The relation (2.32) can be read as

$$\omega^k_{\tau+1}(\mu+1)=-\sum_{\zeta=0}^{\tau}\binom{\mu+1}{\mu-\tau+\zeta}\omega^k_\zeta(\mu-\tau+\zeta).$$

This can be written as

$$\omega^k_{\tau+1}(\mu+1)=-\binom{\mu+1}{\mu-\tau}-\sum_{\zeta=0}^{\tau-1}\binom{\mu+1}{\mu+1-\tau+\zeta}\omega^k_{\zeta+1}(\mu+1-\tau+\zeta). \tag{2.33}$$

Suppose $\omega^k_{\tau+1}(\mu)=(-1)^{\tau+1}\binom{\mu}{\mu-1-\tau}$, $\tau+1\leqslant\mu$. Then (2.33) becomes like

$$\begin{aligned}\omega^k_{\tau+1}(\mu+1)&=-\sum_{\nu=\mu-\tau}^{\mu}\binom{\mu+1}{\nu}(-1)^{\nu-(\mu-\tau)}\binom{\nu}{\mu-\tau}\\&=-\binom{\mu+1}{\mu-\tau}-\sum_{\zeta=0}^{\tau-1}(-1)^{\zeta+1}\binom{\mu+1}{\mu+1-\tau+\zeta}\binom{\mu+1-\tau+\zeta}{\zeta+1}\\&=(-1)^{\tau+1}\binom{\mu+1}{\mu-\tau}.\end{aligned}$$

Consequently, (2.29) holds. Now, from (2.11) and (2.29), we have

$$\Lambda_\nu(k;x)=\left\{\frac{1}{\nu!}\sum_{\mu=0}^{\nu}\lambda^k_\mu(-1)^{\nu-\mu}\binom{\nu}{\mu}\right\}x^{k+\nu}+\cdots.$$

For the Hermite and Laguerre cases, the coefficients λ_n^k do not depend on n, since they are respectively given by

$$\lambda_n^k=(-2)^k\quad\text{(Hermite)} \tag{2.34}$$

and

$$\lambda_n^k=(-1)^k\frac{\Gamma(\alpha+1)}{\Gamma(\alpha+1+k)}\quad\text{(Laguerre)}, \tag{2.35}$$

therefore (2.30) holds.

In the Bessel case, we easily obtain

$$\lambda_n^k = C(k,\alpha)\frac{\Gamma(2\alpha-1+2k+n)}{\Gamma(2\alpha-1+k+n)} \tag{2.36}$$

with

$$C(k,\alpha) = \frac{4^{-k}\Gamma(2\alpha+2k)}{\Gamma(2\alpha)}.$$

Consider

$$\Lambda_\nu(k;x) = C(k,\alpha)\frac{1}{\nu!}b_\nu^k(\alpha)x^{k+\nu} + \cdots$$

with

$$b_\nu^k(\alpha) = \sum_{\mu=0}^{\nu}(-1)^{\nu-\mu}\binom{\nu}{\mu}\frac{\Gamma(2\alpha-1+2k+\mu)}{\Gamma(2\alpha-1+k+\mu)}.$$

After some calculations, we get

$$\frac{b_{\nu+1}^k(\alpha)}{b_\nu^k(\alpha)} = -\frac{\nu-k}{\nu+2\alpha-1+k}, \quad \nu \geqslant 0.$$

It follows $b_\nu^k(\alpha)=0$, $\nu \geqslant k+1$, and

$$b_\nu^k(\alpha) = b_0(\alpha)\frac{\Gamma(k+\nu)}{\Gamma(k)}\frac{\Gamma(2\alpha-1+k)}{\Gamma(2\alpha-1+k+\nu)}, \quad 0 \leqslant \nu \leqslant k.$$

In the Jacobi case, we have

$$\lambda_n^k = C(k,\alpha)\frac{\Gamma(\alpha+\beta+1+2k+n)}{\Gamma(\alpha+\beta+1+k+n)} \tag{2.37}$$

with

$$C(k,\alpha) = \frac{(-4)^{-k}\Gamma(\alpha+1)\Gamma(\beta+1)\Gamma(\alpha+\beta+2+2k)}{\Gamma(\alpha+1+k)\Gamma(\beta+1+k)\Gamma(\alpha+\beta+2)}.$$

With analogous results as above, we finally obtain (2.31). □

Remark 2.4. In 1993, Maroni [10] showed that if $\{P_n^{[2]}\}_{n\geqslant 0}$ is an orthogonal sequence, then each P_{n+2} fulfills a fourth-order differential equation. Actually, when we take $k=2$ in (2.10), we recover the fourth-order differential equation achieved by Maroni in [10, §7] and also by Lesky in [6].

3. Characterization of the classical polynomials

In the previous section we have shown that any polynomial from a classical sequence fulfils the differential equation (2.10). Let $k \geqslant 2$ be an arbitrary integer and let $\{P_n\}_{n\geqslant 0}$ be a MOPS. We now propose to prove that when any polynomial P_n fulfils a certain differential equation of order $2k$, then the sequence $\{P_n\}_{n\geqslant 0}$ is classical.

We begin by recalling two important results to our work.

Lemma 3.1. [10, p. 144]*Consider a semiclassical form u such that*

$$D(\Phi_1 u) + \Psi_1 u = 0, \tag{3.38}$$
$$D(\Phi_2 u) + \Psi_2 u = 0, \tag{3.39}$$

where $\deg \Phi_i = t_i$ *and* $\deg \Psi_i = p_i$, *for* $i = 1, 2$. *If* Φ *is the highest common factor between* Φ_1 *and* Φ_2, *there exists a polynomial* Ψ *such that*

$$D(\Phi u) + \Psi u = 0.$$

We give here a more accurate proof of Lemma 2.1 of [12], which already exists in a not published version.

