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On the inverse problem of the product of a form by a polynomial: The cubic case

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Abstract

A form (linear functional) u is called regular if there exists a sequence of polynomials $\{P_n\}_{n\geqslant 0}$, $\deg P_n = n$, which is orthogonal with respect to u. On certain regularity conditions, the product of a regular form by a polynomial is still a regular form. In this paper, we consider the inverse problem: given a regular form v, find all the regular forms u which satisfy the relation $x^3u = -\lambda v$, $\lambda \in \mathbb{C} - \{0\}$. We give the second-order recurrence relation of the orthogonal polynomial sequence with respect to u. Some examples are studied.

Keywords: Inverse problem; Forms; Orthogonal polynomials

Introduction

The product of a linear form by a polynomial, is one of construction process of linear forms. In 1858, Christoffel proved that the product of a positive definite form by a positive polynomial is still a positive definite form [3,4]. This result has been generalized in [7], where it was proved that, under certain regularity conditions, the product of a regular form u by a polynomial R is a regular form. In particular, if u is a semi-classical form or a second degree form, then Ru is also a semi-classical form or a second degree form, respectively. It is also interesting to consider the inverse problem, which consists of the determination of all regular forms u satisfying $Ru = -\lambda v$, where v is a given regular form and $\lambda \in \mathbb{C} - \{0\}$. In [9], the first author has considered the case $R(x) = x - c$, with $c \in \mathbb{C}$, and in [13] the case $R(x) = x^2$. He gave the regularity conditions and the coefficients of the second-order recurrence relation satisfied by the monic orthogonal polynomial sequence (MOPS) with respect to u. The structure relation of the MOPS relatively to u, in the symmetric case, is given in [1] and in the non-symmetric case is given

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in [2]. Many examples were also treated in [13,2,1]. In this paper, we consider the case $R(x) = x^3$. In the first section we will give the regularity conditions and the coefficients of the second-order recurrence relation satisfied by the MOPS with respect to u. We will also prove that if v is semi-classical then u is semi-classical and some results concerned to the class of u are given. Section 2 is devoted to the particular case of $(u)_2 = 0$ and Section 3 to the case $(u)_2 \neq 0$: regularity conditions more simple and some examples are given if the form v is symmetric and if v is symmetric definite positive, respectively. The regular forms u founded in the examples are semi-classical of class $s \in \{1, 2, 3\}$ and we present there integral representations and the coefficients of the second-order recurrence satisfied by the MOPS with respect to u.

1. The problem $x^3 u = -\lambda v$

1.1. The main problem

Let $\mathcal{P}$ be the vector space of polynomials with coefficients in $\mathbb{C}$ and $\mathcal{P}'$ its dual. We denote by $\langle u, f\rangle$ the action of $u \in \mathcal{P}'$ on $f \in \mathcal{P}$. Let us recall that a form u is called regular if there exists a polynomial sequence $\{P_n\}_{n\geqslant 0}$, $\deg P_n = n$, such that $\langle u, P_n P_m\rangle = k_n \delta_{n,m}$, $n, m \geqslant 0$, $k_n \neq 0$, $n \geqslant 0$; the left-multiplication hw is defined by $\langle hw, p\rangle := \langle w, hp\rangle$. We consider the following problem: given a regular form v, find all regular forms u which satisfy the next equation:

$$x^3 u = -\lambda v, \quad \lambda \in \mathbb{C} - \{0\}, \tag{1.1}$$

with constraints $(u)_0 = 1$, $(v)_0 = 1$, where $(u)_n := \langle u, x^n\rangle$, $n \geqslant 0$, are the moments of u. Equivalently,

$$u = \delta - (u)_1\delta' + \frac{1}{2}(u)_2\delta'' - \lambda x^{-3} v, \tag{1.2}$$

where $\langle \delta, f\rangle = f(0)$, the derivative $w' = Dw$ of the form w is defined by $\langle w', f\rangle := -\langle w, f'\rangle$ and the form $x^{-1}w$ by $\langle x^{-1}w, f\rangle := \langle w, \theta_0 f\rangle$, with in general

$$(\theta_c f)(x) := \frac{f(x) - f(c)}{x - c}, \quad c \in \mathbb{C}.$$

So the form u depends on three arbitrary parameters $(u)_1$, $(u)_2$ and $\lambda = -(u)_3$.

If we suppose that the form v possesses the following integral representation:

$$\langle v, f\rangle = \int_{-\infty}^{+\infty} V(x) f(x)\,\mathrm{d}x, \quad \text{for each polynomial } f,$$

where V is a locally integrable function with rapid decay, continuous at the origin and with derivative continuous at the origin, then the form u is represented by

$$\begin{aligned}\langle u, f\rangle \;=\; & f(0)\left\{1 + \lambda Pf \int_{-\infty}^{+\infty} \frac{V(x)}{x^3}\,\mathrm{d}x\right\} + f'(0)\left\{(u)_1 + \lambda Pf \int_{-\infty}^{+\infty} \frac{V(x)}{x^2}\,\mathrm{d}x\right\} \\ & + \frac{1}{2} f''(0)\left\{(u)_2 + \lambda P \int_{-\infty}^{+\infty} \frac{V(x)}{x}\,\mathrm{d}x\right\} - \lambda Pf \int_{-\infty}^{+\infty} \frac{V(x) f(x)}{x^3}\,\mathrm{d}x,\end{aligned} \tag{1.3}$$

where, [5,8]

$$P\int_{-\infty}^{+\infty}\frac{V(x)}{x}\,\mathrm{d}x=\lim_{\varepsilon\to 0^+}\left(\int_{-\infty}^{-\varepsilon}\frac{V(x)}{x}\,\mathrm{d}x+\int_{\varepsilon}^{+\infty}\frac{V(x)}{x}\,\mathrm{d}x\right),$$

$$Pf\int_{-\infty}^{+\infty}\frac{V(x)}{x^2}\,\mathrm{d}x=\lim_{\varepsilon\to 0^+}\left(\int_{-\infty}^{-\varepsilon}\frac{V(x)}{x^2}\,\mathrm{d}x+\int_{\varepsilon}^{+\infty}\frac{V(x)}{x^2}\,\mathrm{d}x-\frac{2}{\varepsilon}V(0)\right),$$

and

$$Pf\int_{-\infty}^{+\infty}\frac{V(x)}{x^3}\,\mathrm{d}x=\lim_{\varepsilon\to 0^+}\left(\int_{-\infty}^{-\varepsilon}\frac{V(x)}{x^3}\,\mathrm{d}x+\int_{\varepsilon}^{+\infty}\frac{V(x)}{x^3}\,\mathrm{d}x-\frac{2}{\varepsilon}V'(0)\right).$$

Let $\{S_n\}_{n\geqslant 0}$ denote the sequence of orthogonal polynomials with respect to v

$$\begin{aligned}&S_0(x)=1,\qquad S_1(x)=x-\zeta_0,\\&S_{n+2}(x)=(x-\zeta_{n+1})S_{n+1}(x)-\sigma_{n+1}S_n(x),\quad n\geqslant 0.\end{aligned}\tag{1.4}$$

When u is regular, let $\{Z_n\}_{n\geqslant 0}$ be the corresponding orthogonal sequence

$$\begin{aligned}&Z_0(x)=1,\qquad Z_1(x)=x-\beta_0,\\&Z_{n+2}(x)=(x-\beta_{n+1})Z_{n+1}(x)-\gamma_{n+1}Z_n(x),\quad n\geqslant 0.\end{aligned}\tag{1.5}$$

From (1.1) we know that the sequence $\{Z_n\}_{n\geqslant 0}$, when it exists, is among the strictly quasi-orthogonal sequences of order three with respect to v [6,10], see also [12, pp. 127, 128], this is

$$\begin{aligned}&Z_0(x)=1,\qquad Z_1(x)=S_1(x)+c_0,\qquad Z_2(x)=S_2(x)+c_1S_1(x)+b_0,\\&Z_{n+3}(x)=S_{n+3}(x)+c_{n+2}S_{n+2}(x)+b_{n+1}S_{n+1}(x)+a_nS_n(x),\quad n\geqslant 0,\end{aligned}\tag{1.6}$$

with $a_n\neq 0,\ n\geqslant 0$. Moreover, the sequence $\{Z_n\}_{n\geqslant 0}$ is orthogonal with respect to u if and only if

$$\begin{aligned}&\langle u,Z_n\rangle=0,\quad n\geqslant 1;\\&\big\langle u,xZ_n(x)\big\rangle=0,\quad n\geqslant 2,\qquad \big\langle u,xZ_1(x)\big\rangle\neq 0;\\&\big\langle u,x^2Z_n(x)\big\rangle=0,\quad n\geqslant 3,\qquad \big\langle u,x^2Z_2(x)\big\rangle\neq 0;\end{aligned}\tag{1.7}$$

since the other orthogonality conditions have been used in (1.6). From (1.6) and (1.7) we have

$$\begin{aligned}0&=\langle u,Z_{n+3}\rangle\\&=\langle u,S_{n+3}\rangle+c_{n+2}\langle u,S_{n+2}\rangle+b_{n+1}\langle u,S_{n+1}\rangle+a_n\langle u,S_n\rangle,\quad n\geqslant 0,\\0&=\big\langle u,xZ_{n+3}(x)\big\rangle\\&=\big\langle u,xS_{n+3}(x)\big\rangle+c_{n+2}\big\langle u,xS_{n+2}(x)\big\rangle+b_{n+1}\big\langle u,xS_{n+1}(x)\big\rangle+a_n\big\langle u,xS_n(x)\big\rangle,\quad n\geqslant 0,\\0&=\big\langle u,x^2Z_{n+3}(x)\big\rangle\\&=\big\langle u,x^2S_{n+3}(x)\big\rangle+c_{n+2}\big\langle u,x^2S_{n+2}(x)\big\rangle+b_{n+1}\big\langle u,x^2S_{n+1}(x)\big\rangle+a_n\big\langle u,x^2S_n(x)\big\rangle,\quad n\geqslant 0,\end{aligned}\tag{1.8}$$

with the initial conditions:

$$
\begin{aligned}
&0 = \langle u, Z_1\rangle = \langle u, S_1\rangle + c_0,\\
&0 = \langle u, Z_2\rangle = \langle u, S_2\rangle + c_1\langle u, S_1\rangle + b_0,\\
&0 = \big\langle u, xZ_2(x)\big\rangle = \big\langle u, xS_2(x)\big\rangle + c_1\big\langle u, xS_1(x)\big\rangle + b_0(u)_1,\\
&0 \neq \big\langle u, xZ_1(x)\big\rangle = \big\langle u, xS_1(x)\big\rangle + c_0(u)_1,\\
&0 \neq \big\langle u, x^2Z_2(x)\big\rangle = \big\langle u, x^2S_2(x)\big\rangle + c_1\big\langle u, x^2S_1(x)\big\rangle + b_0(u)_2.
\end{aligned}
\tag{1.9}
$$

If we denote

$$
\Delta_n := \begin{vmatrix} \langle u, S_{n+2}\rangle & \langle u, S_{n+1}\rangle & \langle u, S_n\rangle \\ \langle u, xS_{n+2}(x)\rangle & \langle u, xS_{n+1}(x)\rangle & \langle u, xS_n(x)\rangle \\ \langle u, x^2S_{n+2}(x)\rangle & \langle u, x^2S_{n+1}(x)\rangle & \langle u, x^2S_n(x)\rangle \end{vmatrix}, \quad n \geqslant 0, \tag{1.10}
$$

then the system (1.8) is equivalent to

$$
\Delta_n a_n = -\Delta_{n+1}, \quad n \geqslant 0, \tag{1.11}
$$

$$
\Delta_n b_{n+1} = \begin{vmatrix} \langle u, S_{n+2}\rangle & -\langle u, S_{n+3}\rangle & \langle u, S_n\rangle \\ \langle u, xS_{n+2}(x)\rangle & -\langle u, xS_{n+3}(x)\rangle & \langle u, xS_n(x)\rangle \\ \langle u, x^2S_{n+2}(x)\rangle & -\langle u, x^2S_{n+3}(x)\rangle & \langle u, x^2S_n(x)\rangle \end{vmatrix}, \quad n \geqslant 0, \tag{1.12}
$$

$$
\Delta_n c_{n+2} = \begin{vmatrix} -\langle u, S_{n+3}\rangle & \langle u, S_{n+1}\rangle & \langle u, S_n\rangle \\ -\langle u, xS_{n+3}(x)\rangle & \langle u, xS_{n+1}(x)\rangle & \langle u, xS_n(x)\rangle \\ -\langle u, x^2S_{n+3}(x)\rangle & \langle u, x^2S_{n+1}(x)\rangle & \langle u, x^2S_n(x)\rangle \end{vmatrix}, \quad n \geqslant 0. \tag{1.13}
$$

Proposition 1.1. *The form u is regular if and only if $\Delta_n \neq 0,\ n \geqslant 0$ and $(u)_2 - (u)_1^2 \neq 0$. In this case, the coefficients of the second-order recurrence relation of $\{Z_n\}_{n\geqslant 0}$ are given by*

$$
\beta_0 = (u)_1, \qquad \beta_{n+1} = \zeta_{n+1} + c_n - c_{n+1}, \quad n \geqslant 0, \tag{1.14}
$$

$$
\begin{aligned}
&\gamma_1 = (u)_2 - (u)_1^2, \qquad \gamma_2 = -\Delta_0\big[(u)_2 - (u)_1^2\big]^{-2},\\
&\gamma_3 = -\lambda\frac{\Delta_1[(u)_2 - (u)_1^2]}{\Delta_0^2}, \qquad \gamma_{n+4} = \frac{\Delta_{n+2}\Delta_n}{\Delta_{n+1}^2}\sigma_{n+1}, \quad n \geqslant 0.
\end{aligned}
\tag{1.15}
$$

Proof. Necessity. If $\{Z_n\}_{n\geqslant 0}$ is orthogonal, it is strictly quasi-orthogonal with respect to v and then $a_n \neq 0, n \geqslant 0$. This implies $\Delta_n \neq 0,\ n \geqslant 0$. Assuming the contrary, that there exists an $n_0 \geqslant 1$ such that $\Delta_{n_0} = 0$, then from (1.11), $\Delta_0 = 0 = -\langle u, x^2Z_2(x)\rangle\langle u, xZ_1(x)\rangle \neq 0$, which is a contradiction. Moreover $(u)_2 - (u)_1^2 = \langle u, xZ_1(x)\rangle \neq 0$.

