

# A variable metric extension of the forward–backward–forward algorithm for monotone operators \*

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## Abstract

We propose a variable metric extension of the forward–backward–forward algorithm for finding a zero of the sum of a maximally monotone operator and a monotone Lipschitzian operator in Hilbert spaces. In turn, this framework provides a variable metric splitting algorithm for solving monotone inclusions involving sums of composite operators. Monotone operator splitting methods recently proposed in the literature are recovered as special cases.

**Keywords:** variable metric, composite operator, duality, monotone inclusion, monotone operator, operator splitting, primal-dual algorithm

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## 1 Introduction

A basic problem in applied monotone operator theory is to find a zero of a maximally monotone operator  $A$  on a real Hilbert space  $\mathcal{H}$ . This problem can be solved by the proximal point algorithm proposed in [17] which requires only the resolvent of  $A$ , provided it is easy to implement numerically. In order to get more efficient proximal algorithms, some authors have proposed the use of variable metric or preconditioning in such algorithms [3, 5, 6, 10, 13, 15, 16].

This problem was then extended to the problem of finding a zero of the sum of a maximally monotone operator  $A$  and a cocoercive operator  $B$  (i.e.,  $B^{-1}$  is strongly monotone). In such

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instances, the forward-backward splitting algorithm [1, 8, 12, 18] can be used. Recently, this algorithm has been investigated in the context of variable metric [11]. In the case when  $B$  is only Lipschitzian and not cocoercive, the problem can be solved by the forward-backward-forward splitting algorithm [4, 19]. New applications of this basic algorithm to more complex monotone inclusions are presented in [4, 9].

In the present paper, we propose a variable metric version of the forward-backward-forward splitting algorithm. In Section 2, we recall notation and background on convex analysis and monotone operator theory. In Section 3, we present our variable metric forward-backward-forward splitting algorithm. In Section 4, the results of Section 3 are used to develop a variable metric primal–dual algorithm for solving the type of composite inclusions considered in [9].

## 2 Notation and background

Throughout,  $\mathcal{H}$ ,  $\mathcal{G}$ , and  $(\mathcal{G}_i)_{1 \leq i \leq m}$  are real Hilbert spaces. Their scalar products and associated norms are respectively denoted by  $\langle \cdot | \cdot \rangle$  and  $\| \cdot \|$ . We denote by  $\mathcal{B}(\mathcal{H}, \mathcal{G})$  the space of bounded linear operators from  $\mathcal{H}$  to  $\mathcal{G}$ . The adjoint of  $L \in \mathcal{B}(\mathcal{H}, \mathcal{G})$  is denoted by  $L^*$ . We set  $\mathcal{B}(\mathcal{H}) = \mathcal{B}(\mathcal{H}, \mathcal{H})$ . The symbols  $\rightharpoonup$  and  $\rightarrow$  denote respectively weak and strong convergence, and  $\text{Id}$  denotes the identity operator, and  $B(x; \rho)$  denotes the closed ball of center  $x \in \mathcal{H}$  and radius  $\rho \in ]0, +\infty[$ . The interior of  $C \subset \mathcal{H}$  is denoted by  $\text{int } C$ . We denote by  $\ell_+^1(\mathbb{N})$  the set of summable sequences in  $[0, +\infty[$ .

Let  $M_1$  and  $M_2$  be self-adjoint operators in  $\mathcal{B}(\mathcal{H})$ , we write  $M_1 \succcurlyeq M_2$  if and only if  $(\forall x \in \mathcal{H}) \langle M_1 x | x \rangle \geq \langle M_2 x | x \rangle$ . Let  $\alpha \in ]0, +\infty[$ . We set

$$\mathcal{P}_\alpha(\mathcal{H}) = \{M \in \mathcal{B}(\mathcal{H}) \mid M^* = M \quad \text{and} \quad M \succcurlyeq \alpha \text{Id}\}. \quad (2.1)$$

Moreover, for every  $M \in \mathcal{P}_\alpha(\mathcal{H})$ , we define respectively a scalar product and a norm by

$$(\forall x \in \mathcal{H})(\forall y \in \mathcal{H}) \quad \langle x | y \rangle_M = \langle Mx | y \rangle \quad \text{and} \quad \|x\|_M = \sqrt{\langle Mx | x \rangle}. \quad (2.2)$$

Let  $A: \mathcal{H} \rightarrow 2^{\mathcal{H}}$  be a set-valued operator. The domain is  $\text{dom } A = \{x \in \mathcal{H} \mid Ax \neq \emptyset\}$ , and the graph of  $A$  is  $\text{gra } A = \{(x, u) \in \mathcal{H} \times \mathcal{H} \mid u \in Ax\}$ . The set of zeros of  $A$  is  $\text{zer } A = \{x \in \mathcal{H} \mid 0 \in Ax\}$ , and the range of  $A$  is  $\text{ran } A = \{u \in \mathcal{H} \mid (\exists x \in \mathcal{H}) u \in Ax\}$ . The inverse of  $A$  is  $A^{-1}: \mathcal{H} \mapsto 2^{\mathcal{H}}: u \mapsto \{x \in \mathcal{H} \mid u \in Ax\}$ , and the resolvent of  $A$  is

$$J_A = (\text{Id} + A)^{-1}. \quad (2.3)$$

Moreover,  $A$  is monotone if

$$(\forall (x, y) \in \mathcal{H} \times \mathcal{H})(\forall (u, v) \in Ax \times Ay) \quad \langle x - y | u - v \rangle \geq 0, \quad (2.4)$$

and maximally monotone if it is monotone and there exists no monotone operator  $B: \mathcal{H} \rightarrow 2^{\mathcal{H}}$  such that  $\text{gra } A \subset \text{gra } B$  and  $A \neq B$ . We say that  $A$  is uniformly monotone at  $x \in \text{dom } A$  if there exists an increasing function  $\phi_A: [0, +\infty[ \rightarrow [0, +\infty[$  vanishing only at 0 such that

$$(\forall u \in Ax)(\forall (y, v) \in \text{gra } A) \quad \langle x - y | u - v \rangle \geq \phi_A(\|x - y\|). \quad (2.5)$$

### 3 Variable metric forward-backward-forward splitting algorithm

The forward-backward-forward splitting algorithm was first proposed in [19] to solve inclusion involving the sum of a maximally monotone operator and a monotone Lipschitzian operator. In [4], it was revisited to include computational errors. Below, we extend it to a variable metric setting.

**Theorem 3.1** *Let  $\mathcal{K}$  be a real Hilbert space with the scalar product  $\langle\langle \cdot | \cdot \rangle\rangle$  and the associated norm  $||| \cdot |||$ . Let  $\alpha$  and  $\beta$  be in  $]0, +\infty[$ , let  $(\eta_n)_{n \in \mathbb{N}}$  be a sequence in  $\ell_+^1(\mathbb{N})$ , and let  $(\mathbf{U}_n)_{n \in \mathbb{N}}$  be a sequence in  $\mathcal{B}(\mathcal{K})$  such that*

$$\mu = \sup_{n \in \mathbb{N}} \|\mathbf{U}_n\| < +\infty \quad \text{and} \quad (1 + \eta_n)\mathbf{U}_{n+1} \succeq \mathbf{U}_n \in \mathcal{P}_\alpha(\mathcal{K}). \quad (3.1)$$

*Let  $\mathbf{A}: \mathcal{K} \rightarrow 2^{\mathcal{K}}$  be maximally monotone, let  $\mathbf{B}: \mathcal{K} \rightarrow \mathcal{K}$  be a monotone and  $\beta$ -Lipschitzian operator on  $\mathcal{K}$  such that  $\text{zer}(\mathbf{A} + \mathbf{B}) \neq \emptyset$ . Let  $(\mathbf{a}_n)_{n \in \mathbb{N}}$ ,  $(\mathbf{b}_n)_{n \in \mathbb{N}}$ , and  $(\mathbf{c}_n)_{n \in \mathbb{N}}$  be absolutely summable sequences in  $\mathcal{K}$ . Let  $\mathbf{x}_0 \in \mathcal{K}$ , let  $\varepsilon \in ]0, 1/(\beta\mu + 1)[$ , let  $(\gamma_n)_{n \in \mathbb{N}}$  be a sequence in  $[\varepsilon, (1 - \varepsilon)/(\beta\mu)]$ , and set*

$$(\forall n \in \mathbb{N}) \quad \begin{cases} \mathbf{y}_n = \mathbf{x}_n - \gamma_n \mathbf{U}_n(\mathbf{B}\mathbf{x}_n + \mathbf{a}_n) \\ \mathbf{p}_n = J_{\gamma_n \mathbf{U}_n \mathbf{A}} \mathbf{y}_n + \mathbf{b}_n \\ \mathbf{q}_n = \mathbf{p}_n - \gamma_n \mathbf{U}_n(\mathbf{B}\mathbf{p}_n + \mathbf{c}_n) \\ \mathbf{x}_{n+1} = \mathbf{x}_n - \mathbf{y}_n + \mathbf{q}_n. \end{cases} \quad (3.2)$$