Lemma 3.2. [12] *Let* $\{P_n\}_{n\geqslant 0}$ *be a semi-classical sequence, orthogonal with respect to* u_0. *Suppose that* u_0 *fulfills the next two functional equations*

$$D(\Phi_1 u_0) + \Psi_1 u_0 = 0,$$
$$D(\Phi_2 u_0) + \Psi_2 u_0 = 0 \tag{3.40}$$

and there exists an integer $m \geqslant 0$ *and four polynomials* E, F, G, H *such that*

$$\Phi_1(x) = E(x)P_{m+1}(x) + F(x)P_m(x),$$
$$\Phi_2(x) = G(x)P_{m+1}(x) + H(x)P_m(x). \tag{3.41}$$

Let Δ *be the determinant of the system* (3.41)

$$\Delta(x) = \begin{vmatrix} E(x) & F(x) \\ G(x) & H(x) \end{vmatrix}. \tag{3.42}$$

Then if one of the following conditions is fulfilled, the form u_0 *is classical*:

(a) $\exists i = 1, 2$, *such that* $\deg(\Psi_i) \leqslant \deg(\Phi_i) - 1$ *and* $\deg(\Delta) = 2$;
(b) $\exists i = 1, 2$, *such that* $\deg(\Psi_i) = \deg(\Phi_i)$ *and* $\deg(\Delta) = 1$;
(c) $\exists i = 1, 2$, *such that* $\deg(\Psi_i) = \deg(\Phi_i) + 1$ *and* $\deg(\Delta) = 0$.

Proof. Applying the Cramer's rule to the system (3.41), we get that

$$\Delta_k(x)P_{m+1}(x) = \begin{vmatrix} \Phi_1(x) & F(x) \\ \Phi_2(x) & H(x) \end{vmatrix} = \Phi_1(x)H(x) - \Phi_2(x)F(x),$$

$$\Delta_k(x)P_m(x) = \begin{vmatrix} \Phi_1(x) & E(x) \\ \Phi_2(x) & G(x) \end{vmatrix} = \Phi_1(x)E(x) - \Phi_2(x)G(x), \quad m \geqslant 0.$$

Since $\{P_n\}_{n\geqslant 0}$ is an OPS, P_m and P_{m+1} have no common zeros. As a result, any common factor of Φ_1 and Φ_2, is also a factor of Δ. In particular, the highest common factor of Φ_1 and Φ_2, say Φ, is a factor of Δ. Hence, we may express these polynomials as

$$\Phi_i = \Phi\widetilde{\Phi}_i \quad (\text{with } i = 1, 2) \quad \text{and} \quad \Delta = \Phi\widetilde{\Delta}. \tag{3.43}$$

Lemma 3.1 assures us the existence of a polynomial, Ψ, such that $D(\Phi u_0) + \Psi u_0 = 0$. Moreover, in its proof we see that such a polynomial satisfy the equalities given by

$$\widetilde{\Phi}_i\Psi = \Psi_i + \widetilde{\Phi}_i'\Phi, \quad i = 1, 2.$$

Analyzing the degrees of the polynomials presented in both sides of the previous equation, we get that

$$\deg(\Phi_i)+\deg(\Psi)-\deg(\Phi)=\max\{\deg(\Psi_i),\deg(\Phi_i)-1\}. \tag{3.44}$$

Since, by hypothesis, u_0 is a semiclassical form, then $\deg\Psi\geqslant 1$. Furthermore, if $\deg\Delta\leqslant 2$, necessarily $\deg\Phi\leqslant 2$. It suffices now to show that $\deg\Psi=1$, which allows us to say that u_0 is a semiclassical form of class $s=0$ (i.e., a classical form).

In the case (a), we get that (3.44) becomes like $\deg\Phi=\deg\Psi+1$. It follows $\deg\Phi\geqslant 2$, then $\deg\Phi=2$ and consequently $\deg\Psi=1$. The form u_0 is either a Bessel or a Jacobi form.

In the case (b), we have, from (3.44), $\deg\Psi=\deg\Phi$, hence $\deg\Phi\geqslant 1$. But, $\deg\Phi\leqslant 1$, therefore $\deg\Phi=1$ and $\deg\Psi=1$. It is the Laguerre case.

Finally, in case (c), on attempt of (3.44), we get $\deg\Psi=\deg\Phi+1$ with $\deg\Phi=0$. It is the Hermite case. □

We claim the main result:

Theorem 3.3. *Let $k\geqslant 1$ be an integer and $\{P_n\}_{n\geqslant 0}$ be a MOPS whose any polynomial P_{n+k}, $n\geqslant 0$, fulfills the differential equation*

$$\sum_{\nu=0}^{k}\Lambda_\nu(k;x)(P_{n+k})^{(k+\nu)}(x)=\Xi_n(k)P_{n+k}(x),\quad n\geqslant 0, \tag{3.45}$$

where

$$\Lambda_\nu(k;x)=\sum_{\tau=k-\nu}^{k+\nu}\xi_\tau^\nu P_\tau(x), \tag{3.46}$$

with $\xi_\tau^\nu\in\mathbb{C}$ and $\xi_{k-\nu}^\nu\neq 0$, $0\leqslant\nu\leqslant k$ and $\Xi_n(k)\in\mathbb{C}\backslash\{0\}$.

Then $\{P_n\}_{n\geqslant 0}$ is a classical sequence.

Proof. Let m be an integer such that $0\leqslant m\leqslant k-1$. If we multiply both sides of (3.45) and, afterwards, we consider the action of u_0 over the resulting equation, then we get:

$$\left\langle u_0,\sum_{\nu=0}^{k}\Lambda_\nu(k;x)P_m(x)(P_{n+k})^{(k+\nu)}(x)\right\rangle=\langle u_0,\Xi_n(k)P_m(x)P_{n+k}(x)\rangle,\quad n\geqslant 0. \tag{3.47}$$

Since $\{P_n\}_{n\geqslant 0}$ is a MOPS, from (3.47) we have

$$\left\langle\sum_{\nu=0}^{k}(-1)^{k+\nu}D^{k+\nu}\big(\Lambda_\nu(k;x)P_m(x)u_0\big),P_{n+k}(x)\right\rangle=0,\quad n\geqslant 0. \tag{3.48}$$

It can be easily seen that

$$\left\langle\sum_{\nu=0}^{k}(-1)^{k+\nu}D^{k+\nu}\big(\Lambda_\nu(k;x)P_m(x)u_0\big),P_j(x)\right\rangle=0,\quad 0\leqslant j\leqslant k-1, \tag{3.49}$$

due to the fact that $D^{k+\nu}P_j(x)=0$, $0\leqslant j\leqslant k-1$.