Sufficiency. Using (1.2), the conditions $\langle u, Z_1\rangle = 0,\ \langle u, Z_2\rangle = 0$ and $\langle u, xZ_2(x)\rangle = 0$ are satisfied for

$$
\begin{aligned}
&c_0 = \zeta_0 - (u)_1,\\
&c_1 = \{\zeta_1 + \zeta_0\} + \frac{\lambda + (u)_1(u)_2}{(u)_2 - {(u)_1}^2},\\
&b_0 = \sigma_1 + {\zeta_0}^2 - (u)_2 + \big\{\zeta_0 - (u)_1\big\}\frac{\lambda + (u)_1(u)_2}{(u)_2 - {(u)_1}^2}.
\end{aligned}
\tag{1.16}
$$

Taking account of (1.16), we also have $\langle u, xZ_1(x)\rangle = (u)_2 - {(u)_1}^2 \neq 0$ and

$$
\big\langle u, x^2Z_2(x)\big\rangle = \frac{-\lambda^2 - 2(u)_1(u)_2\lambda - \zeta_0\{(u)_2 - {(u)_1}^2\}\lambda - {(u)_2}^3}{(u)_2 - {(u)_1}^2} = -\Delta_0\big[(u)_2 - (u)_1^2\big]^{-1} \neq 0.
$$

We had just proved that the initial conditions (1.9) are satisfied. Further, the system (1.8) is a Cramer system whose solution is given by (1.11), (1.12) and (1.13). Moreover, from (1.5) for $n \to n+2$ and (1.6) for $n \to n+1$ we can write

$$S_{n+4}(x) + c_{n+3}S_{n+3}(x) + b_{n+2}S_{n+2}(x) + a_{n+1}S_{n+1}(x) \\ = (x - \beta_{n+3})Z_{n+3}(x) - \gamma_{n+3}Z_{n+2}(x), \quad n \geqslant 0.$$

Multiplying the above equation by $Z_{n+3}(x)$ and applying the form u, we obtain

$$\langle u, S_{n+4}Z_{n+3}\rangle + c_{n+3}\langle u, S_{n+3}Z_{n+3}\rangle = \langle u, xZ^2_{n+3}(x)\rangle - \beta_{n+3}\langle u, Z^2_{n+3}\rangle, \quad n \geqslant 0,$$

because $\{Z_n\}_{n\geqslant 0}$ is orthogonal with respect to u. Now, we will replace $S_{n+4}(x)$ for its expression given by (1.4) for $n \to n+2$, and then

$$\langle u, xS_{n+3}(x)Z_{n+3}(x)\rangle - \zeta_{n+3}\langle u, S_{n+3}Z_{n+3}\rangle + c_{n+3}\langle u, S_{n+3}Z_{n+3}\rangle \\ = \langle u, xZ^2_{n+3}(x)\rangle - \beta_{n+3}\langle u, Z^2_{n+3}\rangle, \quad n \geqslant 0.$$

Finally, replacing S_{n+3} by its expression given by (1.6)

$$\langle u, xZ^2_{n+3}(x)\rangle - c_{n+2}\langle u, xS_{n+2}(x)Z_{n+3}(x)\rangle - \zeta_{n+3}\langle u, Z^2_{n+3}\rangle + c_{n+3}\langle u, Z^2_{n+3}\rangle \\ = \langle u, xZ^2_{n+3}(x)\rangle - \beta_{n+3}\langle u, Z^2_{n+3}\rangle, \quad n \geqslant 0,$$

that is

$$-c_{n+2}\langle u, Z^2_{n+3}(x)\rangle - \zeta_{n+3}\langle u, Z^2_{n+3}\rangle + c_{n+3}\langle u, Z^2_{n+3}\rangle = -\beta_{n+3}\langle u, Z^2_{n+3}\rangle, \quad n \geqslant 0.$$

As, $\langle u, Z^2_{n+3}\rangle \neq 0$, we have

$$\beta_{n+3} = \zeta_{n+3} + c_{n+2} - c_{n+3}, \quad n \geqslant 0.$$

Using the same proceeding, we easily prove that

$$\beta_{n+1} = \zeta_{n+1} + c_n - c_{n+1}, \quad n = 0, 1,$$

that is, we have proved (1.14). Let us see now that

$$\langle u, Z^2_{n+3}\rangle = \langle u, x^{n+3}Z_{n+3}(x)\rangle = \langle x^3u, x^nZ_{n+3}(x)\rangle = -\lambda\langle v, x^nZ_{n+3}(x)\rangle, \quad n \geqslant 0,$$

from (1.1). Using (1.6) this becomes

$$\langle u, Z^2_{n+3}\rangle = -\lambda a_n\langle v, x^nS_n(x)\rangle = -\lambda a_n\langle v, S^2_n\rangle, \quad n \geqslant 0.$$

Then

$$\gamma_{n+4} = \frac{\langle u, Z^2_{n+4}\rangle}{\langle u, Z^2_{n+3}\rangle} = \frac{a_{n+1}\langle v, S^2_{n+1}\rangle}{a_n\langle v, S^2_n\rangle} = \frac{a_{n+1}}{a_n}\sigma_{n+1}, \quad n \geqslant 0.$$

By virtue of (1.11), we successively obtain

$$\gamma_{n+4} = \frac{\Delta_{n+2}\Delta_n}{\Delta^2_{n+1}}\sigma_{n+1}, \quad n \geqslant 0,$$

$$\gamma_1 = \langle u, xZ_1(x)\rangle = (u)_2 - (u)^2_1,$$

$$\gamma_2 = \frac{\langle u, x^2Z_2(x)\rangle}{\langle u, xZ_1(x)\rangle} = -\frac{\Delta_0}{[(u)_2 - (u)^2_1]^2},$$

$$\gamma_3 = \frac{\langle u, x^3Z_3(x)\rangle}{\langle u, x^2Z_2(x)\rangle} = \frac{-\lambda a_0}{-\Delta_0[(u)_2 - (u)^2_1]^{-1}} = -\lambda\frac{\Delta_1}{\Delta^2_0}\big[(u)_2 - (u)^2_1\big].$$

We have proved (1.15). □

1.2. *The computation of Δ_n*

As we had seen in Proposition 1.1, it is very important to have an explicit expression for Δ_n. In first place, let us note that, from (1.2), we have

$$\langle u, S_{n+1}(x)\rangle = S_{n+1}(0) + (u)_1 S'_{n+1}(0) + \frac{1}{2}(u)_2 S''_{n+1}(0) - \frac{1}{2}\lambda\big(S_n^{(1)}\big)''(0), \quad n \geqslant 0, \tag{1.17}$$

with

$$S_n^{(1)}(x) := \left\langle v, \frac{S_{n+1}(x) - S_{n+1}(\xi)}{x-\xi}\right\rangle, \quad n \geqslant 0,$$

$$\langle u, xS_{n+1}(x)\rangle = (u)_1 S_{n+1}(0) + (u)_2 S'_{n+1}(0) - \lambda\big(S_n^{(1)}\big)'(0), \quad n \geqslant 0, \tag{1.18}$$

$$\langle u, x^2 S_{n+1}(x)\rangle = (u)_2 S_{n+1}(0) - \lambda S_n^{(1)}(0), \quad n \geqslant 0, \tag{1.19}$$

$$\langle u, x^3 S_{n+1}(x)\rangle = 0, \quad n \geqslant 0. \tag{1.20}$$

Using (1.4) and (1.20), we obtain for (1.10):

$$\Delta_n = \begin{vmatrix} \langle u, xS_{n+1}(x)\rangle & \langle u, S_{n+1}\rangle & \langle u, S_n\rangle \\ \langle u, x^2 S_{n+1}(x)\rangle & \langle u, xS_{n+1}(x)\rangle & \langle u, xS_n(x)\rangle \\ 0 & \langle u, x^2 S_{n+1}(x)\rangle & \langle u, x^2 S_n(x)\rangle \end{vmatrix}, \quad n \geqslant 0, \tag{1.21}$$

that is

$$\begin{aligned}\Delta_n &= \langle u, xS_{n+1}(x)\rangle \begin{vmatrix} \langle u, xS_{n+1}(x)\rangle & \langle u, xS_n(x)\rangle \\ \langle u, x^2 S_{n+1}(x)\rangle & \langle u, x^2 S_n(x)\rangle \end{vmatrix} \\ &\quad - \langle u, x^2 S_{n+1}(x)\rangle \begin{vmatrix} \langle u, S_{n+1}\rangle & \langle u, S_n\rangle \\ \langle u, x^2 S_{n+1}(x)\rangle & \langle u, x^2 S_n(x)\rangle \end{vmatrix}, \quad n \geqslant 0. \end{aligned} \tag{1.22}$$

From (1.17), (1.18) and (1.19), we have

$$\begin{aligned}&\begin{vmatrix} \langle u, xS_{n+1}(x)\rangle & \langle u, xS_n(x)\rangle \\ \langle u, x^2 S_{n+1}(x)\rangle & \langle u, x^2 S_n(x)\rangle \end{vmatrix} \\ &\quad = X_{n-1}^{(1)}(0)\lambda^2 + \big\{\langle v, S_n^2\rangle (u)_1 + \big[\widetilde{Y}_n(0) - \widetilde{X}_n(0)\big](u)_2\big\}\lambda + X_n(0)(u)_2^2, \quad n \geqslant 0, \end{aligned} \tag{1.23}$$

and

$$\begin{aligned}&\begin{vmatrix} \langle u, S_{n+1}\rangle & \langle u, S_n\rangle \\ \langle u, x^2 S_{n+1}(x)\rangle & \langle u, x^2 S_n(x)\rangle \end{vmatrix} \\ &\quad = \frac{1}{2}\big(X_{n-1}^{(1)}\big)'(0)\lambda^2 + \left\{\langle v, S_n^2\rangle - \widetilde{X}_n(0)(u)_1 + \frac{1}{2}\big[\widetilde{Y}'_n(0) - \widetilde{X}'_n(0)\big](u)_2\right\}\lambda \\ &\qquad + \left\{X_n(0)(u)_1(u)_2 + \frac{1}{2}X'_n(0)(u)_2^2\right\}, \quad n \geqslant 0, \end{aligned}$$

with

$$X_n(x) = S_n(x)S'_{n+1}(x) - S_{n+1}(x)S'_n(x), \quad n \geqslant 0, \tag{1.24}$$

$$X_n^{(1)}(x) = S_n^{(1)}(x)\big(S_{n+1}^{(1)}\big)'(x) - S_{n+1}^{(1)}(x)\big(S_n^{(1)}\big)'(x), \quad n \geqslant 0, \tag{1.25}$$

$$\widetilde{X}_n(x) = S'_{n+1}(x)S_{n-1}^{(1)}(x) - S'_n(x)S_n^{(1)}(x), \quad n \geqslant 0, \tag{1.26}$$

$$\widetilde{Y}_n(x) = \big(S_{n-1}^{(1)}\big)'(x)S_{n+1}(x) - S_n(x)\big(S_n^{(1)}\big)'(x), \quad n \geqslant 0. \tag{1.27}$$

Therefore,

$$\begin{aligned}
\Delta_n = {} & W_{n-1}^{(1)}(0)\lambda^3 \\
& + \big\{S_n^{(1)}(0)\langle v, S_n^2\rangle + \big[\widetilde{W}_n(0) - \big(S_n^{(1)}\big)'(0)\langle v, S_n^2\rangle\big](u)_1 + \big[\widetilde{T}_n(0) - \widetilde{U}_n(0)\big](u)_2\big\}\lambda^2 \\
& + \big\{S_{n+1}(0)\langle v, S_n^2\rangle\big((u)_1^2 - (u)_2\big) + \big[\widetilde{V}_n(0) - \widetilde{R}_n(0)\big](u)_2^2 \\
& \quad + \big[S'_{n+1}(0)\langle v, S_n^2\rangle + S_n^{(1)}(0)X_n(0) + \widetilde{Y}_n(0)S_{n+1}(0)\big](u)_1(u)_2\big\}\lambda \\
& - W_n(0)(u)_2^3, \quad n \geqslant 0,
\end{aligned} \tag{1.28}$$

with

$$W_n(x) = \frac{1}{2}S_{n+1}(x)X'_n(x) - S'_{n+1}(x)X_n(x), \quad n \geqslant 0, \tag{1.29}$$

$$W_n^{(1)}(x) = \frac{1}{2}S_{n+1}^{(1)}(x)\big(X_n^{(1)}\big)'(x) - \big(S_{n+1}^{(1)}\big)'(x)X_n^{(1)}(x), \quad n \geqslant 0, \tag{1.30}$$

$$\widetilde{W}_n(x) = S_{n+1}(x)X_{n-1}^{(1)}(x) - S_n^{(1)}(x)\widetilde{X}_n(x), \quad n \geqslant 0, \tag{1.31}$$

$$\widetilde{U}_n(x) = \frac{1}{2}S_{n+1}(x)\big(X_{n-1}^{(1)}\big)'(x) - S'_{n+1}(x)X_{n-1}^{(1)}(x), \quad n \geqslant 0, \tag{1.32}$$

$$\widetilde{V}_n(x) = \frac{1}{2}S_n^{(1)}(x)X'_n(x) - \big(S_n^{(1)}\big)'(x)X_n(x), \quad n \geqslant 0, \tag{1.33}$$

$$\widetilde{T}_n(x) = \frac{1}{2}S_n^{(1)}(x)\big[\widetilde{Y}'_n(x) - \widetilde{X}'_n(x)\big] - \big(S_n^{(1)}\big)'(x)\big[\widetilde{Y}_n(x) - \widetilde{X}_n(x)\big], \quad n \geqslant 0, \tag{1.34}$$

$$\widetilde{R}_n(x) = \frac{1}{2}S_{n+1}(x)\big[\widetilde{Y}'_n(x) - \widetilde{X}'_n(x)\big] - S'_{n+1}(x)\big[\widetilde{Y}_n(x) - \widetilde{X}_n(x)\big], \quad n \geqslant 0. \tag{1.35}$$

Moreover, if the form u is regular we have for (1.11), (1.12) and (1.13)

$$a_n = -\frac{\Delta_{n+1}}{\Delta_n}, \quad n \geqslant 0, \tag{1.36}$$

$$b_{n+1} = \sigma_{n+2} - \begin{vmatrix} \langle u, S_{n+2}\rangle & \langle u, xS_{n+2}(x)\rangle & \langle u, S_n\rangle \\ \langle u, xS_{n+2}(x)\rangle & \langle u, x^2S_{n+2}(x)\rangle & \langle u, xS_n(x)\rangle \\ \langle u, x^2S_{n+2}(x)\rangle & 0 & \langle u, x^2S_n(x)\rangle \end{vmatrix} \Delta_n^{-1}, \quad n \geqslant 0, \tag{1.37}$$

$$c_{n+2} = \zeta_{n+2} - \begin{vmatrix} \langle u, xS_{n+2}(x)\rangle & \langle u, S_{n+1}\rangle & \langle u, S_n\rangle \\ \langle u, x^2S_{n+2}(x)\rangle & \langle u, xS_{n+1}(x)\rangle & \langle u, xS_n(x)\rangle \\ 0 & \langle u, x^2S_{n+1}(x)\rangle & \langle u, x^2S_n(x)\rangle \end{vmatrix} \Delta_n^{-1}, \quad n \geqslant 0, \tag{1.38}$$

by virtue of (1.4).

1.3. Some results on the semi-classical particular case

Let us recall that a form u is called semi-classical if there exists two polynomials Φ and ψ such that u satisfies the functional equation

$$D(\Phi u)+\psi u=0.$$

The class of the semi-classical form u is $s=\max(\deg\Phi-2,\deg\psi-1)$ if and only if the following condition is satisfied

$$\prod_c\big(\big|\psi(c)+\Phi'(c)\big|+\big|\big\langle u,\theta_c\psi+\theta_c^2\Phi\big\rangle\big|\big)>0, \tag{1.39}$$

where $c\in\{x\colon\ \Phi(x)=0\}$ [11].

