The Γ-limit of the two-dimensional Ohta-Kawasaki energy. II. Droplet arrangement at the sharp interface level via the renormalized energy.

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Abstract

This is the second in a series of papers in which we derive a Γ-expansion for the two-dimensional non-local Ginzburg-Landau energy with Coulomb repulsion known as the Ohta-Kawasaki model in connection with diblock copolymer systems. In this model, two phases appear, which interact via a nonlocal Coulomb type energy. Here we focus on the sharp interface version of this energy in the regime where one of the phases has very small volume fraction, thus creating small "droplets" of the minority phase in a "sea" of the majority phase. In our previous paper, we computed the Γ-limit of the leading order energy, which yields the averaged behavior for almost minimizers, namely that the density of droplets should be uniform. Here we go to the next order and derive a next order Γ-limit energy, which is exactly the Coulombian renormalized energy obtained by Sandier and Serfaty as a limiting interaction energy for vortices in the magnetic Ginzburg-Landau model. The derivation is based on the abstract scheme of Sandier-Serfaty that serves to obtain lower bounds for 2-scale energies and express them through some probabilities on patterns via the multiparameter ergodic theorem. Without thus appealing to the Euler-Lagrange equation, we establish for all configurations which have "almost minimal energy" the asymptotic roundness and radius of the droplets, and the fact that they asymptotically shrink to points whose arrangement minimizes the renormalized energy in some averaged sense. Via a kind of Γ-equivalence, the obtained results also yield an expansion of the minimal energy for the original Ohta-Kawasaki energy. This leads to expecting to see triangular lattices of droplets as energy minimizers.

1 Introduction

This is our second paper devoted to the Γ-convergence study of the two-dimensional Ohta-Kawasaki energy functional [28] in two space dimensions in the regime near the onset of non-trivial minimizers. The energy functional has the following form:

$$\mathcal{E}[u] = \int_\Omega \left(\frac{\varepsilon^2}{2} |\nabla u|^2 + V(u) \right) dx + \frac{1}{2} \int_\Omega \int_\Omega (u(x) - \bar{u}) G_0(x, y) (u(y) - \bar{u}) \, dx \, dy, \tag{1.1}$$

where Ω is the domain occupied by the material, $u : \Omega \to \mathbb{R}$ is the scalar order parameter, $V(u)$ is a symmetric double-well potential with minima at $u = \pm 1$, such as the usual Ginzburg-Landau potential $V(u) = \frac{9}{32}(1 - u^2)^2$ (for simplicity, the overall coefficient in V is chosen to make the associated surface tension constant to be equal to ε, i.e., we have $\int_{-1}^{1} \sqrt{2V(u)} \, du = 1$), $\varepsilon > 0$ is a parameter characterizing interfacial thickness, $\bar{u} \in (-1, 1)$ is the background charge density, and G_0 is the Neumann Green's function of the Laplacian, i.e., G_0 solves

$$-\Delta G_0(x, y) = \delta(x - y) - \frac{1}{|\Omega|}, \qquad \int_\Omega G_0(x, y) \, dx = 0, \tag{1.2}$$

where Δ is the Laplacian in x and $\delta(x)$ is the Dirac delta-function, with Neumann boundary conditions. Note that u is also assumed to satisfy the "charge neutrality" condition

$$\frac{1}{|\Omega|} \int_\Omega u \, dx = \bar{u}. \tag{1.3}$$

For a discussion of the motivation and the main quantitative features of this model, see our first paper [19], as well as [25, 26]. For specific applications to physical systems, we refer the reader to [16, 18, 22, 24, 25, 27, 28, 40].

In our first paper [19], we established the leading order term in the Γ-expansion of the energy in (1.1) in the scaling regime corresponding to the threshold between trivial and non-trivial minimizers. More precisely, we studied the behavior of the energy as $\varepsilon \to 0$ when

$$\bar{u}^\varepsilon := -1 + \varepsilon^{2/3} |\ln \varepsilon|^{1/3} \bar{\delta}, \tag{1.4}$$

for some fixed $\bar{\delta} > 0$ and when Ω is a flat two-dimensional torus of side length ℓ, i.e., when $\Omega = \mathbb{T}_\ell^2 = [0, \ell)^2$, with periodic boundary conditions. As follows from [19, Theorem 2] and the arguments in the proof of [19, Theorem 3], in this regime minimizers of $\mathcal{E}$ consist of many small "droplets" (regions where $u > 0$) and their number blows up as $\varepsilon \to 0$. We showed that, after a suitable rescaling the energy functional in (1.1) Γ-converges in the sense of convergence of the (suitably normalized) droplet densities, to the limit functional $E^0[\mu]$ defined for all densities $\mu \in \mathcal{M}(\mathbb{T}_\ell^2) \cap H^{-1}(\mathbb{T}_\ell^2)$ by:

$$E^0[\mu] = \frac{\bar{\delta}^2 \ell^2}{2\kappa^2} + \left(3^{2/3} - \frac{2\bar{\delta}}{\kappa^2} \right) \int_{\mathbb{T}_\ell^2} d\mu + 2 \iint_{\mathbb{T}_\ell^2 \times \mathbb{T}_\ell^2} G(x - y) d\mu(x) d\mu(y), \tag{1.5}$$

where $G(x)$ is the *screened* Green's function of the Laplacian, i.e., it solves the periodic problem for the equation

$$-\Delta G + \kappa^2 G = \delta(x) \quad \text{in} \quad \mathbb{T}_\ell^2, \tag{1.6}$$

and $\kappa = 1/\sqrt{V''(1)} = \frac{2}{3}$. Here we noted that the double integral in (1.5) is well defined in the sense $\iint_{\mathbb{T}_\ell^2 \times \mathbb{T}_\ell^2} G(x-y) d\mu(x) d\mu(y) := \int_{\mathbb{T}_\ell^2} v d\mu$, where the latter is interpreted as the Hahn-Banach extension of the corresponding linear functional defined by the integral on smooth test functions (see also [34, Sec. 7.3.1] and [9] for further discussion). Indeed, $v := G * d\mu$ is the convolution understood distributionally, i.e., $\langle G * d\mu, f\rangle := \langle G * f, d\mu\rangle = \int_{\mathbb{T}_\ell^2} \left(\int_{\mathbb{T}_\ell^2} G(x-y) f(y) dy \right) d\mu(x)$ for every $f \in C^\infty(\mathbb{T}_\ell^2)$ and, hence, by elliptic regularity $\|v\|_{H^1(\mathbb{T}_\ell^2)} \leq C \|f\|_{H^{-1}(\mathbb{T}_\ell^2)}$ for some $C > 0$, so $v \in H^1(\mathbb{T}_\ell^2)$.

In particular, for $\bar\delta > \bar\delta_c$, where

$$\bar\delta_c := \frac{1}{2} 3^{2/3} \kappa^2, \tag{1.7}$$

the limit energy $E^0[\mu]$ is minimized by $d\mu(x) = \bar\mu \, dx$, where

$$\bar\mu = \tfrac{1}{2}(\bar\delta - \bar\delta_c) \qquad \text{and} \qquad E^0[\bar\mu] = \tfrac{\bar\delta_c}{2\kappa^2}(2\bar\delta - \bar\delta_c). \tag{1.8}$$

When $\bar\delta \leq \bar\delta_c$, the limit energy is minimized by $\mu = 0$, with $E^0[0] = \bar\delta^2/(2\kappa^2)$. The value of $\bar\delta = \bar\delta_c$ thus serves as the threshold separating the trivial and the non-trivial minimizers of the energy in (1.1) together with (1.4) for sufficiently small ε. Above that threshold, the droplet density of energy-minimizers converges to the uniform density $\bar\mu$.

The key point that enables the analysis above is a kind of Γ-equivalence between the energy functional in (1.1) and its screened sharp interface analog (for general notions of Γ-equivalence or variational equivalence, see [3, 8]):

$$E^\varepsilon[u] = \frac{\varepsilon}{2} \int_{\mathbb{T}_\ell^2} |\nabla u| \, dx + \frac{1}{2} \int_{\mathbb{T}_\ell^2} \int_{\mathbb{T}_\ell^2} (u(x) - \bar u^\varepsilon) G(x-y) (u(y) - \bar u^\varepsilon) \, dx \, dy. \tag{1.9}$$

Here, G is the screened potential as in (1.6), and $u \in \mathcal{A}$, where

$$\mathcal{A} := BV(\mathbb{T}_\ell^2; \{-1, 1\}), \tag{1.10}$$

and we note that on the level of E^ε the neutrality condition in (1.3) has been removed. As we showed in [19], following the approach of [26], for $\mathcal{E}^\varepsilon$ given by (1.1) in which $\bar u = \bar u^\varepsilon$ and $\bar u^\varepsilon$ is defined in (1.4), we have

$$\min \mathcal{E}^\varepsilon = \min E^\varepsilon + O(\varepsilon^\alpha \min E^\varepsilon), \tag{1.11}$$

for some $\alpha > 0$. Therefore, in order to understand the leading order asymptotic expansion of the minimal energy $\min \mathcal{E}^\varepsilon$ in terms of $|\ln \varepsilon|^{-1}$, it is sufficient to obtain such an expansion for $\min E^\varepsilon$. This is precisely what we will do in the present paper.

In view of the discussion above, in this paper we concentrate our efforts on the analysis of the sharp interface energy E^ε in (1.9). An extension of our results to the original diffuse interface energy $\mathcal{E}^\varepsilon$ would lead to further technical complications that lie beyond the scope of the present paper and will be treated elsewhere. Here we wish to extract the next order non-trivial term in the Γ-expansion of the sharp interface energy E^ε after (1.5). In contrast to [26], we will not use the Euler-Lagrange equation associated to (1.9), so our results about minimizers will also be valid for "almost minimizers" (cf. Theorem 2).

We recall that for $\varepsilon \ll 1$ the energy minimizers for E^ε and $\bar{\delta} > \bar{\delta}_c$ consist of $O(|\ln \varepsilon|)$ nearly circular droplets of radius $r \simeq 3^{1/3}\varepsilon^{1/3}|\ln\varepsilon|^{-1/3}$ uniformly distributed throughout $\mathbb{T}^2_\ell$ [26, Theorem 2.2]. This is in contrast with the study of [12, 13] for a closely related energy, where the number of droplets remains bounded as $\varepsilon \to 0$, and the authors extract a limiting interaction energy for a finite number of points.

By Γ-convergence, we obtained in [19, Theorem 1] the convergence of the droplet density of almost minimizers (u^ε) of E^ε:

$$\mu^\varepsilon(x) := \frac{1}{2}\varepsilon^{-2/3}|\ln\varepsilon|^{-1/3}(1 + u^\varepsilon(x)), \tag{1.12}$$

to the uniform density $\bar{\mu}$ defined in (1.8). However, this result does not say anything about the microscopic placement of droplets in the limit $\varepsilon \to 0$. In order to understand the asymptotic arrangement of droplets in an energy minimizer, our plan is to blow-up the coordinates by a factor of $\sqrt{|\ln\varepsilon|}$, which is the inverse of the scale of the typical inter-droplet distance, and to extract the next order term in the Γ-expansion of the energy in terms of the limits as $\varepsilon \to 0$ of the blown-up configurations (which will consist of an infinite number of point charges in the plane with identical charge).

We will show that the arrangement of the limit point configurations is governed by the Coulombic renormalized energy W, which was introduced in [34]. That energy W was already derived as a next order Γ-limit for the magnetic Ginzburg-Landau model of superconductivity [34, 35], and also for two-dimensional Coulomb gases [37]. Our results here follow the same method of [35], and yield almost identical conclusions.

The "Coulombic renormalized energy" is a way of computing a total Coulomb interaction between an infinite number of point charges in the plane, neutralized by a uniform background charge (for more details see Section 2). It is shown in [35] that its minimum is achieved. It is also shown there that the minimum among simple lattice patterns (of fixed volume) is uniquely achieved by the triangular lattice (for a closely related result, see [10]), and it is conjectured that the triangular lattice is also a global minimizer. This triangular lattice is called "Abrikosov lattice" in the context of superconductivity and is observed in experiments in superconductors [41].

The next order limit of E^ε that we shall derive below is in fact the average of the energy W over all limits of blown-up configurations (i.e. average with respect to the blow up center). Our result says that limits of blow-ups of (almost) minimizers should minimize this average of W. This permits one to distinguish between different patterns at the microscopic scale and it leads, in view of the conjecture above, to expecting to see triangular lattices of droplets (in the limit $\varepsilon \to 0$), around almost every blow-up center (possibly with defects). Note that the selection of triangular lattices was also considered in the context of the Ohta-Kawasaki energy by Chen and Oshita [10], but there they were only obtained as minimizers among simple lattice configurations consisting of non-overlapping ideally circular droplets.

It is somewhat expected that minimizers of the Ohta-Kawasaki energy in the macroscopic setting are periodic patterns in all space dimensions (in fact in the original paper [28] only periodic patterns are considered as candidates for minimizers). This fact has never been proved rigorously, except in one dimension by Müller [23] (see also [31, 42]), and at the moment seems very difficult. For higher-dimensional problems, some recent results in this direction were obtained in [2, 26, 38] establishing equidistribution of energy in various versions of the Ohta-Kawasaki model on macroscopically large domains. Several other results [12, 13, 15, 39] were also obtained to characterize the geometry of minimizers on smaller domains. The results we obtain here, in the regime of small volume fraction and in dimension two, provide more quantitative and qualitative information (since we are able to distinguish between the cost of various patterns, and have an idea of what the minimizers should be like) and a first setting where periodicity can be expected to be proved.

The Ohta-Kawasaki setting differs from that of the magnetic Ginzburg-Landau model in the fact that the droplet "charges" (i.e., their volume) are all positive, in contrast with the vortex degrees in Ginzburg-Landau, which play an analogous role and can be both positive and negative integers. It also differs in the fact that the droplet volumes are not quantized, contrary to the degrees in the Ginzburg-Landau model. This creates difficulties and the major difference in the proofs. In particular we have to account for the possibility of many very small droplets, and we have to show that the isoperimetric terms in the energy suffice to force (almost) all the droplets to be round and of fixed volume. This has to be done at the same time as the lower bound for the other term in the energy, for example an adapted "ball construction" for non-quantized quantities has to be re-implemented, and the interplay between these two effects turns out to be delicate.

Our paper is organized as follows. In Section 2 we formulate the problem and state our main results concerning the Γ-limit of the next order term in the energy (1.9) after the zeroth order energy derived in [19] is subtracted off. In Section 3, we derive a lower bound on this next order energy via an energy expansion as done in [19] however isolating lower order terms obtained via the process. We then proceed via a ball construction as in [20,33,34] to obtain lower bounds on this energy in Section 4 and consequently obtain an energy density bounded from below with almost the same energy via energy displacement as in [35] in Section 5. In Section 6 we obtain explicit lower bounds on this density on bounded sets in the plane in terms of the renormalized energy for a finite number of points. We are then in the appropriate setting to apply the multiparameter ergodic theorem as in [35] to extend the lower bounds obtained to global bounds, which we present at the end of Section 6. Finally the corresponding upper bound (cf. Part (ii) of Theorem 1) is presented in Section 7.

Some notations. We use the notation $(u^\varepsilon) \in \mathcal{A}$ to denote sequences of functions $u^\varepsilon \in \mathcal{A}$ as $\varepsilon = \varepsilon_n \to 0$, where $\mathcal{A}$ is an admissible class. We also use the notation $\mu \in \mathcal{M}(\Omega)$ to denote a positive finite Radon measure $d\mu$ on the domain Ω. With a slight abuse of notation, we will often speak of μ as the "density" on Ω and set $d\mu(x) = \mu(x)dx$ whenever $\mu \in L^1(\Omega)$. With some more abuse of notation, for a measurable set E we use $|E|$ to denote its Lebesgue measure, $|\partial E|$ to denote its perimeter (in the sense of De Giorgi), and $\mu(E)$ to denote $\int_E d\mu$. The symbols $H^1(\Omega)$, $BV(\Omega)$, $C^k(\Omega)$ and $H^{-1}(\Omega)$ denote the usual Sobolev space, the space of functions of bounded variation, the space of k-times continuously differentiable functions, and the dual of $H^1(\Omega)$, respectively. The symbol $o_\varepsilon(1)$ stands for the quantities that tend to zero as $\varepsilon \to 0$ with the rate of convergence depending only on ℓ, $\bar{\delta}$ and κ.