Therefore, once $\{P_n\}_{n\geqslant 0}$ is a PS, (3.48) together with (3.49) imply that u_0 satisfies the following functional equations:

$$\sum_{\nu=0}^{k}(-1)^{\nu}D^{\nu}\big(\Lambda_\nu(k;x)P_m(x)u_0\big)=0,\quad 0\leqslant m\leqslant k-1. \tag{3.50}$$

For the sake of simplicity, let us write

$$\Lambda_\nu=\Lambda_\nu(k;x),\quad 0\leqslant \nu\leqslant k,$$
$$P_n=P_n(x),\quad n\geqslant 0.$$

By virtue of (1.2), the system of k functional equations given by (3.50) is equivalent to

$$\sum_{\mu=0}^{k}(P_m)^{(\mu)}\sum_{\nu=\mu}^{k}(-1)^{\nu}\binom{\nu}{\mu}D^{\nu-\mu}(\Lambda_\nu u_0)=0,\quad 0\leqslant m\leqslant k-1. \tag{3.51}$$

The goal is to simplify the system of Eqs. (3.51) into one of k differential equations of order one. This simplification can be done by means of Lemmas 3.4 and 3.5, see below. Thus, following Lemma 3.4, (3.51) may be written as

$$\sum_{\nu=m}^{k}(-1)^{\nu}\binom{\nu}{m}D^{\nu-m}(\Lambda_\nu u_0)=0,\quad 0\leqslant m\leqslant k-1. \tag{3.52}$$

Now, in accordance with Lemma 3.5 (see below), (3.52) imply

$$(k-\mu)D(\Lambda_{k-\mu}u_0)-(\mu+1)\Lambda_{k-\mu-1}u_0=0,\quad 0\leqslant \mu\leqslant k-1. \tag{3.53}$$

This means that u_0 is a semiclassical form. In particular, when we take $\mu=k-1$ and $\mu=k-2$ in (3.53), we have that u_0 satisfies the next two functional equations:

$$\begin{cases} D(\Lambda_1 u_0)+(-k\Lambda_0)u_0=0,\\ D(\Lambda_2 u_0)+\big(-\frac{k-1}{2}\Lambda_1\big)u_0=0, \end{cases} \tag{3.54}$$

where the polynomials Λ_ν, $0\leqslant \nu\leqslant 2$, are given by

$$\Lambda_0=\xi_k^0P_k,$$
$$\Lambda_1=\xi_{k+1}^1P_{k+1}+\xi_k^1P_k+\xi_{k-1}^1P_{k-1},$$
$$\Lambda_2=\xi_{k+2}^2P_{k+2}+\xi_{k+1}^2P_{k+1}+\xi_k^2P_k+\xi_{k-1}^2P_{k-1}+\xi_{k-2}^2P_{k-2}. \tag{3.55}$$

Let us now consider $N_1\Phi_1=\Lambda_1$ and $N_2\Phi_2=\Lambda_2$, where N_1 and N_2 are two normalization constants. Thus, we may write (3.54) like

$$\begin{cases} D(\Phi_1 u_0)+\Psi_1 u_0=0,\\ D(\Phi_2 u_0)+\Psi_2 u_0=0 \end{cases} \tag{3.56}$$

with

$$\Psi_1=-k\big(N_1^{-1}\Lambda_0\big)=-kN_1^{-1}\xi_k^0P_k \tag{3.57}$$

and $\Psi_2=-\frac{k-1}{2}(N_2^{-1}\Lambda_1)$. Since $\{P_n\}_{n\geqslant 0}$ is MOPS by virtue of (3.55), it is possible to write Ψ_2, Φ_1 and Φ_2 as

$$\begin{aligned}
\Psi_2 &= -\frac{(k-1)N_2^{-1}}{2}(E_k P_{k+1} + F_k P_k),\\
\Phi_1 &= N_1^{-1}(E_k P_{k+1} + F_k P_k),\\
\Phi_2 &= N_2^{-1}(G_k P_{k+1} + H_k P_k),
\end{aligned} \tag{3.58}$$

where

$$\begin{aligned}
E_k &= \xi_{k+1}^1 - \frac{\xi_{k-1}^1}{\gamma_k},\\
F_k &= \left(\frac{\xi_{k-1}^1}{\gamma_k}\right)x + \left(\xi_k^1 - \frac{\xi_{k-1}^1}{\gamma_k}\beta_k\right),\\
G_k &= \left(\xi_{k+2}^2 - \frac{\xi_{k-2}^2}{\gamma_k\gamma_{k-1}}\right)x + \left(-\xi_{k+2}^2\beta_{k+1} + \frac{\xi_{k-2}^2}{\gamma_k\gamma_{k-1}}\beta_{k-1} + \xi_{k+1}^2 - \frac{\xi_{k-1}^2}{\gamma_k}\right),\\
H_k &= \left(\xi_{k-2}^2\frac{1}{\gamma_{k-1}\gamma_k}\right)x^2 + \left(\xi_{k-1}^2\frac{1}{\gamma_k} + \xi_{k-2}^2\frac{1}{\gamma_{k-1}\gamma_k}(-\beta_{k-1} - \beta_k)\right)x\\
&\quad + \left(-\xi_{k+2}^2\gamma_{k+1} + \xi_k^2 - \xi_{k-1}^2\frac{1}{\gamma_k}\beta_k + \xi_{k-2}^2\frac{1}{\gamma_{k-1}\gamma_k}\beta_{k-1}\beta_k - \xi_{k-2}^2\frac{1}{\gamma_{k-1}}\right).
\end{aligned}$$

If we denote by Δ_k the determinant of the last two equations of (3.58), that is,

$$\Delta_k(x) = \begin{vmatrix} E_k & F_k \\ G_k & H_k \end{vmatrix},$$

then, by hypothesis, $\deg(\Delta_k) \leqslant 2$. After some straightforward calculations, we can write Δ_k as

$$\Delta_k = \delta_k^2 x^2 + \delta_k^1 x + \delta_k^0,$$

where

$$\begin{aligned}
\delta_k^2 &= \frac{1}{\gamma_k}\left\{\frac{\xi_{k-2}^2\xi_{k+1}^1}{\gamma_{k-1}} - \xi_{k+2}^2\xi_{k-1}\right\},\\
\delta_k^1 &= -(\beta_k + \beta_{k+1})\delta_k^2 - \xi_k^1\xi_{k+2}^2 + \frac{1}{\gamma_k}\left\{\xi_{k+1}^1\xi_{k-1}^2 - \xi_{k-1}^1\xi_{k+1}^2\right\}\\
&\quad + \frac{1}{\gamma_k\gamma_{k-1}}\left\{\xi_k^1\xi_{k-2}^2 + (\beta_{k+1} - \beta_{k-1})\xi_{k+1}^1\xi_{k-2}^2\right\},\\
\delta_k^0 &= -\beta_k\delta_k^1 - \left(\beta_k^2 + \gamma_{k+1}\right)\delta_k^2 - \gamma_{k+1}\xi_{k+1}^1\xi_{k+2}^2 + \xi_{k+1}^1\xi_k^2 - \xi_k^1\xi_{k+1}^2\\
&\quad + (\beta_{k+1} - \beta_k)\xi_k^1\xi_{k+2}^2 + \frac{1}{\gamma_k}\left(\xi_k^1\xi_{k-1}^2 - \xi_{k-1}^1\xi_k^2\right)\\
&\quad + \frac{1}{\gamma_k\gamma_{k-1}}\left(\xi_{k-1}^1 + \gamma_{k+1}\xi_{k+1}^1 + (\beta_k - \beta_{k-1})\xi_k^1\right)\xi_{k-2}^2.
\end{aligned}$$