In the sequel the form v will be supposed semi-classical of class s satisfying $D(\Phi v)+\psi v=0$.

From (1.1), it is clear that the form u, when it is regular, is also semi-classical and satisfies

$$D(\widetilde{\Phi}u)+\tilde{\psi}u=0,$$

with

$$\widetilde{\Phi}(x)=x^3\Phi(x)\quad\text{and}\quad\tilde{\psi}(x)=x^3\psi(x). \tag{1.40}$$

The class of u is at most $\tilde{s}=s+3$.

Proposition 1.2. *The class of u depends only on the zero $x=0$.*

For the proof we use the following lemma:

Lemma 1.3. *For all root c of Φ we have*

$$\big\langle u,\theta_c\tilde{\psi}+\theta_c^2\widetilde{\Phi}\big\rangle=\big(\psi(c)+\Phi'(c)\big)\big((u)_2+c(u)_1+c^2\big)-\lambda\big\langle v,\big(\theta_c^2\Phi+\theta_c\psi\big)\big\rangle \tag{1.41}$$

and

$$\tilde{\psi}(c)+\widetilde{\Phi}'(c)=c^3\big(\psi(c)+\Phi'(c)\big). \tag{1.42}$$

Proof. Let c be a root of Φ, then we can write

$$\widetilde{\Phi}(x)=x^3(x-c)\Phi_c(x)\quad\text{with } \Phi_c(x)=(\theta_c\Phi)(x). \tag{1.43}$$

So from (1.40) and (1.43) we have

$$\big\langle u,\theta_c\tilde{\psi}+\theta_c^2\widetilde{\Phi}\big\rangle=\big\langle u,\theta_c\big(\xi^3\psi\big)\big\rangle+\big\langle u,\theta_c\big(\xi^3\Phi_c\big)\big\rangle. \tag{1.44}$$

Using the definition of the operator θ_c, it is easy to prove that, for two polynomials f and g, we have

$$\theta_c(fg)(x)=g(x)(\theta_c f)(x)+f(c)(\theta_c g)(x). \tag{1.45}$$

Taking $g(x)=x^3$ and $f(x)=\Phi_c(x)$, we obtain

$$\begin{aligned}\big\langle u,\theta_c\big(\xi^3\Phi_c\big)\big\rangle&=\big\langle u,x^3(\theta_c\Phi_c)(x)+\Phi_c(c)\big(\theta_c\xi^3\big)(x)\big\rangle\\&=\big\langle u,x^3\big(\theta_c^2\Phi\big)(x)\big\rangle+\big((u)_2+c(u)_1+c^2\big)\Phi'(c),\end{aligned}$$

because

$$(\theta_c\Phi_c)(x) = \big(\theta_c(\theta_c\Phi)\big)(x) = \big(\theta_c^2\Phi\big)(x), \quad \Phi_c(c) = (\theta_c\Phi)(c) = \Phi'(c) \text{ and } \big(\theta_c\xi^3\big)(x) = x^2 + cx + c^2.$$

Using (1.2), we obtain

$$\big\langle u, \theta_c\big(\xi^3\Phi_c\big)\big\rangle = \left\langle \delta - (u)_1\delta' + \frac{1}{2}(u)_2\delta'' - \lambda x^{-3}v, x^3\big(\theta_c^2\Phi\big)(x)\right\rangle + \big((u)_2 + c(u)_1 + c^2\big)\Phi'(c),$$

hence

$$\big\langle u, \theta_c\big(\xi^3\Phi_c\big)\big\rangle = \big((u)_2 + c(u)_1 + c^2\big)\Phi'(c) - \lambda\big\langle v, \theta_c^2\Phi\big\rangle. \tag{1.46}$$

Now, taking $g(x) = x^3$ and $f(x) = \psi(x)$ in (1.45), we obtain

$$\begin{aligned}\big\langle u, \theta_c\big(\xi^3\psi\big)\big\rangle &= \big\langle u, x^3(\theta_c\psi)(x) + \psi(c)\big(\theta_c\xi^3\big)(x)\big\rangle \\ &= \big\langle u, x^3(\theta_c\psi)(x)\big\rangle + \big((u)_2 + c(u)_1 + c^2\big)\psi(c).\end{aligned}$$

Using (1.2), we obtain

$$\big\langle u, \theta_c\big(\xi^3\psi\big)\big\rangle = \left\langle \delta - (u)_1\delta' + \frac{1}{2}(u)_2\delta'' - \lambda x^{-3}v, x^3(\theta_c\psi)(x)\right\rangle + \big((u)_2 + c(u)_1 + c^2\big)\psi(c).$$

Then

$$\big\langle u, \theta_c\big(\xi^3\psi\big)\big\rangle = \big((u)_2 + c(u)_1 + c^2\big)\psi(c) - \lambda\langle v, \theta_c\psi\rangle. \tag{1.47}$$

Replacing (1.46) and (1.47) in (1.44), we obtain (1.41).

From (1.40), we have $\widetilde{\Phi}'(c) = c^3\Phi'(c)$ and $\tilde{\psi}(c) = c^3\psi(c)$, hence (1.42). □

Proof of Proposition 1.2. Let c be a root of Φ such that $c \neq 0$.

If $\psi(c) + \Phi'(c) = 0$, using (1.41), we have $\langle u, \theta_c\tilde{\psi} + \theta_c^2\widetilde{\Phi}\rangle \neq 0$, since v is semi-classical and so satisfies (1.39).

If $\psi(c) + \Phi'(c) \neq 0$ then $\tilde{\psi}(c) + \widetilde{\Phi}'(c) \neq 0$, from (1.42).

In any case, we cannot simplify by $x - c$. □

Proposition 1.4. *Let v be a semi-classical form of class s satisfying $D(\Phi v) + \psi v = 0$,*

$$\chi_1 := (u)_1\Phi(0) + (u)_2\big(\Phi'(0) + \psi(0)\big) - \lambda\big\langle v, \theta_0\psi + \theta_0^2\Phi\big\rangle, \tag{1.48}$$

and

$$\chi_2 := 2\Phi(0) + (u)_1\big(\psi(0) + 2\Phi'(0)\big) + (u)_2\big(\Phi''(0) + \psi'(0)\big) - \lambda\big\langle v, \theta_0^2\psi + 2\theta_0^3\Phi\big\rangle. \tag{1.49}$$

The form u given by (1.1) *is also semi-classical of class $\tilde{s}$ satisfying $D(\widetilde{\Phi}u) + \tilde{\psi}u = 0$. Moreover,*

(1) *if $\chi_1 \neq 0$ then $\tilde{s} = s + 3$, $\widetilde{\Phi}(x) = x^3\Phi(x)$ and $\tilde{\psi}(x) = x^3\psi(x)$;*
(2) *if $\chi_1 = 0$ and $\chi_2 \neq 0$ then $\tilde{s} = s + 2$, $\widetilde{\Phi}(x) = x^2\Phi(x)$ and $\tilde{\psi}(x) = x^2\psi(x) + x\Phi(x)$;*
(3) *if $\chi_1 = 0$, $\chi_2 = 0$ and $\Phi(0) \neq 0$ then $\tilde{s} = s + 1$, $\widetilde{\Phi}(x) = x\Phi(x)$ and $\tilde{\psi}(x) = x\psi(x) + 2\Phi(x)$.*

Proof. (1) Indeed,

$$\tilde{\psi}(0) + \widetilde{\Phi}'(0) = 0,$$

and

$$\langle u, (\theta_0\tilde{\psi} + \theta_0^2\widetilde{\Phi})(x)\rangle = \langle u, x^2\psi(x) + x\Phi(x)\rangle$$
$$= \left\langle \delta - (u)_1\delta' + \frac{1}{2}(u)_2\delta'' - \lambda x^{-3}v, x^2\psi(x) + x\Phi(x)\right\rangle,$$

using (1.2), that is,

$$\langle u, (\theta_0\tilde{\psi} + \theta_0^2\widetilde{\Phi})(x)\rangle = (u)_1\Phi(0) + (u)_2(\Phi'(0) + \psi(0)) - \lambda\langle v, \theta_0\psi + \theta_0^2\Phi\rangle = \chi_1.$$

Therefore, if $\chi_1 \neq 0$ it is not possible to simplify, which means that the class of u is $s+3$ and u satisfies $D(\widetilde{\Phi}u) + \tilde{\psi}u = 0$, with $\widetilde{\Phi}(x) = x^3\Phi(x)$ and $\tilde{\psi}(x) = x^3\psi(x)$.

(2) If $\chi_1 = 0$ then let $\widetilde{\Phi}_0(x) = x^2\Phi(x)$ and $\tilde{\psi}_0(x) = x^2\psi(x) + x\Phi(x)$. Then

$$\tilde{\psi}_0(0) + \widetilde{\Phi}_0'(0) = 0,$$

and

$$\langle u, (\theta_0\tilde{\psi}_0 + \theta_0^2\widetilde{\Phi}_0)(x)\rangle = \langle u, x\psi(x) + 2\Phi(x)\rangle$$
$$= \left\langle \delta - (u)_1\delta' + \frac{1}{2}(u)_2\delta'' - \lambda x^{-3}v, x\psi(x) + 2\Phi(x)\right\rangle,$$

using (1.2), that is,

$$\langle u, (\theta_0\tilde{\psi}_0 + \theta_0^2\widetilde{\Phi}_0)(x)\rangle$$
$$= 2\Phi(0) + (u)_1(\psi(0) + 2\Phi'(0)) + (u)_2(\Phi''(0) + \psi'(0)) - \lambda\langle v, \theta_0^2\psi + 2\theta_0^3\Phi\rangle = \chi_2.$$

If $\chi_2 \neq 0$ it is not possible to simplify, which means that the class of u is $s+2$ and u satisfies $D(\widetilde{\Phi}u) + \tilde{\psi}u = 0$, with $\widetilde{\Phi}(x) = \widetilde{\Phi}_0(x) = x^2\Phi(x)$ and $\tilde{\psi}(x) = \tilde{\psi}_0(x) = x^2\psi(x) + x\Phi(x)$.

(3) If $\chi_1 = 0$ and $\chi_2 = 0$ then let $\widetilde{\Phi}_1(x) = x\Phi(x)$ and $\tilde{\psi}_1(x) = x\psi(x) + 2\Phi(x)$. Then

$$\tilde{\psi}_1(0) + \widetilde{\Phi}_1'(0) = 3\Phi(0).$$

If $\Phi(0) \neq 0$ it is not possible to simplify, which means that the class of u is $s+1$ and u satisfies $D(\widetilde{\Phi}u) + \tilde{\psi}u = 0$, with $\widetilde{\Phi}(x) = \widetilde{\Phi}_1(x) = x\Phi(x)$ and $\tilde{\psi}(x) = \tilde{\psi}_1(x) = x\psi(x) + 2\Phi(x)$. □

2. The form $u = \delta - (u)_1\delta' - \lambda x^{-3}v$, $(u)_2 = 0$

When v is a symmetric form, we have the following result:

Theorem 2.1. *If v is a symmetric form, $(u)_2 = 0$, $(u)_1 \neq 0$ and $(u)_1 + \lambda\Lambda_n \neq 0$, $n \geqslant 0$, then the form u is regular, where*

$$\Lambda_n = \sum_{\nu=0}^{n}\prod_{\mu=0}^{\nu}\frac{\sigma_{2\mu}}{\sigma_{2\mu+1}}, \quad n \geqslant 0, \quad \sigma_0 = 1. \tag{2.1}$$

For the proof we use the following lemmas:

Lemma 2.2. *If* $\{y_n\}_{n\geqslant 0}$, $\{a_n\}_{n\geqslant 0}$ *and* $\{b_n\}_{n\geqslant 0}$ *are sequences of complex numbers fulfilling*

$$\begin{cases} y_{n+1}+a_n y_n=b_{n+1}, & n\geqslant 0,\ a_n\neq 0,\ n\geqslant 0,\\ y_0=b_0, \end{cases}$$

then

$$y_n=(-1)^n a_n^{-1}\left(\prod_{\mu=0}^{n}a_\mu\right)\sum_{\nu=0}^{n}(-1)^\nu a_\nu\left(\prod_{\mu=0}^{\nu}a_\mu^{-1}\right)b_\nu,\quad n\geqslant 0.$$

Lemma 2.3. *When* $\{S_n\}_{n\geqslant 0}$ *given by* (1.4) *is symmetric we have*

$$S_{2n+2}(0)=(-1)^{n+1}\prod_{\mu=0}^{n}\sigma_{2\mu+1},\quad n\geqslant 0,\qquad S_{2n+1}(0)=0,\quad n\geqslant 0,$$

$$S^{(1)}_{2n+2}(0)=(-1)^{n+1}\prod_{\mu=0}^{n}\sigma_{2\mu+2},\quad n\geqslant 0,\qquad S^{(1)}_{2n+1}(0)=0,\quad n\geqslant 0,$$

$$S'_{2n}(0)=0,\quad n\geqslant 0,\qquad \big(S^{(1)}_{2n}\big)'(0)=0,\quad n\geqslant 0,$$

$$\big(S^{(1)}_{2n+1}\big)'(0)=(-1)^n\prod_{\mu=0}^{n}\sigma_{2\mu+1}\Lambda_n,\quad n\geqslant 0\quad \textit{and}\quad \big(S^{(1)}_{2n+1}\big)''(0)=0,\quad n\geqslant 0.$$

Proof of Lemma 2.3. As v is symmetric then $\zeta_n=0$, $n\geqslant 0$ and so, from (1.4), we have

$$S_0(0)=1,\qquad S_1(0)=0,\qquad S^{(1)}_0(0)=1,\qquad S^{(1)}_1(0)=0,$$

$$S_{n+2}(0)=-\sigma_{n+1}S_n(0),\quad n\geqslant 0,\qquad S^{(1)}_{n+2}(0)=-\sigma_{n+2}S^{(1)}_n(0),\quad n\geqslant 0,$$

$$S'_0(0)=0,\qquad S'_1(0)=1,\qquad \big(S^{(1)}_0\big)'(0)=0,\qquad \big(S^{(1)}_1\big)'(0)=1,$$

$$S'_{n+2}(0)=-\sigma_{n+1}S'_n(0)+S_{n+1}(0),\quad n\geqslant 0,\qquad \big(S^{(1)}_{n+2}\big)'(0)=-\sigma_{n+2}\big(S^{(1)}_n\big)'(0)+S^{(1)}_{n+1}(0),\quad n\geqslant 0,$$

and

$$\big(S^{(1)}_0\big)''(0)=0,\qquad \big(S^{(1)}_1\big)''(0)=0,$$

$$\big(S^{(1)}_{n+2}\big)''(0)=-\sigma_{n+2}\big(S^{(1)}_n\big)''(0)+2\big(S^{(1)}_{n+1}\big)'(0),\quad n\geqslant 0.$$