2 Problem formulation and main results

In the following, we fix the parameters $\kappa > 0$, $\bar{\delta} > 0$ and $\ell > 0$, and work with the energy E^ε in (1.9), which can be equivalently rewritten in terms of the connected components Ω_i^ε of the family of sets of finite perimeter $\Omega^\varepsilon := \{u^\varepsilon = +1\}$, where $(u^\varepsilon) \in \mathcal{A}$ are almost minimizers of E^ε, for sufficiently small ε (cf. the discussion at the beginning of Sec. 2 in [19]). The sets Ω^ε can be decomposed into countable unions of connected disjoint sets, i.e., $\Omega^\varepsilon = \bigcup_i \Omega_i^\varepsilon$, whose boundaries $\partial\Omega_i^\varepsilon$ are rectifiable and can be decomposed (up to negligible sets) into countable unions of disjoint simple closed curves. Then the density μ^ε in (1.12) can be rewritten as

$$\mu^\varepsilon(x) := \varepsilon^{-2/3}|\ln\varepsilon|^{-1/3}\sum_i \chi_{\Omega_i^\varepsilon}(x), \tag{2.1}$$

where $\chi_{\Omega_i^\varepsilon}$ are the characteristic functions of Ω_i^ε. Motivated by the scaling analysis in the discussion preceding equation (1.12), we define the rescaled areas and perimeters of the droplets:

$$A_i^\varepsilon := \varepsilon^{-2/3}|\ln\varepsilon|^{2/3}|\Omega_i^\varepsilon|, \qquad P_i^\varepsilon := \varepsilon^{-1/3}|\ln\varepsilon|^{1/3}|\partial\Omega_i^\varepsilon|. \tag{2.2}$$

Using these definitions, we obtain (see [19, 26]) the following equivalent definition of the energy of the family (u^ε):

$$E^\varepsilon[u^\varepsilon] = \varepsilon^{4/3}|\ln\varepsilon|^{2/3}\left(\frac{\bar{\delta}^2\ell^2}{2\kappa^2} + \bar{E}^\varepsilon[u^\varepsilon]\right), \tag{2.3}$$

where

$$\bar{E}^\varepsilon[u^\varepsilon] := \frac{1}{|\ln\varepsilon|}\sum_i \left(P_i^\varepsilon - \frac{2\bar\delta}{\kappa^2}A_i^\varepsilon\right) + 2\iint_{\mathbb{T}_\ell^2\times\mathbb{T}_\ell^2} G(x-y)\,d\mu^\varepsilon(x)\,d\mu^\varepsilon(y). \tag{2.4}$$

Also note the relation

$$\mu^\varepsilon(\mathbb{T}_\ell^2) = \frac{1}{|\ln\varepsilon|}\sum_i A_i^\varepsilon. \tag{2.5}$$

As was shown in [19, 26], in the limit $\varepsilon \to 0$ the minimizers of E^ε are non-trivial if and only if $\bar\delta > \bar\delta_c$, and we have asymptotically

$$\min E^\varepsilon \simeq \frac{\bar\delta_c}{2\kappa^2}(2\bar\delta - \bar\delta_c)\varepsilon^{4/3}|\ln\varepsilon|^{2/3}\ell^2 \qquad \text{as } \varepsilon \to 0. \tag{2.6}$$

Furthermore, if μ^ε is as in (2.1) and we let v^ε be the unique solution of

$$-\Delta v^\varepsilon + \kappa^2 v^\varepsilon = \mu^\varepsilon \qquad \text{in} \quad W^{2,p}(\mathbb{T}_\ell^2), \tag{2.7}$$

for any $p < \infty$, then we have

$$v^\varepsilon \rightharpoonup \bar v := \frac{1}{2\kappa^2}(\bar\delta - \bar\delta_c) \qquad \text{in} \quad H^1(\mathbb{T}_\ell^2). \tag{2.8}$$

To extract the next order terms in the Γ-expansion of E^ε we, therefore, subtract this contribution from E^ε to define a new rescaled energy F^ε (per unit area):

$$F^\varepsilon[u] := \varepsilon^{-4/3}|\ln\varepsilon|^{1/3}\ell^{-2}E^\varepsilon[u] - |\ln\varepsilon|\frac{\bar\delta_c}{2\kappa^2}(2\bar\delta-\bar\delta_c) + \frac{1}{4\cdot 3^{1/3}}(\bar\delta-\bar\delta_c)(\ln|\ln\varepsilon| + \ln 9). \tag{2.9}$$

Note that we also added the third term into the bracket in the right-hand side of (2.9) to subtract the next-to-leading order contribution of the droplet self-energy, and we have scaled F^ε in a way that allows to extract a non-trivial $O(1)$ contribution to the minimal energy (see details in Section 3). The main result of this paper in fact is to establish Γ-convergence of F^ε to the renormalized energy W which we now define.

In [35], the renormalized energy W was introduced and defined in terms of the superconducting current j, which is particularly convenient for the studies of the magnetic Ginzburg-Landau model of superconductivity. Here, instead, we give an equivalent definition, which is expressed in terms of the limiting electrostatic potential of the charged droplets, after blow-up, which is the limit of some proper rescaling of v^ε (see below). However, this limiting electrostatic potential will only be known up to additive constants, due to the fact that we will take limits over larger and larger tori. This issue can be dealt with in a natural way by considering *equivalence classes* of potentials, whereby two potentials differing by a constant are not distinguished:

$$[\varphi] := \{\varphi + c \mid c \in \mathbb{R}\}. \tag{2.10}$$

This definition turns the homogeneous spaces $\dot{W}^{1,p}(\mathbb{R}^d)$ into Banach spaces of equivalence classes of functions in $W^{1,p}_{loc}(\mathbb{R}^d)$ defined in (2.10) (see, e.g., [29]). Here we similarly define the local analog of the homogeneous Sobolev spaces as

$$\dot{W}^{1,p}_{loc}(\mathbb{R}^2) := \left\{ [\varphi] \mid \varphi \in W^{1,p}_{loc}(\mathbb{R}^2) \right\}, \tag{2.11}$$

with the notion of convergence to be that of the L^p_{loc} convergence of gradients. In the following, we will omit the brackets in $[\cdot]$ to simplify the notation and will write $\varphi \in \dot{W}^{1,p}_{loc}(\mathbb{R}^2)$ to imply that φ is any member of the equivalence class in (2.10).

We define the admissible class of the renormalized energy as follows :

Definition 2.1. *For given $m > 0$ and $p \in (1, 2)$, we say that φ belongs to the admissible class $\mathcal{A}_m$, if $\varphi \in \dot{W}^{1,p}_{loc}(\mathbb{R}^2)$ and φ solves distributionally*

$$-\Delta\varphi = 2\pi \sum_{a\in\Lambda} \delta_a - m, \tag{2.12}$$

where $\Lambda \subset \mathbb{R}^2$ is a discrete set and

$$\lim_{R\to\infty} \frac{2}{R^2} \int_{B_R(0)} \sum_{a\in\Lambda} \delta_a(x)dx = m. \tag{2.13}$$

Remark 2.2. *Observe that if $\varphi \in \mathcal{A}_m$, then for every $x \in B_R(0)$ we have*

$$\varphi(x) = \sum_{a\in\Lambda_R} \ln|x - a|^{-1} + \varphi_R(x), \tag{2.14}$$

where $\Lambda_R := \Lambda \cap \bar{B}_R(0)$ is a finite set of distinct points and $\varphi_R \in C^\infty(\mathbb{R}^2)$ is analytic in $B_R(0)$. In particular, the definition of $\mathcal{A}_m$ is independent of p.

We next define the renormalized energy.

Definition 2.3. *For a given $\varphi \in \bigcup_{m>0} \mathcal{A}_m$, the renormalized energy W of φ is defined as*

$$W(\varphi) := \limsup_{R\to\infty} \lim_{\eta\to 0} \frac{1}{|K_R|} \left(\int_{\mathbb{R}^2\setminus\cup_{a\in\Lambda}B_\eta(a)} \frac{1}{2}|\nabla\varphi|^2\chi_R dx + \pi \ln\eta \sum_{a\in\Lambda} \chi_R(a) \right), \tag{2.15}$$

where $K_R = [-R, R]^2$, χ_R is a smooth cutoff function with the properties that $0 < \chi_R < 1$, in $K_R \setminus (\partial K_R \cup K_{R-1})$, $\chi_R(x) = 1$ for all $x \in K_{R-1}$, $\chi_R(x) = 0$ for all $x \in \mathbb{R}^2 \setminus K_R$, and $|\nabla\chi_R| \leq C$ for some $C > 0$ independent of R.

Various properties of W are established in [35], we refer the reader to that paper. The most relevant to us here are

1. $\min_{\mathcal{A}_m} W$ is achieved for each $m > 0$.

2. If $\varphi \in \mathcal{A}_m$ and $\varphi'(x) := \varphi(\frac{x}{\sqrt{m}})$, then $\varphi' \in \mathcal{A}_1$ and

$$W(\varphi) = m\left(W(\varphi') - \frac{1}{4}\log m\right), \tag{2.16}$$

hence

$$\min_{\mathcal{A}_m} W = m\left(\min_{\mathcal{A}_1} W - \frac{1}{4}\log m\right).$$

3. W is minimized over potentials in $\mathcal{A}_1$ generated by charge configurations Λ consisting of simple lattices by the potential of a triangular lattice, i.e. [35, Theorem 2 and Remark 1.5],

$$\min_{\substack{\varphi\in\mathcal{A}_1 \\ \Lambda \text{ simple lattice}}} W(\varphi) = W(\varphi^{\triangle}) = -\frac{1}{2}\ln(\sqrt{2\pi b}\,|\eta(\tau)|^2) \simeq -0.2011,$$

where $\tau = a + ib$, $\eta(\tau) = q^{1/24}\prod_{n\geq 1}(1-q^n)$ is the Dedekind eta function, $q = e^{2\pi i\tau}$, a and b are real numbers such that $\Lambda^*_{\triangle} = \frac{1}{\sqrt{2\pi b}}\Big((1,0)\mathbb{Z} \oplus (a,b)\mathbb{Z}\Big)$ is the dual lattice to a triangular lattice $\Lambda^{\triangle}$ whose unit cell has area 2π, and $\varphi^{\triangle}$ solves (2.12) with $\Lambda = \Lambda^{\triangle}$.

In particular, from property 2 above it is easy to see that the role of m in the definition of W is inconsequential.

We are now ready to state our main result. Let $\ell^\varepsilon := |\ln\varepsilon|^{1/2}\ell$. For a given $u^\varepsilon \in \mathcal{A}$, we then introduce the potential (recall that φ^ε is a representative in the equivalence class defined in (2.10))

$$\varphi^\varepsilon(x) := 2\cdot 3^{-2/3}|\ln\varepsilon|\ \tilde{v}^\varepsilon(x|\ln\varepsilon|^{-1/2}), \tag{2.17}$$

where $\tilde{v}^\varepsilon$ is a periodic extension of v^ε from $\mathbb{T}^2_{\ell^\varepsilon}$ to the whole of $\mathbb{R}^2$. We also define $\mathcal{P}$ to be the family of translation-invariant probability measures on $\dot{W}^{1,p}_{loc}(\mathbb{R}^2)$ concentrated on $\mathcal{A}_m$ with $m = 3^{-2/3}(\bar{\delta} - \bar{\delta}_c)$.

Theorem 1. **(Γ-convergence of F^ε)** *Fix $\kappa > 0$, $\bar{\delta} > \bar{\delta}_c$, $p \in (1,2)$ and $\ell > 0$, and let F^ε be defined by (2.9). Then, as $\varepsilon \to 0$ we have*

$$F^\varepsilon \xrightarrow{\Gamma} F^0[P] := 3^{4/3}\int W(\varphi)dP(\varphi) + \frac{3^{2/3}(\bar{\delta} - \bar{\delta}_c)}{8}, \tag{2.18}$$

where $P \in \mathcal{P}$. More precisely:

i) (Lower Bound) Let $(u^\varepsilon) \in \mathcal{A}$ *be such that*

$$\limsup_{\varepsilon \to 0} F^\varepsilon[u^\varepsilon] < +\infty, \tag{2.19}$$

and let P^ε *be the probability measure on* $\dot{W}^{1,p}_{loc}(\mathbb{R}^2)$ *which is the pushforward of the normalized uniform measure on* $\mathbb{T}^2_{\ell^\varepsilon}$ *by the map* $x \mapsto \varphi^\varepsilon(x+\cdot)$*, where* φ^ε *is as in* (2.17)*. Then, upon extraction of a subsequence,* (P^ε) *converges weakly to some* $P \in \mathcal{P}$*, in the sense of measures on* $\dot{W}^{1,p}_{loc}(\mathbb{R}^2)$ *and*

$$\liminf_{\varepsilon \to 0} F^\varepsilon[u^\varepsilon] \geq F^0[P]. \tag{2.20}$$

ii) (Upper Bound) Conversely, for any probability measure $P \in \mathcal{P}$*, letting* Q *be its pushforward under* $-\Delta$*, there exists* $(u^\varepsilon) \in \mathcal{A}$ *such that letting* Q^ε *be the pushforward of the normalized Lebesgue measure on* $\mathbb{T}^2_{\ell^\varepsilon}$ *by* $x \mapsto -\Delta\varphi^\varepsilon(x+\cdot)$*, where* φ^ε *is as in* (2.17)*, we have* $Q^\varepsilon \rightharpoonup Q$*, in the sense of measures on* $W^{-1,p}_{loc}(\mathbb{R}^2)$*, and*

$$\limsup_{\varepsilon \to 0} F^\varepsilon[u^\varepsilon] \leq F^0[P], \tag{2.21}$$

as $\varepsilon \to 0$*.*

We will prove that the minimum of F^0 is achieved. Moreover, it is achieved for any $P \in \mathcal{P}$ which is concentrated on minimizers of $\mathcal{A}_m$ with $m = 3^{-2/3}(\bar{\delta} - \bar{\delta}_c)$.

Remark 2.4. *The phrasing of the theorem does not exactly fit the framework of* Γ*-convergence, since the lower bound result and the upper bound result are not expressed with the same notion of convergence. However, since weak convergence of* P_ε *to* P *implies weak convergence of* Q_ε *to* Q*, the theorem implies a result of* Γ*-convergence where the sense of convergence from* P_ε *to* P *is taken to be the weak convergence of their push-forwards* Q_ε *to the corresponding* Q*.*

The next theorem expresses the consequence of Theorem 1 for almost minimizers:

Theorem 2. *Let* $m = 3^{-2/3}(\bar{\delta} - \bar{\delta}_c)$ *and let* $(u^\varepsilon) \in \mathcal{A}$ *be a family of almost minimizers of* F^0*, i.e., let*

$$\lim_{\varepsilon \to 0} F^\varepsilon[u^\varepsilon] = \min_{\mathcal{P}} F^0.$$

Then, if P *is the limit measure from Theorem 1,* P*-almost every* φ *minimizes* W *over* $\mathcal{A}_m$*. In addition*

$$\min_{\mathcal{P}} F^0 = 3^{4/3} \min_{\mathcal{A}_m} W + \frac{3^{2/3}(\bar{\delta} - \bar{\delta}_c)}{8}. \tag{2.22}$$

Note that the formula in (2.22) is not totally obvious, since the probability measure concentrated on a single minimizer $\varphi \in \mathcal{A}_m$ of W does not belong to $\mathcal{P}$.

The result in Theorem 2 allows us to establish the expansion of the minimal value of the original energy $\mathcal{E}^\varepsilon$ by combining it with (2.9) and (1.11).

Theorem 3. (Asymptotic expansion of $\min \mathcal{E}^\varepsilon$**)** *Let* $V = \frac{9}{32}(1-u^2)^2$*,* $\kappa = \frac{2}{3}$ *and* $m = 3^{-2/3}(\bar{\delta} - \bar{\delta}_c)$*. Fix* $\bar{\delta} > \bar{\delta}_c$ *and* $\ell > 0$*, and let* $\mathcal{E}^\varepsilon$ *be defined by (1.1) with* $\bar{u} = \bar{u}^\varepsilon$ *from (1.4). Then, as* $\varepsilon \to 0$ *we have*

$$\ell^{-2} \min \mathcal{E}^\varepsilon = \frac{\bar{\delta}_c}{2\kappa^2}(2\bar{\delta} - \bar{\delta}_c)\varepsilon^{4/3}|\ln \varepsilon|^{2/3} - \frac{1}{4 \cdot 3^{1/3}}(\bar{\delta} - \bar{\delta}_c)\varepsilon^{4/3}|\ln \varepsilon|^{-1/3}(\ln|\ln \varepsilon| + \ln 9)$$
$$+\varepsilon^{4/3}|\ln \varepsilon|^{-1/3}\left(3^{4/3} \min_{\mathcal{A}_m} W + \frac{3^{2/3}(\bar{\delta} - \bar{\delta}_c)}{8}\right) + o(\varepsilon^{4/3}|\ln \varepsilon|^{-1/3}). \tag{2.23}$$

As mentioned above, the Γ-limit in Theorem 1 cannot be expressed in terms of a single limiting function φ, but rather it effectively averages W over all the blown-up limits of φ^ε, with respect to all the possible blow-up centers. Consequently, for almost minimizers of the energy, we cannot guarantee that each blown-up potential φ^ε converges to a minimizer of W, but only that this is true after blow-up except around points that belong to a set with asymptotically vanishing volume fraction. Indeed, one could easily imagine a configuration with some small regions where the configuration does not ressemble any minimizer of W, and this would not contradict the fact of being an almost minimizer since these regions would contribute only a negligible fraction to the energy. Near all the good blow-up centers, we will know some more about the droplets: it will be shown in Theorem 4 that they are asymptotically round and of optimal radii.