*Then the following hold for some  $\bar{\mathbf{x}} \in \text{zer}(\mathbf{A} + \mathbf{B})$ .*

- (i)  $\sum_{n \in \mathbb{N}} |||\mathbf{x}_n - \mathbf{p}_n|||^2 < +\infty$  and  $\sum_{n \in \mathbb{N}} |||\mathbf{y}_n - \mathbf{q}_n|||^2 < +\infty$ .
- (ii)  $\mathbf{x}_n \rightarrow \bar{\mathbf{x}}$  and  $\mathbf{p}_n \rightarrow \bar{\mathbf{x}}$ .
- (iii) *Suppose that one of the following is satisfied:*
  - (a)  $\underline{\lim} d_{\text{zer}(\mathbf{A} + \mathbf{B})}(\mathbf{x}_n) = 0$ .
  - (b)  $\mathbf{A} + \mathbf{B}$  is demiregular (see [1, Definition 2.3]) at  $\bar{\mathbf{x}}$ .
  - (c)  $\mathbf{A}$  or  $\mathbf{B}$  is uniformly monotone at  $\bar{\mathbf{x}}$ .
  - (d)  $\text{int zer}(\mathbf{A} + \mathbf{B}) \neq \emptyset$  and there exists  $(\nu_n)_{n \in \mathbb{N}} \in \ell_+^1(\mathbb{N})$  such that  $(\forall n \in \mathbb{N}) (1 + \nu_n)\mathbf{U}_n \succeq \mathbf{U}_{n+1}$ .

*Then  $\mathbf{x}_n \rightarrow \bar{\mathbf{x}}$  and  $\mathbf{p}_n \rightarrow \bar{\mathbf{x}}$ .*

*Proof.* It follows from [11, Lemma 3.7] that the sequences  $(\mathbf{x}_n)_{n \in \mathbb{N}}$ ,  $(\mathbf{y}_n)_{n \in \mathbb{N}}$ ,  $(\mathbf{p}_n)_{n \in \mathbb{N}}$  and  $(\mathbf{q}_n)_{n \in \mathbb{N}}$  are well defined. Moreover, using [10, Lemma 2.1(i)(ii)] and (3.1), we obtain

$$(\forall (\mathbf{z}_n)_{n \in \mathbb{N}} \in \mathcal{K}^{\mathbb{N}}) \quad \sum_{n \in \mathbb{N}} |||\mathbf{z}_n||| < +\infty \quad \Leftrightarrow \quad \sum_{n \in \mathbb{N}} |||\mathbf{z}_n|||_{\mathbf{U}_n^{-1}} < +\infty \quad (3.3)$$

and

$$(\forall (\mathbf{z}_n)_{n \in \mathbb{N}} \in \mathcal{K}^{\mathbb{N}}) \quad \sum_{n \in \mathbb{N}} |||\mathbf{z}_n||| < +\infty \quad \Leftrightarrow \quad \sum_{n \in \mathbb{N}} |||\mathbf{z}_n|||_{\mathbf{U}_n} < +\infty. \quad (3.4)$$

Let us set

$$(\forall n \in \mathbb{N}) \quad \begin{cases} \tilde{\mathbf{y}}_n = \mathbf{x}_n - \gamma_n \mathbf{U}_n \mathbf{B} \mathbf{x}_n \\ \tilde{\mathbf{p}}_n = J_{\gamma_n} \mathbf{U}_n \mathbf{A} \tilde{\mathbf{y}}_n \\ \tilde{\mathbf{q}}_n = \tilde{\mathbf{p}}_n - \gamma_n \mathbf{U}_n \mathbf{B} \tilde{\mathbf{p}}_n \\ \tilde{\mathbf{x}}_{n+1} = \mathbf{x}_n - \tilde{\mathbf{y}}_n + \tilde{\mathbf{q}}_n, \end{cases} \quad \text{and} \quad \begin{cases} \mathbf{u}_n = \gamma_n^{-1} \mathbf{U}_n^{-1} (\mathbf{x}_n - \tilde{\mathbf{p}}_n) + \mathbf{B} \tilde{\mathbf{p}}_n - \mathbf{B} \mathbf{x}_n \\ \mathbf{e}_n = \tilde{\mathbf{x}}_{n+1} - \mathbf{x}_{n+1} \\ \mathbf{d}_n = \mathbf{q}_n - \tilde{\mathbf{q}}_n + \tilde{\mathbf{y}}_n - \mathbf{y}_n. \end{cases} \quad (3.5)$$

Then (3.5) yields

$$(\forall n \in \mathbb{N}) \quad \mathbf{u}_n = \gamma_n^{-1} \mathbf{U}_n^{-1} (\tilde{\mathbf{y}}_n - \tilde{\mathbf{p}}_n) + \mathbf{B} \tilde{\mathbf{p}}_n \in \mathbf{A} \tilde{\mathbf{p}}_n + \mathbf{B} \tilde{\mathbf{p}}_n, \quad (3.6)$$

and (3.5), (3.2), Lemma [11, Lemma 3.7(ii)], and the Lipschitzianity of  $\mathbf{B}$  on  $\mathcal{K}$  yield

$$(\forall n \in \mathbb{N}) \quad \begin{cases} \|\mathbf{y}_n - \tilde{\mathbf{y}}_n\|_{\mathbf{U}_n^{-1}} \leq (\beta\mu)^{-1} \|\mathbf{a}_n\|_{\mathbf{U}_n} \\ \|\mathbf{p}_n - \tilde{\mathbf{p}}_n\|_{\mathbf{U}_n^{-1}} \leq \|\mathbf{b}_n\|_{\mathbf{U}_n^{-1}} + (\beta\mu)^{-1} \|\mathbf{a}_n\|_{\mathbf{U}_n} \\ \|\mathbf{q}_n - \tilde{\mathbf{q}}_n\|_{\mathbf{U}_n^{-1}} \leq 2 \left( \|\mathbf{b}_n\|_{\mathbf{U}_n^{-1}} + (\beta\mu)^{-1} \|\mathbf{a}_n\|_{\mathbf{U}_n} \right) + (\beta\mu)^{-1} \|\mathbf{c}_n\|_{\mathbf{U}_n}. \end{cases} \quad (3.7)$$

Since  $(\mathbf{a}_n)_{n \in \mathbb{N}}$ ,  $(\mathbf{b}_n)_{n \in \mathbb{N}}$ , and  $(\mathbf{c}_n)_{n \in \mathbb{N}}$  are absolutely summable sequences in  $\mathcal{K}$ , we derive from (3.3), (3.4), (3.5), and (3.7) that

$$\begin{cases} \sum_{n \in \mathbb{N}} \|\mathbf{p}_n - \tilde{\mathbf{p}}_n\| < +\infty & \text{and} & \sum_{n \in \mathbb{N}} \|\mathbf{p}_n - \tilde{\mathbf{p}}_n\|_{\mathbf{U}_n^{-1}} < +\infty \\ \sum_{n \in \mathbb{N}} \|\mathbf{q}_n - \tilde{\mathbf{q}}_n\| < +\infty & \text{and} & \sum_{n \in \mathbb{N}} \|\mathbf{q}_n - \tilde{\mathbf{q}}_n\|_{\mathbf{U}_n^{-1}} < +\infty \\ \sum_{n \in \mathbb{N}} \|\mathbf{d}_n\| < +\infty & \text{and} & \sum_{n \in \mathbb{N}} \|\mathbf{d}_n\|_{\mathbf{U}_n^{-1}} < +\infty. \end{cases} \quad (3.8)$$