Now, it is enough to use Lemma 2.2 in order to obtain the results. □

Proof of Theorem 2.1. Following Lemma 2.3 we have for (1.25), (1.26), (1.30) and (1.31):

$$\begin{aligned} &\widetilde{X}_n(0)=0,\qquad X^{(1)}_{2n}(0)=S^{(1)}_{2n}(0)\big(S^{(1)}_{2n+1}\big)'(0),\\ &X^{(1)}_{2n+1}(0)=-S^{(1)}_{2n+2}(0)\big(S^{(1)}_{2n+1}\big)'(0),\quad n\geqslant 0,\\ &\widetilde{W}_{2n+1}(0)=S_{2n+2}(0)S^{(1)}_{2n}(0)\big(S^{(1)}_{2n+1}\big)'(0),\quad \widetilde{W}_{2n}(0)=0,\quad n\geqslant 0,\\ &W^{(1)}_{2n}(0)=-\big[\big(S^{(1)}_{2n+1}\big)'(0)\big]^2S^{(1)}_{2n}(0),\quad W^{(1)}_{2n+1}(0)=0,\quad n\geqslant 0. \end{aligned}\tag{2.2}$$

Therefore we have for (1.28):

$$\Delta_{2n}=S^{(1)}_{2n}(0)\langle v,S^2_{2n}\rangle\lambda^2,\quad n\geqslant 0,$$

or

$$\Delta_{2n} = (-1)^n \prod_{\mu=0}^{n} \sigma_{2\mu} \langle v, S_{2n}^2 \rangle \lambda^2, \quad n \geqslant 0, \tag{2.3}$$

and

$$\begin{aligned}\Delta_{2n+1} = &-\big[\big(S_{2n+1}^{(1)}\big)'(0)\big]^2 S_{2n}^{(1)}(0)\lambda^3 \\ &+ \big[S_{2n+2}(0) S_{2n}^{(1)}(0)\big(S_{2n+1}^{(1)}\big)'(0) - \big(S_{2n+1}^{(1)}\big)'(0)\langle v, S_{2n+1}^2\rangle\big](u)_1 \lambda^2 \\ &+ S_{2n+2}(0)\langle v, S_{2n+1}^2\rangle (u)_1^2 \lambda, \quad n \geqslant 0.\end{aligned}$$

As

$$\langle v, S_{2n+1}^2 \rangle = -S_{2n}^{(1)}(0) S_{2n+2}(0), \quad n \geqslant 0, \tag{2.4}$$

we have

$$\Delta_{2n+1} = -S_{2n}^{(1)}(0)\big[\big(S_{2n+1}^{(1)}\big)'(0)\lambda - S_{2n+2}(0)(u)_1\big]^2 \lambda, \quad n \geqslant 0,$$

and using Lemma 2.3, we obtain

$$\Delta_{2n+1} = (-1)^{n+1} \prod_{\mu=0}^{n} \sigma_{2\mu+1} \big\{\lambda \Lambda_n + (u)_1\big\}^2 \langle v, S_{2n+1}^2 \rangle \lambda, \quad n \geqslant 0. \tag{2.5}$$

Under the hypothesis of the theorem it is evident that $\Delta_n \neq 0,\ n \geqslant 0$ and therefore u is regular by virtue of Proposition 1.1. □

Let us define ω by

$$(u)_1 = \omega \lambda. \tag{2.6}$$

Therefore we obtain for (2.5)

$$\Delta_{2n+1} = (-1)^{n+1} \prod_{\mu=0}^{n} \sigma_{2\mu+1} \{\Lambda_n + \omega\}^2 \langle v, S_{2n+1}^2 \rangle \lambda^3, \quad n \geqslant 0. \tag{2.7}$$

Corollary 2.4. *If $(u)_2 = 0$ and v is a symmetric positive definite form then the form u is regular when $\omega \in \mathbb{C} -]-\infty, 0]$.*

Proof. If v is positive definite then $\sigma_{n+1} > 0,\ n \geqslant 0$. Therefore $\Lambda_n > 0,\ n \geqslant 0$ and so $\omega + \Lambda_n \neq 0,\ n \geqslant 0$, under the hypothesis of the corollary. □

Moreover, if v is symmetric positive definite, $(u)_2 = 0$ and $\omega \in \mathbb{C} -]-\infty, 0]$, we have from (1.16)

$$b_0 = \sigma_1 + \frac{\lambda}{(u)_1} = \sigma_1 + \frac{1}{\omega}, \tag{2.8}$$

$$c_0 = -(u)_1 = -\omega\lambda, \tag{2.9}$$

$$c_1 = -\frac{\lambda}{(u)_1^2} = -\frac{1}{\omega^2 \lambda}, \tag{2.10}$$

from (1.37)

$$\begin{aligned} b_{n+1} &= \sigma_{n+2} - \begin{vmatrix} \langle u, xS_{n+1}(x)\rangle & \langle u, xS_{n+2}\rangle & \langle u, S_n\rangle \\ \langle u, x^2S_{n+1}(x)\rangle & \langle u, x^2S_{n+2}(x)\rangle & \langle u, xS_n(x)\rangle \\ 0 & 0 & \langle u, x^2S_n(x)\rangle \end{vmatrix} \Delta_n^{-1} \\ &= \sigma_{n+2} - \langle u, x^2S_n(x)\rangle \begin{vmatrix} \langle u, xS_{n+1}(x)\rangle & \langle u, xS_{n+2}\rangle \\ \langle u, x^2S_{n+1}(x)\rangle & \langle u, x^2S_{n+2}(x)\rangle \end{vmatrix} \Delta_n^{-1} \\ &= \sigma_{n+2} - \langle u, x^2S_n(x)\rangle \left\{ \begin{vmatrix} \langle u, xS_{n+1}(x)\rangle & \langle u, x^2S_{n+1}\rangle \\ \langle u, x^2S_{n+1}(x)\rangle & 0 \end{vmatrix} \right. \\ &\qquad \left. - \sigma_{n+1} \begin{vmatrix} \langle u, xS_{n+1}(x)\rangle & \langle u, xS_n\rangle \\ \langle u, x^2S_{n+1}(x)\rangle & \langle u, x^2S_n(x)\rangle \end{vmatrix} \right\} \Delta_n^{-1}, \quad n \geqslant 0, \end{aligned}$$

by virtue of (1.4). From (1.19) and (1.23), we obtain

$$b_{n+1} = \sigma_{n+2} - S_{n-1}^{(1)}(0)\bigl\{\bigl(S_n^{(1)}\bigr)^2(0) + \sigma_{n+1}\bigl[\langle v, S_n^2\rangle\omega + X_{n-1}^{(1)}(0)\bigr]\bigr\}\Delta_n^{-1}\lambda^3, \quad n \geqslant 0,$$

that is,

$$b_{2n+1} = \sigma_{2n+2}, \quad n \geqslant 0 \tag{2.11}$$

and

$$b_{2n+2} = \sigma_{2n+3} - \sigma_{2n+2}S_{2n}^{(1)}(0)\bigl\{\langle v, S_{2n+1}^2\rangle\omega + X_{2n}^{(1)}(0)\bigr\}\Delta_{2n+1}^{-1}\lambda^3, \quad n \geqslant 0,$$

following Lemma 2.3. Therefore, using (2.2) and (2.4), we have

$$b_{2n+2} = \sigma_{2n+3} - \sigma_{2n+2}\bigl(S_{2n}^{(1)}(0)\bigr)^2\bigl\{-S_{2n+2}(0)\omega + \bigl(S_{2n+1}^{(1)}\bigr)'(0)\bigr\}\Delta_{2n+1}^{-1}\lambda^3, \quad n \geqslant 0.$$

From Lemma 2.3, we obtain

$$b_{2n+2} = \sigma_{2n+3} + (-1)^{n+1}\prod_{\mu=0}^{n}\sigma_{2\mu+2}(\omega + \Lambda_n)\langle v, S_{2n+1}^2\rangle\Delta_{2n+1}^{-1}\lambda^3, \quad n \geqslant 0,$$

whence

$$b_{2n+2} = \sigma_{2n+3} + \frac{1}{\omega + \Lambda_n}\prod_{\mu=0}^{n}\frac{\sigma_{2\mu+2}}{\sigma_{2\mu+1}}, \quad n \geqslant 0, \tag{2.12}$$

by virtue of (2.7). Finally, from (1.38), we have

$$\begin{aligned} c_{n+2} &= - \begin{vmatrix} \langle u, xS_{n+2}(x)\rangle & \langle u, S_{n+1}\rangle & \langle u, S_n\rangle \\ \langle u, x^2S_{n+2}(x)\rangle & \langle u, xS_{n+1}(x)\rangle & \langle u, xS_n(x)\rangle \\ 0 & \langle u, x^2S_{n+1}(x)\rangle & \langle u, x^2S_n(x)\rangle \end{vmatrix} \Delta_n^{-1} \\ &= - \begin{vmatrix} \langle u, x^2S_{n+1}(x)\rangle & \langle u, S_{n+1}\rangle & \langle u, S_n\rangle \\ 0 & \langle u, xS_{n+1}(x)\rangle & \langle u, xS_n(x)\rangle \\ 0 & \langle u, x^2S_{n+1}(x)\rangle & \langle u, x^2S_n(x)\rangle \end{vmatrix} \Delta_n^{-1} \\ &\quad + \sigma_{n+1} \begin{vmatrix} \langle u, xS_n(x)\rangle & \langle u, S_{n+1}\rangle & \langle u, S_n\rangle \\ \langle u, x^2S_n(x)\rangle & \langle u, xS_{n+1}(x)\rangle & \langle u, xS_n(x)\rangle \\ 0 & \langle u, x^2S_{n+1}(x)\rangle & \langle u, x^2S_n(x)\rangle \end{vmatrix} \Delta_n^{-1}, \quad n \geqslant 0, \end{aligned}$$

by virtue of (1.4). Or,

$$c_2 = (u)_1 + \sigma_1 \frac{(u)_1^2}{\lambda} = (1 + \omega\sigma_1)\omega\lambda, \tag{2.13}$$

and

$$c_{n+3} = -\langle u, x^2 S_{n+2}(x)\rangle \begin{vmatrix} \langle u, xS_{n+2}(x)\rangle & \langle u, xS_{n+1}(x)\rangle \\ \langle u, x^2 S_{n+2}(x)\rangle & \langle u, x^2 S_{n+1}(x)\rangle \end{vmatrix} \Delta_{n+1}^{-1} + \sigma_{n+2}\sigma_{n+1}\Delta_n\Delta_{n+1}^{-1}, \quad n \geqslant 0.$$

Following (1.19) and (1.23), this becomes

$$c_{n+3} = S_{n+1}^{(1)}(0)\big[X_n^{(1)}(0) + \langle v, S_{n+1}^2\rangle\omega\big]\lambda^3\Delta_{n+1}^{-1} + \sigma_{n+2}\sigma_{n+1}\Delta_n\Delta_{n+1}^{-1}, \quad n \geqslant 0.$$

Using Lemma 2.3 we obtain

$$c_{2n+3} = \sigma_{2n+2}\sigma_{2n+1}\frac{\Delta_{2n}}{\Delta_{2n+1}}, \quad n \geqslant 0$$

and

$$c_{2n+4} = S_{2n+2}^{(1)}(0)\big[X_{2n+1}^{(1)}(0) + \langle v, S_{2n+2}^2\rangle\omega\big]\lambda^3\Delta_{2n+2}^{-1} + \sigma_{2n+3}\sigma_{2n+2}\Delta_{2n+1}\Delta_{2n+2}^{-1}, \quad n \geqslant 0.$$

As $\langle v, S_{2n+2}^2\rangle = S_{2n+2}^{(1)}(0)S_{2n+2}(0)$ and by virtue of (2.2) we have

$$c_{2n+4} = (-1)^{n+1}\prod_{\mu=0}^{n}\sigma_{2\mu+2}[\omega + \Lambda_n]\langle v, S_{2n+2}^2\rangle\lambda^3\Delta_{2n+2}^{-1} + \sigma_{2n+3}\sigma_{2n+2}\frac{\Delta_{2n+1}}{\Delta_{2n+2}}, \quad n \geqslant 0,$$

or

$$c_{2n+4} = (\omega + \Lambda_n)\lambda + \sigma_{2n+3}\sigma_{2n+2}\frac{\Delta_{2n+1}}{\Delta_{2n+2}}, \quad n \geqslant 0,$$

by virtue of (2.3). Taking (1.11) into account, we may also write

$$c_{2n+3} = -\frac{\sigma_{2n+2}\sigma_{2n+1}}{a_{2n}}, \quad n \geqslant 0 \tag{2.14}$$

and

$$c_{2n+4} = (\omega + \Lambda_n)\lambda - \frac{\sigma_{2n+3}\sigma_{2n+2}}{a_{2n+1}}, \quad n \geqslant 0. \tag{2.15}$$

Examples. (1) Let us describe the case $v := \mathcal{H}$, where $\mathcal{H}$ is the Hermite form. Here [3]

$$\zeta_n = 0, \quad n \geqslant 0, \qquad \sigma_{n+1} = \frac{1}{2}(n+1) > 0, \quad n \geqslant 0. \tag{2.16}$$

We want

$$\Lambda_n = \sum_{\nu=0}^{n}\prod_{\mu=0}^{\nu}\frac{\sigma_{2\mu}}{\sigma_{2\mu+1}} = \frac{1}{\sigma_1}\left\{1 + \sum_{\nu=0}^{n-1}\prod_{\mu=0}^{\nu}\frac{\sigma_{2\mu+2}}{\sigma_{2\mu+3}}\right\}, \quad n \geqslant 0.$$

But from (2.16)

$$\frac{\sigma_{2\mu+2}}{\sigma_{2\mu+3}} = \frac{\mu+1}{\mu+3/2}.$$

Therefore,

$$\prod_{\mu=0}^{\nu}\frac{\sigma_{2\mu+2}}{\sigma_{2\mu+3}}=\frac{\Gamma(\nu+2)\Gamma(3/2)}{\Gamma(3/2+\nu+1)}=\sqrt{\pi}\frac{1/2}{\nu+3/2}h_{\nu+1},$$

with

$$h_n=\frac{\Gamma(n+1)}{\Gamma(n+1/2)},\quad n\geqslant 0,$$

fulfilling

$$h_{n+1}=\frac{n+1}{n+1/2}h_n,\quad n\geqslant 0.$$

Therefore,

$$h_{n+1}-h_n=\frac{1/2}{n+1/2}h_n,\quad n\geqslant 0,$$

and so

$$\sum_{\nu=0}^{n-1}\prod_{\mu=0}^{\nu}\frac{\sigma_{2\mu+2}}{\sigma_{2\mu+3}}=\sum_{\nu=0}^{n-1}\sqrt{\pi}(h_{\nu+2}-h_{\nu+1})=\sqrt{\pi}\frac{\Gamma(n+2)}{\Gamma(n+3/2)}-2,\quad n\geqslant 0.$$

Finally,

$$\Lambda_n=2\{\sqrt{\pi}h_{n+1}-1\},\quad n\geqslant 0.$$

Therefore, (2.3) and (2.7) becomes

$$\Delta_{2n}=(-1)^n\Gamma(n+1)\langle v,S_{2n}^2\rangle\lambda^2,\quad n\geqslant 0,$$

and

$$\Delta_{2n+1}=\frac{(-1)^{n+1}}{\sqrt{\pi}}\Gamma(n+3/2)(\Lambda_n+\omega)^2\langle v,S_{2n+1}^2\rangle\lambda^3,\quad n\geqslant 0.$$