We finish this section with a short sketch of the proof. Most of the proof consists in proving the lower bound, i.e. Part (i) of Theorem 1. The first step, accomplished in Section 3 is, following the ideas of [26], to extract from F^ε some positive terms involving the sizes and shapes of the droplets and which are minimized by round droplets of fixed appropriate radius. These positive terms, gathered in what will be called M_ε, can be put aside and will serve to control the discrepancy between the droplets and the ideal round droplets of optimal sizes. We then consider what remains when this M_ε is subtracted off from F^ε and express it in blown-up coordinates $x' = x\sqrt{|\ln \varepsilon|}$. It is then an energy functional, expressed in terms of some rescaling of φ^ε which has no sign and which ressembles that studied in [35]. Thus we apply to it the strategy of [35]. The main point is to show that, even though the energy density is not bounded below, it can be transformed into one that is by absorbing the negative terms into positive terms in the energy in the sense of energy displacement [35], while making only a small error. In order to prove that this is possible, we first need to establish sharp lower bounds for the energy carried by the droplets (with an error $o(1)$ per droplet). These lower bounds contain possible errors which will later be controlled via the M_ε term. This is done in Section 4 via a ball construction as in [20,33,34]. In Section 5 we use these lower bounds to perform the energy displacement as in [35]. Once the energy density has been replaced this way by an essentially equivalent energy density which is bounded below, we can apply the abstract scheme of [35] that serves to obtain lower bounds for "two-scale energies" which Γ-converge at the microscopic scale, via the multiparameter ergodic theorem. This is achieved is Section 6. Prior to this we obtain explicit lower bounds at the microscopic scale in terms of the renormalized energy for a finite number of points. It is then these lower bounds that get integrated out, or averaged out at the macroscopic scale to provide a global lower bound.

Finally, there remains to obtain the corresponding upper bound. This is done via an explicit construction of a periodic test-configuration, following again the method of [35].

3 Derivation of the leading order energy

In preparation for the proof of Theorem 1, we define

$$\rho_\varepsilon := 3^{1/3}\varepsilon^{1/3}|\ln\varepsilon|^{1/6} \quad \text{and} \quad \bar{r}_\varepsilon := \left(\frac{|\ln\varepsilon|}{|\ln\rho_\varepsilon|}\right)^{1/3}. \tag{3.1}$$

Recall that to leading order the droplets are expected to be circular with radius $3^{1/3}\varepsilon^{1/3}|\ln\varepsilon|^{-1/3}$. Thus ρ_ε is the expected radius, once we have blown up coordinates by the factor of $\sqrt{|\ln\varepsilon|}$, which will be done below. Also, we know that the expected normalized area A_i is $3^{2/3}\pi$, but this is only true up to lower order terms which were negligible in [19]; as we show below, a more precise estimate is $A_i \simeq \pi\bar{r}_\varepsilon^2$, so $\bar{r}_\varepsilon$ above can be viewed as a "corrected" normalized droplet radius. Since our estimates must be accurate up to $o_\varepsilon(1)$ per droplet and the self-energy of a droplet is of order $A_i^2 \ln\rho_\varepsilon$, we can no longer ignore these corrections.

The goal of the next subsection is to obtain an explicit lower bound for F_ε defined by (2.9) in terms of the droplet areas and perimeters, which will then be studied in Sections 4 and onward. We follow the analysis of [19], but isolate higher order terms.

3.1 Energy extraction

We begin with the original energy $\bar{E}^\varepsilon$ (cf. (2.4)) while adding and subtracting the *truncated* self interaction: first we define, for $\gamma \in (0,1)$, truncated droplet volumes by

$$\tilde{A}_i^\varepsilon := \begin{cases} A_i^\varepsilon & \text{if } A_i^\varepsilon < 3^{2/3}\pi\gamma^{-1}, \\ (3^{2/3}\pi\gamma^{-1}|A_i^\varepsilon|)^{1/2} & \text{if } A_i^\varepsilon \geq 3^{2/3}\pi\gamma^{-1}, \end{cases} \tag{3.2}$$

as in [19]. The motivation for this truncation will become clear in the proof of Proposition 5.1, when we obtain lower bounds on the energy on annuli. In [19] the self-interaction energy of each droplet extracted from $\bar{E}^\varepsilon$ was $\frac{|\tilde{A}_i^\varepsilon|^2}{3\pi|\ln\varepsilon|}$, yielding in the end the leading order energy $E^0[\mu]$ in (1.5). A more precise calculation of the self-interaction energy corrects the coefficient of $|\tilde{A}_i^\varepsilon|^2$ by an $O(\ln|\ln\varepsilon|/|\ln\varepsilon|)$ term, yielding the following corrected leading order energy for E^ε:

$$E_\varepsilon^0[\mu] := \frac{\bar{\delta}^2\ell^2}{2\kappa^2} + \left(\frac{3}{\bar{r}_\varepsilon} - \frac{2\bar{\delta}}{\kappa^2}\right)\int_{\mathbb{T}_\ell^2} d\mu + 2\iint_{\mathbb{T}_\ell^2\times\mathbb{T}_\ell^2} G(x-y)d\mu(x)d\mu(y). \tag{3.3}$$

The energy in (3.3) is explicitly minimized by $d\mu(x) = \bar{\mu}_\varepsilon\,dx$ (again a correction to the previously known $\bar{\mu}$ from (1.8)) where

$$\bar{\mu}_\varepsilon := \frac{1}{2}\left(\bar{\delta} - \frac{3\kappa^2}{2\bar{r}_\varepsilon}\right) \qquad \text{for} \qquad \bar{\delta} > \frac{3\kappa^2}{2\bar{r}_\varepsilon}, \tag{3.4}$$

and

$$\min E_\varepsilon^0 = \frac{\bar{\delta}_c\ell^2}{2\kappa^2}\left\{2\bar{\delta}\left(\frac{3}{\bar{r}_\varepsilon^3}\right)^{1/3} - \bar{\delta}_c\left(\frac{3}{\bar{r}_\varepsilon^3}\right)^{2/3}\right\}. \tag{3.5}$$

Observing that $\bar{r}_\varepsilon \to 3^{1/3}$ we immediately check that

$$\bar{\mu}_\varepsilon \to \bar{\mu} \quad \text{as } \varepsilon \to 0, \tag{3.6}$$

and in addition that (3.5) converges to the second expression in (1.8). To obtain the next order term, we Taylor-expand the obtained formulas upon substituting the definition of $\bar{r}_\varepsilon$. After some algebra, we obtain

$$\ell^{-2}\min E_\varepsilon^0 = \frac{\bar{\delta}_c}{2\kappa^2}\left(2\bar{\delta} - \bar{\delta}_c\right) - \frac{1}{4\cdot 3^{1/3}}(\bar{\delta} - \bar{\delta}_c)\frac{\ln|\ln\varepsilon| + \ln 9}{|\ln\varepsilon|} + O\left(\frac{(\ln|\ln\varepsilon|)^2}{|\ln\varepsilon|^2}\right). \tag{3.7}$$

Recalling once again the definition of F^ε from (2.9), we then find

$$F^\varepsilon[u^\varepsilon] = |\ln\varepsilon|\left(\varepsilon^{-4/3}|\ln\varepsilon|^{-2/3}\ell^{-2}E^\varepsilon[u^\varepsilon] - \ell^{-2}\min E_\varepsilon^0\right) + O\left(\frac{(\ln|\ln\varepsilon|)^2}{|\ln\varepsilon|}\right),$$

and in view of the definition of $\bar{E}^\varepsilon$ from (2.3), we thus may write

$$F^\varepsilon[u^\varepsilon] = |\ln\varepsilon|\ell^{-2}\left(\bar{E}^\varepsilon[u^\varepsilon] + \frac{\bar{\delta}^2\ell^2}{2\kappa^2} - \min E_\varepsilon^0\right) + O\left(\frac{(\ln|\ln\varepsilon|)^2}{|\ln\varepsilon|}\right). \tag{3.8}$$

Thus obtaining a lower bound for the first term in the right-hand side of (3.8) implies, up to $o_\varepsilon(1)$, a lower bound for F^ε. This is how we proceed to prove Lemma 3.1 below.

With this in mind, we begin by setting

$$v^\varepsilon = \bar{v}^\varepsilon + \frac{h_\varepsilon}{|\ln\varepsilon|}, \qquad \bar{v}^\varepsilon = \frac{1}{2\kappa^2}\left(\bar{\delta} - \frac{3\kappa^2}{2\bar{r}_\varepsilon}\right), \tag{3.9}$$

where $\bar{v}^\varepsilon$ is the solution to (2.7) with right side equal to $\bar{\mu}_\varepsilon$ in (3.4).

3.2 Blowup of coordinates

We now rescale the domain $\mathbb{T}_\ell^2$ by making the change of variables

$$\begin{aligned} x' &= x\sqrt{|\ln\varepsilon|}, \\ h'_\varepsilon(x') &= h_\varepsilon(x), \\ \Omega'_{i,\varepsilon} &= \Omega_i^\varepsilon\sqrt{|\ln\varepsilon|}, \\ \ell^\varepsilon &= \ell\sqrt{|\ln\varepsilon|}. \end{aligned} \tag{3.10}$$

Observe that

$$\varphi^\varepsilon(x') = 2\cdot 3^{-2/3}h'_\varepsilon(x') \qquad \forall x' \in \mathbb{T}^2_{\ell^\varepsilon}, \tag{3.11}$$

where φ^ε is defined by (2.17). It turns out to be more convenient to work with h'_ε and rescale only at the end back to φ^ε.

3.3 Main result

We are now ready to state the main result of this section, which provides an explicit lower bound on F^ε. The strategy, in particular for dealing with droplets that are too small or too large is the same as [19], except that we need to go to higher order terms.

Proposition 3.1. *There exist universal constants $\gamma \in (0, \frac{1}{6})$, $c_1 > 0, c_2 > 0$, $c_3 > 0$ and $\varepsilon_0 > 0$ such that if $\bar\delta > \bar\delta_c$ and $(u^\varepsilon) \in \mathcal{A}$ with $\Omega^\varepsilon := \{u^\varepsilon > 0\}$, then for all $\varepsilon < \varepsilon_0$*

$$\ell^2 F^\varepsilon[u^\varepsilon] \geq M_\varepsilon + \frac{2}{|\ln\varepsilon|}\int_{\mathbb{T}^2_{\ell^\varepsilon}} \left(|\nabla h'_\varepsilon|^2 + \frac{\kappa^2}{|\ln\varepsilon|}|h'_\varepsilon|^2\right)dx' - \frac{1}{\pi\bar r_\varepsilon^3}\sum_{A_i^\varepsilon \geq 3^{2/3}\pi\gamma} |\tilde A_i^\varepsilon|^2 + o_\varepsilon(1), \tag{3.12}$$

where $M_\varepsilon \geq 0$ is defined by

$$\begin{aligned} M_\varepsilon := \sum_i \left(P_i^\varepsilon - \sqrt{4\pi A_i^\varepsilon}\right) + c_1 \sum_{A_i^\varepsilon > 3^{2/3}\pi\gamma^{-1}} A_i^\varepsilon \\ + c_2 \sum_{3^{2/3}\pi\gamma \leq A_i^\varepsilon \leq 3^{2/3}\pi\gamma^{-1}} (A_i^\varepsilon - \pi\bar r_\varepsilon^2)^2 + c_3 \sum_{A_i^\varepsilon < 3^{2/3}\pi\gamma} A_i^\varepsilon. \end{aligned} \tag{3.13}$$

Remark 3.2. *Defining $\beta := 3^{2/3}\pi\gamma$, by isoperimetric inequality applied to each connected component of Ω^ε separately every term in the first sum in the definition of M_ε in (3.13) is non-negative. In particular, M_ε measures the discrepancy between the droplets Ω_i^ε with $A_i^\varepsilon \geq \beta$ and disks of radius $\bar r_\varepsilon$.*

The proposition will be proved below, but before let us examine some of its further consequences. The result of the proposition implies that our a priori assumption $\limsup_{\varepsilon\to 0} F^\varepsilon[u^\varepsilon] < +\infty$ translates into

$$M_\varepsilon + \frac{2}{|\ln\varepsilon|}\int_{\mathbb{T}^2_{\ell^\varepsilon}} \left(|\nabla h'_\varepsilon|^2 + \frac{\kappa^2}{|\ln\varepsilon|}|h'_\varepsilon|^2\right) dx' - \frac{1}{\pi\bar{r}^3_\varepsilon}\sum_{A^\varepsilon_i\ge\beta}|\tilde{A}^\varepsilon_i|^2 \le C,$$

for some $C>0$ independent of $\varepsilon \ll 1$, which, in view of (3.1) is also

$$M_\varepsilon + \frac{2}{|\ln\varepsilon|}\left(\int_{\mathbb{T}^2_{\ell^\varepsilon}} \left(|\nabla h'_\varepsilon|^2 + \frac{\kappa^2}{|\ln\varepsilon|}|h'_\varepsilon|^2\right) dx' - \frac{1}{2\pi}|\ln\rho_\varepsilon|\sum_{A^\varepsilon_i\ge\beta}|\tilde{A}^\varepsilon_i|^2\right) \le C. \qquad (3.14)$$

A major goal of the next sections is to obtain the following estimate

$$\frac{1}{|\ln\varepsilon|}\left(\int_{\mathbb{T}^2_{\ell^\varepsilon}} \left(|\nabla h'_\varepsilon|^2 + \frac{\kappa^2}{|\ln\varepsilon|}|h'_\varepsilon|^2\right) dx' - \frac{1}{2\pi}|\ln\rho_\varepsilon|\sum_{A^\varepsilon_i\ge\beta}|\tilde{A}^\varepsilon_i|^2\right) \ge -C\ln^2(M_\varepsilon+2), \quad (3.15)$$

for some $C>0$ independent of $\varepsilon \ll 1$, so that the a priori bound (3.14) in fact implies that M_ε is uniformly bounded independently of ε for small ε. This will be used crucially in Section 6.2.

We note that $h'_\varepsilon(x')$ satisfies the equation

$$-\Delta h'_\varepsilon + \frac{\kappa^2}{|\ln\varepsilon|}h'_\varepsilon = \mu'_\varepsilon - \bar{\mu}^\varepsilon \quad \text{in } W^{2,p}(\mathbb{T}^2_{\ell^\varepsilon}) \qquad (3.16)$$

where we define in $\mathbb{T}^2_{\ell^\varepsilon}$

$$\mu'_\varepsilon(x') := \sum_i A^\varepsilon_i \tilde{\delta}^\varepsilon_i(x'), \qquad (3.17)$$

and

$$\tilde{\delta}^\varepsilon_i(x') := \frac{\chi_{\Omega'_{i,\varepsilon}}(x')}{|\Omega'_{i,\varepsilon}|}, \qquad (3.18)$$

which will be used in what follows. Notice that each $\tilde{\delta}^\varepsilon_i(x')$ approximates the Dirac delta concentrated on some point in the support of $\Omega'_{i,\varepsilon}$ and, hence, $\mu'_\varepsilon(x')dx'$ approximates the measure associated with the collection of point charges with magnitude A^ε_i. In particular, the measure $d\mu'_\varepsilon$ evaluated over the whole torus equals the total charge: $\mu'_\varepsilon(\mathbb{T}^2_{\ell^\varepsilon}) = \sum_i A^\varepsilon_i$.

