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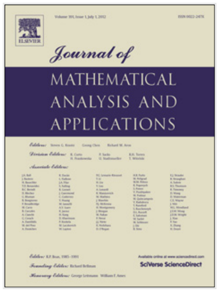

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Global Growth of Bandlimited Local Approximations

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Abstract

We obtain explicit estimates showing how the global norm of a band-limited function blows up, as it locally approximates a function with a jump in a derivative. As an application, we obtain bounds on how well a function, with a jump in a derivative, can be essentially time- and band-limited.

Key words: Fourier transform, Bandlimited approximation, Uncertainty Principle, Mean-value inequality, Legendre Polynomials, Approximate Concentration

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1 Introduction

Let U and V be closed vector subspaces of a Hilbert space $\mathcal{H}$ with respective orthogonal projections P_U and P_V. Existence of solutions $g \in \mathcal{H}$ to the system

$$P_U g = u, \quad P_V g = v \tag{1.1}$$

has been well studied, for example, in [7, p. 88] and [4]. It is shown in [4] that the condition $U \cap V = \{0\}$ is equivalent to the existence of approximate solutions to (1.1) given any pair $(u, v) \in U \times V$; i.e., there exists a sequence $\{g_k\}_{k=1}^{\infty}$ in $\mathcal{H}$ such that

$$||P_U g_k - u|| + ||P_V g_k - v|| \longrightarrow 0. \tag{1.2}$$

Suppose that for some particular $(u, v) \in U \times V$, there exists no $g \in \mathcal{H}$ that satisfies (1.1) but there exists a sequence $\{g_k\}_{k=1}^{\infty}$ satisfying (1.2). A compactness argument shows that $||g_k|| \to \infty$. This raises the general question of finding explicit estimates describing how $||g_k||$ tends to infinity in relation to how fast $||P_U g_k - u|| + ||P_V g_k - v||$ tends to zero.

In this note, we shall consider the important special case with $\mathcal{H} = L^2(\mathbb{R})$,

$$U = \{u \in \mathcal{H} : \operatorname{supp} u \subset I\} \quad \text{and} \quad V = \{v \in \mathcal{H} : \operatorname{supp} \widehat{v} \subset \mathbb{R} \setminus [-\Omega, \Omega]\},$$

where $\Omega > 0$ is fixed, I is a given compact interval, and $\widehat{v}$ denotes the Fourier transform of v. Since $U \cap V = \{0\}$, approximate solutions to (1.1) always exist.

Now, fix $u \in U$ and $\rho > 0$. Classical arguments show there exists $g \in L^2(\mathbb{R})$ such that $||P_U g - u|| < \rho$ and $P_V g = 0$; i.e.,

$$||u - g||_{L^2(I)} < \rho \quad \text{and} \quad \operatorname{supp} \widehat{g} \subset [-\Omega, \Omega].$$

Suppose $a \in I$ and $\delta > 0$ such that $]a - \delta, a + \delta[\subset I$ and u is n times differentiable on both $]a - \delta, a[$ and on $]a, a + \delta[$. Our main result states that, under reasonable conditions,

$$C_n \, \Omega^{n+\frac{1}{2}} ||u - g||^p_{L^2(I)} ||g||_{L^2(\mathbb{R})} \;\geq\; \Delta_n(u) \tag{1.3}$$

where $\Delta_n(u) := \inf\{|u^{(n)}(t^+) - u^{(n)}(t^-)| : \; t^- \in \,]a - \delta, a[\;, \quad t^+ \in \,]a, a + \delta[\;\}$, $0 < p < 2/(2n + 1)$, and C_n is a constant depending only on n.