And so (1.36), (2.8), (2.11), (2.12), (2.9), (2.10), (2.13), (2.14) and (2.15) become

$$a_{2n}=\frac{(n+1/2)^2(\Lambda_n+\omega)^2}{\sqrt{\pi}h_n}\lambda,\quad n\geqslant 0,\tag{2.17}$$

$$a_{2n+1}=-\frac{\sqrt{\pi}(n+1)h_{n+1}}{(\Lambda_n+\omega)^2}\frac{1}{\lambda},\quad n\geqslant 0,\tag{2.18}$$

$$b_0=\frac{1}{2}+\frac{1}{\omega},\tag{2.19}$$

$$b_{2n+1}=n+1,\quad n\geqslant 0,\tag{2.20}$$

$$b_{2n+2}=(n+3/2)+\frac{\sqrt{\pi}h_{n+1}}{\Lambda_n+\omega},\quad n\geqslant 0,\tag{2.21}$$

$$c_0=-\omega\lambda,\qquad c_1=-\frac{1}{\omega^2\lambda},\qquad c_2=\left(1+\frac{1}{2}\omega\right)\omega\lambda,\tag{2.22}$$

$$c_{2n+3} = -\frac{\sqrt{\pi}}{\lambda}\frac{h_{n+1}}{(\Lambda_n+\omega)^2}, \quad n \geqslant 0, \tag{2.23}$$

$$c_{2n+4} = \frac{\sqrt{\pi}h_{n+1} + (n+3/2)(\Lambda_n+\omega)}{\sqrt{\pi}h_{n+1}}(\Lambda_n+\omega)\lambda, \quad n \geqslant 0. \tag{2.24}$$

Therefore, we have for (1.14) and (1.15)

$$\begin{cases} \beta_0 = \omega\lambda, \qquad \beta_1 = \dfrac{1-\omega^3\lambda^2}{\omega^2\lambda}, \\ \beta_2 = -\dfrac{1+(1+\omega/2)\omega^3\lambda^2}{\omega^2\lambda}, \qquad \beta_3 = \dfrac{1+2(1+\omega/2)^3\omega\lambda^2}{2(1+\omega/2)^2\lambda}, \\ \gamma_1 = -\omega^2\lambda^2, \qquad \gamma_2 = -\dfrac{1}{\omega^4\lambda^2}, \qquad \gamma_3 = -(1+\omega/2)^2\omega^2\lambda^2, \\ \beta_{2n+5} = \dfrac{[\sqrt{\pi}h_{n+1} + (n+3/2)(\Lambda_n+\omega)](\Lambda_n+\omega)(\Lambda_{n+1}+\omega)^2\lambda^2 + \pi h_{n+1}h_{n+2}}{\sqrt{\pi}h_{n+1}(\Lambda_{n+1}+\omega)^2\lambda}, \quad n \geqslant 0, \\ \beta_{2n+4} = -\dfrac{[\sqrt{\pi}h_{n+1} + (n+3/2)(\Lambda_n+\omega)](\Lambda_n+\omega)^3\lambda^2 + \pi h_{n+1}^2}{\sqrt{\pi}h_{n+1}(\Lambda_n+\omega)^2\lambda}, \quad n \geqslant 0, \\ \gamma_{2n+5} = -\dfrac{(n+3/2)^2(\Lambda_{n+1}+\omega)^2(\Lambda_n+\omega)^2}{\pi h_{n+1}^2}\lambda^2, \quad n \geqslant 0, \\ \gamma_{2n+4} = -\dfrac{\pi h_{n+1}^2}{(\Lambda_n+\omega)^4\lambda^2}, \quad n \geqslant 0. \end{cases} \tag{2.25}$$

Proposition 2.5. *The form* u *given by* (1.1) *possesses the following integral representation*

$$\langle u, f\rangle = f(0) + f'(0)(\omega-2)\lambda - \frac{\lambda}{\sqrt{\pi}} Pf\int_{-\infty}^{+\infty} \frac{e^{-x^2}}{x^3} f(x)\,dx, \quad \forall f \in \mathcal{P}. \tag{2.26}$$

It is a quasi-antisymmetric and semi-classical form of class s *satisfying the following functional equation*:

$$\begin{array}{lll} \omega \neq 2 & \begin{cases} D(x^3u) + 2x^4u = 0 \\ (u)_1 = \omega\lambda, \quad (u)_2 = 0, \quad (u)_3 = -\lambda \end{cases} & s = 3 \\ \omega = 2 & \begin{cases} D(x^2u) + (2x^3+x)u = 0 \\ (u)_1 = 2\lambda, \quad (u)_2 = 0, \quad (u)_3 = -\lambda \end{cases} & s = 2 \end{array} \tag{2.27}$$

Proof. It is well known that the Hermite form possesses the following integral representation

$$\langle v, f\rangle = \int_{-\infty}^{+\infty} V(x)f(x)\,dx, \quad \forall f \in \mathcal{P},$$

with $V(x) = \frac{1}{\sqrt{\pi}} e^{-x^2}$. Following (1.3), we easily obtain (2.26). Also the form u is quasi-antisymmetric because it satisfies

$$\langle u, x^{2n+4}\rangle = -\lambda\langle v, x^{2n+1}\rangle = 0, \quad n \geqslant 0,$$

since v is symmetric and $(u)_2 = 0$, by hypotheses. The Hermite form is classical and satisfies [11]

$$D(\Phi v) + \psi v = 0,$$

with $\Phi(x) = 1$ and $\psi(x) = 2x$. Therefore, (1.48) and (1.49) become

$$\chi_1 = (u)_1 - \lambda\langle v, 2\rangle = (\omega - 2)\lambda,$$

using (2.6) and

$$\chi_2 = 2.$$

Now, it is enough to use Proposition 1.4 in order to obtain (2.27).

(2) Let us describe the case $v := \mathcal{J}_{(0,0)}$, where $\mathcal{J}_{(0,0)}$ is the Legendre form, a particular case of the Jacobi form $\mathcal{J}_{(\alpha,\beta)}$ for $\alpha = \beta = 0$. Here [3]

$$\zeta_n = 0, \quad n \geqslant 0, \qquad \sigma_{n+1} = \frac{(n+1)^2}{(2n+1)(2n+3)} < 0, \quad n > 0. \tag{2.28}$$

We have [13]

$$\Lambda_n = \sum_{\nu=0}^{n} \prod_{\mu=0}^{\nu} \frac{\sigma_{2\mu}}{\sigma_{2\mu+1}} = \pi t_{n+1} - 1, \quad n \geqslant 0,$$

with

$$t_n = \left(\frac{\Gamma(n+1)}{\Gamma(n+1/2)}\right)^2, \quad n \geqslant 0.$$

Consequently, (2.3) and (2.7) become

$$\Delta_{2n} = \frac{(-1)^n}{4^n}\bigl(\Gamma(n+1)\bigr)^2 \frac{\Gamma(3/4)}{\Gamma(n+3/4)} \frac{\Gamma(5/4)}{\Gamma(n+5/4)}\langle v, S_{2n}^2\rangle\lambda^2, \quad n \geqslant 0 \tag{2.29}$$

and

$$\Delta_{2n+1} = \frac{(-1)^{n+1}}{4^{n+1}\pi}\bigl(\Gamma(n+3/2)\bigr)^2 \frac{\Gamma(1/4)}{\Gamma(n+5/4)} \frac{\Gamma(3/4)}{\Gamma(n+7/4)}\{\Lambda_n + \omega\}^2\langle v, S_{2n+1}^2\rangle\lambda^3, \quad n \geqslant 0. \tag{2.30}$$

And so (1.36), (2.8), (2.11), (2.12), (2.9), (2.10), (2.13), (2.14) and (2.15) become

$$a_{2n} = \frac{(2n+1)^4(\Lambda_n + \omega)^2}{\pi(4n+1)(4n+3)^2 t_n}\lambda, \quad n \geqslant 0, \tag{2.31}$$

$$a_{2n+1} = -\frac{4\pi(n+1)^2 t_{n+1}}{(4n+3)(4n+5)^2(\Lambda_n + \omega)^2}\frac{1}{\lambda}, \quad n \geqslant 0, \tag{2.32}$$

$$b_0 = \frac{1}{3} + \frac{1}{\omega}, \tag{2.33}$$

$$b_{2n+1}=\frac{4(n+1)^2}{(4n+3)(4n+5)},\quad n\geqslant 0, \tag{2.34}$$

$$b_{2n+2}=\frac{(2n+3)^2(\Lambda_n+\omega)+\pi(4n+7)t_{n+1}}{(4n+5)(4n+7)(\Lambda_n+\omega)},\quad n\geqslant 0, \tag{2.35}$$

$$c_0=-\omega\lambda,\qquad c_1=-\frac{1}{\omega^2\lambda},\qquad c_2=(1+\frac{1}{3}\omega)\omega\lambda,$$

$$c_{2n+3}=-\frac{\pi t_{n+1}}{(4n+5)(\Lambda_n+\omega)^2\lambda},\quad n\geqslant 0, \tag{2.36}$$

$$c_{2n+4}=(\omega+\Lambda_n)\lambda+\frac{(2n+3)^2(\Lambda_n+\omega)^2\lambda}{\pi(4n+7)t_{n+1}},\quad n\geqslant 0. \tag{2.37}$$

Therefore, we have for (1.14) and (1.15)

$$\begin{cases}\beta_0=\omega\lambda,\qquad \beta_1=\dfrac{1-\omega^3\lambda^2}{\omega^2\lambda},\\ \beta_2=-\dfrac{1+(1+\omega/3)\omega^3\lambda^2}{\omega^2\lambda},\qquad \beta_3=\dfrac{4+45(1+\omega/3)^3\omega\lambda^2}{45(1+\omega/3)^2\lambda},\\ \gamma_1=-\omega^2\lambda^2,\qquad \gamma_2=-\dfrac{1}{\omega^4\lambda^2},\qquad \gamma_3=-(1+\omega/3)^2\omega^2\lambda^3,\\ \beta_{2n+5}=(\Lambda_n+\omega)\lambda+\dfrac{\pi^2(4n+7)t_{n+1}^2+(2n+3)^2(4n+5)(\Lambda_n+\omega)^4\lambda^2}{\pi(4n+5)(4n+7)(\Lambda_n+\omega)^2t_{n+1}\lambda},\quad n\geqslant 0,\\ \beta_{2n+4}=-(\Lambda_n+\omega)\lambda-\dfrac{\pi(4n+7)t_{n+1}^2+(2n+3)^2(4n+5)(\Lambda_n+\omega)^4\lambda^2}{\pi(4n+5)(4n+7)(\Lambda_n+\omega)^2t_{n+1}\lambda},\quad n\geqslant 0,\\ \gamma_{2n+5}=-\dfrac{(2n+3)^4(\Lambda_{n+1}+\omega)^2(\Lambda_n+\omega)^2\lambda^2}{\pi^2(4n+7)^2t_{n+1}^2},\quad n\geqslant 0,\\ \gamma_{2n+4}=-\dfrac{\pi^2t_{n+1}^2}{(4n+5)^2(\Lambda_n+\omega)^4\lambda^2},\quad n\geqslant 0.\end{cases} \tag{2.38}$$

Proposition 2.6. *The form u given by* (1.1) *possesses the following integral representation*

$$\langle u,f\rangle=f(0)+f'(0)(\omega-1)\lambda-\frac{\lambda}{2}Pf\int_{-1}^{+1}\frac{f(x)}{x^3}\,\mathrm{d}x,\quad \forall f\in\mathcal{P}. \tag{2.39}$$

It is a quasi-antisymmetric and semi-classical form of class s satisfying the following functional equation:

$$\begin{array}{lll} \omega\neq 1 & \begin{cases} D(x^3(x^2-1)u)-2x^4u=0\\ (u)_1=\omega\lambda,\quad (u)_2=0,\quad (u)_3=-\lambda\end{cases} & s=3\\ \omega=1 & \begin{cases} D(x^2(x^2-1)u)-(x^3+x)u=0\\ (u)_1=\lambda,\quad (u)_2=0,\quad (u)_3=-\lambda\end{cases} & s=2\end{array} \tag{2.40}$$

Proof. It is well known that the Legendre form possesses the following integral representation

$$\langle v, f\rangle = \int_{-\infty}^{+\infty} V(x)f(x)\,\mathrm{d}x, \quad \forall f \in \mathcal{P},$$

with $V(x) = \frac{Y(1-x^2)}{2}$, where $Y(x) = \begin{cases} 0, & x \leqslant 0 \\ 1, & x > 0 \end{cases}$. Following (1.3), we easily obtain (2.39). Also the form u is quasi-antisymmetric because it satisfies

$$\langle u, x^{2n+4}\rangle = -\lambda\langle v, x^{2n+1}\rangle = 0, \quad n \geqslant 0,$$

since v is symmetric and $(u)_2 = 0$ by hypotheses. The Legendre form is classical and satisfies [11]

$$D(\Phi v) + \psi v = 0,$$

with $\Phi(x) = x^2 - 1$ and $\psi(x) = -2x$. Therefore, (1.48) and (1.49) become

$$\chi_1 = -(u)_1 + \lambda = (1-\omega)\lambda,$$

using (2.6), and

$$\chi_2 = -2.$$

Now it is enough to use Proposition 1.4 in order to obtain (2.40). □

In the next section, we treat the case in which $(u)_2 \neq 0$.