3.4 Proof of Proposition 3.1

- *Step 1:* We are first going to show that for universally small $\varepsilon > 0$ and all $\gamma \in (0, \frac{1}{6})$ we have

$$\ell^2 F^\varepsilon[u^\varepsilon] \geq T_1 + T_2 + T_3 + T_4 + T_5 + o_\varepsilon(1), \tag{3.19}$$

where

$$T_1 = \sum_i \left(P_i^\varepsilon - \sqrt{4\pi A_i^\varepsilon} \right), \tag{3.20}$$

$$T_2 = \frac{\gamma^{7/2}}{4\pi} \sum_{3^{2/3}\pi\gamma \leq A_i^\varepsilon \leq 3^{2/3}\pi\gamma^{-1}} (A_i^\varepsilon - \pi \bar{r}_\varepsilon^2)^2, \tag{3.21}$$

$$T_3 = \frac{\gamma^{-5/2}}{4\pi^2 \cdot 3^{2/3}} \sum_{A_i^\varepsilon < 3^{2/3}\pi\gamma} A_i^\varepsilon (A_i^\varepsilon - \pi \bar{r}_\varepsilon^2)^2, \tag{3.22}$$

$$T_4 = \sum_{A_i^\varepsilon > 3^{2/3}\pi\gamma^{-1}} \left(6^{-1}\gamma^{-1} - 1\right) A_i^\varepsilon, \tag{3.23}$$

$$T_5 = \frac{2}{|\ln \varepsilon|} \int_{\mathbb{T}_\ell^2} \left(|\nabla h_\varepsilon|^2 + \kappa^2 |h_\varepsilon|^2 \right) dx - \frac{1}{\pi \bar{r}_\varepsilon^3} \sum_i |\tilde{A}_i^\varepsilon|^2. \tag{3.24}$$

To bound $F^\varepsilon[u^\varepsilon]$ from below, we start from (3.8). In particular, in view of (2.7) we may rewrite (2.4) as

$$\begin{aligned}
\bar{E}^\varepsilon[u^\varepsilon] &= \frac{1}{|\ln \varepsilon|} \sum_i \left(P_i^\varepsilon - \frac{2\bar{\delta}}{\kappa^2} A_i^\varepsilon \right) + 2 \int_{\mathbb{T}_\ell^2} \left(|\nabla v^\varepsilon|^2 + \kappa^2 |v^\varepsilon|^2 \right) dx \\
&= \frac{1}{|\ln \varepsilon|} \sum_i \left(P_i^\varepsilon - \sqrt{4\pi A_i^\varepsilon} \right) \\
&\quad + \frac{1}{|\ln \varepsilon|} \sum_i \left(\sqrt{4\pi A_i^\varepsilon} - \frac{2\bar{\delta}}{\kappa^2} A_i^\varepsilon + \frac{1}{\pi \bar{r}_\varepsilon^3} |\tilde{A}_i^\varepsilon|^2 \right) \qquad (3.25) \\
&\quad + 2 \int_{\mathbb{T}_\ell^2} \left(|\nabla v^\varepsilon|^2 + \kappa^2 |v^\varepsilon|^2 \right) dx - \frac{1}{\pi \bar{r}_\varepsilon^3 |\ln \varepsilon|} \sum_i |\tilde{A}_i^\varepsilon|^2. \qquad (3.26)
\end{aligned}$$

We start by focusing on (3.25). First, in the case $A_i^\varepsilon > 3^{2/3}\pi\gamma^{-1}$ we have $|\tilde{A}_i^\varepsilon|^2 = 3^{2/3}\pi\gamma^{-1} A_i^\varepsilon$ and hence, recalling that $\bar{r}_\varepsilon = 3^{1/3} + o_\varepsilon(1)$, where $o_\varepsilon(1)$ depends only on ε, we have for ε universally small and $\gamma < \frac{1}{6}$:

$$\begin{aligned}
\frac{|\tilde{A}_i^\varepsilon|^2}{\pi \bar{r}_\varepsilon^3} = \frac{A_i^\varepsilon}{\pi \bar{r}_\varepsilon^3} \left(3^{2/3}\pi\gamma^{-1} - 3\pi \bar{r}_\varepsilon^2 + 3\pi \bar{r}_\varepsilon^2 \right) &= A_i^\varepsilon \left(\frac{3}{\bar{r}_\varepsilon} + \frac{3^{2/3}}{\bar{r}_\varepsilon^3} \left(\gamma^{-1} - 3 \left(\frac{\bar{r}_\varepsilon}{3^{1/3}} \right)^2 \right) \right) \\
&\geq A_i^\varepsilon \left(\frac{3}{\bar{r}_\varepsilon} + \frac{1}{6} \left(\gamma^{-1} - 6 \right) \right). \qquad (3.27)
\end{aligned}$$

We conclude that for $A_i^\varepsilon > 3^{2/3}\pi\gamma^{-1}$, we have

$$\left(\sqrt{4\pi A_i^\varepsilon} + \frac{|\tilde{A}_i^\varepsilon|^2}{\pi\bar{r}_\varepsilon^3} - \frac{2\bar{\delta}}{\kappa^2}A_i^\varepsilon\right) \geq \left(\frac{3}{\bar{r}_\varepsilon} - \frac{2\bar{\delta}}{\kappa^2} + \frac{1}{6}\left(\gamma^{-1} - 6\right)\right) A_i^\varepsilon. \tag{3.28}$$

On the other hand, when $A_i^\varepsilon \leq 3^{2/3}\pi\gamma^{-1}$ we have $\tilde{A}_i^\varepsilon = A_i^\varepsilon$ and we proceed as follows. Let us begin by defining, similarly to [19], the function

$$f(x) = \frac{2\sqrt{\pi}}{\sqrt{x}} + \frac{x}{\pi\bar{r}_\varepsilon^3}$$

for $x \in (0, +\infty)$ and observe that f is convex and attains its minimum of $\frac{3}{\bar{r}_\varepsilon}$ at $x = \pi\bar{r}_\varepsilon^2$, with

$$f''(x) = \frac{3\sqrt{\pi}}{2x^{5/2}} > 0.$$

By a second order Taylor expansion of f around $\pi\bar{r}_\varepsilon^2$, using the fact that f'' is decreasing on $(0, +\infty)$, we then have for all $x \leq x_0$

$$\sqrt{4\pi x} + \frac{x^2}{\pi\bar{r}_\varepsilon^3} = xf(x) \geq x\left(\frac{3}{\bar{r}_\varepsilon} + \frac{3\sqrt{\pi}}{4x_0^{5/2}}\left(x - \pi\bar{r}_\varepsilon^2\right)^2\right). \tag{3.29}$$

We, hence, conclude that when $3^{2/3}\pi\gamma \leq A_i^\varepsilon \leq 3^{2/3}\pi\gamma^{-1}$, we have

$$\sqrt{4\pi A_i^\varepsilon} + \frac{|\tilde{A}_i^\varepsilon|^2}{\pi\bar{r}_\varepsilon^3} - \frac{2\bar{\delta}}{\kappa^2}A_i^\varepsilon \geq \left(\frac{3}{\bar{r}_\varepsilon} - \frac{2\bar{\delta}}{\kappa^2}\right) A_i^\varepsilon + \frac{\gamma^{5/2}}{4\pi^2 \cdot 3^{2/3}} A_i^\varepsilon (A_i^\varepsilon - \pi\bar{r}_\varepsilon^2)^2, \tag{3.30}$$

and when $A_i^\varepsilon < 3^{2/3}\pi\gamma$, we have

$$\sqrt{4\pi A_i^\varepsilon} + \frac{|\tilde{A}_i^\varepsilon|^2}{\pi\bar{r}_\varepsilon^3} - \frac{2\bar{\delta}}{\kappa^2}A_i^\varepsilon \geq \left(\frac{3}{\bar{r}_\varepsilon} - \frac{2\bar{\delta}}{\kappa^2}\right) A_i^\varepsilon + \frac{\gamma^{-5/2}}{4\pi^2 \cdot 3^{2/3}} A_i^\varepsilon (A_i^\varepsilon - \pi\bar{r}_\varepsilon^2)^2, \tag{3.31}$$

Combining (3.28), (3.30) and (3.31), summing over all i, and distinguishing the different cases, we can now bound (3.25) from below as follows:

$$\begin{aligned}\sum_i \left(\sqrt{4\pi A_i^\varepsilon} - \frac{2\bar{\delta}}{\kappa^2}A_i^\varepsilon + \frac{1}{\pi\bar{r}_\varepsilon^3}|\tilde{A}_i^\varepsilon|^2\right) \geq{}& \left(\frac{3}{\bar{r}_\varepsilon} - \frac{2\bar{\delta}}{\kappa^2}\right)\sum_i A_i^\varepsilon \\ &+ \frac{\gamma^{7/2}}{4\pi} \sum_{3^{2/3}\pi\gamma \leq A_i^\varepsilon \leq 3^{2/3}\pi\gamma^{-1}} (A_i^\varepsilon - \pi\bar{r}_\varepsilon^2)^2 \\ &+ \frac{\gamma^{-5/2}}{4\pi^2 \cdot 3^{2/3}} \sum_{A_i^\varepsilon < 3^{2/3}\pi\gamma} A_i^\varepsilon (A_i^\varepsilon - \pi\bar{r}_\varepsilon^2)^2 \\ &+ \sum_{A_i^\varepsilon > 3^{2/3}\pi\gamma^{-1}} \left(6^{-1}\gamma^{-1} - 1\right) A_i^\varepsilon.\end{aligned} \tag{3.32}$$

We now focus on the term in (3.26). Using (3.9), we can write the integral in (3.26) as:

$$
\begin{aligned}
2 \int_{\mathbb{T}^2_\ell} & \left(\nabla v^\varepsilon |^2 + \kappa^2 |v^\varepsilon|^2 \right) dx \\
&= \frac{2}{|\ln \varepsilon|^2} \int_{\mathbb{T}^2_\ell} \left(|\nabla h_\varepsilon|^2 + \kappa^2 h_\varepsilon^2 \right) dx \ + \frac{4\kappa^2 \bar{v}^\varepsilon}{|\ln \varepsilon|} \int_{\mathbb{T}^2_\ell} h_\varepsilon dx + 2\kappa^2 |\bar{v}^\varepsilon|^2 \ell^2. \qquad (3.33)
\end{aligned}
$$

Integrating (2.7) over $\mathbb{T}^2_\ell$ and recalling the definition of h_ε in (3.9), as well as (2.5), leads to

$$
\frac{4\kappa^2 \bar{v}^\varepsilon}{|\ln \varepsilon|} \int_{\mathbb{T}^2_\ell} h_\varepsilon dx = \frac{4\bar{v}^\varepsilon}{|\ln \varepsilon|} \sum_i A_i^\varepsilon - 4\kappa^2 |\bar{v}^\varepsilon|^2 \ell^2. \qquad (3.34)
$$

Combining (3.33) and (3.34), we then find

$$
\begin{aligned}
2 \int_{\mathbb{T}^2_\ell} \left(|\nabla v^\varepsilon|^2 + \kappa^2 |v^\varepsilon|^2 \right) dx &= \frac{2}{|\ln \varepsilon|^2} \int_{\mathbb{T}^2_\ell} \left(|\nabla h_\varepsilon|^2 + \kappa^2 h_\varepsilon^2 \right) dx \\
&\quad - \frac{1}{|\ln \varepsilon|} \left(\frac{3}{\bar{r}_\varepsilon} - \frac{2\bar{\delta}}{\kappa^2} \right) \sum_i A_i^\varepsilon - 2\kappa^2 |\bar{v}^\varepsilon|^2 \ell^2. \qquad (3.35)
\end{aligned}
$$

Also, by direct computation using (3.5) and (3.9) we have

$$
2\kappa^2 |\bar{v}^\varepsilon|^2 \ell^2 = \frac{\bar{\delta}^2 \ell^2}{2\kappa^2} - \min E^0_\varepsilon. \qquad (3.36)
$$

Therefore, combining this with (3.8), (3.32) and (3.35), after passing to the rescaled coordinates and performing the cancellations we find that

$$
\begin{aligned}
\ell^2 F^\varepsilon[u^\varepsilon] \geq T_1 + T_2 + T_3 + T_4 &+ \frac{2}{|\ln \varepsilon|} \int_{\mathbb{T}^2_{\ell\varepsilon}} \left(|\nabla h'_\varepsilon(x')|^2 + \frac{\kappa^2}{|\ln \varepsilon|} |h'_\varepsilon(x')|^2 \right) dx' \\
&- \frac{1}{\pi \bar{r}_\varepsilon^3} \sum_i |\tilde{A}_i^\varepsilon|^2 + o_\varepsilon(1), \qquad (3.37)
\end{aligned}
$$

which is nothing but (3.19).

- *Step 2:* We proceed to absorbing the contributions of the small droplets in (3.24) by (3.21). To that effect, we observe that, for the function

$$
\Phi_\varepsilon(x) := \frac{\gamma^{-5/2}}{4\pi^2 \cdot 3^{2/3}} x (x - \pi \bar{r}_\varepsilon^2)^2 - \frac{1}{\bar{r}_\varepsilon^3} x^2 \geq \frac{\gamma^{-5/2} x}{4\pi^2 \cdot 3^{2/3}} \left\{ \pi^2 \bar{r}_\varepsilon^4 - \left(2\pi \bar{r}_\varepsilon^2 + \frac{\gamma^{5/2}}{\bar{r}_\varepsilon^3} \right) x \right\}, \quad (3.38)
$$

there exists a universal $\gamma \in (0, \frac{1}{6})$ such that $\Phi_\varepsilon(x) \geq x$ whenever $0 \leq x < 3^{2/3}\pi\gamma$ and ε is universally small. Using this observation, we may absorb all the terms with $A_i^\varepsilon < 3^{2/3}\pi\gamma$ appearing in the second term in (3.24) into (3.22) by suitably reducing the coefficient in front of the latter. This proves the result. $\square$

4 Ball construction

The goal of this section is to show (3.15) using the abstract framework of Theorem 3 in [35]. The difficulty in doing this, as in the case of the Ginzburg-Landau model treated in [35], is that the energy density $e'_\varepsilon - \frac{1}{\pi}|\ln\rho_\varepsilon|\sum_{A_i^\varepsilon\geq\beta}|\tilde{A}_i^\varepsilon|^2\tilde{\delta}_i^\varepsilon$ is not positive (or bounded below independently of (u^ε)). The next two subsections are meant to go around this difficulty by showing that this energy density can be modified, by displacing a part of the energy from the regions where the energy density is positive into regions where the energy density is negative in order to bound the modified energy density from below while making only a small enough error. This is achieved by obtaining sharp lower bounds on the energy of the droplets. Since their volumes and shapes are a priori unknown, the terms in M_ε are used to control in a quantitative way the deviations from the droplets being balls of fixed volume.

In this section we perform a ball construction which follows the procedure of [35]. The goal is to cover the droplets $\{\Omega'_{i,\varepsilon}\}$ whose volumes are bounded from below by a given $\beta > 0$ with a finite collection of disjoint closed balls whose radii are smaller than 1, on which we have a good lower bound for the energy in the left-hand side of (3.15). This is possible for sufficiently small ε in view of the fact that $\ell^\varepsilon \to \infty$ and that the leading order asymptotic behavior of the energy from (2.6) yields control on the perimeter and, therefore, the essential diameter of each of $\Omega'_{i,\varepsilon}$. The precise statements are given below. We will also need the following basic result, which holds for sufficiently small ε ensuring that the droplets are smaller than the sidelength of the torus (see the discussion at the beginning of Sec. 2 in [19]).

Lemma 4.1. *There exists $\varepsilon_0 > 0$ depending only on ℓ, κ, $\bar{\delta}$ and $\sup_{\varepsilon>0} F^\varepsilon[u^\varepsilon]$ such that for all $\varepsilon \leq \varepsilon_0$ we have*

$$\operatorname{ess\,diam}(\Omega'_{i,\varepsilon}) \leq c|\partial\Omega'_{i,\varepsilon}|, \tag{4.1}$$

for some universal $c > 0$.

From now on and for the rest of the paper we fix γ to be the constant given in Proposition 3.1 and, as in the previous section, we define $\beta = 3^{2/3}\pi\gamma$. We also introduce the following notation which will be used repeatedly below. To index the droplets, we will use the following definitions:

$$I_\beta := \{i \in \mathbb{N} : A_i^\varepsilon \ \geq \ \beta\}, \quad I_E := \{i \in \mathbb{N} : |\Omega'_{i,\varepsilon} \cap (\mathbb{T}^2_{\ell^\varepsilon} \backslash E)| = 0\}, \quad I_{\beta,E} := I_\beta \cap I_E, \tag{4.2}$$

where $E \subset \mathbb{T}^2_{\ell^\varepsilon}$. For a collection of balls $\mathcal{B}$, the number $r(\mathcal{B})$ (also called the total radius of the collection) denotes the sum of the radii of the balls in $\mathcal{B}$. For simplicity, we will say that a ball B *covers* $\Omega'_{i,\varepsilon}$, if $i \in I_B$.