As an application of (1.3), we obtain bounds on how well a function, with a jump in a derivative, can be essentially time- and band-limited. Taking $I = [-\delta, \delta]$ for simplicity, we obtain under reasonable conditions that

$$A_n (7\varepsilon)^p \;\geq\; \frac{\Delta_n(u)}{\Omega^{n+\frac{1}{2}} ||u||^{p+1}_{L^2(I)}}$$

whenever u and $\widehat{u}$ are ε-concentrated on $[-T, T]$ and $[-\Omega, \Omega]$ respectively, and A_n is a constant depending only on n. We recall that a function f is ε-concentrated on $A \subset \mathbb{R}$ if

$$\int_{\mathbb{R} \setminus A} |f|^2 \;\leq\; \varepsilon^2 \int_{\mathbb{R}} |f|^2.$$

This complements a result of Donoho and Stark in [5]. They showed that if a function f of unit norm is ε-concentrated on A and $\widehat{f}$ is ρ-concentrated on B, then $|A| \cdot |B| \geq (1 - \varepsilon - \rho)^2$ (see also [8] for a slight improvement).

The estimate (1.3) can be viewed as an uncertainty principle inequality. It describes how the global norm $||g||_{L^2(\mathbb{R})}$ grows as the local approximation error $||u-g||_{L^2(I)}$ tends to zero. It is remarkable to note that the growth of $||g||_{L^2(\mathbb{R})}$ is manifested outside I. The reader may consult [1], [2], [3], [6], [7], [8] and [12] for background on uncertainty principle inequalities and on essentially time- and band-limited functions.

For an interval I and a measurable function $u : I \longrightarrow \mathbb{C}$, we let

$$||u||_{L^2(I)} \;=\; \left(\int_I |u(t)|^2 dt\right)^{1/2}.$$

If I is the entire real line, we simply write $||u||_2$ in place of $||u||_{L^2(I)}$. The Fourier transform of an integrable function $f : \mathbb{R} \longrightarrow \mathbb{C}$ is defined by

$$\widehat{f}(\xi) = \int_{\mathbb{R}} f(t)e^{-2\pi i\xi t}dt.$$

If $\Omega > 0$, we define the Paley-Wiener class $PW(\Omega)$ as the set of all $f \in L^2(\mathbb{R})$ such that $\operatorname{supp}\widehat{f} \subset [-\Omega, \Omega]$.

2 Growth of norms of approximate solutions

When the problem (1.1) has no solution for a fixed $(u, v) \in U \times V$, then approximate solutions blow up.

Proposition 2.1 *Let U and V be closed vector subspaces of a Hilbert space $\mathcal{H}$ with respective orthogonal projections P_U and P_V. Let $(u, v) \in U \times V$ such that the system*

$$P_U g = u, \quad P_V g = v$$

has no solution $g \in \mathcal{H}$. Suppose $\{g_k\}_{k\in\mathbb{N}}$ satisfies $||P_U g_k - u|| + ||P_V g_k - v|| \to 0$ as $k \to \infty$. Then $||g_k|| \to \infty$.

Proof. Suppose otherwise. Then some subsequence $\{g_{n_k}\}_{k=1}^{\infty}$ converges weakly, say to $g \in \mathcal{H}$. Thus

$$\langle g - g_{n_k},\, P_U g - u\rangle \longrightarrow 0. \tag{2.1}$$

Moreover, since $\{g_{n_k}\}$ is a bounded sequence,

$$|\langle g - g_{n_k},\, u - P_U g_{n_k}\rangle| \le ||u - P_U g_{n_k}|| \sup_{j\in\mathbb{N}} ||g - g_{n_j}|| \longrightarrow 0. \tag{2.2}$$

Hence, from (2.1) and (2.2), we obtain

$$||P_U(g - g_{n_k})||^2 = \langle g - g_{n_k},\, P_U g - u\rangle + \langle g - g_{n_k},\, u - P_U g_{n_k}\rangle \longrightarrow 0. \tag{2.3}$$

Thus,

$$\frac{1}{2}||P_U g - u||^2 \leq ||P_U(g - g_{n_k})||^2 + ||P_U g_{n_k} - u||^2 \longrightarrow 0. \tag{2.4}$$

Hence, $P_U g = u$. Likewise, we conclude that $P_V g = v$. □

In the context of bandlimited local approximations, Proposition 2.1 translates into the following corollary.