3. The form $u = \delta - (u)_1\delta' + \frac{1}{2}(u)_2\delta'' - \lambda x^{-3}v$, $(u)_2 \neq 0$

We can treat (1.1) in two steps. Let w denote the form which fulfills

$$x^2u = -\lambda_1 w, \quad \lambda_1 \in \mathbb{C} - \{0\}, \tag{3.1}$$

with $(w)_0 = 1$, what implies $(u)_2 = -\lambda_1 \neq 0$. Then (1.1) becomes $-\lambda_1 xw = -\lambda v$, or

$$xw = \theta v, \tag{3.2}$$

with

$$\theta = \frac{\lambda}{\lambda_1} = \frac{(u)_3}{(u)_2}. \tag{3.3}$$

We have for (3.1) [2]:

$$u = \delta - (u)_1\delta' - \lambda_1 x^{-2} w, \tag{3.4}$$

and for (3.2) [9]:

$$w = \delta + \theta x^{-1} v. \tag{3.5}$$

When w is regular, let $\{P_n\}_{n\geqslant 0}$ be the corresponding orthogonal sequence,

$$\begin{aligned} &P_0(x) = 1, \qquad P_1(x) = x - \xi_0, \\ &P_{n+2}(x) = (x - \xi_{n+1})P_{n+1}(x) - \rho_{n+1}P_n(x), \quad n \geqslant 0. \end{aligned} \tag{3.6}$$

Both the two above problems had been treated in [13,2,1] and [9]. For the sake of completeness, we give the following results which will be useful in the sequel:

Theorem 3.1 [9, pp. 228–229]. *Let w be a form such that $(w)_0 = 1$. Then the following claims are equivalent*:

(i) *The form w is quasi-antisymmetric and the form xw is regular*;
(ii) *The form xw is symmetric and regular*;
(iii) *There exists a form v regular and symmetric such that $w = \delta + (w)_1 x^{-1} v$, $(v)_0 = 1$*;
(iv) *There exists one sequence $\{d_n\}_{n\geqslant 0}$ fulfilling $d_n \neq 0$, $n \geqslant 0$ such that the form w is quasi-antisymmetric and regular and then*

$$\sigma_{n+1} = -d_n d_{n+1}, \quad n \geqslant 0, \tag{3.7}$$

or equivalently

$$\xi_0 = -d_0, \qquad \xi_{n+1} = d_n - d_{n+1}, \qquad \rho_{n+1} = -d_n^2, \quad n \geqslant 0. \tag{3.8}$$

Moreover, the sequence $\{P_n\}_{n\geqslant 0}$ orthogonal with respect to w is such that

$$P_n(0) = \prod_{\mu=0}^{n-1} d_\mu, \quad n \geqslant 1. \tag{3.9}$$

Theorem 3.2 [13, pp. 245–246]. *Let w be a regular form with $(w)_0 = 1$. The form u, such that $x^2 u = -\lambda_1 w$, $\lambda_1 \neq 0$, is regular if and only if*

$$\Theta_n := \langle u, P_{n+1}\rangle\big\langle u, xP_n(x)\big\rangle - \langle u, P_n\rangle\langle u, xP_{n+1}(x)\rangle \neq 0, \quad n \geqslant 0. \tag{3.10}$$

Further, we have

$$\Theta_{n+1} - \rho_{n+1}\Theta_n = \left\{(u)_1 P_{n+1}(0) + (u)_2 P_n^{(1)}(0)\right\}^2, \quad n \geqslant 0 \tag{3.11}$$

and consequently,

$$\frac{\Theta_n}{\langle w, P_n^2\rangle} = -(u)_2 + \sum_{\nu=0}^{n} \frac{((u)_1 P_\nu(0) + (u)_2 P_{\nu-1}^{(1)}(0))^2}{\langle w, P_\nu^2\rangle}, \quad n \geqslant 0. \tag{3.12}$$

Moreover, if u is regular, the coefficients of the second-order recurrence relation are given by:

$$\beta_0 = (u)_1, \qquad \beta_{n+1} = \xi_{n+1} + \tilde{b}_n - \tilde{b}_{n+1}, \quad n \geqslant 0, \tag{3.13}$$

where

$$\begin{aligned} &\tilde{b}_0 = \xi_0 - (u)_1, \\ &\tilde{b}_{n+1} = \xi_{n+1} - \big[(u)_1 P_{n+1}(0) + (u)_2 P_n^{(1)}(0)\big]\big[(u)_1 P_n(0) + (u)_2 P_{n-1}^{(1)}(0)\big]\Theta_n^{-1}, \quad n \geqslant 0 \end{aligned} \tag{3.14}$$

and

$$\gamma_1 = -\Theta_0, \qquad \gamma_2 = -(u)_2 \frac{\Theta_1}{\Theta_0^2}, \qquad \gamma_{n+3} = \frac{\Theta_{n+2}\Theta_n}{\Theta_{n+1}^2}\rho_{n+1}, \quad n \geqslant 0. \tag{3.15}$$

From Theorem 3.1, we know that the form w is regular if and only if v is regular and symmetric. Then w is quasi-antisymmetric. In these conditions, what can we say about u? First a definition:

Definition. We will say that a linear form u is an r-quasi-antisymmetric form, when there exists an integer $r \geqslant 0$ such that,

$$(u)_{2n} \neq 0, \quad 0 \leqslant n \leqslant r \quad \text{and} \quad (u)_{2n} = 0, \quad n \geqslant r+1.$$

Remark. When $r = 0$, we say that u is a quasi-antisymmetric form [9].

From (3.4) we have

$$(u)_n = \delta_{0,n} + (u)_1 n\delta_{0,n-1} - \lambda_1 (w)_{n-2}, \quad n \geqslant 0.$$

Therefore

$$(u)_0 = 1, \qquad (u)_2 = -\lambda_1 \neq 0, \qquad (u)_{2n+4} = 0, \quad n \geqslant 0$$

and so u is a 1-quasi-antisymmetric form. Is it regular? From Theorem 3.2, the form u will be regular whenever $\Theta_n \neq 0,\ n \geqslant 0$. In the following we will search for sufficient conditions in order to prove that u is regular supposing that w, given by (3.1) is a quasi-antisymmetric and regular form and so obtained from a symmetric form v.

First a lemma:

Lemma 3.3. *If $\{P_n\}_{n\geqslant 0}$ is the MOPS with respect to $w = \delta - \theta x^{-1} v,\ \theta \neq 0$ and v is symmetric then*

$$P_n^{(1)}(0) = \sum_{\nu=0}^{n} \frac{(-1)^\nu}{d_\nu} \prod_{\mu=0}^{n} d_\mu, \quad n \geqslant 0. \tag{3.16}$$

Proof. Following (3.6) and (3.8) we know that the associated sequence of $\{P_n\}_{n\geqslant 0}$ fulfills the following second-order recurrence

$$P_0^{(1)}(0) = 1, \qquad P_1^{(1)}(0) = d_1 - d_0,$$
$$P_{n+2}^{(1)}(0) = (d_{n+2} - d_{n+1}) P_{n+1}^{(1)}(0) + d_{n+1}^2 P_n^{(1)}(0), \quad n \geqslant 0.$$

For $n = 1$, (3.16) becomes

$$P_1^{(1)}(0) = \sum_{\nu=0}^{1} \frac{(-1)^\nu}{d_\nu} \prod_{\mu=0}^{1} d_\mu = \left(\frac{1}{d_0} - \frac{1}{d_1}\right) d_0 d_1 = d_1 - d_0.$$

Let us suppose that (3.16) is true for $n+1$. Let us see that for $n+2$ is still true. Replacing (3.16) on the recurrence above we obtain

$$\begin{aligned} P_{n+2}^{(1)}(0) &= (d_{n+2} - d_{n+1}) \sum_{\nu=0}^{n+1} \frac{(-1)^\nu}{d_\nu} \prod_{\mu=0}^{n+1} d_\mu + d_{n+1}^2 \sum_{\nu=0}^{n} \frac{(-1)^\nu}{d_\nu} \prod_{\mu=0}^{n} d_\mu \\ &= \left\{ (d_{n+2} - d_{n+1}) \sum_{\nu=0}^{n+1} \frac{(-1)^\nu}{d_\nu} + d_{n+1} \sum_{\nu=0}^{n} \frac{(-1)^\nu}{d_\nu} \right\} \prod_{\mu=0}^{n+1} d_\mu \\ &= \left(d_{n+2} \sum_{\nu=0}^{n+1} \frac{(-1)^\nu}{d_\nu} - (-1)^{n+1} \right) \prod_{\mu=0}^{n+1} d_\mu = \sum_{\nu=0}^{n+2} \frac{(-1)^\nu}{d_\nu} \prod_{\mu=0}^{n+2} d_\mu, \quad n \geqslant 0. \qquad \square \end{aligned}$$

If we put

$$d_n = (-1)^n \alpha_n, \quad n \geqslant 0, \tag{3.17}$$

we obtain for (3.7)

$$\sigma_{n+1} = \alpha_n \alpha_{n+1}, \quad n \geqslant 0, \tag{3.18}$$

equivalently,

$$\begin{cases} \alpha_{2n+2} = \alpha_0 \displaystyle\prod_{\nu=0}^{n} \frac{\sigma_{2\nu+2}}{\sigma_{2\nu+1}}, \\ \alpha_{2n+1} = \frac{1}{\alpha_0} \displaystyle\prod_{\nu=0}^{n} \frac{\sigma_{2\nu+1}}{\sigma_{2\nu}}, \quad \sigma_0 = 1, \end{cases} \quad n \geqslant 0. \tag{3.19}$$

Therefore, $\alpha_n > 0$, $n \geqslant 0$, if and only if $\alpha_0 > 0$ arbitrary and the form v is positive definite. On these conditions and from (3.12) we have

$$\Theta_0 = (u)_1^2 - (u)_2,$$
$$\frac{\Theta_{n+1}}{\langle w, P_{n+1}^2 \rangle} = \Theta_0 + \sum_{\nu=0}^{n} \frac{\Psi_\nu^2}{\langle w, P_{\nu+1}^2 \rangle}, \quad n \geqslant 0,$$

with

$$\Psi_n = (u)_1 P_{n+1}(0) + (u)_2 P_n^{(1)}(0), \quad n \geqslant 0. \tag{3.20}$$

Using (3.9) and (3.16) we obtain

$$\Psi_n = \prod_{\mu=0}^{n} d_\mu \left\{ (u)_1 + (u)_2 \sum_{\nu=0}^{n} \frac{(-1)^\nu}{d_\nu} \right\}, \quad n \geqslant 0,$$

that is

$$\Psi_n = \prod_{\mu=0}^{n} (-1)^\mu \alpha_\mu \left\{ (u)_1 + (u)_2 \sum_{\nu=0}^{n} \frac{1}{\alpha_\nu} \right\}, \quad n \geqslant 0, \tag{3.21}$$

taking (3.17) into account. And so, we have

$$\frac{\Theta_{n+1}}{\langle w, P_{n+1}^2 \rangle} = \Theta_0 - \sum_{\nu=0}^{n} (-1)^\nu \Omega_\nu, \quad n \geqslant 0, \tag{3.22}$$

with

$$\Omega_n = \left((u)_1 + (u)_2 \sum_{\mu=0}^{n} \frac{1}{\alpha_\mu} \right)^2, \quad n \geqslant 0. \tag{3.23}$$

Lemma 3.4. *If $\{\eta_n\}_{n \geqslant 0}$ is a increasing positive sequence and $\Sigma_n = \sum_{\nu=0}^{n} (-1)^\nu \eta_\nu$ then the subsequence $\{\Sigma_{2n}\}_{n \geqslant 0}$ is positive and strictly increasing and $\{\Sigma_{2n+1}\}_{n \geqslant 0}$ is negative and strictly decreasing.*

Lemma 3.5. *If $\alpha_0 > 0$ and v is a positive definite form, then if*

(1) $(u)_1 + \frac{(u)_2}{\alpha_0} > 0$ *and* $(u)_2 > 0$, *or*
(2) $(u)_1 - \frac{|(u)_2|}{\alpha_0} < 0$ *and* $(u)_2 < 0$

we have that $\{\Omega_n\}_{n\geqslant 0}$, *given by* (3.23), *is a positive increasing sequence.*

Proof. In first place, let us defined $\{\tau_n\}_{n\geqslant 0}$ by $\tau_n = \sum_{\mu=0}^{n} \frac{1}{\alpha_\mu}$, $n \geqslant 0$, which is a positive and increasing sequence. Therefore,

$$\Omega_n = \big((u)_1 + (u)_2\tau_n\big)^2, \quad n \geqslant 0$$

and so

$$\begin{aligned}\Omega_{n+1} - \Omega_n &= 2(u)_1(u)_2(\tau_{n+1} - \tau_n) + (u)_2^2(\tau_{n+1} - \tau_n)^2 \\ &= (u)_2(\tau_{n+1} - \tau_n)\big[2(u)_1 + (u)_2(\tau_{n+1} + \tau_n)\big], \quad n \geqslant 0.\end{aligned}$$

But,
(1) if $(u)_2 > 0$ then

$$2(u)_1 + (u)_2(\tau_{n+1} + \tau_n) \geqslant 2\big((u)_1 + (u)_2\tau_0\big) = 2\left((u)_1 + \frac{(u)_2}{\alpha_0}\right), \quad n \geqslant 0,$$

and
(2) if $(u)_2 < 0$ then

$$\begin{aligned}&2(u)_1 + (u)_2(\tau_{n+1} + \tau_n) \\ &\quad = 2(u)_1 - \big|(u)_2\big|(\tau_{n+1} + \tau_n) \leqslant 2\big((u)_1 - \big|(u)_2\big|\tau_0\big) = 2\left((u)_1 - \frac{|(u)_2|}{\alpha_0}\right), \quad n \geqslant 0.\end{aligned}$$

In any case, we proved that $\Omega_{n+1} - \Omega_n > 0$, $n \geqslant 0$, under the hypotheses of the lemma. □

We are now able to write the following theorem:

Theorem 3.6. *If $\alpha_0 > 0$, v is a positive definite form, $\Theta_0 \neq 0$ and if*

(1a) $(u)_1 > \frac{\alpha_1}{2}\frac{1}{\sqrt{1+2\frac{\alpha_1}{\alpha_0}}+1}$ *and* $(u)_2 > 0$; *or*
(1b) $(u)_1 < -\frac{\alpha_1}{2}\frac{1}{\sqrt{1+2\frac{\alpha_1}{\alpha_0}}-1}$ *and* $\frac{(u)_2}{\alpha_0^2} > \max\big\{2\frac{|(u)_1|}{\alpha_0} - 1, \frac{|(u)_1|}{\alpha_0}\big\}$; *or*
(2) $(u)_2 < 0$ *and* $(u)_1 < \min\big\{\frac{1}{2}(\frac{|(u)_2|}{\alpha_0} - \alpha_0), \frac{|(u)_2|}{\alpha_0}\big\}$,

then u is a regular form.