The principle of the ball construction introduced by Jerrard [20] and Sandier [33] and adapted to the present situation is to start from an initial set, here $\bigcup_{i\in I_{\beta,U}} \Omega'_{i,\varepsilon}$ for a given

$U \subseteq \mathbb{T}^2_{\ell^\varepsilon}$ and cover it by a union of finitely many closed balls with sufficiently small radii. This collection can then be transformed into a collection of *disjoint* closed balls by the procedure, whereby every pair of intersecting balls is replaced by a larger ball whose radius equals the sum of the radii of the smaller balls and which contains the smaller balls. This process is repeated until all the balls are disjoint. The obtained collection will be denoted $\mathcal{B}_0$, its total radius is $r(\mathcal{B}_0)$. Then each ball is dilated by the same factor with respect to its corresponding center. As the dilation factor increases, some balls may touch. If that happens, the above procedure of ball merging is applied again to obtain a new collection of disjoint balls of the same total radius. The construction can be stopped when any desired total radius r is reached, provided that r is universally small compared to ℓ^ε. This yields a collection $\mathcal{B}_r$ covering the initial set and containing a logarithmic energy [20,33].

We now give the statement of our result concerning the ball construction and the associated lower bounds. Throughout the rest of the paper we use the notation $f^+ := \max(f,0)$ and $f^- := -\min(f,0)$.

Proposition 4.2. *Let $U \subseteq \mathbb{T}^2_{\ell^\varepsilon}$ be an open set such that $I_{\beta,U} \neq \varnothing$, and assume that* (2.19) *holds.*

- *There exists $\varepsilon_0 > 0$, $r_0 \in (0,1)$ and $C > 0$ depending only on ℓ, κ, $\bar{\delta}$ and $\sup_{\varepsilon>0} F^\varepsilon[u^\varepsilon]$ such that for all $\varepsilon < \varepsilon_0$ there exists a collection of finitely many disjoint closed balls $\mathcal{B}_0$ whose union covers $\bigcup_{i\in I_{\beta,U}} \Omega'_{i,\varepsilon}$ and such that*

$$r(\mathcal{B}_0) \leq c\varepsilon^{1/3}|\ln\varepsilon|^{1/6} \sum_{i\in I_{\beta,U}} P_i^\varepsilon < r_0, \tag{4.3}$$

  for some universal $c > 0$. Furthermore, for every $r \in [r(\mathcal{B}_0), r_0]$ there is a family of disjoint closed balls $\mathcal{B}_r$ of total radius r covering $\mathcal{B}_0$.

- *For every $B \in \mathcal{B}_r$ such that $B \subset U$ we have*

$$\int_B \left(|\nabla h'_\varepsilon|^2 \, dx' + \frac{\kappa^2}{4|\ln\varepsilon|} |h'_\varepsilon|^2 \right) dx' \geq \frac{1}{2\pi} \left(\ln \frac{r}{r(\mathcal{B}_0)} - cr \right)^+ \sum_{i\in I_{\beta,B}} |\tilde{A}_i^\varepsilon|^2,$$

  for some $c > 0$ depending only on κ and $\bar{\delta}$.

- *If $B \in \mathcal{B}_r$, for any non-negative Lipschitz function χ with support in U, we have*

$$\begin{aligned}\int_B \chi \left(|\nabla h'_\varepsilon|^2 \, dx' + \frac{\kappa^2}{4|\ln\varepsilon|} |h'_\varepsilon|^2 \right) dx' - \frac{1}{2\pi} \left(\ln \frac{r}{r(\mathcal{B}_0)} - cr \right)^+ \sum_{i\in I_{\beta,B}} \chi_i |\tilde{A}_i^\varepsilon|^2 \\ \geq -C\|\nabla\chi\|_\infty \sum_{i\in I_{\beta,B}} |\tilde{A}_i^\varepsilon|^2,\end{aligned}$$

  where $\chi_i := \int_U \chi \tilde{\delta}_i^\varepsilon dx'$, with $\tilde{\delta}_i^\varepsilon(x')$ defined in (3.18)*, for some $c > 0$ depending only on κ and $\bar{\delta}$, and a universal $C > 0$.*

Remark 4.3. *The explanation for the factor of $\frac{1}{4}$ in front of $\frac{\kappa^2}{|\ln\varepsilon|}|h'_\varepsilon|^2$ is that we must 'save' a fraction of this term for the mass displacement argument in Section 5 and in the convergence result in Section 6.*

Proof of the first item. Choose an arbitrary $r_0 \in (0,1)$. As in [19], from the basic lower bound on $\bar{E}^\varepsilon$ (see [19, Equations (2.12) and (2.15)]):

$$\bar{E}^\varepsilon[u^\varepsilon] \geq \frac{1}{|\ln\varepsilon|}\sum_i P_i^\varepsilon - \frac{2\bar{\delta}}{\kappa^2|\ln\varepsilon|}\sum_i A_i^\varepsilon + \frac{2}{\kappa^2\ell^2|\ln\varepsilon|^2}\left(\sum_i A_i^\varepsilon\right)^2, \tag{4.4}$$

where A_i^ε and P_i^ε are defined in (2.2), we obtain with the help of (2.19) that

$$\limsup_{\varepsilon\to 0}\frac{1}{|\ln\varepsilon|}\sum_i A_i^\varepsilon \leq C, \qquad \limsup_{\varepsilon\to 0}\frac{1}{|\ln\varepsilon|}\sum_i P_i^\varepsilon \leq C, \tag{4.5}$$

for some $C>0$ depending only on ℓ, κ, $\bar{\delta}$ and $\sup_{\varepsilon>0} F^\varepsilon[u^\varepsilon]$.

As is well-known, the essential diameter of a connected component of a set of finite perimeter on a torus can be bounded by its perimeter, provided that the latter is universally small compared to the size of the torus (see, e.g., [4]). Therefore, in view of the the definition of P_i^ε in (2.2) and the second of (4.5), for sufficiently small ε it is possible to cover each $\Omega'_{i,\varepsilon}$ with $i \in I_{\beta,U}$ by a closed ball B_i, so that the collection $\tilde{\mathcal{B}}_0$ consisting of all B_i's (possibly intersecting) has total radius

$$r_0(\tilde{\mathcal{B}}_0) \leq C\varepsilon^{1/3}|\ln\varepsilon|^{1/6}\sum_{i\in I_{\beta,U}} P_i^\varepsilon, \tag{4.6}$$

for some universal $C>0$. Furthermore, by the first inequality in (4.5) and the fact that $A_i^\varepsilon \geq \beta$ for all $i \in I_{\beta,U}$ the collection $\tilde{\mathcal{B}}_0$ consists of only finitely many balls. Therefore, we can apply the construction à la Jerrard and Sandier outlined at the beginning of this section to obtain the desired family of balls $\mathcal{B}_0$ and $\mathcal{B}_r$, with $r(\mathcal{B}_0) = r(\tilde{\mathcal{B}}_0)$. The estimate on the radii follows by combining the second of (4.5) and (4.6) and the fact that $\ell^\varepsilon \to \infty$ with the rate depending only on ℓ, for sufficiently small ε depending on ℓ, κ, $\bar{\delta}$, $\sup_{\varepsilon>0} F^\varepsilon[u^\varepsilon]$ and r_0. □

Proof of the second item. Let $B \subset U$ be a ball in the collection $\mathcal{B}_r$. Denote the radius of B by r_B and set

$$X_\varepsilon := \frac{\kappa^2}{|\ln\varepsilon|}\int_B h'_\varepsilon dx'.$$

Integrating (3.16) over B and applying the divergence theorem, we have

$$\int_{\partial B}\frac{\partial h'_\varepsilon}{\partial\nu}\, d\mathcal{H}^1(x') = m_{B,\varepsilon} - X_\varepsilon, \tag{4.7}$$

where

$$m_{B,\varepsilon} := \int_B (\mu'_\varepsilon(x') - \bar{\mu}^\varepsilon)dx' = \sum_{i\in I_B} A_i^\varepsilon + \sum_{i\notin I_B} \theta_i A_i^\varepsilon - \bar{\mu}^\varepsilon |B|,$$

for some $\theta_i \in [0,1)$ representing the volume fraction in B of those droplets that are not covered completely by B, and ν is the inward normal to ∂B. Using the Cauchy-Schwarz inequality, we then deduce from (4.7) that

$$\int_{\partial B} |\nabla h'_\varepsilon|^2 \, d\mathcal{H}^1(x') \geq \frac{1}{2\pi r_B}(m_{B,\varepsilon} - X_\varepsilon)^2 \geq \frac{m_{B,\varepsilon}^2 - 2m_{B,\varepsilon}X_\varepsilon}{2\pi r_B}. \tag{4.8}$$

By another application of the Cauchy-Schwarz inequality, we may write

$$\frac{\kappa^2}{4|\ln\varepsilon|}\int_B |h'_\varepsilon|^2 \, dx' \geq \frac{X_\varepsilon^2}{4\pi r_B^2}\frac{|\ln\varepsilon|}{\kappa^2}. \tag{4.9}$$

We now add (4.8) and (4.9) and optimize the right-hand side over X_ε. We obtain

$$\int_{\partial B} |\nabla h'_\varepsilon|^2 \, d\mathcal{H}^1(x') + \frac{\kappa^2}{4|\ln\varepsilon|}\int_B |h'_\varepsilon|^2 \, dx' \geq \frac{m_{B,\varepsilon}^2}{2\pi r_B}\left(1 - \frac{Cr_B}{|\ln\varepsilon|}\right), \tag{4.10}$$

for $C = \kappa^4$. Recalling that $r_B \leq r \leq r_0 < 1$, we can choose ε sufficiently small depending only on κ so that the term in parentheses above is positive.

Inserting the definition of $m_{B,\varepsilon}$ into (4.10) and discarding some positive terms yields

$$\begin{aligned}
&\int_{\partial B} |\nabla h'_\varepsilon|^2 d\mathcal{H}^1(x') + \frac{\kappa^2}{4|\ln\varepsilon|}\int_B |h'_\varepsilon|^2 \, dx' \\
&\qquad \geq \frac{1}{2\pi r_B}\Big(\sum_{i\in I_B} A_i^\varepsilon + \sum_{i\notin I_B} \theta_i A_i^\varepsilon - \bar{\mu}^\varepsilon|B|\Big)^2 \left(1 - \frac{Cr_B}{|\ln\varepsilon|}\right) \\
&\qquad \geq \frac{1}{2\pi r_B}\Big(\sum_{i\in I_B} A_i^\varepsilon + \sum_{i\notin I_B} \theta_i A_i^\varepsilon\Big)^2 \left(1 - 2\bar{\mu}^\varepsilon|B|\Big(\sum_{i\in I_B} A_i^\varepsilon\Big)^{-1} - \frac{Cr_B}{|\ln\varepsilon|}\right).
\end{aligned} \tag{4.11}$$

We now use the fact that by construction B covers at least one $\Omega'_{i,\varepsilon}$ with $A_i^\varepsilon \geq \beta$. This leads us to

$$\begin{aligned}
&\int_{\partial B} |\nabla h'_\varepsilon|^2 d\mathcal{H}^1(x') + \frac{\kappa^2}{4|\ln\varepsilon|}\int_B |h'_\varepsilon|^2 \, dx' \\
&\qquad \geq \frac{1}{2\pi r_B}\Big(\sum_{i\in I_B} A_i^\varepsilon + \sum_{i\notin I_B} \theta_i A_i^\varepsilon\Big)^2 \left(1 - \frac{2\pi\bar{\mu}^\varepsilon r_B^2}{\beta} - \frac{Cr_B}{|\ln\varepsilon|}\right) \\
&\qquad \geq \frac{1}{2\pi r_B}\sum_{i\in I_{\beta,B}} |\tilde{A}_i^\varepsilon|^2 \, (1 - cr_B)\,,
\end{aligned} \tag{4.12}$$

for some $c > 0$ depending only on κ and $\bar{\delta}$, where in the last line we used that $A_i^\varepsilon \geq \tilde{A}_i^\varepsilon$. Hence there exists $r_0 \in (0,1)$ depending only on κ, and $\bar{\delta}$ such that the right-hand side of (4.12) is positive.

Finally, let us define $\mathcal{F}(x,r) := \int_{B(x,r)} |\nabla h'_\varepsilon|^2 \, dx' + \frac{r\kappa^2}{4|\ln\varepsilon|} \int_{B(x,r)} |h'_\varepsilon|^2 \, dx'$, where $B(x,r)$ is the ball centered at x of radius r. The relation (4.12) then reads for $B(x,r) = B \in \mathcal{B}_r$ and a.e. $r \in (r(\mathcal{B}_0), r_0]$:

$$\frac{\partial \mathcal{F}}{\partial r} \geq \frac{1}{2\pi r} \sum_{i \in I_{\beta,B}} |\tilde{A}_i^\varepsilon|^2 (1 - cr), \tag{4.13}$$

with c as before. Then using [34, Proposition 4.1], for every $B \in \mathcal{B}(s) := \mathcal{B}_r$ with $r = e^s r(\mathcal{B}_0)$ (using the notation of [34, Theorem 4.2]) we have

$$\begin{aligned}
\int_{B \setminus \mathcal{B}_0} |\nabla h'_\varepsilon|^2 \, dx' + \frac{r_B \kappa^2}{4|\ln\varepsilon|} \int_B |h'_\varepsilon|^2 \, dx' &\geq \int_0^s \sum_{\substack{B' \in \mathcal{B}(t) \\ B' \subset B}} \frac{1}{2\pi} \sum_{i \in I_{\beta,B'}} |\tilde{A}_i^\varepsilon|^2 \, (1 - cr(\mathcal{B}(t))) \, dt \\
&= \int_0^s \sum_{\substack{B' \in \mathcal{B}(t) \\ B' \subset B}} \frac{1}{2\pi} \sum_{i \in I_{\beta,B'}} |\tilde{A}_i^\varepsilon|^2 \left(1 - ce^t r(\mathcal{B}_0)\right) dt \\
&\geq \frac{1}{2\pi} \sum_{i \in I_{\beta,B}} |\tilde{A}_i^\varepsilon|^2 \left(\ln \frac{r}{r(\mathcal{B}_0)} - cr \right),
\end{aligned} \tag{4.14}$$

where we observed that the double summation appearing in the first and second lines is simply the summation over $I_{\beta,B}$. Once again, in view of the fact that $r_B \leq 1$ and that both terms in the integrand of the left-hand side of (4.14) are non-negative, this completes the proof of the second item. $\square$

Proof of the third item. This follows [35]. Let χ be a non-negative Lipschitz function with support in U. By the "layer-cake" theorem [21], for any $B \in \mathcal{B}_r$ we have

$$\int_B \chi \left(|\nabla h'_\varepsilon|^2 + \frac{\kappa^2}{4|\ln\varepsilon|} |h'_\varepsilon|^2 \right) dx' = \int_0^{+\infty} \int_{E_t \cap B} \left(|\nabla h'_\varepsilon|^2 + \frac{\kappa^2}{4|\ln\varepsilon|} |h'_\varepsilon|^2 \right) dx' \, dt, \tag{4.15}$$

where $E_t := \{\chi > t\}$. If $i \in I_{\beta,B}$, then by construction for any $s \in [r(\mathcal{B}_0), r]$ there exists a unique closed ball $B_{i,s} \in \mathcal{B}_s$ containing $\Omega'_{i,\varepsilon}$. Therefore, for $t > 0$ we can define

$$s(i,t) := \sup \left\{ s \in [r(\mathcal{B}_0), r] \, : \, B_{i,s} \subset E_t \right\},$$

with the convention that $s(i,t) = r(\mathcal{B}_0)$ if the set is empty. We also let $B_i^t := B_{i,s(i,t)}$ whenever $s(i,t) > r(\mathcal{B}_0)$. Note that for each $i \in I_{\beta,B}$ we have that $t \mapsto s(i,t)$ is a non-increasing function. In particular, we can define $t_i \geq 0$ to be the supremum of the set of t's at which $s(i,t) = r$ (or zero, if this set is empty).

If $t > t_i$ and $s(i,t) > r(\mathcal{B}_0)$, then for any $x \in \Omega'_{i,\varepsilon}$ and any $y \in B_i^t \setminus E_t$ (which is not empty) we have

$$\chi(x) - t \leq \chi(x) - \chi(y) \leq 2s(i,t)\|\nabla\chi\|_\infty. \tag{4.16}$$

Averaging over all $x \in \Omega'_{i,\varepsilon}$, we hence deduce

$$\chi_i - t \leq 2s(i,t)\|\nabla\chi\|_\infty. \tag{4.17}$$

Now, for any $t \geq 0$ the collection $\{B_i^t\}_{i\in I_{\beta,B,t}}$, where $I_{\beta,B,t} := \{i \in I_{\beta,B} : s(i,t) > r(\mathcal{B}_0)\}$ is disjoint. Indeed if $i,j \in I_{\beta,B,t}$ and $s(i,t) \geq s(j,t)$ then, since $\mathcal{B}_{s(i,t)}$ is disjoint, the balls $B_{i,s(i,t)}$ and $B_{j,s(i,t)}$ are either equal or disjoint. If they are disjoint we note that $s(i,t) \geq s(j,t)$ implies that $B_{j,s(j,t)} \subseteq B_{j,s(i,t)}$, and, therefore, $B_j^t = B_{j,s(j,t)}$ and $B_i^t = B_{i,s(i,t)}$ are disjoint. If they are equal and $s(i,t) > s(j,t)$, then $B_{j,s(j,t)} \subset E_t$, contradicting the definition of $s(j,t)$. So $s(j,t) = s(i,t)$ and then $B_j^t = B_i^t$.