Corollary 2.2 *Let $\Omega > 0$ and I be a compact interval of the real line. Suppose $u \in L^2(I)$ is not the restriction of a function from $PW(\Omega)$. Let $\{g_k\}_{k=1}^{\infty}$ be a sequence in $PW(\Omega)$ such that $||u - g_k||_{L^2(I)} \to 0$. Then $||g_k||_2 \to \infty$.*

Proof. We shall apply Proposition 2.1 with $\mathcal{H} = L^2(\mathbb{R})$, $U = \{u \in \mathcal{H} : \operatorname{supp} u \subset I\}$, $V = \{v \in \mathcal{H} : \operatorname{supp} \widehat{v} \subset \mathbb{R} \setminus [-\Omega, \Omega]\}$, $v = 0$ and u as given in the statement of the corollary. Note that for all $f \in \mathcal{H}$,

$$P_U f = f \cdot 1_I \quad \text{and} \quad \widehat{P_V f} = \widehat{f} \cdot 1_{[-\Omega,\Omega]^c}.$$

Thus, the system

$$P_U g = u, \quad P_V g = 0$$

has no solution $g \in \mathcal{H}$. Moreover, since $P_V g_k \equiv 0 \equiv v$, $||P_U g_k - u||_2 + ||P_V g_k - v||_2 = ||g_k - u||_{L^2(I)} \to 0$ by hypothesis. By Proposition 2.1, we conclude that $||g_k||_2 \to \infty$. □

We emphasize that to a locally square-integrable function u, it is always possible to obtain local approximations with Fourier transforms having arbitrarily small supports. In [13], they showed that this follows from the density of suitable prolate spheroidal wave functions in $L^2(I)$. We give a simpler constructive proof below.

Proposition 2.3 *Set $I = [-1, 1]$ and fix $u \in L^2(I)$. Let Ω and ϵ be arbitrary positive numbers. Then there exists $g \in PW(\Omega)$ such that $||u - g||_{L^2(I)} < \epsilon$.*

Proof. Let ϕ be a function from the Schwartz class such that $\operatorname{supp} \widehat{\phi} \subset [-1, 1]$ with $\phi(0) = 1$. Choose a polynomial P such that $||u - P||_{L^2(I)} < \epsilon/2$. Let $\omega > 0$ be small enough such that $\omega < \Omega$ and $\omega^2 ||\phi'||_\infty^2 \int_{-1}^{1} |tP(t)|^2 dt < \epsilon^2/4$. Define $g(t) = P(t)\phi(\omega t)$. Then

$$\int_{-1}^{1} |P(t) - g(t)|^2 dt \leq \omega^2 ||\phi'||_\infty^2 \int_{-1}^{1} |tP(t)|^2 dt < \epsilon^2/4$$

and therefore $||u - g||_{L^2(I)} < \epsilon$. Moreover, if $P(t) = \sum_{k=0}^{n} a_k t^k$, then

$$\widehat{g}(\xi) \;=\; \omega^{-1} \sum_{k=0}^{n} \frac{a_k}{(-2\pi i w)^k} \; \widehat{\phi}^{\,(k)}(\omega^{-1}\xi)$$

and therefore, $\operatorname{supp} \widehat{g} \subset [-\omega, \omega] \subset [-\Omega, \Omega]$. □

3 Mean Value Inequality

Our main result in section 4 makes use of the following inequality, which bounds a local L^2-norm by a pointwise derivative.

Lemma 3.1 *Fix $n \in \mathbb{N} \cup \{0\}$ and real numbers α, β with $\alpha < \beta$. Let $\phi : [\alpha, \beta] \to \mathbb{R}$ be n times continuously differentiable. Then there exists $\tau \in]\alpha, \beta[$ such that*

$$(\beta - \alpha)^{n+\frac{1}{2}} |\phi^{(n)}(\tau)| \;\leq\; B_n \; ||\phi||_{L^2([\alpha,\beta])} \tag{3.1}$$

where $B_n = \dfrac{(2n)!\sqrt{2n+1}}{n!}$.