In accordance with (3.43) presented in the proof of Lemma 3.2, we have that $\deg(\Phi) \leqslant \deg(\Delta_k)$. Thus, no matter which the expressions of the coefficients δ_k^i $(i = 0, 1, 2)$ are, we will always have $\deg\Phi \leqslant 2$. Yet, this is not sufficient to say that u_0 is a classical form. We will absolutely need to show that there exists a polynomial Ψ such that u_0 fulfills $D(\Phi u_0) + \Psi u_0 = 0$ and $\deg\Psi = 1$. Actually, this can be done by making use of Lemma 3.2. So, our analysis will consist on studying what happens when $\deg\Delta_k$ is equal to 2, 1 or 0.

Suppose that $\delta_k^2 \neq 0$, which implies that $\deg \Delta_k = 2$. Moreover, if $\xi_{k+1}^1 \neq 0$, then $\deg \Phi_1 = k+1$ and $\deg \Psi_1 = k = \deg \Phi_1 - 1$, in accordance with (3.58) and (3.57). So, the condition (a) of Lemma 3.2 is satisfied. On the other hand, if $\xi_{k+1}^1 = 0$, then, on account of (3.58), $\deg \Psi_2 \leqslant k$ and we will necessarily have $\xi_{k+2}^2 \neq 0$, due to $\delta_k^2 \neq 0$, which means that $\deg \Psi_2 \leqslant k \leqslant k+1 = \deg \Phi_2 - 1$. Once more, we are in the condition (a) of Lemma 3.2. In both of these cases, u_0 is either a Bessel form or a Jacobi form.

Now, suppose that $\delta_k^2 = 0$ and $\delta_k^1 \neq 0$, that is, $\deg \Delta_k = 1$. We will necessarily have $\xi_{k+1}^1 = 0$. Otherwise, we would have, from (3.57), $\deg \Psi_1 = k$ and from (3.58) $\deg \Phi_1 = k+1$, so, on attempt of (3.44), this would imply $\deg \Psi = \deg \Phi - 1$, which contradicts the hypothesis $\deg \Phi \leqslant \deg \Delta_k \leqslant 1$, since the regularity conditions of u_0 imply $\deg \Psi \geqslant 1$, and therefore we would have $\deg \Phi \geqslant 2$. As a consequence, we will have $\xi_{k+1}^1 = \xi_{k+2}^2 = 0$. Thus, in these conditions, the expression of δ_k^1 becomes

$$\delta_k^1 = \frac{1}{\gamma_k}\left\{-\xi_{k-1}^1 \xi_{k+1}^2\right\} + \frac{1}{\gamma_k \gamma_{k-1}}\left\{\xi_k^1 \xi_{k-2}^2\right\}.$$

Actually, we will necessarily have $\xi_k^1 \neq 0$. If $\xi_k^1 = 0$, then $\xi_{k+1}^2 \neq 0$ (since $\delta_k^1 \neq 0$), and consequently, from (3.58), $\deg \Psi_2 = k-1$ and $\deg \Phi_2 = k+1$. As a result, the regularity conditions of u_0 ($\deg \Psi \geqslant 1$) together with (3.44), imply $\deg \Phi \geqslant 2$, which contradicts the hypothesis $\deg \Phi \leqslant \deg \Delta_k \leqslant 1$. Thus, $\deg \Psi_1 = k = \deg \Phi_1$, and Lemma 3.2 assures that u_0 is a classical form. More precisely it is a Laguerre form.

To finalize our discussion, let us suppose that $\delta_k^2 = \delta_k^1 = 0$. Then $\Delta_k = \delta_k^0$ and the two following equalities hold:

$$\begin{aligned}
&\xi_{k-2}^2 \xi_{k+1}^1 = \gamma_{k-1} \xi_{k+2}^2 \xi_{k-1}^1, \\
&\xi_{k-1}^1 \xi_{k+1}^2 = \xi_{k+1}^1 \xi_{k-1}^2 + \left(\frac{1}{\gamma_{k-1}} \xi_{k-2}^2 - \gamma_k \xi_{k+2}^2\right) \xi_k^1 - (\beta_{k-1} - \beta_{k+1}) \xi_{k+1}^1 \xi_{k-2}^2.
\end{aligned} \tag{3.59}$$

On account of the previous discussion, necessarily, $\xi_{k+1}^1 = 0$, therefore, from (3.59), $\xi_{k+2}^2 = 0$. If we suppose $\xi_k^1 \neq 0$, then (3.57) and (3.58) would, respectively, imply $\deg \Psi_1 = k$ and $\deg \Phi_1 = k$. Therefore $\deg \Psi = \deg \Phi = 0$, due to (3.44). But this contradicts the regularity condition of u_0: $\deg \Psi \geqslant 1$. So $\xi_k^1 = 0$. Therefore, one has $\deg \Phi_1 = k-1$, thus $\deg \Psi = \deg \Phi + 1 = 1$, it is the Hermite case. On the other hand, $\deg \Psi_2 = k-1$ and $\deg \Phi_2 \leqslant k$, since $\xi_{k+1}^2 = 0$. But $\deg \Phi_2 = k$ implies $\deg \Psi = \deg \Phi - 1 = -1$ which is not possible. Consequently, $\xi_k^2 = 0$ and $\delta_k^0 \neq 0$, since $\delta_k^0 = \gamma_{k-1}^{-1} \gamma_k^{-1} \xi_{k-1}^1 \xi_{k-2}^2$. Now, Lemma 3.2 allows us to conclude that, in this case, u_0 is a Hermite classical form. □

To end this section, we present the two lemmas that were already mentioned in the proof of the previous result.