Now assume that $B' \in \{B_i^t\}_{i\in I_{\beta,B,t}}$ and let s be the common value of $s(i,t)$ for i's in $I_{\beta,B'}$. Then, the previous item of the proposition yields

$$\int_{B'} \left(|\nabla h'_\varepsilon|^2 + \frac{\kappa^2}{4|\ln\varepsilon|}|h'_\varepsilon|^2\right) dx' \geq \frac{1}{2\pi}\left(\ln\frac{s}{r(\mathcal{B}_0)} - cs\right)^+ \sum_{i\in I_{\beta,B',t}} |\tilde{A}_i^\varepsilon|^2.$$

Summing over $B' \in \{B_i^t\}_{i\in I_{\beta,B,t}}$, we deduce

$$\begin{aligned}\int_{B\cap E_t} \left(|\nabla h'_\varepsilon|^2 + \frac{\kappa^2}{4|\ln\varepsilon|}|h'_\varepsilon|^2\right) dx' &\geq \frac{1}{2\pi}\sum_{i\in I_{\beta,B,t}} |\tilde{A}_i^\varepsilon|^2 \left(\ln\frac{s(i,t)}{r(\mathcal{B}_0)} - cs(i,t)\right)^+ \\ &= \frac{1}{2\pi}\sum_{i\in I_{\beta,B}} |\tilde{A}_i^\varepsilon|^2 \left(\ln\frac{s(i,t)}{r(\mathcal{B}_0)} - cs(i,t)\right)^+,\end{aligned} \tag{4.18}$$

where in the last inequality we took into consideration that all the terms corresponding to $i \in I_{\beta,B} \setminus I_{\beta,B,t}$ give no contribution to the sum in the right-hand side. Integrating the above expression over t and using the fact that $r_0(\mathcal{B}_0) \leq s(i,t) \leq r$ yields

$$\begin{aligned}\int_0^{+\infty}\int_{E_t\cap B} \left(|\nabla h'_\varepsilon|^2 + \frac{\kappa^2}{4|\ln\varepsilon|}|h'_\varepsilon|^2\right) dx'\,dt &\geq \frac{1}{2\pi}\sum_{i\in I_{\beta,B}} |\tilde{A}_i^\varepsilon|^2 \int_0^{\chi_i}\left(\ln\frac{s(i,t)}{r(\mathcal{B}_0)} - cr\right)^+ dt \\ \geq \frac{1}{2\pi}\sum_{i\in I_{\beta,B}} \chi_i|\tilde{A}_i^\varepsilon|^2\left(\ln\frac{r}{r(\mathcal{B}_0)} - cr\right)^+ &+ \frac{1}{2\pi}\sum_{i\in I_{\beta,B}} |\tilde{A}_i^\varepsilon|^2 \int_0^{\chi_i}\ln\frac{s(i,t)}{r}dt.\end{aligned} \tag{4.19}$$

We now concentrate on the last term in (4.19). Using the estimate in (4.17) and the definition of t_i, we can bound the integral in this term as follows

$$\int_0^{\chi_i}\ln\frac{s(i,t)}{r}dt \geq \int_{t_i}^{\chi_i}\ln\left(\frac{\chi_i - t}{2r\|\nabla\chi\|_\infty}\right)dt \geq -C\|\nabla\chi\|_\infty, \tag{4.20}$$

for some universal $C > 0$, which is obtained by an explicit computation and the fact that $r \leq r_0 < 1$. Finally, combining (4.20) with (4.19), the statement follows from (4.15). $\square$

Remark 4.4. *Inspecting the proof, we note that the statements of the proposition are still true with the left-hand sides replaced by* $\int_{B\setminus\mathcal{B}_0} \chi|\nabla h'_\varepsilon|^2\, dx' + \frac{\kappa^2}{4|\ln\varepsilon|}\int_B \chi|h'_\varepsilon|^2 dx'$ *(with* $\chi \equiv 1$ *or* χ *Lipschitz, respectively).*

5 Energy displacement

In this section, we follow the idea of [35] of localizing the ball construction and combine it with a "energy displacement" which allows to reduce to the situation where the energy density in (3.15) is bounded below. For the proposition below we define for all $x' \in \mathbb{T}^2_{\ell^\varepsilon}$:

$$\nu^\varepsilon(x') := \sum_{i\in I_\beta} |\tilde{A}^\varepsilon_i|^2 \tilde{\delta}^\varepsilon_i(x'), \tag{5.1}$$

where $\tilde{\delta}^\varepsilon_i(x')$ is given by (3.18). We also recall that ρ_ε defined in (3.1) is the expected radius of droplets in a minimizing configuration in the blown up coordinates.

We cover $\mathbb{T}^2_{\ell^\varepsilon}$ by the balls of radius $\frac{1}{4}r_0$ whose centers are in $\frac{r_0}{8}\mathbb{Z}^2$. We call this cover $\{U_\alpha\}_\alpha$ and $\{x_\alpha\}_\alpha$ the centers. We also introduce $D_\alpha := B(x_\alpha, \frac{3r_0}{4})$.

Proposition 5.1. *Let* h'_ε *satisfy* (3.16)*, assume* (2.19) *holds, and set*

$$f_\varepsilon := |\nabla h'_\varepsilon|^2 + \frac{\kappa^2}{2|\ln\varepsilon|}|h'_\varepsilon|^2 - \frac{1}{2\pi}|\ln\rho_\varepsilon|\,\nu^\varepsilon. \tag{5.2}$$

Then there exist $\varepsilon_0 > 0$ *as in Proposition 4.2 and constants* $c, C > 0$ *depending only on* $\bar{\delta}$ *and* κ *such that for all* $\varepsilon < \varepsilon_0$*, there exists a family of integers* $\{n_\alpha\}_\alpha$ *and a density* g_ε *on* $\mathbb{T}^2_{\ell^\varepsilon}$ *with the following properties.*

- g_ε *is bounded below:*
$$g_\varepsilon \geq -c\ln^2(M_\varepsilon + 2) \quad \text{on } \mathbb{T}^2_{\ell^\varepsilon}.$$

- *For any* α,
$$n_\alpha^2 \leq C\left(g_\varepsilon(D_\alpha) + c\ln^2(M_\varepsilon+2)\right).$$

- *For any Lipschitz function* χ *on* $\mathbb{T}^2_{\ell^\varepsilon}$ *we have*

$$\left|\int_{\mathbb{T}^2_{\ell^\varepsilon}} \chi(f_\varepsilon - g_\varepsilon)dx'\right| \leq C\sum_\alpha \left(\nu^\varepsilon(U_\alpha) + (n_\alpha + M_\varepsilon)\ln(n_\alpha + M_\varepsilon + 2)\right)\|\nabla\chi\|_{L^\infty(D_\alpha)}. \tag{5.3}$$

Proof. The proof follows the method of [32], involving a localization of the ball construction followed by energy displacement. Here we follow [35, Proposition 4.9]. One key difference is the restriction to I_β which means we cover only those $\Omega'_{i,\varepsilon}$ satisfying $A_i^\varepsilon \geq \beta$ as in Proposition (4.2).

- Step 1: Localization of the ball construction.
We use U_α defined above as the cover on $\mathbb{T}^2_{\ell^\varepsilon}$. For each U_α covering at least one droplet whose volume is greater or equal than β and for any $r \in (r(\mathcal{B}_0), \frac{1}{4}r_0)$ we construct disjoint balls $\mathcal{B}_r^\alpha$ covering all $\Omega'_{i,\varepsilon}$ with $i \in I_{\beta,U_\alpha}$, using Proposition 4.2. Then choosing a small enough $\rho \in (r(\mathcal{B}_0), \frac{1}{4}r_0)$ independent of ε (to be specified below), we may extract from $\cup_\alpha \mathcal{B}_\rho^\alpha$ a disjoint family which covers $\cup_{i\in I_\beta}\Omega'_{i,\varepsilon}$ as follows: Denoting by $\mathcal{C}$ a connected component of $\cup_\alpha \mathcal{B}_\rho^\alpha$, we claim that there exists α_0 such that $\mathcal{C} \subset U_{\alpha_0}$. Indeed if $x \in \mathcal{C}$ and letting λ be a Lebesgue number[1] of the covering of $\mathbb{T}^2_{\ell^\varepsilon}$ by $\{U_\alpha\}_\alpha$ (it is easy to see that in our case $\frac{1}{4}r_0 < \lambda < \frac{1}{2}r_0$), there exists α_0 such that $B(x,\lambda) \subset U_{\alpha_0}$. If $\mathcal{C}$ intersected the complement of U_{α_0}, there would exist a chain of balls connecting x to $(U_{\alpha_0})^c$, each of which would intersect U_{α_0}. Each of the balls in the chain would belong to some $\mathcal{B}_\rho^{\alpha'}$ with α' such that dist $(U_{\alpha'}, U_{\alpha_0}) \leq 2\rho < \frac{1}{2}r_0$. Thus, calling k the universal maximum number of α''s such that dist $(U_{\alpha'}, U_{\alpha_0}) < \frac{1}{2}r_0$, the length of the chain is at most $2k\rho$ and thus $\lambda \leq 2k\rho$. If we choose $\rho < \lambda/(2k)$, this is impossible and the claim is proved. Let us then choose $\rho = \lambda/(4k)$. By the above, each $\mathcal{C}$ is included in some U_α.

We next obtain a disjoint cover of $\cup_{i\in I_\beta}\Omega'_{i,\varepsilon}$ from $\cup_\alpha \mathcal{B}_\rho^\alpha$. Let $\mathcal{C}$ be a connected component of $\cup_\alpha \mathcal{B}_\rho^\alpha$. By the discussion of the preceding paragraph, there exists an index α_0 such that $\mathcal{C} \subset U_{\alpha_0}$. We then remove from $\mathcal{C}$ all the balls which do not belong to $\mathcal{B}_\rho^{\alpha_0}$ and still denote by $\mathcal{B}_\rho^{\alpha_0}$ the obtained collection. We repeat this process for all the connected components and obtain a disjoint cover $\mathcal{B}_\rho = \cup_\alpha \mathcal{B}_\rho^\alpha$ of $\cup_{i\in I_\beta}\Omega'_{i,\varepsilon}$. Note that this procedure uniquely associates an α to a given $B \in \mathcal{B}_\rho$, as well as to each $\Omega'_{i,\varepsilon}$ for a given $i \in I_\beta$ by assigning to it the ball in $\mathcal{B}_\rho$ that covers it, and then the α of this ball. We will use this repeatedly below. We also slightly abuse the notation by sometimes using $\mathcal{B}_\rho^\alpha$ to denote the union of the balls in the family $\mathcal{B}_\rho^\alpha$.

We now proceed to the energy displacement.

- Step 2: Energy displacement in the balls.
Note that by construction every ball in $\mathcal{B}_\rho^\alpha$ is included in U_α. From the last item of Proposition 4.2 applied to a ball $B \in \mathcal{B}_\rho^\alpha$, if ε is small enough then, for any Lipschitz non-negative χ we have for some $c > 0$ depending only on κ and $\bar{\delta}$ and a universal $C > 0$

$$\int_B \chi \left(|\nabla h'_\varepsilon|^2 + \frac{\kappa^2}{4|\ln\varepsilon|}|h'_\varepsilon|^2 \right) dx' - \frac{1}{2\pi}\left(\ln \frac{\rho}{r(\mathcal{B}_0^\alpha)} - c\rho \right)^+ \sum_{i\in I_{\beta,B}} \chi_i |\tilde{A}_i^\varepsilon|^2$$
$$\geq -C\nu^\varepsilon(B)\|\nabla\chi\|_{L^\infty(B)},$$

[1] A Lebesgue number of a covering of a compact set is a number $\lambda > 0$ such that every subset of diameter less than λ is contained in some element of the covering.

where ν^ε is defined by (5.1). Rewriting the above, recalling the definition (3.1) and defining $n_\alpha \geq 1$ to be the number of droplets included in $U_\alpha \supset B$ and satisfying $A_i^\varepsilon \geq \beta$, we have

$$\int_B \chi \left(|\nabla h_\varepsilon'|^2 + \frac{\kappa^2}{4|\ln \varepsilon|} |h_\varepsilon'|^2 \right) dx' - \frac{1}{2\pi} \left(\ln \frac{\rho}{n_\alpha \rho_\varepsilon} - c \right) \sum_{i \in I_{\beta,B}} \chi_i |\tilde{A}_i^\varepsilon|^2 + \int_B \chi \omega_\varepsilon dx' \\ \geq -C\nu^\varepsilon(B) \|\nabla \chi\|_{L^\infty(B)},$$

where we set $\bar{r}_\alpha := \frac{r(\mathcal{B}_0^\alpha)}{\rho_\varepsilon}$ and define (recall that α implicitly depends on $i \in I_\beta$)

$$\omega_\varepsilon(x') := \frac{1}{2\pi} \sum_{i \in I_\beta} |\tilde{A}_i^\varepsilon|^2 \ln \left(\frac{r(\mathcal{B}_0^\alpha)}{n_\alpha \rho_\varepsilon} \right) \tilde{\delta}_i^\varepsilon(x') = \frac{1}{2\pi} \sum_{i \in I_\beta} |\tilde{A}_i^\varepsilon|^2 \ln \left(\frac{\bar{r}_\alpha}{n_\alpha} \right) \tilde{\delta}_i^\varepsilon(x'). \tag{5.4}$$

The quantity ω_ε in some sense measures the discrepancy between the droplets $\Omega_{i,\varepsilon}'$ and balls of radius ρ_ε. We will thus naturally use M_ε in (3.13) to control it. Note also that it is only supported in the droplets, hence in the balls of $\mathcal{B}_\rho$.

Applying Lemma 3.1 of [32] to

$$f_{B,\varepsilon} = \left(|\nabla h_\varepsilon'|^2 + \frac{\kappa^2}{4|\ln \varepsilon|} |h_\varepsilon'|^2 - \frac{1}{2\pi} \left(\ln \frac{\rho}{\rho_\varepsilon n_\alpha} - c \right) \sum_{i \in I_{\beta,B}} |\tilde{A}_i^\varepsilon|^2 \tilde{\delta}_i^\varepsilon + \omega_\varepsilon \right) \mathbf{1}_B$$

we deduce the existence of a positive measure $g_{B,\varepsilon}$ such that

$$\|f_{B,\varepsilon} - g_{B,\varepsilon}\|_{\mathrm{Lip}^*} \leq C\nu^\varepsilon(B), \tag{5.5}$$

where Lip* denotes the dual norm to the space of Lipschitz functions and $C > 0$ is universal.

- Step 3: Energy displacement on annuli and definition of g_ε.
We define a set C_α as follows: recall that ρ was assumed equal to $\lambda/(4k)$, where $\lambda \leq \frac{1}{4} r_0$ and k bounds the number of α''s such that dist $(U_{\alpha'}, U_\alpha) < \frac{1}{2} r_0$ for any given α. Therefore the total radius of the balls in $\mathcal{B}_\rho$ which are at distance less than r_0 from U_α is at most $k\rho = \frac{1}{16} r_0$. In particular, letting T_α denote the set of $t \in (\frac{r_0}{2}, \frac{3r_0}{4})$ such that the circle of center x_α (where we recall x_α is the center of U_α) and radius t does not intersect $\mathcal{B}_\rho^\alpha$, we have $|T_\alpha| \geq \frac{3}{16} r_0$. We let $C_\alpha = \{x \mid |x - x_\alpha| \in T_\alpha\}$ and recall that $D_\alpha = B(x_\alpha, \frac{3r_0}{4})$.