Proof. (1) Let us note that in both the two cases (a) and (b) the conditions $(u)_2 > 0$ and $(u)_1 + \frac{(u)_2}{\alpha_0} > 0$ are satisfied and so we may use Lemma 3.5 to conclude that $\{\Omega_n\}_{n\geqslant 0}$ is a positive and increasing sequence. Therefore, using Lemma 3.4 for $\eta_n = \Omega_n$, we have for (3.22)

$$\frac{\Theta_{2n+1}}{\langle w, P_{2n+1}^2\rangle} = (u)_1^2 - (u)_2 - \Sigma_{2n} < (u)_1^2 - \big((u)_2 + \Sigma_0\big), \quad n \geqslant 0$$

and

$$\frac{\Theta_{2n+2}}{\langle w, P_{2n+2}^2\rangle} = (u)_1^2 - (u)_2 - \Sigma_{2n+1} = (u)_1^2 + |\Sigma_{2n+1}| - (u)_2 > (u)_1^2 + |\Sigma_1| - (u)_2, \quad n \geqslant 0.$$

But

$$\Sigma_0 = \Omega_0 = \left((u)_1 + \frac{(u)_2}{\alpha_0}\right)^2$$

and

$$\Sigma_1 = \Omega_0 - \Omega_1 = \left((u)_1 + \frac{(u)_2}{\alpha_0}\right)^2 - \left((u)_1 + \frac{(u)_2}{\alpha_0} + \frac{(u)_2}{\alpha_1}\right)^2$$
$$= -\frac{(u)_2}{\alpha_1}\left\{2(u)_1 + \left(\frac{2}{\alpha_0} + \frac{1}{\alpha_1}\right)(u)_2\right\}$$

therefore

$$(u)_2 + \Sigma_0 > (u)_1^2$$

is equivalent to

$$1 + 2\frac{(u)_1}{\alpha_0} + \frac{(u)_2}{\alpha_0^2} > 0,$$

which is fulfilled for (a) and (b) and so

$$\frac{\Theta_{2n+1}}{\langle w, P_{2n+1}^2\rangle} < 0, \quad n \geqslant 0$$

and $(u)_1^2 + |\Sigma_1| > (u)_2$ is equivalent to

$$\left(\frac{1}{\alpha_1} + \frac{2}{\alpha_0}\right)(u)_2^2 + \big(2(u)_1 - \alpha_1\big)(u)_2 + \alpha_1 (u)_1^2 > 0.$$

When the discriminant is negative this condition is realized, that is

$$\big(2(u)_1 - \alpha_1\big)^2 - 4\left(\frac{1}{\alpha_1} + \frac{2}{\alpha_0}\right)\alpha_1 (u)_1^2 < 0,$$

or

$$(u)_1^2 + \frac{1}{2}\alpha_0 (u)_1 - \frac{1}{8}\alpha_0\alpha_1 > 0,$$

which is fulfilled for $(u)_1 < x_1$ and $(u)_1 > x_2$, where

$$x_1 = -\frac{\alpha_1}{2}\frac{1}{\sqrt{1 + 2\frac{\alpha_1}{\alpha_0}} - 1} \quad \text{and} \quad x_2 = \frac{\alpha_1}{2}\frac{1}{\sqrt{1 + 2\frac{\alpha_1}{\alpha_0}} + 1},$$

are the roots of $(u)_1^2 + \frac{1}{2}\alpha_0 (u)_1 - \frac{1}{8}\alpha_0\alpha_1 = 0$. So, for (a) and (b), we have

$$\frac{\Theta_{2n+2}}{\langle w, P_{2n+2}^2\rangle} > 0, \quad n \geqslant 0.$$

(2) The conditions $(u)_2 < 0$ and $(u)_1 - \frac{|(u)_2|}{\alpha_0} < 0$ are satisfied and so we may use Lemma 3.5 to conclude that $\{\Omega_n\}_{n\geqslant 0}$, is a positive and increasing sequence. Therefore, using Lemma 3.4 for $\eta_n = \Omega_n,\ n \geqslant 0$, we have for (3.22)

$$\frac{\Theta_{2n+1}}{\langle w, P_{2n+1}^2\rangle} = (u)_1^2 + \big|(u)_2\big| - \Sigma_{2n} < (u)_1^2 + \big|(u)_2\big| - \Sigma_0,$$

and

$$\frac{\Theta_{2n+2}}{\langle w, P_{2n+2}^2\rangle} = (u)_1^2 + \big|(u)_2\big| - \Sigma_{2n+1} = (u)_1^2 + \big|(u)_2\big| + |\Sigma_{2n+1}| > 0.$$

In this case

$$\Sigma_0 = \Omega_0 = \left((u)_1 - \frac{|(u)_2|}{\alpha_0}\right)^2,$$

therefore

$$\Sigma_0 > (u)_1^2 + \big|(u)_2\big|,$$

is equivalent to

$$\frac{|(u)_2|}{\alpha_0^2} - 2\frac{(u)_1}{\alpha_0} - 1 > 0,$$

which is fulfilled and so

$$\frac{\Theta_{2n+1}}{\langle w, P_{2n+1}^2\rangle} < 0, \quad n \geqslant 0.$$

In any case, we have proved that $\Theta_n \neq 0,\ n \geqslant 0$, which means that u is regular. □

Moreover, from (3.9) and (3.16), we obtain for (3.14)

$$\tilde{b}_{n+1} = \xi_{n+1} - \prod_{\mu=0}^{n} d_\mu^2 \left\{(u)_1 + (u)_2 \sum_{\nu=0}^{n} \frac{(-1)^\nu}{d_\nu}\right\}\left\{(u)_1 + (u)_2 \sum_{\nu=0}^{n-1} \frac{(-1)^\nu}{d_\nu}\right\} \frac{1}{d_n}\Theta_n^{-1}, \quad n \geqslant 0,$$

that is

$$\tilde{b}_{n+1} = \xi_{n+1} + (-1)^{n+1} \prod_{\mu=0}^{n} (-\rho_{\mu+1})\sqrt{\Omega_n}\left\{\sqrt{\Omega_n} - \frac{(u)_2}{\alpha_n}\right\}\frac{1}{\alpha_n\Theta_n}, \quad n \geqslant 0,$$

by virtue of (3.8), (3.17) and (3.23). Therefore

$$\tilde{b}_{n+1} = \xi_{n+1} + \langle w, P_{n+1}^2\rangle \left\{\Omega_n - \frac{(u)_2}{\alpha_n}\sqrt{\Omega_n}\right\}\frac{1}{\alpha_n\Theta_n}, \quad n \geqslant 0. \tag{3.24}$$

Therefore (3.13) becomes

$$\begin{aligned} \beta_1 &= \xi_0 - (u)_1 + \frac{\alpha_0\Omega_0 - (u)_2\sqrt{\Omega_0}}{\Theta_0} \\ \beta_{n+2} &= \xi_{n+1} + \langle w, P_{n+1}^2\rangle \left\{\frac{1}{\alpha_n\Theta_n}\left(\Omega_n - \frac{(u)_2}{\alpha_n}\sqrt{\Omega_n}\right) + \frac{\alpha_{n+1}}{\Theta_{n+1}}\left(\Omega_{n+1} - \frac{(u)_2}{\alpha_{n+1}}\sqrt{\Omega_{n+1}}\right)\right\}, \quad n \geqslant 0, \end{aligned} \tag{3.25}$$

by virtue of (3.8) and (3.17), that is

$$\beta_{n+2} = \xi_{n+1} + \langle w, P_{n+1}^2 \rangle \left\{ \frac{1}{\alpha_n \Theta_n} \left(\Omega_n - \frac{(u)_2}{\alpha_n} \sqrt{\Omega_n} \right) + \frac{\alpha_{n+1}}{\Theta_{n+1}} \left(\Omega_n + \frac{(u)_2}{\alpha_{n+1}} \sqrt{\Omega_n} \right) \right\}, \quad n \geqslant 0, \tag{3.26}$$

by virtue of (3.23).

Examples. In the following we will suppose that $\alpha_0 > 0$, $(u)_2 \neq (u)_1^2$ and that $(u)_1$ and $(u)_2$ satisfy one of the three conditions of Theorem 3.6.

(1) Let us describe the case $v := \mathcal{U}$, where $\mathcal{U}$ is the Tchebychev form of second kind. This form is positive definite and so we may apply Theorem 3.6 and conclude that the form u given by (1.1) is regular.

Here [3]

$$\zeta_n = 0, \quad n \geqslant 0, \qquad \sigma_{n+1} = \frac{1}{4}, \quad n \geqslant 0. \tag{3.27}$$

Therefore, (3.19) becomes

$$\alpha_{2n} = \alpha_0, \quad \alpha_{2n+1} = \frac{1}{4\alpha_0}, \quad n \geqslant 0. \tag{3.28}$$

Using (3.28), (3.17) becomes

$$d_{2n} = \alpha_0, \quad d_{2n+1} = -\frac{1}{4\alpha_0}, \quad n \geqslant 0. \tag{3.29}$$

So (3.8) and (3.9) become, respectively

$$\xi_0 = -\alpha_0, \qquad \xi_{2n+1} = \frac{4\alpha_0^2 + 1}{4\alpha_0}, \quad \xi_{2n+2} = -\frac{4\alpha_0^2 + 1}{4\alpha_0}, \quad n \geqslant 0, \tag{3.30}$$

$$\rho_{2n+1} = -\alpha_0^2, \quad \rho_{2n+2} = -\frac{1}{16\alpha_0^2}, \quad n \geqslant 0, \tag{3.31}$$

$$P_0(0) = 1, \qquad P_1(0) = \alpha_0,$$

$$P_{2n+2}(0) = \prod_{\mu=0}^{2n+1} d_\mu = \prod_{\mu=0}^{n} d_{2\mu} d_{2\mu+1} = \frac{(-1)^{n+1}}{4^{n+1}}, \quad n \geqslant 0$$

and

$$P_{2n+3}(0) = \prod_{\mu=0}^{2n+2} d_\mu = \frac{(-1)^{n+1}}{4^{n+1}} d_{2n+2} = \frac{(-1)^{n+1}}{4^{n+1}} \alpha_0, \quad n \geqslant 0,$$

that is

$$P_{2n}(0) = \frac{(-1)^n}{4^n}, \quad P_{2n+1}(0) = \frac{(-1)^n}{4^n} \alpha_0, \quad n \geqslant 0. \tag{3.32}$$

Using (3.31), we also have

$$\langle w, P_1^2 \rangle = \rho_1 = -\alpha_0^2,$$

$$\langle w, P_{2n+2}^2\rangle = \prod_{\mu=0}^{2n+1} \rho_{\mu+1} = \prod_{\mu=0}^{n} \rho_{2\mu+1}\rho_{2\mu+2} = \frac{1}{16^{n+1}}, \quad n \geqslant 0$$

and

$$\langle w, P_{2n+3}^2\rangle = \prod_{\mu=0}^{2n+2} \rho_{\mu+1} = \frac{1}{16^{n+1}}\rho_{2n+3} = -\frac{1}{16^{n+1}}\alpha_0^2, \quad n \geqslant 0,$$

that is

$$\langle w, P_{2n+1}^2\rangle = -\frac{1}{4^{2n}}\alpha_0^2, \quad \langle w, P_{2n}^2\rangle = \frac{1}{4^{2n}}, \quad n \geqslant 0. \tag{3.33}$$

Finally, from (3.28), we have for (3.23)

$$\left.\begin{aligned} \Omega_{2n} &= \left\{(u)_1 + \left(\frac{1+4\alpha_0^2}{\alpha_0}n + \frac{1}{\alpha_0}\right)(u)_2\right\}^2, \\ \Omega_{2n+1} &= \left\{(u)_1 + \frac{1+4\alpha_0^2}{\alpha_0}(n+1)(u)_2\right\}^2, \end{aligned}\right\} \quad n \geqslant 0. \tag{3.34}$$

After some calculations and taking account of (3.33) and (3.34) we obtain for (3.22)

$$\left.\begin{aligned} \Theta_{2n} &= \frac{1}{16^n}\left[(u)_1^2 - (u)_2 + 8n\alpha_0(u)_1(u)_2 + 4n\left(4n\alpha_0^2 + n + 1\right)(u)_2^2\right], \\ \Theta_{2n+1} &= \frac{1}{16^n}\left[\alpha_0^2(u)_2 + 2(n+1)\alpha_0(u)_1(u)_2 + (n+1)\left(4n\alpha_0^2 + n + 1\right)(u)_2^2\right], \end{aligned}\right\} \quad n \geqslant 0 \tag{3.35}$$

equivalently,

$$\left.\begin{aligned} \Theta_{2n} &= \frac{1}{4^{2n-1}}(u)_2^2\left(4\alpha_0^2+1\right)(n+x_1)(n+x_2), \\ \Theta_{2n+1} &= \frac{1}{4^{2n}}(u)_2^2\left(4\alpha_0^2+1\right)(n+x_3)(n+x_4), \end{aligned}\right\} \quad n \geqslant 0, \tag{3.36}$$

with

$$\begin{aligned} (u)_1 &= \frac{\alpha_0(4\alpha_0^2+1)(x_3+x_4) - 2(1+2\alpha_0^2)\alpha_0}{2(4\alpha_0^2+1)(x_3x_4 - (x_3+x_4)+1)}, \\ (u)_2 &= \frac{\alpha_0^2}{(4\alpha_0^2+1)(x_3x_4 - (x_3+x_4)+1)}, \\ x_1 + x_2 &= x_3 + x_4 - 1, \\ x_1x_2 &= \frac{1}{4}\left\{(x_3+x_4)(x_3+x_4-2) + \frac{1}{4\alpha_0^2}(x_3-x_4)^2 + \frac{4\alpha_0^2}{4\alpha_0^2+1}\right\}. \end{aligned} \tag{3.37}$$

Remark. In this example, we found necessary and sufficient conditions such that u is regular. Indeed, u is regular if and only if x_1, x_2, x_3 and x_4 are not negatives integers or zero (there is no need to exclude the complex numbers).

Therefore, we have for (3.25), (3.26) and (3.15):

$$\begin{cases} \beta_0 = (u)_1, \qquad \beta_1 = \dfrac{\alpha_0 (u)_2 + 2(u)_1(u)_2 - (u)_1^3}{(u)_1^2 - (u)_2}, \\ \beta_{2n+2} = -\dfrac{[(n+x_3)(n+x_4) + (n+x_1)(n+x_2)]\alpha_0^2 \Omega_{2n}}{4\alpha_0 (u)_2^2 (4\alpha_0^2+1)(n+x_1)(n+x_2)(n+x_3)(n+x_4)} \\ \quad + \dfrac{[(n+x_3)(n+x_4) - \alpha_0^2(n+x_1)(n+x_2)]4\alpha_0 (u)_2 \sqrt{\Omega_{2n}}}{4\alpha_0 (u)_2^2 (4\alpha_0^2+1)(n+x_1)(n+x_2)(n+x_3)(n+x_4)} + \dfrac{4\alpha_0^2+1}{4\alpha_0}, \quad n \geqslant 0, \\ \beta_{2n+3} = \dfrac{[(n+x_1+1)(n+x_2+1) + (n+x_3)(n+x_4)]\alpha_0^2 \Omega_{2n+1}}{4\alpha_0 (u)_2^2 (4\alpha_0^2+1)(n+x_1+1)(n+x_2+1)(n+x_3)(n+x_4)} \\ \quad - \dfrac{[16\alpha_0^2(n+x_1+1)(n+x_2+1) - (n+x_3)(n+x_4)]\alpha_0 (u)_2 \sqrt{\Omega_{2n+1}}}{4\alpha_0 (u)_2^2 (4\alpha_0^2+1)(n+x_1+1)(n+x_2+1)(n+x_3)(n+x_4)} - \dfrac{4\alpha_0^2+1}{4\alpha_0}, \quad n \geqslant 0, \\ \gamma_1 = (u)_2 - (u)_1^2, \qquad \gamma_2 = -(u)_2^3 \dfrac{(4\alpha_0^2+1)x_3 x_4}{((u)_1^2 - (u)_2)^2}, \\ \gamma_{2n+3} = -\dfrac{(n+x_1)(n+x_2)(n+x_1+1)(n+x_2+1)}{(n+x_3)^2(n+x_4)^2}\alpha_0^2, \quad n \geqslant 0, \\ \gamma_{2n+4} = -\dfrac{(n+x_3)(n+x_4)(n+x_3+1)(n+x_4+1)}{16(n+x_1+1)^2(n+x_2+1)^2}\dfrac{1}{\alpha_0^2}, \quad n \geqslant 0 \end{cases} \tag{3.38}$$

with Ω_{2n}, $n \geqslant 0$ and Ω_{2n+1} $n \geqslant 0$ given by (3.34).