Lemma 3.4. *The system of k equations given by*

$$\sum_{\mu=0}^{k} (P_m)^{(\mu)} \sum_{\nu=\mu}^{k} (-1)^\nu \binom{\nu}{\mu} D^{\nu-\mu}(\Lambda_\nu u_0) = 0, \quad 0 \leqslant m \leqslant k-1, \tag{3.60}$$

is equivalent to

$$\sum_{\nu=m}^{k} (-1)^\nu \binom{\nu}{m} D^{\nu-m}(\Lambda_\nu u_0) = 0, \quad 0 \leqslant m \leqslant k-1. \tag{3.61}$$

Proof. We begin with the proof that (3.60) implies (3.61). For $m=0$, (3.60) becomes

$$\sum_{\nu=0}^{k}(-1)^{\nu}\binom{\nu}{0}D^{\nu}(\Lambda_{\nu}u_0)=0.$$

For $1\leqslant m\leqslant k-1$ ($k\geqslant 2$), suppose that

$$\sum_{\nu=\mu}^{k}(-1)^{\nu}\binom{\nu}{\mu}D^{\nu-\mu}(\Lambda_{\nu}u_0)=0,\quad 0\leqslant\mu\leqslant m-1.$$

Since $(P_m)^{(\mu)}(x)=0$, $\mu\geqslant m+1$ and $(P_m)^{(m)}(x)=m!$, we have

$$m!\sum_{\nu=m}^{k}(-1)^{\nu}\binom{\nu}{m}D^{\nu-m}(\Lambda_{\nu}u_0)+\sum_{\mu=0}^{m-1}(P_m)^{(\mu)}\sum_{\nu=\mu}^{k}(-1)^{\nu}\binom{\nu}{\mu}D^{\nu-\mu}(\Lambda_{\nu}u_0)=0.$$

Therefore,

$$\sum_{\nu=m}^{k}(-1)^{\nu}\binom{\nu}{m}D^{\nu-m}(\Lambda_{\nu}u_0)=0.$$

It is evident that (3.61) implies (3.60). □

The next lemma shows that it is possible to write (3.61) as a system of k differential functional equations of order one.

Lemma 3.5. *If a form u_0 fulfills the k equations given by* (3.61)*, then it also fulfills the following k equations*:

$$(k-\mu)D(\Lambda_{k-\mu}u_0)-(\mu+1)\Lambda_{k-\mu-1}u_0=0,\quad 0\leqslant\mu\leqslant k-1. \tag{3.62}$$

Proof. If we take $m=k-1$ in (3.61), we naturally have

$$kD(\Lambda_k u_0)-\Lambda_{k-1}u_0=0.$$

Thus, (3.62) is valid for $\mu=0$.

When $1\leqslant\mu\leqslant k-2$, we suppose that (3.62) holds for $0\leqslant\nu\leqslant\mu$:

$$(k-\mu+\nu)D(\Lambda_{k-\mu+\nu}u_0)=(\mu-\nu+1)\Lambda_{k-\mu+\nu-1}u_0. \tag{3.63}$$

Now, for $m=k-\mu-2$, it is possible to write (3.61) as

$$(-1)^{k-\mu-2}\Lambda_{k-\mu-2}u_0+(-1)^{k-\mu-1}(k-\mu-1)D(\Lambda_{k-\mu-1}u_0)+S_\mu=0, \tag{3.64}$$

where

$$S_\mu=\sum_{\nu=k-\mu}^{k}(-1)^{\nu}\binom{\nu}{k-\nu-2}D^{\nu-k+\mu+2}(\Lambda_{\nu}u_0),$$

i.e.,

$$S_\mu=\sum_{\nu=0}^{\mu}(-1)^{k-\mu+\nu}\binom{k-\mu+\nu}{\nu+2}D^{\nu+2}(\Lambda_{k-\mu+\nu}u_0). \tag{3.65}$$

We shall be transforming S_μ. Indeed, we get

$$S_\mu = (-1)^{k-\mu}\binom{k-\mu}{2}D^2(\Lambda_{k-\mu}u_0) + \sum_{\nu=1}^{\mu}(-1)^{k-\mu+\nu}\binom{k-\mu+\nu}{\nu+2}D^2\left(\frac{\mu-\nu+1}{k-\mu+\nu}D^{\nu-1}(\Lambda_{k-\mu+\nu-1}u_0)\right),$$

since from (3.63) we have

$$(k-\mu+\nu)D^\nu(\Lambda_{k-\mu+\nu}u_0) = (\mu-\nu+1)D^{\nu-1}(\Lambda_{k-\mu+\nu-1}u_0), \quad \nu \geqslant 1. \tag{3.66}$$

Therefore,

$$\begin{aligned} S_\mu &= (-1)^{k-\mu}\binom{k-\mu}{2}D^2(\Lambda_{k-\mu}u_0) \\ &\quad + \sum_{\nu=0}^{\mu-1}(-1)^{k-\mu+\nu}\binom{k-\mu+\nu}{\nu+2}D^{\nu+2}(\Lambda_{k-\mu+\nu}u_0) \\ &= (-1)^{k-\mu}\left(1-\frac{\mu}{3}\right)\binom{k-\mu}{2}D^2(\Lambda_{k-\mu}u_0) \\ &\quad + \sum_{\nu=0}^{\mu-2}\binom{k-\mu+\nu+2}{\nu+4}\frac{(-1)^{k-\mu+\nu+2}(\mu-\nu-1)(\mu-\nu)}{(k-\mu+\nu+2)(k-\mu+\nu+1)}D^{2+\nu}(\Lambda_{k-\mu+\nu}u_0). \end{aligned}$$