Now, let  $\mathbf{x} \in \text{zer}(\mathbf{A} + \mathbf{B})$ . Then, for every  $n \in \mathbb{N}$ ,  $(\mathbf{x}, -\gamma_n \mathbf{U}_n \mathbf{B} \mathbf{x}) \in \text{gra}(\gamma_n \mathbf{U}_n \mathbf{A})$  and (3.5) yields  $(\tilde{\mathbf{p}}_n, \tilde{\mathbf{y}}_n - \tilde{\mathbf{p}}_n) \in \text{gra}(\gamma_n \mathbf{U}_n \mathbf{A})$ . Hence, by monotonicity of  $\mathbf{U}_n \mathbf{A}$  with respect to the scalar product  $\langle \langle \cdot | \cdot \rangle \rangle_{\mathbf{U}_n^{-1}}$ , we have  $\langle \langle \tilde{\mathbf{p}}_n - \mathbf{x} | \tilde{\mathbf{p}}_n - \tilde{\mathbf{y}}_n - \gamma_n \mathbf{U}_n \mathbf{B} \mathbf{x} \rangle \rangle_{\mathbf{U}_n^{-1}} \leq 0$ . Moreover, by monotonicity of  $\mathbf{U}_n \mathbf{B}$  with respect to the scalar product  $\langle \langle \cdot | \cdot \rangle \rangle_{\mathbf{U}_n^{-1}}$ , we also have  $\langle \langle \tilde{\mathbf{p}}_n - \mathbf{x} | \gamma_n \mathbf{U}_n \mathbf{B} \mathbf{x} - \gamma_n \mathbf{U}_n \mathbf{B} \tilde{\mathbf{p}}_n \rangle \rangle_{\mathbf{U}_n^{-1}} \leq 0$ . By adding the last two inequalities, we obtain

$$(\forall n \in \mathbb{N}) \quad \langle \langle \tilde{\mathbf{p}}_n - \mathbf{x} | \tilde{\mathbf{p}}_n - \tilde{\mathbf{y}}_n - \gamma_n \mathbf{U}_n \mathbf{B} \tilde{\mathbf{p}}_n \rangle \rangle_{\mathbf{U}_n^{-1}} \leq 0. \quad (3.9)$$

In turn, we derive from (3.5) that

$$\begin{aligned} (\forall n \in \mathbb{N}) \quad & 2\gamma_n \langle \langle \tilde{\mathbf{p}}_n - \mathbf{x} | \mathbf{U}_n \mathbf{B} \mathbf{x}_n - \mathbf{U}_n \mathbf{B} \tilde{\mathbf{p}}_n \rangle \rangle_{\mathbf{U}_n^{-1}} \\ & = 2 \langle \langle \tilde{\mathbf{p}}_n - \mathbf{x} | \tilde{\mathbf{p}}_n - \tilde{\mathbf{y}}_n - \gamma_n \mathbf{U}_n \mathbf{B} \tilde{\mathbf{p}}_n \rangle \rangle_{\mathbf{U}_n^{-1}} \\ & \quad + 2 \langle \langle \tilde{\mathbf{p}}_n - \mathbf{x} | \gamma_n \mathbf{U}_n \mathbf{B} \mathbf{x}_n + \tilde{\mathbf{y}}_n - \tilde{\mathbf{p}}_n \rangle \rangle_{\mathbf{U}_n^{-1}} \\ & \leq 2 \langle \langle \tilde{\mathbf{p}}_n - \mathbf{x} | \gamma_n \mathbf{U}_n \mathbf{B} \mathbf{x}_n + \tilde{\mathbf{y}}_n - \tilde{\mathbf{p}}_n \rangle \rangle_{\mathbf{U}_n^{-1}} \\ & = 2 \langle \langle \tilde{\mathbf{p}}_n - \mathbf{x} | \mathbf{x}_n - \tilde{\mathbf{p}}_n \rangle \rangle_{\mathbf{U}_n^{-1}} \\ & = \|\mathbf{x}_n - \mathbf{x}\|_{\mathbf{U}_n^{-1}}^2 - \|\tilde{\mathbf{p}}_n - \mathbf{x}\|_{\mathbf{U}_n^{-1}}^2 - \|\mathbf{x}_n - \tilde{\mathbf{p}}_n\|_{\mathbf{U}_n^{-1}}^2. \end{aligned} \quad (3.10)$$

Hence, using (3.5), (3.10), the  $\beta$ -Lipschitz continuity of  $\mathbf{B}$ , (3.1), and [10, Lemma 2.1(ii)], for every  $n \in \mathbb{N}$ , we obtain

$$\begin{aligned}
\|\|\|\tilde{\mathbf{x}}_{n+1} - \mathbf{x}\|\|_{\mathbf{U}_n^{-1}}^2 &= \|\|\|\tilde{\mathbf{q}}_n + \mathbf{x}_n - \tilde{\mathbf{y}}_n - \mathbf{x}\|\|_{\mathbf{U}_n^{-1}}^2 \\
&= \|\|\|(\tilde{\mathbf{p}}_n - \mathbf{x}) + \gamma_n \mathbf{U}_n(\mathbf{B}\mathbf{x}_n - \mathbf{B}\tilde{\mathbf{p}}_n)\|\|_{\mathbf{U}_n^{-1}}^2 \\
&= \|\|\|\tilde{\mathbf{p}}_n - \mathbf{x}\|\|_{\mathbf{U}_n^{-1}}^2 + 2\gamma_n \langle\langle \tilde{\mathbf{p}}_n - \mathbf{x} \mid \mathbf{B}\mathbf{x}_n - \mathbf{B}\tilde{\mathbf{p}}_n \rangle\rangle \\
&\quad + \gamma_n^2 \|\|\|\mathbf{U}_n(\mathbf{B}\mathbf{x}_n - \mathbf{B}\tilde{\mathbf{p}}_n)\|\|_{\mathbf{U}_n^{-1}}^2 \\
&\leq \|\|\|\mathbf{x}_n - \mathbf{x}\|\|_{\mathbf{U}_n^{-1}}^2 - \|\|\|\mathbf{x}_n - \tilde{\mathbf{p}}_n\|\|_{\mathbf{U}_n^{-1}}^2 + \gamma_n^2 \mu \beta^2 \|\|\|\mathbf{x}_n - \tilde{\mathbf{p}}_n\|\|_{\mathbf{U}_n^{-1}}^2 \\
&\leq \|\|\|\mathbf{x}_n - \mathbf{x}\|\|_{\mathbf{U}_n^{-1}}^2 - \mu^{-1} \|\|\|\mathbf{x}_n - \tilde{\mathbf{p}}_n\|\|_{\mathbf{U}_n^{-1}}^2 + \gamma_n^2 \mu \beta^2 \|\|\|\mathbf{x}_n - \tilde{\mathbf{p}}_n\|\|_{\mathbf{U}_n^{-1}}^2. \tag{3.11}
\end{aligned}$$

Hence, it follows from (3.1) and [10, Lemma 2.1(i)] that

$$(\forall n \in \mathbb{N}) \quad \|\|\|\tilde{\mathbf{x}}_{n+1} - \mathbf{x}\|\|_{\mathbf{U}_{n+1}^{-1}}^2 \leq (1 + \eta_n) \|\|\|\mathbf{x}_n - \mathbf{x}\|\|_{\mathbf{U}_n^{-1}}^2 - \mu^{-1} (1 - \gamma_n^2 \beta^2 \mu^2) \|\|\|\mathbf{x}_n - \tilde{\mathbf{p}}_n\|\|_{\mathbf{U}_n^{-1}}^2. \tag{3.12}$$

Consequently,

$$(\forall n \in \mathbb{N}) \quad \|\|\|\tilde{\mathbf{x}}_{n+1} - \mathbf{x}\|\|_{\mathbf{U}_{n+1}^{-1}} \leq (1 + \eta_n) \|\|\|\mathbf{x}_n - \mathbf{x}\|\|_{\mathbf{U}_n^{-1}}. \tag{3.13}$$

For every  $n \in \mathbb{N}$ , set

$$\varepsilon_n = \sqrt{\mu\alpha^{-1}} \left( 2(\|\|\|\mathbf{b}_n\|\|_{\mathbf{U}_n^{-1}} + (\beta\mu)^{-1} \|\|\|\mathbf{a}_n\|\|_{\mathbf{U}_n}) + (\beta\mu)^{-1} \|\|\|\mathbf{c}_n\|\|_{\mathbf{U}_n} + (\beta\mu)^{-1} \|\|\|\mathbf{a}_n\|\|_{\mathbf{U}_n} \right). \tag{3.14}$$

Then  $(\varepsilon_n)_{n \in \mathbb{N}}$  is summable by (3.3) and (3.4). We derive from [10, Lemma 2.1(ii)(iii)], and (3.8) that

$$\begin{aligned}
(\forall n \in \mathbb{N}) \quad \|\|\|\mathbf{e}_n\|\|_{\mathbf{U}_{n+1}^{-1}} &= \|\|\|\tilde{\mathbf{x}}_{n+1} - \mathbf{x}_{n+1}\|\|_{\mathbf{U}_{n+1}^{-1}} \\
&\leq \sqrt{\alpha^{-1}} \|\|\|\tilde{\mathbf{x}}_{n+1} - \mathbf{x}_{n+1}\|\| \\
&\leq \sqrt{\mu\alpha^{-1}} \|\|\|\tilde{\mathbf{x}}_{n+1} - \mathbf{x}_{n+1}\|\|_{\mathbf{U}_n^{-1}} \\
&\leq \sqrt{\mu\alpha^{-1}} (\|\|\|\tilde{\mathbf{y}}_n - \mathbf{y}_n\|\|_{\mathbf{U}_n^{-1}} + \|\|\|\tilde{\mathbf{q}}_n - \mathbf{q}_n\|\|_{\mathbf{U}_n^{-1}}) \\
&\leq \varepsilon_n. \tag{3.15}
\end{aligned}$$