Proof. Define $\psi : [-1, 1] \longrightarrow \mathbb{R}$ by $\psi(x) = \phi(\frac{\beta-\alpha}{2}(x-1)+\beta)$ and

$$L_n(x) = \frac{1}{2^n n!} \cdot \frac{d^n}{dx^n}(x^2-1)^n, \tag{3.2}$$

the Legendre polynomial of degree n (for example, see [11]). Integrating by parts n times yields

$$\begin{aligned} \int_{-1}^{1} \psi(x)\; L_n(x)dx &= \frac{(-1)^n}{2^n n!} \int_{-1}^{1} \psi^{(n)}(x)(x^2-1)^n \, dx \\ &= \frac{\psi^{(n)}(\sigma)}{2^n n!} \int_{-1}^{1} (1-x^2)^n \, dx, \end{aligned} \tag{3.3}$$

for some $\sigma \in]-1, 1[$. Moreover, using the Beta function, we find that

$$\int_{-1}^{1} (1-x^2)^n \, dx = \frac{2^{2n+1}(n!)^2}{(2n+1)!}. \tag{3.4}$$

By combining (3.4), (3.3), the Cauchy-Schwarz inequality, and the fact that $||L_n||_{L^2([-1,1])} = \frac{\sqrt{2}}{\sqrt{2n+1}}$, we obtain

$$\frac{n! 2^{n+\frac{1}{2}}}{(2n)!\sqrt{2n+1}} \cdot |\psi^{(n)}(\sigma)| \;\leq\; ||\psi||_{L^2([-1,1])}. \tag{3.5}$$

Finally, in view of the equalities

$$||\psi||_{L^2([-1,1])} = \frac{\sqrt{2}}{\sqrt{\beta-\alpha}} \cdot ||\phi||_{L^2([\alpha,\beta])} \quad \text{and} \quad \psi^{(n)}(\sigma) = \left(\frac{\beta-\alpha}{2}\right)^n \phi^{(n)}(\tau), \tag{3.6}$$

where $\tau = \frac{\beta-\alpha}{2}(\sigma - 1) + \beta$, we see that (3.5) is equivalent to the desired estimate (3.1). □

4 Bandlimited approximation of functions with a jump in a derivative

For a fixed function u defined on a compact interval I, we obtain an estimate showing how the $L^2(\mathbb{R})$-norm of a function $g \in PW(\Omega)$ blows up, as the local approximation errors $\|u-g\|_{L^2(I)}$ go to zero. Here, $u^{(n)}$ is assumed to have a jump at some interior point a in I.

Theorem 4.1 *Let $n \in \mathbb{N} \cup \{0\}$, I be a compact interval of $\mathbb{R}$ and $u \in L^2(I)$ be real-valued. Suppose that for some $a \in I$ and $\delta_0 > 0$, $]a-\delta_0, a+\delta_0[\subset I$, u is n times continuously differentiable on the intervals $]a-\delta_0, a[$ and $]a, a+\delta_0[$, and*

$$\Delta_n(u) := \inf\{|u^{(n)}(t^+) - u^{(n)}(t^-)| : \ t^- \in]a-\delta_0, a[\,, \ t^+ \in]a, a+\delta_0[\ \} > 0. \tag{4.1}$$

Let $0 < p < 2/(2n+1)$, $q = 1 - p(n+\frac{1}{2})$, and $g \in PW(\Omega)$ such that

$$||g||_2 > D_n \left(\frac{\Omega^{n+\frac{1}{2}}}{\Delta_n(u)}\right)^{\frac{p}{q}-1} \quad \text{with} \quad \delta_0^q\ \Omega\ > \ \left(\frac{\Delta_n(u)}{4B_n}\right)^p. \tag{4.2}$$

where $B_n = (2n)!\sqrt{2n+1}/n!$, $C_n = (2\pi)^{n+1}\sqrt{32}/\sqrt{2n+3}$, and $D_n = (4B_n)^{\frac{p}{q}}/C_n$. Then

$$\Omega^{n+\frac{1}{2}}||g||_2||u-g||^p_{L^2(I)} \geq \frac{\Delta_n(u)}{C_n}. \tag{4.3}$$

Proof. Observe that $\|u-g\|_{L^2(I)} > 0$ since $u^{(n)}$ is discontinuous at a, and therefore cannot be the restriction to I of an entire function. First, assume that

$$4B_n\Omega^{n+\frac{1}{2}}\,\|u-g\|^q_{L^2(I)} < \Delta_n(u). \tag{4.4}$$

Set $\delta = \Omega^{-1}\,\|u-g\|^p_{L^2(I)}$. In view of (4.2) and (4.4), we have $0 < \delta < \delta_0$.