Let $t \in T_\alpha$. Arguing exactly as in the proof of (4.10), we find that

$$\int_{\partial B(x_\alpha, t)} |\nabla h_\varepsilon'|^2 \, d\mathcal{H}^1(x') + \frac{\kappa^2}{4|\ln \varepsilon|} \int_{B(x_\alpha, t)} |h_\varepsilon'|^2 \, dx' \geq \frac{m_{\varepsilon,t}^2}{2\pi t} \left(1 - \frac{\kappa^2 t}{4|\ln \varepsilon|} \right)$$

with $m_{\varepsilon,t} := \int_{B(x_\alpha, t)} (\mu_\varepsilon'(x') - \bar{\mu}^\varepsilon) \, dx'$. Arguing as in (4.12) and using the fact that $B(x_\alpha, \frac{1}{2} r_0)$ contains all the droplets with $i \in I_{\beta, U_\alpha}$, we find that we can take ε sufficiently small depending on κ, and r_0 sufficiently small depending on κ and $\bar{\delta}$ such that for all $t \in T_\alpha$,

$$\int_{\partial B(x_\alpha,t)} |\nabla h'_\varepsilon|^2 \, d\mathcal{H}^1(x') + \frac{\kappa^2}{4|\ln \varepsilon|} \int_{B(x_\alpha,t)} |h'_\varepsilon|^2 \, dx' \geq \frac{1}{4\pi t} \left(\sum_{i \in I_{\beta,U_\alpha}} A_i^\varepsilon \right)^2 .$$

Integrating this over $t \in T_\alpha$, using that $|T_\alpha| \geq \frac{3}{16} r_0$, we obtain that

$$\int_{C_\alpha} |\nabla h'_\varepsilon|^2 dx' + \frac{\kappa^2}{4|\ln \varepsilon|} \int_{D_\alpha} |h'_\varepsilon|^2 dx' \geq c \left(\sum_{i \in I_{\beta,U_\alpha}} A_i^\varepsilon \right)^2 , \tag{5.6}$$

with $c > 0$ depending only on r_0, hence on κ and $\bar{\delta}$.

We now trivially extend the estimate in (5.6) to all α's, including those U_α that contain no droplets of size greater or equal than β. The overlap number of the sets $\{C_\alpha\}_\alpha$, defined as the maximum number of sets to which a given $x' \in \mathbb{T}^2_{\ell^\varepsilon}$ belongs is bounded above by the overlap number of the sets $\{D_\alpha\}_\alpha$, call it k'. Since the latter collection of balls covers the entire $\mathbb{T}^2_{\ell^\varepsilon}$, we have $k' \geq 1$. Then, letting

$$\begin{aligned} f'_\varepsilon := f_\varepsilon - \sum_{B \in \mathcal{B}_\rho} f_{B,\varepsilon} = \left(|\nabla h'_\varepsilon|^2 + \frac{\kappa^2}{2|\ln \varepsilon|} |h'_\varepsilon|^2 \right) \mathbf{1}_{\mathbb{T}^2_{\ell^\varepsilon} \setminus \mathcal{B}_\rho} + \frac{\kappa^2}{4|\ln \varepsilon|} |h'_\varepsilon|^2 \mathbf{1}_{\mathcal{B}_\rho} \\ + \frac{1}{2\pi} \sum_{i \in I_\beta} \left(\ln \frac{\rho}{n_\alpha} - c \right) |\tilde{A}_i^\varepsilon|^2 \tilde{\delta}_i^\varepsilon - \omega_\varepsilon, \end{aligned} \tag{5.7}$$

and

$$f_{\alpha,\varepsilon} := \frac{1}{2k'} \left(|\nabla h'_\varepsilon|^2 + \frac{\kappa^2}{4|\ln \varepsilon|} |h'_\varepsilon|^2 \right) \mathbf{1}_{C_\alpha} + \frac{1}{2\pi} \sum_{i \in I_{\beta,\mathcal{B}_\rho^\alpha}} |\tilde{A}_i^\varepsilon|^2 \left(\ln \frac{\rho}{n_\alpha} - c \right) \tilde{\delta}_i^\varepsilon - \omega_\varepsilon \mathbf{1}_{\mathcal{B}_\rho^\alpha}, \tag{5.8}$$

we have

$$\begin{aligned} f'_\varepsilon - \sum_\alpha f_{\alpha,\varepsilon} &\geq \left(|\nabla h'_\varepsilon|^2 + \frac{\kappa^2}{2|\ln \varepsilon|} |h'_\varepsilon|^2 \right) \mathbf{1}_{\mathbb{T}^2_{\ell^\varepsilon} \setminus \mathcal{B}_\rho} \\ &\quad - \frac{1}{2k'} \sum_\alpha \left(|\nabla h'_\varepsilon|^2 + \frac{\kappa^2}{4|\ln \varepsilon|} |h'_\varepsilon|^2 \right) \mathbf{1}_{C_\alpha} + \frac{\kappa^2}{4|\ln \varepsilon|} |h'_\varepsilon|^2 \mathbf{1}_{\mathcal{B}_\rho} \\ &\geq \frac{1}{2} \left(|\nabla h'_\varepsilon|^2 + \frac{\kappa^2}{2|\ln \varepsilon|} |h'_\varepsilon|^2 \right) \mathbf{1}_{\mathbb{T}^2_{\ell^\varepsilon} \setminus \mathcal{B}_\rho} + \frac{\kappa^2}{4|\ln \varepsilon|} |h'_\varepsilon|^2 \mathbf{1}_{\mathcal{B}_\rho} \geq 0 \end{aligned} \tag{5.9}$$

and from (5.6)

$$\begin{aligned} f_{\alpha,\varepsilon}(D_\alpha) &= \frac{1}{2k'} \int_{C_\alpha} \left(|\nabla h'_\varepsilon|^2 + \frac{\kappa^2}{4|\ln \varepsilon|} |h'_\varepsilon|^2 \right) dx' + \frac{1}{2\pi} \left(\ln \frac{\rho}{n_\alpha} - c \right) \sum_{i \in I_{\beta,\mathcal{B}_\rho^\alpha}} |\tilde{A}_i^\varepsilon|^2 - \omega_\varepsilon(D_\alpha) \\ &\geq c \left(\sum_{i \in I_{\beta,U_\alpha}} A_i^\varepsilon \right)^2 - \frac{1}{2\pi} \ln n_\alpha \sum_{i \in I_{\beta,\mathcal{B}_\rho^\alpha}} |\tilde{A}_i^\varepsilon|^2 - \omega_\varepsilon(D_\alpha) - C \sum_{i \in I_{\beta,\mathcal{B}_\rho^\alpha}} |\tilde{A}_i^\varepsilon|^2, \end{aligned} \tag{5.10}$$

for some $C, c > 0$ depending only on κ and $\bar{\delta}$. Now we combine the middle two terms, using the definition of $\omega_{\varepsilon,\alpha}$ in (5.4), to obtain

$$f_{\alpha,\varepsilon}(D_\alpha) \geq c\left(\sum_{i\in I_{\beta,U_\alpha}} A_i^\varepsilon\right)^2 - \frac{1}{2\pi}\ln \bar{r}_\alpha \sum_{i\in I_{\beta,\mathcal{B}_\rho^\alpha}} |\tilde{A}_i^\varepsilon|^2 - C\sum_{i\in I_{\beta,\mathcal{B}_\rho^\alpha}} |\tilde{A}_i^\varepsilon|^2. \tag{5.11}$$

The next step is to bound $\bar{r}_\alpha$. We separate those $\Omega'_{i,\varepsilon}$ with $A_i^\varepsilon \geq 3^{2/3}\pi\gamma^{-1}$ and those with $A_i^\varepsilon < 3^{2/3}\pi\gamma^{-1}$. We denote (with s for "small" and b for "big")

$$\begin{aligned}
I_{\beta,\alpha}^s &= \left\{ i \in I_{\beta,U_\alpha} : A_i^\varepsilon \leq 3^{2/3}\pi\gamma^{-1} \right\},\\
I_{\beta,\alpha}^b &= I_{\beta,U_\alpha}\backslash I_{\beta,\alpha}^s,\\
n_{\alpha_s} &= \# I_{\beta,\alpha}^s.
\end{aligned}$$

For the small droplets, we use the obvious bound

$$\sum_{i\in I_{\beta,\alpha}^s} |A_i^\varepsilon|^{1/2} \leq \ c n_{\alpha_s}, \tag{5.12}$$

with a universal $c > 0$, while for the large droplets we use that in view of the definition of M_ε in (3.13) we have

$$\sum_{i\in I_{\beta,\alpha}^b} |A_i^\varepsilon|^{1/2} \leq C \sum_{i\in I_{\beta,\alpha}^b} A_i^\varepsilon \leq C' M_\varepsilon, \tag{5.13}$$

for some universal $C, C' > 0$. We can now proceed to controlling $\bar{r}_\alpha$. By (3.1) and (4.3), for universally small ε we have

$$\bar{r}_\alpha \leq C \sum_{i\in I_{\beta,U_\alpha}} P_i^\varepsilon, \tag{5.14}$$

for some universal $C > 0$. In view of (3.13), (5.13) and (5.12), we deduce from Remark 3.2 that for universally small ε we have

$$\begin{aligned}
\bar{r}_\alpha &\leq C\Big(M_\varepsilon + \sqrt{4\pi}\sum_{i\in I_{\beta,U_\alpha}} |A_i^\varepsilon|^{1/2}\Big)\\
&\leq C\Big(M_\varepsilon + c n_{\alpha_s} + C' M_\varepsilon\Big) \leq C''(n_{\alpha_s} + M_\varepsilon) \leq C''(1 + n_{\alpha_s} + M_\varepsilon),
\end{aligned}\tag{5.15}$$

where $c, C, C', C'' > 0$ are universal. Therefore, (5.11) becomes

$$\begin{aligned}
f_{\alpha,\varepsilon}(D_\alpha) &\geq c\left(\sum_{i\in I_{\beta,\alpha}^s} A_i^\varepsilon\right)^2 + c\left(\sum_{i\in I_{\beta,\alpha}^b} A_i^\varepsilon\right)^2 - C\ln \bar{r}_\alpha \sum_{i\in I_{\beta,\mathcal{B}_\rho^\alpha}} |\tilde{A}_i^\varepsilon|^2 - C''' \sum_{i\in I_{\beta,\mathcal{B}_\rho^\alpha}} |\tilde{A}_i^\varepsilon|^2,\\
&\geq c\beta^2 n_{\alpha_s}^2 + c\left(\sum_{i\in I_{\beta,\alpha}^b} A_i^\varepsilon\right)^2 - C'\ln(C''(1 + n_{\alpha_s} + M_\varepsilon))\left(n_{\alpha_s} + \sum_{i\in I_{\beta,\alpha}^b} A_i^\varepsilon\right),
\end{aligned}\tag{5.16}$$

where $C, C' > 0$ are universal, $c, C'', C''' > 0$ depend only on κ and $\bar{\delta}$, and C' was chosen so that $C|\tilde{A}_i^\varepsilon|^2 \le C'(A_i^\varepsilon + 1)$.

We now claim that this implies that

$$f_{\alpha,\varepsilon}(D_\alpha) \ge \frac{c}{2}\beta^2 n_{\alpha_s}^2 + \frac{c}{2}\left(\sum_{i\in I_{\beta,\alpha}^b} A_i^\varepsilon\right)^2 - C''' \ln^2(M_\varepsilon + 2), \tag{5.17}$$

where $C''' > 0$ depends only on κ and $\bar{\delta}$. This is seen by minimization of the right-hand side, as we now detail. For the rest of the proof, all constants will depend only on κ and $\bar{\delta}$. For shortness, we will set $X := \sum_{i\in I_{\beta,\alpha}^b} A_i^\varepsilon$.

First assume $n_{\alpha_s} = 0$. Then (5.16) can be rewritten

$$f_{\alpha,\varepsilon}(D_\alpha) \ge cX^2 - C' \ln(C''(1 + M_\varepsilon)))X,$$

By minimization of the quadratic polynomial in the right-hand side, we easily see that an inequality of the form (5.17) holds. Second, let us consider the case $n_{\alpha_s} \ge 1$. We may use the obvious inequality $\log(1 + x + y) \le \log(1 + x) + \log(1 + y)$ that holds for all $x \ge 0$ and $y \ge 0$ to bound from below

$$\begin{aligned}\frac{c}{2}\beta^2 n_{\alpha_s}^2 + \frac{c}{2}X^2 - C' \ln(C''(1 + n_{\alpha_s} + M_\varepsilon))(n_{\alpha_s} + X) \ge \frac{c}{2}\beta^2 n_{\alpha_s}^2 + \frac{c}{2}X^2 \\ - C(n_{\alpha_s} + X) - Cn_{\alpha_s} \ln(n_{\alpha_s} + 1) - CX \ln(n_{\alpha_s} + 1) - C \ln(M_\varepsilon + 1)(n_{\alpha_s} + X).\end{aligned} \tag{5.18}$$

It is clear that the first three negative terms on the right-hand side can be absorbed into the first two positive terms, at the expense of a possible additive constant, which yields

$$\begin{aligned}\frac{c}{2}\beta^2 n_{\alpha_s}^2 + \frac{c}{2}X^2 - C' \ln(C''(n_{\alpha_s} + M_\varepsilon))(n_{\alpha_s} + X) \\ \ge \frac{c}{4}\beta^2 n_{\alpha_s}^2 + \frac{c}{4}X^2 - C \ln(M_\varepsilon + 1)(n_{\alpha_s} + X) - C.\end{aligned} \tag{5.19}$$

Then by quadratic optimization the right hand side of (5.19) is bounded below by $-C \ln^2(M_\varepsilon + 2)$ (after possibly changing the constant). Inserting this into (5.16), we obtain (5.17).

We then apply [32, Lemma 3.2] over D_α to $f_{\alpha,\varepsilon} + C'''|D_\alpha|^{-1} \ln^2(M_\varepsilon + 2)$, where C''' is the constant in the right-hand side of (5.17). We then deduce the existence of a measure $g_{\alpha,\varepsilon}$ on $\mathbb{T}^2_{\ell^\varepsilon}$ supported in D_α such that $g_{\alpha,\varepsilon} \ge -C'''|D_\alpha|^{-1} \ln^2(M_\varepsilon + 2)$ and such that for every Lipschitz function χ

$$\begin{aligned}\left|\int_{D_\alpha} \chi(f_{\alpha,\varepsilon} - g_{\alpha,\varepsilon})\, dx'\right| \le 2 \operatorname{diam}\,(D_\alpha)\|\nabla\chi\|_{L^\infty(D_\alpha)} f^-_{\alpha,\varepsilon}(D_\alpha) \\ \le C \ln(n_{\alpha_s} + M_\varepsilon + 2)\|\nabla\chi\|_{L^\infty(D_\alpha)} \sum_{i\in I_{\beta,\mathcal{B}_\rho^\alpha}} |\tilde{A}_i^\varepsilon|^2,\end{aligned} \tag{5.20}$$

and we have used the observation that

$$f_{\alpha,\varepsilon} = \frac{1}{2k'}\left(|\nabla h'_\varepsilon|^2 + \frac{\kappa^2}{4|\ln\varepsilon|}|h'_\varepsilon|^2\right)\mathbf{1}_{C_\alpha} + \frac{1}{2\pi}\sum_{i\in I_{\beta,\mathcal{B}_\rho^\alpha}}\left(\ln\frac{\rho}{\bar r_\alpha} - C\right)|\tilde A_i^\varepsilon|^2\delta_i^\varepsilon, \tag{5.21}$$

and (5.15) to bound the negative part of $f_{\alpha,\varepsilon}$. In particular, taking $\chi = 1$, we deduce, in view of (5.17), that

$$g_{\alpha,\varepsilon}(D_\alpha) = f_{\alpha,\varepsilon}(D_\alpha) \geq \frac{c}{2}\beta^2 n_{\alpha_s}^2 + \frac{c}{2}\left(\sum_{i\in I_{\beta,\alpha}^b} A_i^\varepsilon\right)^2 - C''' \ln^2(M_\varepsilon + 2), \tag{5.22}$$

from which it follows that

$$g_{\alpha,\varepsilon}(D_\alpha) \geq c'\left(n_{\alpha_s}^2 + (\#I_{\beta,\alpha})^2\right) - C''' \ln^2(M_\varepsilon+2) \geq \frac{1}{2}c' n_\alpha^2 - C'''\ln^2(M_\varepsilon+2). \tag{5.23}$$

Recalling the positivity of $g_{B,\varepsilon}$ introduced in Step 2, we now let

$$g_\varepsilon := \sum_{B\in\mathcal{B}_\rho} g_{B,\varepsilon} + \sum_\alpha g_{\alpha,\varepsilon} + \left(f'_\varepsilon - \sum_\alpha f_{\alpha,\varepsilon}\right), \tag{5.24}$$

and observe that since $f'_\varepsilon - \sum_\alpha f_{\alpha,\varepsilon}$ is also non-negative by (5.9), and since $\sum_\alpha g_{\alpha,\varepsilon}$ is bounded below by $-k'C'''|D_\alpha|^{-1}\ln^2(M_\varepsilon+2)$, where, as before, k' is the overlap number of $\{D_\alpha\}_\alpha$, we have $g_\varepsilon \geq -c\ln^2(M_\varepsilon+2)$ for some $c>0$ depending only on κ and $\bar\delta$, which proves the first item. The second item follows from (5.23), (5.24) and the positiveness of $g_{B,\varepsilon}$ and $(f'_\varepsilon - \sum_\alpha f_{\alpha,\varepsilon})$.