In turn, we derive from (3.13) that

$$\begin{aligned}
(\forall n \in \mathbb{N}) \quad \|\|\|\mathbf{x}_{n+1} - \mathbf{x}\|\|_{\mathbf{U}_{n+1}^{-1}} &\leq \|\|\|\tilde{\mathbf{x}}_{n+1} - \mathbf{x}\|\|_{\mathbf{U}_{n+1}^{-1}} + \|\|\|\tilde{\mathbf{x}}_{n+1} - \mathbf{x}_{n+1}\|\|_{\mathbf{U}_{n+1}^{-1}} \\
&\leq \|\|\|\tilde{\mathbf{x}}_{n+1} - \mathbf{x}\|\|_{\mathbf{U}_{n+1}^{-1}} + \varepsilon_n \\
&\leq (1 + \eta_n) \|\|\|\mathbf{x}_n - \mathbf{x}\|\|_{\mathbf{U}_n^{-1}} + \varepsilon_n. \tag{3.16}
\end{aligned}$$

This shows that  $(\mathbf{x}_n)_{n \in \mathbb{N}}$  is  $|\cdot|$ -quasi-Fejér monotone with respect to the target set  $\text{zer}(\mathbf{A} + \mathbf{B})$  relative to  $(\mathbf{U}_n^{-1})_{n \in \mathbb{N}}$ . Moreover, by [10, Proposition 3.2],  $(\|\|\|\mathbf{x}_n - \mathbf{x}\|\|_{\mathbf{U}_n^{-1}})_{n \in \mathbb{N}}$  is bounded. In turn, since  $\mathbf{B}$  and  $(J_{\gamma_n \mathbf{U}_n \mathbf{A}})_{n \in \mathbb{N}}$  are Lipschitzian, and  $(\forall n \in \mathbb{N}) \mathbf{x} = J_{\gamma_n \mathbf{U}_n \mathbf{A}}(\mathbf{x} - \gamma_n \mathbf{U}_n \mathbf{B} \mathbf{x})$ , we deduce from (3.5) that  $(\tilde{\mathbf{y}}_n)_{n \in \mathbb{N}}$ ,  $(\tilde{\mathbf{p}}_n)_{n \in \mathbb{N}}$ , and  $(\tilde{\mathbf{q}}_n)_{n \in \mathbb{N}}$  are bounded. Therefore,

$$\tau = \sup_{n \in \mathbb{N}} \{ \|\|\|\mathbf{x}_n - \tilde{\mathbf{y}}_n + \tilde{\mathbf{q}}_n - \mathbf{x}\|\|_{\mathbf{U}_n^{-1}}, \|\|\|\mathbf{x}_n - \mathbf{x}\|\|_{\mathbf{U}_n^{-1}}, 1 + \eta_n \} < +\infty. \tag{3.17}$$

Hence, using (3.5), Cauchy-Schwarz for the norms  $(\|\cdot\|_{\mathcal{U}_n^{-1}})_{n \in \mathbb{N}}$ , and (3.11), we get

$$\begin{aligned}
(\forall n \in \mathbb{N}) \quad \|\mathbf{x}_{n+1} - \mathbf{x}\|_{\mathcal{U}_n^{-1}}^2 &= \|\mathbf{x}_n - \mathbf{y}_n + \mathbf{q}_n - \mathbf{x}\|_{\mathcal{U}_n^{-1}}^2 \\
&= \|\tilde{\mathbf{q}}_n + \mathbf{x}_n - \tilde{\mathbf{y}}_n - \mathbf{x} + \mathbf{d}_n\|_{\mathcal{U}_n^{-1}}^2 \\
&\leq \|\tilde{\mathbf{q}}_n + \mathbf{x}_n - \tilde{\mathbf{y}}_n - \mathbf{x}\|_{\mathcal{U}_n^{-1}}^2 + 2\tau \|\mathbf{d}_n\|_{\mathcal{U}_n^{-1}} + \|\mathbf{d}_n\|_{\mathcal{U}_n^{-1}}^2 \\
&\leq \|\mathbf{x}_n - \mathbf{x}\|_{\mathcal{U}_n^{-1}}^2 - \mu^{-1}(1 - \gamma_n^2 \beta^2 \mu^2) \|\mathbf{x}_n - \tilde{\mathbf{p}}_n\|^2 + \varepsilon_{1,n},
\end{aligned} \tag{3.18}$$

where  $(\forall n \in \mathbb{N}) \varepsilon_{1,n} = 2\tau \|\mathbf{d}_n\|_{\mathcal{U}_n^{-1}} + \|\mathbf{d}_n\|_{\mathcal{U}_n^{-1}}^2$ . In turn, for every  $n \in \mathbb{N}$ , by (3.1) and [10, Lemma 2.1(i)],

$$\begin{aligned}
\|\mathbf{x}_{n+1} - \mathbf{x}\|_{\mathcal{U}_{n+1}^{-1}}^2 &\leq (1 + \eta_n) \|\mathbf{x}_{n+1} - \mathbf{x}\|_{\mathcal{U}_n^{-1}}^2 \\
&\leq \|\mathbf{x}_n - \mathbf{x}\|_{\mathcal{U}_n^{-1}}^2 - \mu^{-1}(1 - \gamma_n^2 \beta^2 \mu^2) \|\mathbf{x}_n - \tilde{\mathbf{p}}_n\|^2 + \tau \varepsilon_{1,n} + \tau^2 \eta_n.
\end{aligned} \tag{3.19}$$

Since  $(\tau \varepsilon_{1,n} + \tau^2 \eta_n)_{n \in \mathbb{N}} \in \ell_+^1(\mathbb{N})$  by (3.8), it follows from [7, Lemma 3.1] that

$$\sum_{n \in \mathbb{N}} \|\mathbf{x}_n - \tilde{\mathbf{p}}_n\|^2 < +\infty. \tag{3.20}$$

(i): It follows from (3.20) and (3.8) that

$$\sum_{n \in \mathbb{N}} \|\mathbf{x}_n - \mathbf{p}_n\|^2 \leq 2 \sum_{n \in \mathbb{N}} \|\mathbf{x}_n - \tilde{\mathbf{p}}_n\|^2 + 2 \sum_{n \in \mathbb{N}} \|\mathbf{p}_n - \tilde{\mathbf{p}}_n\|^2 < +\infty. \tag{3.21}$$

Furthermore, we derive from (3.8) and (3.5) that

$$\begin{aligned}
\sum_{n \in \mathbb{N}} \|\mathbf{y}_n - \mathbf{q}_n\|^2 &= \sum_{n \in \mathbb{N}} \|\tilde{\mathbf{q}}_n - \tilde{\mathbf{y}}_n + \mathbf{d}_n\|^2 \\
&= \sum_{n \in \mathbb{N}} \|\tilde{\mathbf{p}}_n - \mathbf{x}_n + \gamma_n \mathbf{U}_n(\mathbf{B}\mathbf{x}_n - \mathbf{B}\tilde{\mathbf{p}}_n) + \mathbf{d}_n\|^2 \\
&\leq 3 \left( \sum_{n \in \mathbb{N}} \|\mathbf{x}_n - \tilde{\mathbf{p}}_n\|^2 + \|\gamma_n \mathbf{U}_n(\mathbf{B}\mathbf{x}_n - \mathbf{B}\tilde{\mathbf{p}}_n)\|^2 + \|\mathbf{d}_n\|^2 \right) \\
&< +\infty.
\end{aligned} \tag{3.22}$$

(ii): Let  $\mathbf{x}$  be a weak cluster point of  $(\mathbf{x}_n)_{n \in \mathbb{N}}$ . Then there exists a subsequence  $(\mathbf{x}_{k_n})_{n \in \mathbb{N}}$  that converges weakly to  $\mathbf{x}$ . Therefore  $\tilde{\mathbf{p}}_{k_n} \rightharpoonup \mathbf{x}$  by (3.20). Furthermore, it follows from (3.5) that  $\mathbf{u}_{k_n} \rightarrow 0$ . Hence, since  $(\forall n \in \mathbb{N}) (\tilde{\mathbf{p}}_{k_n}, \mathbf{u}_{k_n}) \in \text{gra}(\mathbf{A} + \mathbf{B})$ , we obtain,  $\mathbf{x} \in \text{zer}(\mathbf{A} + \mathbf{B})$  [2, Proposition 20.33(ii)]. Altogether, it follows [10, Lemma 2.3(ii)] and [10, Theorem 3.3] that  $\mathbf{x}_n \rightharpoonup \bar{\mathbf{x}}$  and hence that  $\mathbf{p}_n \rightharpoonup \bar{\mathbf{x}}$  by (i).