Let g_1 denote the real part of g. Let $0 < \rho < \delta$. By Lemma 3.1 applied to $\phi = u - g_1$, there exist $t^+ \in]a+\rho, a+\delta[$ and $t^- \in]a-\delta, a-\rho[$ such that

$$|\phi^{(n)}(t^\pm)| \ \leq \ B_n\,(\delta-\rho)^{-n-\frac{1}{2}}||\phi||_{L^2(I)} \ \leq \ B_n\,(\delta-\rho)^{-n-\frac{1}{2}}||u-g||_{L^2(I)}. \tag{4.5}$$

Meanwhile, the inversion formula for the Fourier transform implies,

$$g^{(n)}(t^+) - g^{(n)}(t^-) = \int_{-\Omega}^{\Omega} \widehat{g}(w)(2\pi i w)^n(e^{2\pi i w t^+} - e^{2\pi i w t^-})\,dw.$$

Since $|t^+ - t^-| < 2\delta$, it follows that

$$|g^{(n)}(t^+) - g^{(n)}(t^-)| \leq 2\delta \int_{-\Omega}^{\Omega} |\widehat{g}(w)| \cdot |2\pi w|^{n+1}\,dw \leq \frac{1}{2}C_n\delta||g||_2\Omega^{n+\frac{3}{2}}. \tag{4.6}$$

With our choice of δ, (4.6) implies

$$|g^{(n)}(t^+) - g^{(n)}(t^-)| \ \leq \ \frac{1}{2} C_n \Omega^{n+\frac{1}{2}} ||g||_2 ||u-g||^p_{L^2(I)} \ . \tag{4.7}$$

Since $\Delta_n(u) \leq |u^{(n)}(t^+) - u^{(n)}(t^-)|$, adding (4.5) and (4.7) gives

$$\begin{aligned} \Delta_n(u) \leq & \ |\phi^{(n)}(t^+)| + |g_1^{(n)}(t^+) - g_1^{(n)}(t^-)| + |\phi^{(n)}(t^-)| \\ \leq & \ \frac{2B_n \, ||u-g||_{L^2(I)}}{(\delta-\rho)^{n+\frac{1}{2}}} \ + \ \frac{1}{2} C_n \Omega^{n+\frac{1}{2}} ||g||_2 ||u-g||^p_{L^2(I)}. \end{aligned} \tag{4.8}$$

Note that by (4.4), $2B_n ||u-g||_{L^2(I)} \delta^{-n-\frac{1}{2}} < \frac{1}{2}\Delta_n(u)$. Letting ρ tend to zero in (4.8), we obtain

$$\frac{\Delta_n(u)}{C_n} \ \leq \ \Omega^{n+\frac{1}{2}} ||g||_2 ||u-g||^p_{L^2(I)}. \tag{4.9}$$

On the other hand, assume that $4B_n \Omega^{n+\frac{1}{2}} \, \|u-g\|^q_{L^2(I)} \geq \Delta_n(u)$. Then

$$\Omega^{n+\frac{1}{2}} ||g||_2 ||u-g||^p_{L^2(I)} \geq ||g||_2 \Omega^\gamma \left(\frac{\Delta_n(u)}{4B_n} \right)^{\frac{p}{q}} \ \text{ where } \ \gamma = (n+\frac{1}{2})(1-\frac{p}{q}).$$

By (4.2), the right-hand side of the last inequality above is greater than $\Delta_n(u)/C_n$. Thus, (4.9) also holds. □