We are proceeding by induction. Suppose that

$$\begin{aligned} S_\mu &= (-1)^{k-\mu}a_{\tau-1}(\mu)\binom{k-\mu}{2}D^2(\Lambda_{k-\mu}u_0) \\ &\quad + \sum_{\nu=0}^{\mu-\tau}(-1)^{k-\mu+\nu+\tau}\binom{k-\mu+\nu+\tau}{\nu+\tau+2}\prod_{\xi=0}^{\tau-1}\frac{\mu-\nu-\xi}{k-\mu+\nu+\xi+1} \\ &\quad \times D^{2+\nu}(\Lambda_{k-\mu+\nu}u_0), \end{aligned} \tag{3.67}$$

for $1 \leqslant \tau \leqslant \mu - 1$ and with $a_0(\mu) = 1$. As above, we have

$$\begin{aligned} S_\mu &= (-1)^{k-\mu}\left\{a_{\tau-1}(\mu)\binom{k-\mu}{2} + (-1)^\tau\binom{k-\mu+\tau}{\tau+2}\prod_{\xi=0}^{\tau-1}\frac{\mu-\xi}{k-\mu+\xi+1}\right\}D^2(\Lambda_{k-\mu}u_0) \\ &\quad + \sum_{\nu=1}^{\mu-\tau}\left\{(-1)^{k-\mu+\nu+\tau}\binom{k-\mu+\nu+\tau}{\nu+\tau+2}\prod_{\xi=0}^{\tau-1}\left(\frac{\mu-\nu-\xi}{k-\mu+\nu+\xi+1}\right)\right. \\ &\quad \left. \times D^2\left(\frac{\mu-\nu+1}{k-\mu+\nu}D^{\nu-1}(\Lambda_{k-\mu+\nu-1}u_0)\right)\right\}, \end{aligned}$$

if we take (3.67) into account. Consequently,

$$S_\mu = (-1)^{k-\mu} a_\tau(\mu) \binom{k-\mu}{2} D^2(\Lambda_{k-\mu} u_0) + \sum_{\nu=0}^{\mu-\tau-1} \left\{ (-1)^{k-\mu+\nu+\tau+1} \binom{k-\mu+\nu+\tau+1}{\nu+\tau+3} \prod_{\xi=0}^{\tau} \left(\frac{\mu-\nu-\xi}{k-\mu+\nu+\xi+1} \right) \times D^{\nu+2}(\Lambda_{k-\mu+\nu} u_0) \right\},$$

where

$$\binom{k-\mu}{2} a_\tau(\mu) = \binom{k-\mu}{2} a_{\tau-1}(\mu) + (-1)^\tau \binom{k-\mu+\tau}{\tau+2} \prod_{\xi=0}^{\tau-1} \frac{\mu-\xi}{k-\mu+\xi+1}.$$

But

$$\binom{k-\mu+\tau}{\tau+2} \prod_{\zeta=0}^{\tau-1} \frac{\mu-\zeta}{k-\mu+\zeta+1} = \binom{k-\mu}{2} \binom{\mu}{\tau} \frac{2}{(\tau+1)(\tau+2)},$$

whence

$$a_\tau(\mu) - a_{\tau-1}(\mu) = (-1)^\tau \binom{\mu}{\tau} \frac{2}{(\tau+1)(\tau+2)}. \tag{3.68}$$

It follows,

$$a_\tau(\mu) = 1 + \sum_{\nu-1}^{\tau} (-1)^\tau \binom{\mu}{\nu} \frac{2}{(\nu+1)(\nu+2)}.$$

As a result, we deduce, in the particular case of $\tau = \mu$, that

$$a_\mu(\mu) = \sum_{\tau=0}^{\mu} (-1)^\tau \binom{\mu}{\tau} \frac{2}{(\tau+1)(\tau+2)}, \quad \mu \geqslant 0.$$

Besides, if we consider the following relation

$$(1-x)^\mu = \sum_{\tau=0}^{\mu} \binom{\mu}{\tau} (-1)^\tau x^\tau,$$

after two integrations, we get

$$\frac{1}{\mu+1} \left\{ x + \frac{(1-x)^{\mu+2} - 1}{\mu+2} \right\} = \sum_{\tau=0}^{\mu} \binom{\mu}{\tau} (-1)^\tau \frac{x^{\tau+2}}{(\tau+1)(\tau+2)}.$$

Thus, taking $x = 1$ in the previous relation, we find

$$a_\mu(\mu) = \frac{2}{\mu+2}.$$

Now, taking $\tau = \mu$ in (3.67), we obtain

$$S_\mu = (-1)^{k-\mu} \binom{k-\mu}{2} a_\mu(\mu) D^2(\Lambda_{k-\nu} u_0),$$

on account of (3.68). Finally, (3.64) becomes

$$\Lambda_{k-\mu-2}u_0 - (k-\mu-1)D(\Lambda_{k-\mu-1}u_0) + \binom{k-\mu}{2}a_\mu(\mu)D^2(\Lambda_{k-\mu}u_0) = 0.$$

As long as

$$(k-\mu)D^2(\Lambda_{k-\mu}u_0) = (\mu+1)D(\Lambda_{k-\mu-1}u_0),$$

we conclude that

$$(k-\mu-1)D(\Lambda_{k-\mu-1}u_0) - \left(1 - \frac{\mu+1}{2}a_\mu(\mu)\right)^{-1}\Lambda_{k-\mu-2}u_0 = 0,$$

which is (3.61) where $\mu \to \mu+1$. □

4. The differential equations of order $2k$, for $k = 1, 2$ and 3

The aim of this section is to write explicitly the generalized differential equation of Theorem 2.1, for the first values of k, say $k = 1, 2$ and 3:

$\Lambda_1(1;x)P''_{n+1}(x) + \Lambda_0(1;x)P'_{n+1}(x) = \Xi_n(1)P_{n+1}(x)$
$\Lambda_2(2;x)(P_{n+2})^{(4)}(x) + \Lambda_1(2;x)(P_{n+2})^{(3)}(x) + \Lambda_0(2;x)(P_{n+2})^{(2)}(x) = \Xi_n(2)P_{n+2}(x)$
$\Lambda_3(3;x)(P_{n+3})^{(6)}(x) + \Lambda_2(3;x)(P_{n+3})^{(5)}(x) + \Lambda_1(3;x)(P_{n+3})^{(4)}(x) + \Lambda_0(3;x)(P_{n+3})^{(3)}(x) = \Xi_n(3)P_{n+3}(x)$

In order to do it, we implement the formulas (2.11)–(2.14) in the *Mathematica* language [13].

We treat separately each family of classical polynomials (Hermite, Laguerre, Bessel and Jacobi). We remember the regularity conditions of the four classical families [9–11]:

Hermite	Laguerre	Bessel	Jacobi
–	$\alpha \neq -n$, $n \geqslant 1$	$\alpha \neq -n/2$, $n \geqslant 0$	$\alpha, \beta \neq -n, \alpha+\beta \neq -(n+1)$, $n \geqslant 1$

In the sequel, we show the results we have obtained for $k = 1, 2$ and 3. When k increases, formulas become more and more bigger. The corresponding results to higher values of k can be obtained, nevertheless we do not show them here just for space reasons.