(iii)(a): Since  $\mathbf{A}$  and  $\mathbf{B}$  are maximally monotone and  $\text{dom } \mathbf{B} = \mathcal{K}$ ,  $\mathbf{A} + \mathbf{B}$  is maximally monotone [2, Corollary 24.4(i)],  $\text{zer}(\mathbf{A} + \mathbf{B})$  is therefore closed [2, Proposition 23.39]. Hence, the claims follow from (i), (3.16), and [10, Proposition 3.4].

(iii)(b): By (i),  $\mathbf{x}_n \rightharpoonup \bar{\mathbf{x}}$ , and hence (3.20) implies that  $\tilde{\mathbf{p}}_n \rightharpoonup \bar{\mathbf{x}}$ . Furthermore, it follows from (3.5) that  $\mathbf{u}_n \rightarrow 0$ . Hence, since  $(\forall n \in \mathbb{N}) (\tilde{\mathbf{p}}_n, \mathbf{u}_n) \in \text{gra}(\mathbf{A} + \mathbf{B})$  and since  $\mathbf{A} + \mathbf{B}$  is demiregular at  $\bar{\mathbf{x}}$ , by [1, Definition 2.3],  $\tilde{\mathbf{p}}_n \rightarrow \bar{\mathbf{x}}$ , and therefore (3.20) implies that  $\mathbf{x}_n \rightarrow \bar{\mathbf{x}}$ .

(iii)(c): If  $\mathbf{A}$  or  $\mathbf{B}$  is uniformly monotone at  $\bar{\mathbf{x}}$ , then  $\mathbf{A} + \mathbf{B}$  is uniformly monotone at  $\bar{\mathbf{x}}$ . Therefore, the result follows from [1, Proposition 2.4(i)].

(iii)(d): Suppose that  $\mathbf{z} \in \text{int zer}(\mathbf{A} + \mathbf{B})$  and fix  $\rho \in ]0, +\infty[$  such that  $B(\mathbf{z}; \rho) \subset \text{zer}(\mathbf{A} + \mathbf{B})$ . It follows from (3.16) and [10, Proposition 3.2] that

$$\varepsilon = \sup_{\mathbf{x} \in B(\mathbf{z}; \rho)} \sup_{n \in \mathbb{N}} \|\|\|\mathbf{x}_n - \mathbf{x}\|\|\|_{\mathbf{U}_n^{-1}} \leq (1/\sqrt{\alpha}) \left( \sup_{n \in \mathbb{N}} \|\|\|\mathbf{x}_n - \mathbf{z}\|\|\| + \sup_{\mathbf{x} \in B(\mathbf{z}; \rho)} \|\|\|\mathbf{x} - \mathbf{z}\|\|\| \right) < +\infty \quad (3.23)$$

and from (3.16) that

$$(\forall n \in \mathbb{N})(\forall \mathbf{x} \in B(\mathbf{z}; \rho)) \|\|\|\mathbf{x}_{n+1} - \mathbf{x}\|\|\|_{\mathbf{U}_{n+1}^{-1}}^2 \leq \|\|\|\mathbf{x}_n - \mathbf{x}\|\|\|_{\mathbf{U}_n^{-1}}^2 + 2\varepsilon(\varepsilon\eta_n + \varepsilon_n) + (\varepsilon\eta_n + \varepsilon_n)^2. \quad (3.24)$$

Hence, the claim follows from (i), [10, Lemma 2.1], and [10, Proposition 4.3].  $\square$

**Remark 3.2** Here are some remarks.

- (i) In the case when  $(\forall n \in \mathbb{N}) \mathbf{U}_n = \mathbf{Id}$ , the standard forward-backward-forward splitting algorithm (3.2) reduces to algorithm proposed in [4, Eq. (2.3)], which was proposed initially in the error-free setting in [19].
- (ii) An alternative variable metric splitting algorithm proposed in [14] can be used to find a zero of the sum of a maximally monotone operator  $\mathbf{A}$  and a Lipschitzian monotone operator  $\mathbf{B}$  in instance when  $\mathcal{K}$  is finite-dimensional. This algorithm uses a different error model and involves more iteration-dependent variables than (3.2).

**Example 3.3** Let  $\mathbf{f}: \mathcal{K} \rightarrow [-\infty, +\infty]$  be a proper lower semicontinuous convex function, let  $\alpha \in ]0, +\infty[$ , let  $\beta \in ]0, +\infty[$ , let  $\mathbf{B}: \mathcal{K} \rightarrow \mathcal{K}$  be a monotone and  $\beta$ -Lipschitzian operator, let  $(\eta_n)_{n \in \mathbb{N}} \in \ell_+^1(\mathbb{N})$ , and let  $(\mathbf{U}_n)_{n \in \mathbb{N}}$  be a sequence in  $\mathcal{P}_\alpha(\mathcal{K})$  that satisfies (3.1). Furthermore, let  $\mathbf{x}_0 \in \mathcal{K}$ , let  $\varepsilon \in ]0, \min\{1, 1/(\mu\beta + 1)\}[$ , where  $\mu$  is defined as in (3.1), let  $(\gamma_n)_{n \in \mathbb{N}}$  be a sequence in  $[\varepsilon, (1 - \varepsilon)/(\beta\mu)]$ . Suppose that the variational inequality

$$\text{find } \bar{\mathbf{x}} \in \mathcal{K} \quad \text{such that} \quad (\forall \mathbf{y} \in \mathcal{K}) \quad \langle \bar{\mathbf{x}} - \mathbf{y} \mid \mathbf{B}\bar{\mathbf{x}} \rangle + \mathbf{f}(\bar{\mathbf{x}}) \leq \mathbf{f}(\mathbf{y}) \quad (3.25)$$

admits at least one solution and set

$$(\forall n \in \mathbb{N}) \quad \begin{cases} \mathbf{y}_n = \mathbf{x}_n - \gamma_n \mathbf{U}_n \mathbf{B} \mathbf{x}_n \\ \mathbf{p}_n = \arg \min_{\mathbf{x} \in \mathcal{K}} \left( \mathbf{f}(\mathbf{x}) + \frac{1}{2\gamma_n} \|\|\|\mathbf{x} - \mathbf{y}_n\|\|\|_{\mathbf{U}_n^{-1}}^2 \right) \\ \mathbf{q}_n = \mathbf{p}_n - \gamma_n \mathbf{U}_n \mathbf{B} \mathbf{p}_n \\ \mathbf{x}_{n+1} = \mathbf{x}_n - \mathbf{y}_n + \mathbf{q}_n. \end{cases} \quad (3.26)$$

Then  $(\mathbf{x}_n)_{n \in \mathbb{N}}$  converges weakly to a solution  $\bar{\mathbf{x}}$  to (3.25).

*Proof.* Set  $\mathbf{A} = \partial \mathbf{f}$  and  $(\forall n \in \mathbb{N}) \mathbf{a}_n = 0, \mathbf{b}_n = 0, \mathbf{c}_n = 0$  in Theorem 3.1(ii).  $\square$

## 4 Monotone inclusions involving Lipschitzian operators

The applications of the forward-backward-forward splitting algorithm considered in [4, 9, 19] can be extended to a variable metric setting using Theorem 3.1. As an illustration, we present a variable metric version of the algorithm proposed in [9, Eq. (3.1)]. Recall that the parallel sum of  $A: \mathcal{H} \rightarrow 2^{\mathcal{H}}$  and  $B: \mathcal{H} \rightarrow 2^{\mathcal{H}}$  is [2]

$$A \square B = (A^{-1} + B^{-1})^{-1}. \quad (4.1)$$

**Problem 4.1** Let  $\mathcal{H}$  be a real Hilbert space, let  $m$  be a strictly positive integer, let  $z \in \mathcal{H}$ , let  $A: \mathcal{H} \rightarrow 2^{\mathcal{H}}$  be maximally monotone operator, let  $C: \mathcal{H} \rightarrow \mathcal{H}$  be monotone and  $\nu_0$ -Lipschitzian for some  $\nu_0 \in ]0, +\infty[$ . For every  $i \in \{1, \dots, m\}$ , let  $\mathcal{G}_i$  be a real Hilbert space, let  $r_i \in \mathcal{G}_i$ , let  $B_i: \mathcal{G}_i \rightarrow 2^{\mathcal{G}_i}$  be maximally monotone operator, let  $D_i: \mathcal{G}_i \rightarrow 2^{\mathcal{G}_i}$  be monotone and such that  $D_i^{-1}$  is  $\nu_i$ -Lipschitzian for some  $\nu_i \in ]0, +\infty[$ , and let  $L_i: \mathcal{H} \rightarrow \mathcal{G}_i$  is a nonzero bounded linear operator. Suppose that

$$z \in \text{ran} \left( A + \sum_{i=1}^m L_i^* ((B_i \square D_i)(L_i \cdot -r_i)) + C \right). \quad (4.2)$$

The problem is to solve the primal inclusion

$$\text{find } \bar{x} \in \mathcal{H} \text{ such that } z \in A\bar{x} + \sum_{i=1}^m L_i^* ((B_i \square D_i)(L_i \bar{x} - r_i)) + C\bar{x}, \quad (4.3)$$

and the dual inclusion

$$\text{find } \bar{v}_1 \in \mathcal{G}_1, \dots, \bar{v}_m \in \mathcal{G}_m \text{ such that } (\exists x \in \mathcal{H}) \begin{cases} z - \sum_{i=1}^m L_i^* \bar{v}_i \in Ax + Cx, \\ (\forall i \in \{1, \dots, m\}) \bar{v}_i \in (B_i \square D_i)(L_i x - r_i). \end{cases} \quad (4.4)$$

As shown in [9], Problem 4.1 covers a wide class of problems in nonlinear analysis and convex optimization problems. However, the algorithm in [9, Theorem 3.1] is studied in the context of a fixed metric. The following result extends this result to a variable metric setting.