5 Bounds on essential time- and band-limitedness

As an application of Theorem 4.1, we obtain bounds on how well a function, with a jump in a derivative, can be essentially time- and band-limited. Given positive numbers T, Ω and ε, let $\mathcal{P}(T, \Omega, \varepsilon)$ denote the set of all $u \in L^2(\mathbb{R})$ such that

$$\int_{|x|\geq T} |u(x)|^2 dx \ \leq \ \varepsilon^2 ||u||_2^2 \quad \text{and} \quad \int_{|\omega|\geq \Omega} |\widehat{u}(\omega)|^2 d\omega \ \leq \ \varepsilon^2 ||u||_2^2. \tag{5.1}$$

Theorem 5.1 *Let $T, \Omega > 0$, $0 < 14\varepsilon \leq 1$ and $u \in \mathcal{P}(T, \Omega, \varepsilon)$ be real-valued. Let $n \in \mathbb{N} \cup \{0\}$ such that u is n times continuously differentiable on the intervals $]-T, 0[$ and $]0, T[$ and suppose*

$$\Delta_n(u) := \inf\{|u^{(n)}(t^+) - u^{(n)}(t^-)| : \ t^- \in]-T, 0[\, , \ t^+ \in]0, T[\ \} > 0. \tag{5.2}$$

Let $0 < p < 2/(2n+1)$, $q = 1 - p(n+\frac{1}{2})$, and assume that

$$||u||_{L^2(I)} \geq 2D_n \left(\frac{\Omega^{n+\frac{1}{2}}}{\Delta_n(u)} \right)^{\frac{p}{q}-1} \quad \text{and} \quad T^q \, \Omega \ > \ \left(\frac{\Delta_n(u)}{4B_n} \right)^p \tag{5.3}$$

where $I = [-T, T]$, and D_n and B_n are defined as in Theorem 4.1. Then

$$A_n(7\varepsilon)^p \Omega^{n+\frac{1}{2}} ||u||_{L^2(I)}^{p+1} \geq \Delta_n(u) \tag{5.4}$$

where $A_n = C_n \left(14/\sqrt{195}\right)^{p+1}$, and C_n is defined in the statement of Theorem 4.1.

Proof. Let $\{\psi_n\}_{n=0}^{\infty}$ denote the sequence of prolate spheroidal wave functions corresponding to (T, Ω). They form an orthonormal basis for $PW(\Omega)$. Moreover, under the conditions in (5.1), they satisfy

$$||u - \mathbb{P}_d u||_2 \leq 7\varepsilon ||u||_2, \tag{5.5}$$

where $\mathbb{P}_d$ denotes the orthogonal projection onto the span of $\{\psi_0, \cdots, \psi_{d-1}\}$ and $d = 1 + \lfloor 4T\Omega \rfloor$ ([9, Theorem 3.6], [10]). Combining (5.3) and (5.5), we see that the first condition in (4.2) is satisfied with $g = \mathbb{P}_d u$. Hence, we may apply Theorem 4.1 and obtain

$$\frac{\Delta_n(u)}{C_n} \leq \Omega^{n+\frac{1}{2}} ||g||_2 ||u - g||_{L^2(I)}^p. \tag{5.6}$$

Combining this with (5.5) and the fact that $||g||_2 \leq ||u||_2$, we obtain

$$\frac{\Delta_n(u)}{C_n} \ \leq \ (7\varepsilon)^p \Omega^{n+\frac{1}{2}} ||u||_2^{p+1}. \tag{5.7}$$

Meanwhile, since we have assumed $14\varepsilon \leq 1$, the first inequality in (5.1) implies

$$\frac{\sqrt{195}}{14} ||u||_2 \ \leq \ \sqrt{1 - \varepsilon^2} ||u||_2 \ \leq \ ||u||_{L^2(I)}. \tag{5.8}$$

Combining this with (5.7) gives the desired estimate (5.4). ☐

Taking $n = 0$ and $p = 1$ in Theorem 5.1, the following example shows that the lower bound for ε in (5.4) is optimal up to a constant factor.