Results are organized by classical sequence and increasing values of k ($k = 1, 2, 3$). For each classical family and each value of k, the equations were divided by a nonzero factor C_k such that $\Lambda_k(k;x) = C_k\Lambda_k^*(k;x)$ and $\Lambda_k^*(k;x)$ is monic. Thus we show here the elements presented in the referred differential equations, that is, the polynomials $\Lambda_\nu^*(k;x) = (C_k)^{-1}\Lambda_\nu(k;x)$, $0 \leqslant \nu \leqslant k$, and $\Xi_n^*(k) = (C_k)^{-1}\Xi_n(k)$, $n \geqslant 0$.

When $k = 1$ we recover the Bochner's differential equation [9–11].

4.1. Hermite

For $\mathbf{k = 1}$,

$$\Lambda_1^*(1;x) = 1, \qquad \Lambda_0^*(1;x) = -2x, \qquad \Xi_n^*(1) = -2(1+n).$$

For $\mathbf{k}=\mathbf{2}$,

$$\Lambda_2^*(2;x)=1,$$
$$\Lambda_1^*(2;x)=-4x,$$
$$\Lambda_0^*(2;x)=-2+4x^2,$$
$$\Xi_n^*(2)=4(1+n)(2+n).$$

For $\mathbf{k}=\mathbf{3}$,

$$\Lambda_3^*(3;x)=1,$$
$$\Lambda_2^*(3;x)=-6x,$$
$$\Lambda_1^*(3;x)=6\big(-1+2x^2\big),$$
$$\Lambda_0^*(3;x)=12x-8x^3,$$
$$\Xi_n^*(3)=-8(1+n)(2+n)(3+n).$$

4.2. Laguerre

For $\mathbf{k}=\mathbf{1}$,

$$\Lambda_1^*(1;x)=x,\qquad \Lambda_0^*(1;x)=1-x+\alpha,\qquad \Xi_n^*(1)=1+n.$$

For $\mathbf{k}=\mathbf{2}$,

$$\Lambda_2^*(2;x)=x^2,$$
$$\Lambda_1^*(2;x)=(4-2x+2\alpha)x,$$
$$\Lambda_0^*(2;x)=x^2-2x(2+\alpha)+(1+\alpha)(2+\alpha),$$
$$\Xi_n^*(2)=(1+n)(2+n).$$

For $\mathbf{k}=\mathbf{3}$,

$$\Lambda_3^*(3;x)=x^3,$$
$$\Lambda_2^*(3;x)=3x^2(3-x+\alpha),$$
$$\Lambda_1^*(3;x)=3x\big(x^2-2x(3+\alpha)+(2+\alpha)(3+\alpha)\big),$$
$$\Lambda_0^*(3;x)=6+11\alpha-(-6+x-\alpha)\big((-3+x)x-2x\alpha+\alpha^2\big),$$
$$\Xi_n^*(3)=-(1+n)(2+n)(3+n).$$

4.3. Bessel

For $\mathbf{k}=\mathbf{1}$,

$$\Lambda_1^*(1;x)=x^2,\qquad \Lambda_0^*(1;x)=2(1+x\alpha),\qquad \Xi_n^*(1)=(1+n)(n+2\alpha).$$

For $\mathbf{k}=\mathbf{2}$,

$$\Lambda_2^*(2;x) = x^4,$$
$$\Lambda_1^*(2;x) = 4x^2(1+x+x\alpha),$$
$$\Lambda_0^*(2;x) = 2\big(2+x(1+2\alpha)(2+x+x\alpha)\big),$$
$$\Xi_n^*(2) = (1+n)(2+n)(1+n+2\alpha)(2+n+2\alpha).$$

For $\mathbf{k}=\mathbf{3}$,

$$\Lambda_3^*(3;x) = x^6,$$
$$\Lambda_2^*(3;x) = 6x^4\big(1+x(2+\alpha)\big),$$
$$\Lambda_1^*(3;x) = 6x^2\big(2+x(3+2\alpha)\big(2+x(2+\alpha)\big)\big),$$
$$\Lambda_0^*(3;x) = 4\big(2+x(1+\alpha)\big(6+x(3+2\alpha)\big(3+x(2+\alpha)\big)\big)\big),$$
$$\Xi_n^*(3) = (1+n)(2+n)(3+n)(2+n+2\alpha)(3+n+2\alpha)(4+n+2\alpha).$$

4.4. Jacobi

For $\mathbf{k}=\mathbf{1}$,

$$\Lambda_1^*(1;x) = -1+x^2, \qquad \Lambda_0^*(1;x) = -\alpha+\beta+x(2+\alpha+\beta),$$
$$\Xi_n^*(1) = (1+n)(2+n+\alpha+\beta).$$

For $\mathbf{k}=\mathbf{2}$,

$$\Lambda_2^*(2;x) = \big(-1+x^2\big)^2,$$
$$\Lambda_1^*(2;x) = 2\big(-1+x^2\big)\big(-\alpha+\beta+x(4+\alpha+\beta)\big),$$
$$\Lambda_0^*(2;x) = -4-\alpha+(\alpha-\beta)^2-\beta-2x(\alpha-\beta)(3+\alpha+\beta) + x^2(3+\alpha+\beta)(4+\alpha+\beta),$$
$$\Xi_n^*(2) = (1+n)(2+n)(3+n+\alpha+\beta)(4+n+\alpha+\beta).$$

For $\mathbf{k}=\mathbf{3}$,

$$\Lambda_3^*(3;x) = \big(-1+x^2\big)^3,$$
$$\Lambda_2^*(3;x) = 3\big(-1+x^2\big)^2\big(-\alpha+\beta+x(6+\alpha+\beta)\big),$$
$$\Lambda_1^*(3;x) = \big(-1+x^2\big)\big(-6-\alpha+(\alpha-\beta)^2-\beta-2x(\alpha-\beta)(5+\alpha+\beta) + x^2(5+\alpha+\beta)(6+\alpha+\beta)\big),$$
$$\Lambda_0^*(3;x) = \bigg(-x^2(\alpha-\beta)(4+\alpha+\beta)(5+\alpha+\beta) + \frac{1}{3}x^3(4+\alpha+\beta)(5+\alpha+\beta)(6+\alpha+\beta) - \frac{1}{3}(\alpha-\beta)\big(-16+(-3+\alpha)\alpha-\beta-2\alpha\beta+\beta^2\big) + x(4+\alpha+\beta)\big(-6+(-1+\alpha)\alpha-\beta-2\alpha\beta+\beta^2\big)\bigg),$$
$$\Xi_n^*(3) = -\frac{1}{3}(1+n)(2+n)(3+n)(4+n+\alpha+\beta)(5+n+\alpha+\beta)(6+n+\alpha+\beta).$$

4.4.1. Legendre polynomials

Taking $\alpha = \beta = 0$ in the Jacobi polynomials, we get the Legendre polynomials.