**Corollary 4.2** *Let  $\alpha$  be in  $]0, +\infty[$ , let  $(\eta_{0,n})_{n \in \mathbb{N}}$  be a sequence in  $\ell_+^1(\mathbb{N})$ , let  $(U_n)_{n \in \mathbb{N}}$  be a sequence in  $\mathcal{P}_\alpha(\mathcal{H})$ , and for every  $i \in \{1, \dots, m\}$ , let  $(\eta_{i,n})_{n \in \mathbb{N}}$  be a sequence in  $\ell_+^1(\mathbb{N})$ , let  $(U_{i,n})_{n \in \mathbb{N}}$  be a sequence in  $\mathcal{P}_\alpha(\mathcal{G}_i)$  such that  $\mu = \sup_{n \in \mathbb{N}} \{\|U_n\|, \|U_{1,n}\|, \dots, \|U_{m,n}\|\} < +\infty$  and*

$$(\forall n \in \mathbb{N}) \quad (1 + \eta_{0,n})U_{n+1} \succcurlyeq U_n, \quad \text{and} \quad (\forall i \in \{1, \dots, m\}) \quad (1 + \eta_{i,n})U_{i,n+1} \succcurlyeq U_{i,n}. \quad (4.5)$$

*Let  $(a_{1,n})_{n \in \mathbb{N}}, (b_{1,n})_{n \in \mathbb{N}}$ , and  $(c_{1,n})_{n \in \mathbb{N}}$  be absolutely summable sequences in  $\mathcal{H}$ , and for every  $i \in \{1, \dots, m\}$ , let  $(a_{2,i,n})_{n \in \mathbb{N}}, (b_{2,i,n})_{n \in \mathbb{N}}$ , and  $(c_{2,i,n})_{n \in \mathbb{N}}$  be absolutely summable sequences in  $\mathcal{G}_i$ . Furthermore, set*

$$\beta = \max\{\nu_0, \nu_1, \dots, \nu_m\} + \sqrt{\sum_{i=1}^m \|L_i\|^2}, \quad (4.6)$$



let  $x_0 \in \mathcal{H}$ , let  $(v_{1,0}, \dots, v_{m,0}) \in \mathcal{G}_1 \oplus \dots \oplus \mathcal{G}_m$ , let  $\varepsilon \in ]0, 1/(1 + \beta\mu)[$ , let  $(\gamma_n)_{n \in \mathbb{N}}$  be a sequence in  $[\varepsilon, (1 - \varepsilon)/(\beta\mu)]$ . Set

$$(\forall n \in \mathbb{N}) \quad \begin{cases} y_{1,n} = x_n - \gamma_n U_n (C x_n + \sum_{i=1}^m L_i^* v_{i,n} + a_{1,n}) \\ p_{1,n} = J_{\gamma_n U_n A} (y_{1,n} + \gamma_n U_n z) + b_{1,n} \\ \text{for } i = 1, \dots, m \\ \quad \begin{cases} y_{2,i,n} = v_{i,n} + \gamma_n U_{i,n} (L_i x_n - D_i^{-1} v_{i,n} + a_{2,i,n}) \\ p_{2,i,n} = J_{\gamma_n U_{i,n} B_i^{-1}} (y_{2,i,n} - \gamma_n U_{i,n} r_i) + b_{2,i,n} \\ q_{2,i,n} = p_{2,i,n} + \gamma_n U_{i,n} (L_i p_{1,n} - D_i^{-1} p_{2,i,n} + c_{2,i,n}) \\ v_{i,n+1} = v_{i,n} - y_{2,i,n} + q_{2,i,n} \end{cases} \\ q_{1,n} = p_{1,n} - \gamma_n U_n (C p_{1,n} + \sum_{i=1}^m L_i^* p_{2,i,n} + c_{1,n}) \\ x_{n+1} = x_n - y_{1,n} + q_{1,n}. \end{cases} \quad (4.7)$$

Then the following hold.

- (i)  $\sum_{n \in \mathbb{N}} \|x_n - p_{1,n}\|^2 < +\infty$  and  $(\forall i \in \{1, \dots, m\}) \sum_{n \in \mathbb{N}} \|v_{i,n} - p_{2,i,n}\|^2 < +\infty$ .
- (ii) There exist a solution  $\bar{x}$  to (4.3) and a solution  $(\bar{v}_1, \dots, \bar{v}_m)$  to (4.4) such that the following hold.
  - (a)  $x_n \rightarrow \bar{x}$  and  $p_{1,n} \rightarrow \bar{x}$ .
  - (b)  $(\forall i \in \{1, \dots, m\}) v_{i,n} \rightarrow \bar{v}_i$  and  $p_{2,i,n} \rightarrow \bar{v}_i$ .
  - (c) Suppose that  $A$  or  $C$  is uniformly monotone at  $\bar{x}$ , then  $x_n \rightarrow \bar{x}$  and  $p_{1,n} \rightarrow \bar{x}$ .
  - (d) Suppose that  $B_j^{-1}$  or  $D_j^{-1}$  is uniformly monotone at  $\bar{v}_j$ , for some  $j \in \{1, \dots, m\}$ , then  $v_{j,n} \rightarrow \bar{v}_j$  and  $p_{2,j,n} \rightarrow \bar{v}_j$ .

*Proof.* All sequences generated by algorithm (4.7) are well defined by [11, Lemma 3.7]. We define  $\mathcal{K} = \mathcal{H} \oplus \mathcal{G}_1 \oplus \dots \oplus \mathcal{G}_m$  the Hilbert direct sum of the Hilbert spaces  $\mathcal{H}$  and  $(\mathcal{G}_i)_{1 \leq i \leq m}$ , the scalar product and the associated norm of  $\mathcal{K}$  respectively defined by

$$\langle \langle \cdot | \cdot \rangle \rangle : ((x, \mathbf{v}), (y, \mathbf{w})) \mapsto \langle x | y \rangle + \sum_{i=1}^m \langle v_i | w_i \rangle \quad \text{and} \quad ||| \cdot ||| : (x, \mathbf{v}) \mapsto \sqrt{\|x\|^2 + \sum_{i=1}^m \|v_i\|^2}, \quad (4.8)$$

where  $\mathbf{v} = (v_1, \dots, v_m)$  and  $\mathbf{w} = (w_1, \dots, w_m)$  are generic elements in  $\mathcal{G}_1 \oplus \dots \oplus \mathcal{G}_m$ . Set

$$\begin{cases} \mathbf{A} : \mathcal{K} \rightarrow 2^{\mathcal{K}} : (x, v_1, \dots, v_m) \mapsto (-z + Ax) \times (r_1 + B_1^{-1} v_1) \times \dots \times (r_m + B_m^{-1} v_m) \\ \mathbf{B} : \mathcal{K} \rightarrow \mathcal{K} : (x, v_1, \dots, v_m) \mapsto \left( Cx + \sum_{i=1}^m L_i^* v_i, D_1^{-1} v_1 - L_1 x, \dots, D_m^{-1} v_m - L_m x \right) \\ (\forall n \in \mathbb{N}) \quad \mathbf{U}_n : \mathcal{K} \rightarrow \mathcal{K} : (x, v_1, \dots, v_m) \mapsto (U_n x, U_{1,n} v_1, \dots, U_{m,n} v_m). \end{cases} \quad (4.9)$$