Proposition 3.7. *The form u given by* (1.1) *possesses the following integral representation*

$$\langle u, f\rangle = f(0) + f'(0)\big((u)_1 - 2\lambda\big) + \frac{1}{2} f''(0)(u)_2 - \frac{2\lambda}{\pi} Pf \int_{-1}^{+1} \frac{\sqrt{1-x^2}}{x^3} f(x)\,\mathrm{d}x, \quad \forall f \in \mathcal{P}. \tag{3.39}$$

It is a 1*-quasi-antisymmetric and semi-classical form of class s satisfying the following functional equation*:

(3.40)

$(u)_1 \neq 2\lambda$	$\begin{cases} D(x^3(x^2-1)u) - 3x^4 u = 0 \\ (u)_3 = -\lambda \end{cases}$	$s = 3$
$(u)_1 = 2\lambda$ *and* $(u)_2 \neq -2$	$\begin{cases} D(x^2(x^2-1)u) - (2x^3+x)u = 0 \\ (u)_1 = 2\lambda,\ (u)_3 = -\lambda,\ (u)_2 \neq -2 \end{cases}$	$s = 2$
$(u)_1 = 2\lambda$ *and* $(u)_2 = -2$	$\begin{cases} D(x(x^2-1)u) - (x^2+2)u = 0 \\ (u)_1 = 2\lambda,\ (u)_3 = -\lambda,\ (u)_2 = -2 \end{cases}$	$s = 1$

Proof. It is well known that the Tchebychev form of second kind possess the following integral representation

$$\langle \mathcal{U}, f(x) \rangle = \int_{-1}^{+1} V(x) f(x)\,dx, \quad \forall f \in \mathcal{P},$$

with $V(x) = \frac{2\sqrt{1-x^2}}{\pi}$. Following (1.3), we easily obtain (3.39).

Also the form u is 1-quasi-antisymmetric because it satisfies

$$(u)_{2n+4} = \langle u, x^{2n+4} \rangle = -\lambda \langle v, x^{2n+1} \rangle = 0, \quad n \geqslant 0,$$

since v is symmetric, by hypotheses.

The Tchebychev form of second kind $\mathcal{U}$ is classical and satisfies [12]

$$D(\Phi v) + \psi v = 0,$$

with $\Phi(x) = x^2 - 1$ and $\psi(x) = -3x$. Therefore, (1.48) and (1.49) become

$$\chi_1 = -(u)_1 + 2\lambda \quad \text{and} \quad \chi_2 = -2 - (u)_2,$$

using (2.6). Moreover, $\Phi(0) = -1 \neq 0$.

Now it is enough to use Proposition 1.4 in order to obtain (2.40). We have just proved (3.40). □

(2) Let us describe the case $v := \mathcal{T}$, where $\mathcal{T}$ is the Tchebychev form of first kind. This form is positive definite and so we may apply Theorem 3.6 and conclude that the form u given by (1.1) is regular.

Here [3]

$$\zeta_n = 0, \quad n \geqslant 0, \qquad \sigma_1 = \frac{1}{2}, \qquad \sigma_{n+2} = \frac{1}{4}, \quad n \geqslant 0. \tag{3.41}$$

Therefore, (3.19) becomes

$$\alpha_{2n+2} = \frac{1}{2}\alpha_0, \quad \alpha_{2n+1} = \frac{1}{2\alpha_0}, \quad n \geqslant 0. \tag{3.42}$$

Using (3.42), (3.17) becomes

$$d_0 = \alpha_0, \qquad d_{2n+2} = \frac{1}{2}\alpha_0, \quad d_{2n+1} = -\frac{1}{2\alpha_0}, \quad n \geqslant 0. \tag{3.43}$$

So (3.8) and (3.9) become, respectively

$$\xi_0 = -\alpha_0, \qquad \xi_1 = \frac{1+2\alpha_0^2}{2\alpha_0}, \quad \xi_{2n+2} = -\frac{1+\alpha_0^2}{2\alpha_0}, \quad \xi_{2n+3} = \frac{1+\alpha_0^2}{2\alpha_0}, \quad n \geqslant 0, \tag{3.44}$$

$$\rho_1 = -\alpha_0^2, \qquad \rho_{2n+2} = -\frac{1}{4\alpha_0^2}, \quad \rho_{2n+3} = -\frac{1}{4}\alpha_0^2, \quad n \geqslant 0, \tag{3.45}$$

$$P_0(0) = 1, \qquad P_1(0) = \alpha_0,$$

$$P_{2n+2}(0) = \prod_{\mu=0}^{2n+1} d_\mu = d_0 d_1 \prod_{\mu=1}^{n} d_{2\mu} d_{2\mu+1} = -\frac{1}{2}\frac{(-1)^n}{4^n} = \frac{(-1)^{n+1}}{2^{2n+1}}, \quad n \geqslant 0$$

and

$$P_{2n+3}(0)=\prod_{\mu=0}^{2n+2}d_\mu=\frac{(-1)^{n+1}}{2^{2n+1}}d_{2n+2}=\frac{(-1)^{n+1}}{2^{2n+2}}\alpha_0,\quad n\geqslant 0,$$

that is

$$P_0(0)=1,\qquad P_{2n+1}(0)=\frac{(-1)^n}{2^{2n}}\alpha_0,\quad P_{2n+2}(0)=\frac{(-1)^{n+1}}{2^{2n+1}},\quad n\geqslant 0. \tag{3.46}$$

From (3.45), we also have

$$\langle w,P_1^2\rangle=\rho_1=-\alpha_0^2,$$

$$\langle w,P_{2n+2}^2\rangle=\prod_{\mu=0}^{2n+1}\rho_{\mu+1}=\rho_1\rho_2\prod_{\mu=1}^{n}\rho_{2\mu+1}\rho_{2\mu+2}=\frac{1}{4^{2n+1}},\quad n\geqslant 0$$

and

$$\langle w,P_{2n+3}^2\rangle=\prod_{\mu=0}^{2n+2}\rho_{\mu+1}=\frac{1}{4^{2n+1}}\rho_{2n+3}=-\frac{1}{4^{2n+2}}\alpha_0^2,\quad n\geqslant 0,$$

that is

$$\langle w,P_{2n+1}^2\rangle=-\frac{1}{4^{2n}}\alpha_0^2,\qquad \langle w,P_{2n+2}^2\rangle=\frac{1}{4^{2n+1}},\quad n\geqslant 0. \tag{3.47}$$

Finally, from (3.42), we have for (3.23)

$$\left.\begin{aligned}\Omega_{2n}&=\left\{(u)_1+\left(\frac{1+\alpha_0^2}{\alpha_0}2n+\frac{1}{\alpha_0}\right)(u)_2\right\}^2,\\ \Omega_{2n+1}&=\left\{(u)_1+\left(\frac{1+\alpha_0^2}{\alpha_0}(2n+1)+\alpha_0\right)(u)_2\right\}^2,\end{aligned}\right\}\quad n\geqslant 0. \tag{3.48}$$

After some calculations and taking account of (3.47) and (3.48) we obtain for (3.22)

$$\begin{aligned}\Theta_{2n+1}&=\frac{1}{4^{2n-1}}\biggl\{(u)_2^2(1+\alpha_0^2)n^2+\bigl[\alpha_0(u)_1(u)_2+(1+\alpha_0^2)(u)_2^2\bigr]n\\&\qquad+\frac{2\alpha_0(u)_1(u)_2+(u)_2^2+\alpha_0^2(u)_2}{4}\biggr\},\quad n\geqslant 0,\\ \Theta_{2n+2}&=\frac{1}{4^{2n}}\biggl\{(u)_2^2(1+\alpha_0^2)n^2+\bigl[\alpha_0(u)_1(u)_2+2(1+\alpha_0^2)(u)_2^2\bigr]n\\&\qquad+\alpha_0(u)_1(u)_2+(1+\alpha_0^2)(u)_2^2+\frac{(u)_1^2-(u)_2}{4}\biggr\},\end{aligned} \tag{3.49}$$

equivalently,

$$\left.\begin{aligned}\Theta_{2n+1}&=\frac{1}{4^{2n-1}}(u)_2^2(1+\alpha_0^2)(n+y_1)(n+y_2),\\ \Theta_{2n+2}&=\frac{1}{4^{2n}}(u)_2^2(1+\alpha_0^2)(n+y_3)(n+y_4),\end{aligned}\right\}\quad n\geqslant 0, \tag{3.50}$$

with

$$(u)_1 = \frac{\alpha_0(1+\alpha_0^2)(y_1+y_2-1)}{(1+\alpha_0^2)(4y_1y_2-2(y_1+y_2-1))-1}, \qquad (u)_2 = \frac{\alpha_0^2}{(1+\alpha_0^2)(4y_1y_2-2(y_1+y_2-1))-1},$$

$$y_3+y_4 = y_1+y_2+1, \tag{3.51}$$

$$y_3y_4 = \frac{1}{4}\left\{(y_1+y_2-1)(y_1+y_2+3) + \frac{1}{\alpha_0^2}\big[(y_1-y_2)^2-1\big] + \frac{(1+2\alpha_0^2)^2}{\alpha_0^2(1+\alpha_0^2)}\right\}.$$

Remark. In this example, we found necessary and sufficient conditions such that u is regular. Indeed, u is regular if and only if y_1, y_2, y_3 and y_4 are not negatives integers or zero (there is no need to exclude the complex numbers).

Therefore, we have for (3.25), (3.26) and (3.15):

$$\begin{cases} \beta_0 = (u)_1, \qquad \beta_1 = \dfrac{\alpha_0(u)_2 + 2(u)_1(u)_2 - (u)_1^3}{(u)_1^2 - (u)_2}, \\[2mm] \beta_2 = \dfrac{-2\alpha_0^3(u)_1^3(u)_2 + 2\alpha_0^2(u)_1^2(u)_2^2 + 2\alpha_0^2(u)_1^2(u)_2 - (1+2\alpha_0^2)(u)_2^3 + (1+\alpha_0^2-2\alpha_0^4)(u)_2^2}{2\alpha_0((u)_1^2-(u)_2)[2\alpha_0(u)_1(u)_2 + (u)_2^2 + \alpha_0^2(u)_2]} \\ \qquad + \dfrac{2\alpha_0(u)_1(u)_2^3 - \alpha_0^2(u)_1^4}{2\alpha_0((u)_1^2-(u)_2)[2\alpha_0(u)_1(u)_2 + (u)_2^2 + \alpha_0^2(u)_2]}, \\[2mm] \beta_{2n+3} = \dfrac{\big[(n+y_3)(n+y_4) + (n+y_1)(n+y_2)\big]\alpha_0^2\Omega_{2n+1}}{8\alpha_0(u)_2^2(\alpha_0^2+1)(n+y_1)(n+y_2)(n+y_3)(n+y_4)} \\ \qquad - \dfrac{[\alpha_0^2(n+y_3)(n+y_4) - (n+y_1)(n+y_2)]2\alpha_0(u)_2\sqrt{\Omega_{2n+1}}}{8\alpha_0(u)_2^2(\alpha_0^2+1)(n+y_1)(n+y_2)(n+y_3)(n+y_4)} - \dfrac{\alpha_0^2+1}{2\alpha_0}, \quad n \geqslant 0, \\[2mm] \beta_{2n+4} = -\dfrac{[(n+y_3)(n+y_4) + (n+y_1+1)(n+y_2+1)]\alpha_0^2\Omega_{2n+2}}{8\alpha_0(u)_2^2(\alpha_0^2+1)(n+y_1+1)(n+y_2+1)(n+y_3)(n+y_4)} \\ \qquad + \dfrac{[(n+y_1+1)(n+y_2+1) - \alpha_0^2(n+y_3)(n+y_4)]2\alpha_0(u)_2\sqrt{\Omega_{2n+2}}}{8\alpha_0(u)_2^2(\alpha_0^2+1)(n+y_1+1)(n+y_2+1)(n+y_3)(n+y_4)} + \dfrac{\alpha_0^2+1}{2\alpha_0}, \quad n \geqslant 0, \\[2mm] \gamma_1 = (u)_2 - (u)_1^2, \qquad \gamma_2 = -4(u)_2^3\dfrac{(1+\alpha_0^2)y_1y_2}{((u)_1^2-(u)_2)^2}, \qquad \gamma_3 = -\dfrac{((u)_1^2-(u)_2)y_3y_4}{16(u)_2^2(1+\alpha_0^2)y_1^2y_2^2}\alpha_0^2, \\[2mm] \gamma_{2n+4} = -\dfrac{(n+y_1)(n+y_2)(n+y_1+1)(n+y_2+1)}{4(n+y_3)^2(n+y_4)^2}\dfrac{1}{\alpha_0^2}, \quad n \geqslant 0, \\[2mm] \gamma_{2n+5} = -\dfrac{(n+y_3)(n+y_4)(n+y_3+1)(n+y_4+1)}{4(n+y_1+1)^2(n+y_2+1)^2}\alpha_0^2, \quad n \geqslant 0 \end{cases} \tag{3.52}$$

with Ω_{2n+1}, $n \geqslant 0$ and Ω_{2n+2} $n \geqslant 0$ given by (3.48).

Proposition 3.8. *The form u given by* (1.1) *possesses the following integral representation*

$$\langle u, f\rangle = f(0) + f'(0)(u)_1 + \frac{1}{2}f''(0)(u)_2 - \frac{\lambda}{\pi} Pf\int_{-1}^{+1} \frac{1}{x^3\sqrt{1-x^2}} f(x)\,\mathrm{d}x, \quad \forall f \in \mathcal{P}. \tag{3.53}$$

It is a 1*-quasi-antisymmetric and semi-classical form of class s satisfying the following functional equation*:

$$\begin{array}{lll} (u)_1 \neq 0 & \begin{cases} D(x^3(x^2-1)u) - x^4 u = 0 \\ (u)_3 = -\lambda \end{cases} & s = 3 \\ (u)_1 = 0 & \begin{cases} D(x^2(x^2-1)u) - xu = 0 \\ (u)_1 = 0,\ (u)_3 = -\lambda,\ (u)_2 \neq 0 \end{cases} & s = 2 \end{array} \tag{3.54}$$

Proof. It is well known that the Tchebychev form of first kind possess the following integral representation

$$\langle \mathcal{T}, f(x)\rangle = \int_{-1}^{+1} V(x) f(x)\,\mathrm{d}x, \quad \forall f \in \mathcal{P},$$

with $V(x) = \frac{1}{\pi\sqrt{1-x^2}}$. Following (1.3), we easily obtain (3.53).

Also the form u is 1-quasi-antisymmetric because it satisfies

$$(u)_{2n+4} = \langle u, x^{2n+4}\rangle = -\lambda\langle v, x^{2n+1}\rangle = 0, \quad n \geqslant 0,$$

since v is symmetric, by hypotheses.

The Tchebychev form of first kind $\mathcal{T}$ is classical and satisfies [12]

$$D(\Phi v) + \psi v = 0,$$

with $\Phi(x) = x^2 - 1$ and $\psi(x) = -x$. Therefore, (1.48) and (1.49) become

$$\chi_1 = -(u)_1 \quad \text{and} \quad \chi_2 = -2,$$

using (2.6).

Now it is enough to use Proposition 1.4 in order to obtain (2.27). We have just proved (3.54) □

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