For $k = 1$,

$$\Lambda_1^*(1;x) = -1 + x^2,$$
$$\Lambda_0^*(1;x) = 2x,$$
$$\Xi_n^*(1) = (1+n)(2+n).$$

For $k = 2$,

$$\Lambda_2^*(2;x) = \left(-1 + x^2\right)^2,$$
$$\Lambda_1^*(2;x) = 8\left(-1 + x^2\right)x,$$
$$\Lambda_0^*(2;x) = 4\left(-1 + 3x^2\right),$$
$$\Xi_n^*(2) = (1+n)(2+n)(3+n)(4+n).$$

For $k = 3$,

$$\Lambda_3^*(3;x) = \left(-1 + x^2\right)^3,$$
$$\Lambda_2^*(3;x) = 18\left(-1 + x^2\right)^2 x,$$
$$\Lambda_1^*(3;x) = 18\left(1 - 6x^2 + 5x^4\right),$$
$$\Lambda_0^*(3;x) = 24x\left(-3 + 5x^2\right),$$
$$\Xi_n^*(3) = (1+n)(2+n)(3+n)(4+n)(5+n)(6+n).$$

4.4.2. Tchebyshev polynomials of the first kind

Taking $\alpha = \beta = -1/2$ in the Jacobi polynomials we get the Tchebyshev polynomials of the first kind.

For $k = 1$,

$$\Lambda_1^*(1;x) = -1 + x^2,$$
$$\Lambda_0^*(1;x) = x,$$
$$\Xi_n^*(1) = (1+n)^2.$$

For $k = 2$,

$$\Lambda_2^*(2;x) = \left(-1 + x^2\right)^2,$$
$$\Lambda_1^*(2;x) = 6\left(-1 + x^2\right)x,$$
$$\Lambda_0^*(2;x) = -3 + 6x^2,$$
$$\Xi_n^*(2) = (1+n)(2+n)^2(3+n).$$

For $k = 3$,

$$\Lambda_3^*(3;x) = \left(-1 + x^2\right)^3,$$
$$\Lambda_2^*(3;x) = 15\left(-1 + x^2\right)^2 x,$$
$$\Lambda_1^*(3;x) = 15\left(1 - 5x^2 + 4x^4\right),$$

$$\Lambda_0^*(3; x) = 15x\left(-3 + 4x^2\right),$$
$$\Xi_n^*(3) = (1 + n)(2 + n)(3 + n)^2(4 + n)(5 + n).$$

4.4.3. Tchebyshev polynomials of the second kind

Taking $\alpha = \beta = 1/2$ in the Jacobi polynomials we get the Tchebyshev polynomials of the second kind.

For $k = 1$,

$$\Lambda_1^*(1; x) = -1 + x^2,$$
$$\Lambda_0^*(1; x) = 3x,$$
$$\Xi_n^*(1) = (1 + n)(3 + n).$$

For $k = 2$,

$$\Lambda_2^*(2; x) = \left(-1 + x^2\right)^2,$$
$$\Lambda_1^*(2; x) = 10\left(-1 + x^2\right)x,$$
$$\Lambda_0^*(2; x) = 5\left(-1 + 4x^2\right),$$
$$\Xi_n^*(2) = (1 + n)(2 + n)(4 + n)(5 + n).$$

For $k = 3$,

$$\Lambda_3^*(3; x) = \left(-1 + x^2\right)^3,$$
$$\Lambda_2^*(3; x) = 15\left(-1 + x^2\right)^2 x,$$
$$\Lambda_1^*(3; x) = 15\left(1 - 5x^2 + 4x^4\right),$$
$$\Lambda_0^*(3; x) = 15x\left(-3 + 4x^2\right),$$
$$\Xi_n^*(3) = (1 + n)(2 + n)(3 + n)^2(4 + n)(5 + n).$$

References

[1] S. Bochner, Über Sturm–Liouville polynomsysteme, Math. Z. 29 (1929) 730–736.
[2] T.S. Chihara, An Introduction to Orthogonal Polynomials, Gordon and Breach, New York, 1978.
[3] W. Hahn, Über die jacobischen polynome und zwei verwandte polynomklassen, Math. Z. 39 (1935) 634–638.
[4] W. Hahn, Über höhere ableitungen von orthogonal polynomen, Math. Z. 43 (1937) 101.
[5] K.H. Kwon, L.L. Littlejohn, B.H. Yoo, New characterizations of classical orthogonal polynomials, Indag. Math. (N.S.) 7 (1996) 199–213.
[6] P.A. Lesky, Eigenwertprobleme mit Differentialgleichungen vierter Ordnung für die kontinuierlichen klassischen Orthogonalpolynome, Österreich. Akad. Wiss. Math.-Natur. Kl. Sitzungsber. II 206 (1997).
[7] A.F. Loureiro, Uma nova caracterização dos polinómios ortogonais clássicos, Tese de Mestrado, Faculdade de Ciências da Universidade do Porto, 2003.
[8] A.F. Loureiro, Caracterização dos polinómios ortogonais clássicos, CMUP, http://www.fc.up.pt/cmup/monograph/NotasPolyOrt.pdf, 2004.
[9] P. Maroni, Une théorie algébrique des polynômes orthogonaux. Application aux polynômes orthogonaux semi-classiques, in: C. Brezinski, et al. (Eds.), Orthogonal Polynomials and Their Applications, in: IMACS Ann. Comput. Appl. Math., vol. 9, Baltzer, Basel, 1991, pp. 95–130.
[10] P. Maroni, Variations around classical orthogonal polynomials. Connected problems, J. Comput. Appl. Math. 48 (1993) 133–155.

[11] P. Maroni, Fonctions Eulériennes. Polynômes orthogonaux classiques. Téchniques de l'Ingénieur, traité Généralités (Sciences Fondamentales), 1994.
[12] P. Maroni, Z. da Rocha, A new characterization of classical forms, Comm. Appl. Anal. 5 (2001) 351–362.
[13] S. Wolfram, The Mathematica Book, fourth ed., Wolfram Media/Cambridge Univ. Press, 1999.