Since  $\mathbf{A}$  is maximally monotone [2, Propositions 20.22 and 20.23],  $\mathbf{B}$  is monotone and  $\beta$ -Lipschitzian [9, Eq. (3.10)] with  $\text{dom } \mathbf{B} = \mathcal{K}$ ,  $\mathbf{A} + \mathbf{B}$  is maximally monotone [2, Corollary 24.24(i)]. Now set  $(\forall n \in \mathbb{N}) \eta_n = \max\{\eta_{0,n}, \eta_{1,n}, \dots, \eta_{m,n}\}$ . Then  $(\eta_n)_{n \in \mathbb{N}} \in \ell_+^1(\mathbb{N})$ . Moreover, we derive from our assumptions on the sequences  $(U_n)_{n \in \mathbb{N}}$  and  $(U_{1,n})_{n \in \mathbb{N}}, \dots, (U_{m,n})_{n \in \mathbb{N}}$  that

$$\mu = \sup_{n \in \mathbb{N}} \|U_n\| < +\infty \quad \text{and} \quad (1 + \eta_n) U_{n+1} \succcurlyeq U_n \in \mathcal{P}_\alpha(\mathcal{K}). \quad (4.10)$$

In addition, [2, Propositions 23.15(ii) and 23.16] yield  $(\forall \gamma \in ]0, +\infty[)(\forall n \in \mathbb{N})(\forall (x, v_1, \dots, v_m) \in \mathcal{K})$

$$J_{\gamma U_n \mathbf{A}}(x, v_1, \dots, v_m) = \left( J_{\gamma U_n \mathbf{A}}(x + \gamma U_n z), (J_{\gamma U_{i,n} B_i^{-1}}(v_i - \gamma U_{i,n} r_i))_{1 \leq i \leq m} \right). \quad (4.11)$$

It is shown in [9, Eq. (3.12)] and [9, Eq. (3.13)] that under the condition (4.2),  $\text{zer}(\mathbf{A} + \mathbf{B}) \neq \emptyset$ . Moreover, [9, Eq. (3.21)] and [9, Eq. (3.22)] yield

$$(\bar{x}, \bar{v}_1, \dots, \bar{v}_m) \in \text{zer}(\mathbf{A} + \mathbf{B}) \Rightarrow \bar{x} \text{ solves (4.3) and } (\bar{v}_1, \dots, \bar{v}_m) \text{ solves (4.4)}. \quad (4.12)$$

Let us next set

$$(\forall n \in \mathbb{N}) \quad \begin{cases} \mathbf{x}_n = (x_n, v_{1,n}, \dots, v_{m,n}) \\ \mathbf{y}_n = (y_{1,n}, y_{2,1,n}, \dots, y_{2,m,n}) \\ \mathbf{p}_n = (p_{1,n}, p_{2,1,n}, \dots, p_{2,m,n}) \\ \mathbf{q}_n = (q_{1,n}, q_{2,1,n}, \dots, q_{2,m,n}) \end{cases} \quad \text{and} \quad \begin{cases} \mathbf{a}_n = (a_{1,n}, a_{2,1,n}, \dots, a_{2,m,n}) \\ \mathbf{b}_n = (b_{1,n}, b_{2,1,n}, \dots, b_{2,m,n}) \\ \mathbf{c}_n = (c_{1,n}, c_{2,1,n}, \dots, c_{2,m,n}). \end{cases} \quad (4.13)$$

Then our assumptions imply that

$$\sum_{n \in \mathbb{N}} \|\mathbf{a}_n\| < \infty, \quad \sum_{n \in \mathbb{N}} \|\mathbf{b}_n\| < \infty, \quad \text{and} \quad \sum_{n \in \mathbb{N}} \|\mathbf{c}_n\| < \infty. \quad (4.14)$$

Furthermore, it follows from the definition of  $\mathbf{B}$ , (4.11), and (4.13) that (4.7) can be rewritten in  $\mathcal{K}$  as

$$(\forall n \in \mathbb{N}) \quad \begin{cases} \mathbf{y}_n = \mathbf{x}_n - \gamma_n \mathbf{U}_n (\mathbf{B} \mathbf{x}_n + \mathbf{a}_n) \\ \mathbf{p}_n = J_{\gamma_n \mathbf{U}_n \mathbf{A}} \mathbf{y}_n + \mathbf{b}_n \\ \mathbf{q}_n = \mathbf{p}_n - \gamma_n \mathbf{U}_n (\mathbf{B} \mathbf{p}_n + \mathbf{c}_n) \\ \mathbf{x}_{n+1} = \mathbf{x}_n - \mathbf{y}_n + \mathbf{q}_n, \end{cases} \quad (4.15)$$

which is (3.2). Moreover, every specific conditions in Theorem 3.1 are satisfied.

(i): By Theorem 3.1(i),  $\sum_{n \in \mathbb{N}} \|\mathbf{x}_n - \mathbf{p}_n\|^2 < \infty$ .

(ii)(a)&(ii)(b): These assertions follow from Theorem 3.1(ii).

(ii)(c): Theorem 3.1(ii) shows that  $(\bar{x}, \bar{v}_1, \dots, \bar{v}_m) \in \text{zer}(\mathbf{A} + \mathbf{B})$ . Hence, it follows from [9, Eq (3.19)] that  $(\bar{x}, \bar{v}_1, \dots, \bar{v}_m)$  satisfies the inclusions

$$\begin{cases} -\sum_{i=1}^m L_i^* \bar{v}_i - C \bar{x} \in -z + A \bar{x} \\ (\forall i \in \{1, \dots, m\}) L_i \bar{x} - D_i^{-1} \bar{v}_i \in r_i + B_i^{-1} \bar{v}_i. \end{cases} \quad (4.16)$$

For every  $n \in \mathbb{N}$  and every  $i \in \{1, \dots, m\}$ , set

$$\begin{cases} \tilde{y}_{1,n} = x_n - \gamma_n U_n (C x_n + \sum_{i=1}^m L_i^* v_{i,n}) \\ \tilde{p}_{1,n} = J_{\gamma_n U_n \mathbf{A}}(\tilde{y}_{1,n} + \gamma_n U_n z) \end{cases} \quad \text{and} \quad \begin{cases} \tilde{y}_{2,i,n} = v_{i,n} + \gamma_n U_{i,n} (L_i x_n - D_i^{-1} v_{i,n}) \\ \tilde{p}_{2,i,n} = J_{\gamma_n U_{i,n} B_i^{-1}}(\tilde{y}_{2,i,n} - \gamma_n U_{i,n} r_i). \end{cases} \quad (4.17)$$

Then, using [11, Lemma 3.7], we get

$$\tilde{p}_{1,n} - p_{1,n} \rightarrow 0 \quad \text{and} \quad (\forall i \in \{1, \dots, m\}) \quad \tilde{p}_{2,i,n} - p_{2,i,n} \rightarrow 0, \quad (4.18)$$

in turn, by (i),(ii)(a), and (ii)(b), we obtain

$$\tilde{p}_{1,n} - x_n \rightarrow 0, \quad \tilde{p}_{1,n} \rightarrow \bar{x}, \quad \text{and} \quad (\forall i \in \{1, \dots, m\}) \quad \tilde{p}_{2,i,n} - v_{i,n} \rightarrow 0, \quad \tilde{p}_{2,i,n} \rightarrow \bar{v}_i. \quad (4.19)$$

Furthermore, we derive from (4.17) that

$$(\forall n \in \mathbb{N}) \quad \begin{cases} \gamma_n^{-1} U_n^{-1}(x_n - \tilde{p}_{1,n}) - \sum_{i=1}^m L_i^* v_{i,n} - Cx_n \in -z + A\tilde{p}_{1,n} \\ (\forall i \in \{1, \dots, m\}) \quad \gamma_n^{-1} U_{i,n}^{-1}(v_{i,n} - \tilde{p}_{2,i,n}) + L_i x_n - D_i^{-1} v_{i,n} \in r_i + B_i^{-1} \tilde{p}_{2,i,n}. \end{cases} \quad (4.20)$$