Example 5.2 Fix $s > 1$. Suppose $u : \mathbb{R} \longrightarrow \mathbb{R}$ is odd such that $u = \alpha 1_{]0,T[} + \beta 1_{]T,T+\delta[}$ on $]0, \infty[$, for some positive α, β, T, δ with $4T^s(\alpha + \beta)^2 = \pi^2$. Let $\Omega = (\delta T^s \beta^2)^{-1}$ and $\varepsilon^2 = (\Omega T^{s+1} \alpha^2)^{-1}$ and assume

$$T \geq \max\left\{ \left(\frac{196}{\alpha^2 \Omega}\right)^{1/(s+1)}, \ \frac{3\Omega}{\pi^2 \alpha^4}, \ \frac{\alpha^2}{4\Omega^2} \right\}. \tag{5.9}$$

Then $u \in \mathcal{P}(T, \Omega, \varepsilon)$ and

$$\frac{6}{7^3 \pi} \cdot \frac{1}{\alpha T \Omega^{1/2}} \ \leq \varepsilon \ = \ \frac{1}{\alpha T^{(s+1)/2} \Omega^{1/2}}. \tag{5.10}$$

Remark 5.3 Observe that with $\Omega = \alpha^2$, (5.9) becomes

$$\frac{\pi^{2/s}}{4^{1/s}(\alpha + \beta)^{2/s}} \ \geq \ \max\left\{ \frac{14^{2/(s+1)}}{\alpha^{4/(s+1)}}, \ \frac{3}{\pi^2 \alpha^2}, \ \frac{1}{4\alpha^2} \right\},$$

which is satisfied for α sufficiently large with β fixed.

Proof. We have $i\pi w\widehat{u}(w) = \alpha\{\cos(2\pi Tw) - 1\} + \beta\{\cos(2\pi(T+\delta)w) - \cos(2\pi Tw)\}$. Therefore, $|\widehat{u}(w)| \leq 2(\alpha+\beta)(\pi|w|)^{-1}$ and

$$\int_{|w|\geq\Omega} |\widehat{u}|^2 \leq \frac{8(\alpha+\beta)^2}{\pi^2\Omega} = \frac{2}{T^s\Omega}. \tag{5.11}$$

Also, we have

$$\int_{|t|\geq T} |u|^2 = 2\delta\beta^2 = \frac{2}{T^s\Omega}. \tag{5.12}$$

Note that $||u||_2^2 = 2(T\alpha^2 + \delta\beta^2)$. Thus, (5.11) and (5.12) show that $u \in \mathcal{P}(T,\Omega,\varepsilon)$.

Meanwhile, with our choice of ε, the condition $14\varepsilon \leq 1$ translates into $T^{s+1} > 196/(\alpha^2\Omega)$. Moreover, with $p = 1$ and $q = 1/2$ in Theorem 4.1, we have $B_0 = 1$, $C_0 = 2\pi\sqrt{32}/\sqrt{3}$, and $D_0 = \sqrt{6}/\pi$. Thus, the conditions in (5.3) translate respectively into

$$T > \frac{3\Omega}{\pi^2\alpha^4} \quad \text{and} \quad T > \frac{\alpha^2}{4\Omega^2}. \tag{5.13}$$

In short, (5.9) ensures that the hypotheses of Theorem 5.1 are satisfied with $p = 1$ and $q = 1/2$. Hence, (5.4) implies

$$7A_0\varepsilon \geq \frac{1}{\alpha T\sqrt{\Omega}},$$

which gives the inequality in (5.10) since $7A_0 \leq 7^3\pi/6$. □

To obtain similar examples for higher smoothness $n \in \mathbb{N}$, one may start with an odd simple function u_0 supported on some compact interval $[-R, R]$ such that $\int_{-R}^{R} u_0 = 0$ and

$$\int_{-R}^{R}\int_{-R}^{x_{k-1}} \cdots \int_{-R}^{x_1} u_0(x_0)dx_0dx_1\cdots dx_{k-1} = 0$$

for $k \in \{2, \cdots, n\}$, in case $n \geq 2$. Then one considers the function u defined by

$$u(x) = \int_{-R}^{x}\int_{-R}^{x_{n-1}} \cdots \int_{-R}^{x_1} u_0(x_0)dx_0dx_1\cdots dx_{n-1}$$

for $x \in [-R, R]$ and $u \equiv 0$ outside $[-R, R]$.

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