Since  $A$  is uniformly monotone at  $\bar{x}$ , using (4.16) and (4.20), there exists an increasing function  $\phi_A: [0, +\infty[ \rightarrow [0, +\infty[$  vanishing only at 0 such that, for every  $n \in \mathbb{N}$ ,

$$\begin{aligned} \phi_A(\|\tilde{p}_{1,n} - \bar{x}\|) &\leq \left\langle \tilde{p}_{1,n} - \bar{x} \mid \gamma_n^{-1} U_n^{-1}(x_n - \tilde{p}_{1,n}) - \sum_{i=1}^m (L_i^* v_{i,n} - L_i^* \bar{v}_i) - (Cx_n - C\bar{x}) \right\rangle \\ &= \langle \tilde{p}_{1,n} - \bar{x} \mid \gamma_n^{-1} U_n^{-1}(x_n - \tilde{p}_{1,n}) \rangle - \sum_{i=1}^m \langle \tilde{p}_{1,n} - \bar{x} \mid L_i^* v_{i,n} - L_i^* \bar{v}_i \rangle - \chi_n, \end{aligned} \quad (4.21)$$

where we denote  $(\forall n \in \mathbb{N}) \quad \chi_n = \langle \tilde{p}_{1,n} - \bar{x} \mid Cx_n - C\bar{x} \rangle$ . Since  $(B_i^{-1})_{1 \leq i \leq m}$  are monotone, for every  $i \in \{1, \dots, m\}$ , we obtain

$$\begin{aligned} (\forall n \in \mathbb{N}) \quad 0 &\leq \left\langle \tilde{p}_{2,i,n} - \bar{v}_i \mid L_i x_n + \gamma_n^{-1} U_{i,n}^{-1}(v_{i,n} - \tilde{p}_{2,i,n}) - L_i \bar{x} - (D_i^{-1} v_{i,n} - D_i^{-1} \bar{v}_i) \right\rangle \\ &= \left\langle \tilde{p}_{2,i,n} - \bar{v}_i \mid L_i(x_n - \bar{x}) + \gamma_n^{-1} U_{i,n}^{-1}(v_{i,n} - \tilde{p}_{2,i,n}) \right\rangle - \beta_{i,n}, \end{aligned} \quad (4.22)$$

where  $(\forall n \in \mathbb{N}) \quad \beta_{i,n} = \langle \tilde{p}_{2,i,n} - \bar{v}_i \mid D_i^{-1} v_{i,n} - D_i^{-1} \bar{v}_i \rangle$ . Now, adding (4.22) from  $i = 1$  to  $i = m$  and (4.21), we obtain, for every  $n \in \mathbb{N}$ ,

$$\begin{aligned} \phi_A(\|\tilde{p}_{1,n} - \bar{x}\|) &\leq \langle \tilde{p}_{1,n} - \bar{x} \mid \gamma_n^{-1} U_n^{-1}(x_n - \tilde{p}_{1,n}) \rangle + \left\langle \tilde{p}_{1,n} - \bar{x} \mid \sum_{i=1}^m L_i^* (\tilde{p}_{2,i,n} - v_{i,n}) \right\rangle \\ &\quad + \sum_{i=1}^m \left\langle \tilde{p}_{2,i,n} - \bar{v}_i \mid L_i(x_n - \tilde{p}_{1,n}) + \gamma_n^{-1} U_{i,n}^{-1}(v_{i,n} - \tilde{p}_{2,i,n}) \right\rangle - \chi_n - \sum_{i=1}^m \beta_{i,n}. \end{aligned} \quad (4.23)$$

For every  $n \in \mathbb{N}$  and every  $i \in \{1, \dots, m\}$ , we expand  $\chi_n$  and  $\beta_{i,n}$  as

$$\begin{cases} \chi_n = \langle x_n - \bar{x} \mid Cx_n - C\bar{x} \rangle + \langle \tilde{p}_{1,n} - x_n \mid Cx_n - C\bar{x} \rangle, \\ \beta_{i,n} = \langle v_{i,n} - \bar{v}_i \mid D_i^{-1} v_{i,n} - D_i^{-1} \bar{v}_i \rangle + \langle \tilde{p}_{2,i,n} - v_{i,n} \mid D_i^{-1} v_{i,n} - D_i^{-1} \bar{v}_i \rangle. \end{cases} \quad (4.24)$$

By monotonicity of  $C$  and  $(D_i^{-1})_{1 \leq i \leq m}$ ,

$$(\forall n \in \mathbb{N}) \quad \begin{cases} \langle x_n - \bar{x} \mid Cx_n - C\bar{x} \rangle \geq 0, \\ (\forall i \in \{1, \dots, m\}) \quad \langle v_{i,n} - \bar{v}_i \mid D_i^{-1} v_{i,n} - D_i^{-1} \bar{v}_i \rangle \geq 0. \end{cases} \quad (4.25)$$

Therefore, for every  $n \in \mathbb{N}$ , we derive from (4.24) and (4.23) that

$$\begin{aligned}
\phi_A(\|\tilde{p}_{1,n} - \bar{x}\|) &\leq \phi_A(\|\tilde{p}_{1,n} - \bar{x}\|) + \langle x_n - \bar{x} \mid Cx_n - C\bar{x} \rangle + \sum_{i=1}^m \langle v_{i,n} - \bar{v}_i \mid D_i^{-1}v_{i,n} - D_i^{-1}\bar{v}_i \rangle \\
&\leq \langle \tilde{p}_{1,n} - \bar{x} \mid \gamma_n^{-1}U_n^{-1}(x_n - \tilde{p}_{1,n}) \rangle + \left\langle \tilde{p}_{1,n} - \bar{x} \mid \sum_{i=1}^m L_i^*(\tilde{p}_{2,i,n} - v_{i,n}) \right\rangle \\
&\quad + \sum_{i=1}^m \left\langle \tilde{p}_{2,i,n} - \bar{v}_i \mid L_i(x_n - \tilde{p}_{1,n}) + \gamma_n^{-1}U_{i,n}^{-1}(v_{i,n} - \tilde{p}_{2,i,n}) \right\rangle \\
&\quad - \langle \tilde{p}_{1,n} - x_n \mid Cx_n - C\bar{x} \rangle - \sum_{i=1}^m \langle \tilde{p}_{2,i,n} - v_{i,n} \mid D_i^{-1}v_{i,n} - D_i^{-1}\bar{v}_i \rangle. \tag{4.26}
\end{aligned}$$

We set

$$\zeta = \max_{1 \leq i \leq m} \sup_{n \in \mathbb{N}} \{ \|x_n - \bar{x}\|, \|\tilde{p}_{1,n} - \bar{x}\|, \|v_{i,n} - \bar{v}_i\|, \|\tilde{p}_{2,i,n} - \bar{v}_i\| \}. \tag{4.27}$$

Then it follows from (ii)(a), (ii)(b), and (4.19) that  $\zeta < \infty$ , and from [10, Lemma 2.1(ii)] that  $(\forall n \in \mathbb{N}) \|\gamma_n^{-1}U_n^{-1}\| \leq (\varepsilon\alpha)^{-1}$  and  $(\forall i \in \{1, \dots, m\}) \|\gamma_n^{-1}U_{i,n}^{-1}\| \leq (\varepsilon\alpha)^{-1}$ . Therefore, using the Cauchy-Schwarz inequality, and the Lipschitzianity of  $C$  and  $(D_i^{-1})_{1 \leq i \leq m}$ , we derive from (4.26) that

$$\begin{aligned}
\phi_A(\|\tilde{p}_{1,n} - \bar{x}\|) &\leq (\varepsilon\alpha)^{-1}\zeta\|x_n - \tilde{p}_{1,n}\| + \zeta \sum_{i=1}^m (\|L_i\| \|x_n - \tilde{p}_{1,n}\| + (\varepsilon\alpha)^{-1}\|v_{i,n} - \tilde{p}_{2,i,n}\|) \\
&\quad + \zeta \left( \sum_{i=1}^m \|L_i^*\| \|\tilde{p}_{2,i,n} - v_{i,n}\| + \nu_0 \|\tilde{p}_{1,n} - x_n\| + \sum_{i=1}^m \nu_i \|\tilde{p}_{2,i,n} - v_{i,n}\| \right) \\
&\rightarrow 0. \tag{4.28}
\end{aligned}$$

We deduce from (4.28) and (4.19) that  $\phi_A(\|\tilde{p}_{1,n} - \bar{x}\|) \rightarrow 0$ , which implies that  $\tilde{p}_{1,n} \rightarrow \bar{x}$ . In turn,  $x_n \rightarrow \bar{x}$  and  $p_n \rightarrow \bar{x}$ . Likewise, if  $C$  is uniformly monotone at  $\bar{x}$ , there exists an increasing function  $\phi_C: [0, +\infty[ \rightarrow [0, +\infty[$  that vanishes only at 0 such that

$$\begin{aligned}
\phi_C(\|x_n - \bar{x}\|) &\leq (\varepsilon\alpha)^{-1}\zeta\|x_n - \tilde{p}_{1,n}\| + \zeta \sum_{i=1}^m (\|L_i\| \|x_n - \tilde{p}_{1,n}\| + (\varepsilon\alpha)^{-1}\|v_{i,n} - \tilde{p}_{2,i,n}\|) \\
&\quad + \zeta \left( \sum_{i=1}^m \|L_i^*\| \|\tilde{p}_{2,i,n} - v_{i,n}\| + \nu_0 \|\tilde{p}_{1,n} - x_n\| + \sum_{i=1}^m \nu_i \|\tilde{p}_{2,i,n} - v_{i,n}\| \right) \\
&\rightarrow 0, \tag{4.29}
\end{aligned}$$

in turn,  $x_n \rightarrow \bar{x}$  and  $p_n \rightarrow \bar{x}$ .

(ii)(d): Proceeding as in the proof of (ii)(c), we obtain the conclusions.  $\square$

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