- *Step 4: Proof of the last item.*

Using the definition of g_ε in (5.24), for any Lipschitz χ we have

$$\int_{\mathbb{T}^2_{\ell^\varepsilon}} \chi g_\varepsilon dx' = \sum_{B\in\mathcal{B}_\rho}\int_{\mathbb{T}^2_{\ell^\varepsilon}} \chi g_{B,\varepsilon} dx' + \sum_\alpha \int_{\mathbb{T}^2_{\ell^\varepsilon}} \chi(g_{\alpha,\varepsilon} - f_{\alpha,\varepsilon})dx' + \int_{\mathbb{T}^2_{\ell^\varepsilon}} \chi f'_\varepsilon dx'.$$

Hence, in view of (5.5), (5.7) and (5.20) we obtain for some $C>0$

$$\begin{aligned}\left|\int_{\mathbb{T}^2_{\ell^\varepsilon}} \chi(f_\varepsilon - g_\varepsilon)dx'\right| &\leq \sum_{B\in\mathcal{B}_\rho}\left|\left(\int_{\mathbb{T}^2_{\ell^\varepsilon}}\chi(g_{B,\varepsilon} - f_{B,\varepsilon})dx'\right)\right| + \sum_\alpha\left|\int_{\mathbb{T}^2_{\ell^\varepsilon}}\chi(g_{\alpha,\varepsilon} - f_{\alpha,\varepsilon})dx'\right| \\ &\leq C\sum_{B\in\mathcal{B}_\rho}\nu^\varepsilon(B)\|\nabla\chi\|_{L^\infty(B)} + C\sum_\alpha \ln(n_{\alpha_s} + M_\varepsilon + 2)\|\nabla\chi\|_{L^\infty(D_\alpha)}\sum_{i\in I_{\beta,\mathcal{B}_\rho^\alpha}}|\tilde A_i^\varepsilon|^2. \end{aligned} \tag{5.25}$$

Using that $|\tilde A_i^\varepsilon|^2 \leq C(A_i^\varepsilon + 1)$ for a universal $C>0$ and (5.13), we have

$$\sum_{i\in I_{\beta,\mathcal{B}_\rho^\alpha}} |\tilde A_i^\varepsilon|^2 \leq C(n_{\alpha_s} + M_\varepsilon).$$

Since $n_{\alpha_s} \leq n_\alpha$, the third item follows from (5.25). □

We now apply Proposition 5.1 to establish uniform bounds on M_ε, which characterizes the deviation of the droplets from the optimal shape.

Proposition 5.2. *If* (2.19) *holds, then M_ε is bounded by a constant depending only on $\sup_{\varepsilon>0} F^\varepsilon[u^\varepsilon]$, κ, $\bar\delta$ and ℓ.*

Proof. From the last item of Proposition 5.1 applied with $\chi \equiv 1$ together with the first item, we have

$$\int_{\mathbb{T}^2_{\ell\varepsilon}} f_\varepsilon dx' = \int_{\mathbb{T}^2_{\ell\varepsilon}} g_\varepsilon dx' \geq -C|\ln\varepsilon| \ln^2(M_\varepsilon + 2),$$

with some $C > 0$ depending only on κ, $\bar\delta$ and ℓ, while from (2.19), (3.12) and (5.2), we have

$$C' \geq \ell^2 F^\varepsilon[u^\varepsilon] \geq M_\varepsilon + \frac{2}{|\ln\varepsilon|} \int_{\mathbb{T}^2_{\ell\varepsilon}} f_\varepsilon dx' + o(1) \geq M_\varepsilon - C\ln^2(M_\varepsilon + 2) + o_\varepsilon(1),$$

for some $C' > 0$ depending only on $\sup_{\varepsilon>0} F^\varepsilon[u^\varepsilon]$, κ, $\bar\delta$ and ℓ. The claimed result easily follows. □

With the help of Proposition 5.2, an immediate consequence of Proposition 5.1 is the following conclusion.

Corollary 5.3. *There exists $C > 0$ depending only on κ, $\bar\delta$, ℓ and $\sup_{\varepsilon>0} F^\varepsilon[u^\varepsilon]$ such that if g_ε is as in Proposition 5.1 and* (2.19) *holds, then $g_\varepsilon \geq -C$.*

In the following, we also define the modified energy density $\bar g_\varepsilon$, in which we include back the positive terms of M_ε and a half of $\frac{\kappa^2}{|\ln\varepsilon|}|h'_\varepsilon|^2$ that had been "kept aside" instead of being included in f_ε:

$$\bar g_\varepsilon := g_\varepsilon + \frac{\kappa^2}{2|\ln\varepsilon|}|h'_\varepsilon|^2 + |\ln\varepsilon| \Bigg\{ \sum_i (P_i^\varepsilon - \sqrt{4\pi A_i^\varepsilon}\,\tilde\delta_i) + c_1 \sum_{A_i^\varepsilon > \pi 3^{2/3}\gamma^{-1}} A_i^\varepsilon \tilde\delta_i \\ + c_2 \sum_{\beta \leq A_i^\varepsilon \leq \pi 3^{2/3}\gamma^{-1}} (A_i^\varepsilon - \pi \bar r_\varepsilon^2)^2 \tilde\delta_i + c_3 \sum_{A_i^\varepsilon < \beta} A_i^\varepsilon \tilde\delta_i \Bigg\} \quad (5.26)$$

where we recall $\bar r_\varepsilon = \left(\frac{|\ln\varepsilon|}{|\ln\rho_\varepsilon|}\right)^{1/3}$ and $\tilde\delta_i^\varepsilon$ is defined by (3.18). These extra terms will be used to control the shapes and sizes of the droplets as well as to control h'_ε. We also point out that in view of (5.2), (5.3) and (3.12), we have

$$\ell^2 F^\varepsilon[u^\varepsilon] \geq \frac{2}{|\ln\varepsilon|} \int_{\mathbb{T}^2_{\ell\varepsilon}} \bar g_\varepsilon dx' + o_\varepsilon(1). \quad (5.27)$$

6 Convergence

In this section we study the consequences of the hypothesis

$$\forall R > 0,\ \mathcal{C}_R := \limsup_{\varepsilon \to 0} \int_{K_R} \bar{g}_\varepsilon(x + x_\varepsilon^0)dx < +\infty, \tag{6.1}$$

where $K_R = [-R, R]^2$ and (x_ε^0) is such that $x_\varepsilon^0 + K_R \subset \mathbb{T}^2_{\ell^\varepsilon}$. This corresponds to "good" blow up centers x_ε^0, and will be satisfied for most of them.

In order to obtain $o_\varepsilon(1)$ estimates on the energetic cost of each droplet under this assumption, we need good quantitative estimates for the deviations of the shape of the droplets from balls of the same volume. A convenient quantity that can be used to characterize these deviations is the *isoperimetric deficit*, defined as (in two space dimensions)

$$D(\Omega'_{i,\varepsilon}) := \frac{|\partial \Omega'_{i,\varepsilon}|}{\sqrt{4\pi |\Omega'_{i,\varepsilon}|}} - 1. \tag{6.2}$$

The isoperimetric deficit may be used to bound several types of geometric characteristics of $\Omega'_{i,\varepsilon}$ that measure their deviations from balls. The quantitative isoperimetric inequality, which holds for any set of finite perimeter, may be used to estimate the measure of the symmetric difference between $\Omega'_{i,\varepsilon}$ and a ball. More precisely, we have [17]

$$\alpha(\Omega'_{i,\varepsilon}) \leq C\sqrt{D(\Omega'_{i,\varepsilon})}, \tag{6.3}$$

where $C > 0$ is a universal constant and $\alpha(\Omega'_{i,\varepsilon})$ is the Fraenkel asymmetry defined as

$$\alpha(\Omega'_{i,\varepsilon}) := \min_B \frac{|\Omega'_{i,\varepsilon} \triangle B|}{|\Omega'_{i,\varepsilon}|}, \tag{6.4}$$

where $\triangle$ denotes the symmetric difference between the two sets, and the infimum is taken over balls B with $|B| = |\Omega'_{i,\varepsilon}|$. In the following, we will use the notation r_i^ε and a_i^ε for the radii and the centers of the balls that minimize $\alpha(\Omega'_{i,\varepsilon})$, respectively.

On the other hand, in two space dimensions the following inequality due originally to Bonnesen [6] (for a review, see [30]) is applicable to $\Omega'_{i,\varepsilon}$:

$$R_i^\varepsilon \leq r_i^\varepsilon \left(1 + c\sqrt{D(\Omega'_{i,\varepsilon})}\right). \tag{6.5}$$

Here R_i^ε is the radius of the circumscribed circle of the measure theoretic interior of $\Omega'_{i,\varepsilon}$ and $c > 0$ is universal. Indeed, apply Bonnesen inequality to the saturation of $\Omega'_{i,\varepsilon}$ (i.e., the set with no holes) for each droplet. Then since the set $\Omega'_{i,\varepsilon}$ is connected and, therefore, its saturation has, up to negligible sets, a Jordan boundary [4], Bonnesen inequality applies to it.

6.1 Main result

We will obtain local lower bounds in terms of the renormalized energy for a finite number of Dirac masses in the manner of [5]:

Definition 6.1. *For any function* χ *and* $\varphi \in \mathcal{A}_m$ *(cf. Definition 2.1), we denote*

$$W(\varphi, \chi) = \lim_{\eta \to 0} \left(\frac{1}{2} \int_{\mathbb{R}^2 \setminus \cup_{p \in \Lambda} B(p,\eta)} \chi |\nabla \varphi|^2 dx + \pi \ln \eta \sum_{p \in \Lambda} \chi(p) \right). \tag{6.6}$$

We now state the main result of this section and postpone its proof to Section 6.2. Throughout the section, we use the notation of Sec. 5. To further simplify the notation, we periodically extend all the measures defined on $\mathbb{T}^2_{\ell^\varepsilon}$ to the whole of $\mathbb{R}^2$, without relabeling them. We also periodically extend the ball constructions to the whole of $\mathbb{R}^2$. This allows us to set, without loss of generality, all $x^0_\varepsilon = 0$.

Theorem 4. *Under assumption* (2.19)*, the following holds.*

1. *Assume that for any* $R > 0$ *we have*

$$\limsup_{\varepsilon \to 0} \bar{g}_\varepsilon(K_R) < +\infty, \tag{6.7}$$

where $K_R = [-R, R]^2$*. Then, up to a subsequence, the measures* μ'_ε*, defined in* (3.17)*, converge in* $(C_0(\mathbb{R}^2))^*$ *to a measure of the form* $\nu = 3^{2/3}\pi \sum_{a \in \Lambda} \delta_a$ *where* Λ *is a discrete subset of* $\mathbb{R}^2$*, and* $\{\varphi^\varepsilon\}_\varepsilon$ *defined in* (2.17) *converge weakly in* $\dot{W}^{1,p}_{loc}(\mathbb{R}^2)$ *for any* $p \in (1, 2)$ *to* φ *which satisfies*

$$-\Delta \varphi = 2\pi \sum_{a \in \Lambda} \delta_a - m \text{ in } \mathbb{R}^2,$$

in the distributional sense, with $m = 3^{-2/3}(\bar{\delta} - \bar{\delta}_c)$*. Moreover, for any sequence* $\{\Omega_{i_\varepsilon,\varepsilon}\}_\varepsilon$ *which remains in* K_R*, up to a subsequence, the following two alternatives hold:*

   i. *Either* $A^\varepsilon_{i_\varepsilon} \leq \frac{C_R}{|\ln \varepsilon|}$ *and* $P^\varepsilon_{i_\varepsilon} \leq \frac{C_R}{\sqrt{|\ln \varepsilon|}}$ *as* $\varepsilon \to 0$*,*

   ii. *Or* $A^\varepsilon_{i_\varepsilon}$ *is bounded below by a positive constant as* $\varepsilon \to 0$*, and*

$$A^\varepsilon_{i_\varepsilon} \to 3^{2/3}\pi \text{ and } P^\varepsilon_{i_\varepsilon} \to 2 \cdot 3^{1/3}\pi \quad \text{as } \varepsilon \to 0,$$

with

$$\alpha(\Omega'_{i_\varepsilon,\varepsilon}) \leq \frac{C_R}{|\ln \varepsilon|^{1/2}} \quad \text{as } \varepsilon \to 0, \tag{6.8}$$

for some $C_R > 0$ independent of ε.

2. *If we replace (6.7) by the stronger assumption*

$$\limsup_{\varepsilon\to 0} \bar{g}_\varepsilon(K_R) < CR^2, \tag{6.9}$$

where $C > 0$ is independent of R, then we have for any $p \in (1,2)$,

$$\limsup_{R\to+\infty} \left(\frac{1}{|K_R|} \int_{K_R} |\nabla\varphi|^p dx \right) < +\infty. \tag{6.10}$$

Moreover, for every family $\{\chi_R\}_{R>0}$ defined in Definition 2.3 we have

$$\liminf_{\varepsilon\to 0} \int_{\mathbb{R}^2} \chi_R \bar{g}_\varepsilon dx \geq \frac{3^{4/3}}{2} W(\varphi, \chi_R) + \frac{3^{4/3}\pi}{8} \sum_{a\in\Lambda} \chi_R(a) + o(|K_R|). \tag{6.11}$$

Remark 6.2. *We point out that it is included in Part 1 of Theorem 4 that at most one droplet $\Omega'_{i_\varepsilon,\varepsilon}$ with $A_{i_\varepsilon,\varepsilon}$ bounded from below converges to $a \in \Lambda$. Indeed otherwise in the first item we would have $\mu'_\varepsilon \to 3^{2/3}\pi n_a \sum_{a\in\Lambda} \delta_a$ where $n_a > 1$ is the number of non-vanishing droplets converging to the point a.*

Theorem 4 relies crucially on the following proposition which establishes bounds needed for compactness. Each of the bounds relies on (6.7). Throughout the rest of this section, all constants are assumed to implicitly depend on κ, $\bar{\delta}$, ℓ and $\sup_{\varepsilon>0} F^\varepsilon[u^\varepsilon]$.

Lemma 6.3. *Let $\bar{g}_\varepsilon$ be as above, assume (6.7) holds and denote $\mathcal{C}_R = \limsup_{\varepsilon\to 0} \bar{g}_\varepsilon(K_R)$. Then for any R and ε small enough depending on R we have*

$$\sum_{\alpha|U_\alpha\subset K_R} n_\alpha^2 \leq C(\mathcal{C}_{R+C} + R^2), \tag{6.12}$$

$$\sum_{i\in I_{\beta,K_R}} A_i^\varepsilon \leq C(\mathcal{C}_{R+C} + R^2), \tag{6.13}$$

$$\left| \int_{K_R} \chi_R (f_\varepsilon - g_\varepsilon) dx \right| \leq C \sum_{\alpha|U_\alpha\subset K_{R+C}\setminus K_{R-C}} (n_\alpha + 1)\ln(n_\alpha + 2) \leq C(\mathcal{C}_{R+C} + R^2), \tag{6.14}$$

where $\{\chi_R\}$ is as in Definition 2.3 and $n_\alpha = \#I_{\beta,U_\alpha}$, with U_α as in the proof of Proposition 5.1, for some $C > 0$ independent of ε or R. Furthermore, for any $p \in (1,2)$ there exists a $C_p > 0$ depending on p such that for any $R > 0$ and ε small enough

$$\int_{K_R} |\nabla h'_\varepsilon|^p dx \leq C_p(\mathcal{C}_{R+C} + R^2). \tag{6.15}$$

Proof. First observe that the rescaled droplet volumes and perimeters A_i^ε and P_i^ε are bounded independently of ε, as follows from Proposition 5.2 and the definition of M_ε. Then, (6.12) and (6.13) are a consequence of (6.7), the second item in Proposition 5.1 together with the upper bound on M_ε. The first inequality appearing in (6.14) follows from item 3 of Proposition 5.1 with the bound on M_ε, where we took into consideration that only those D_α that are in the $O(1)$ neighborhood of the support of $|\nabla\chi_R|$ contribute to the sum, along with the observation that the mass of ν^ε (of (5.1)) is now controlled by n_α (a consequence of the above fact that all droplet volumes are uniformly bounded). The second inequality in (6.14) follows from (6.12). The bound (6.15) is a consequence of Proposition 4.2 and follows as in [32] and [35]. We refer the reader to [32], Lemma 4.6 or [35] Lemma 4.6 for the proof in a slightly simpler setting. □

6.2 Lower bound by the renormalized energy (Proof of Theorem 4)

We start by proving the first assertions of the theorem.

- *Step 1: All limit droplets have optimal sizes.* From (5.26), (6.7) and Corollary 5.3, for all ε sufficiently small depending on R we have

$$\int_{K_R}\Bigg(\sum_i\left(P_i-\sqrt{4\pi|A_i^\varepsilon|}\right)\tilde\delta_i^\varepsilon+c_1\sum_{A_i^\varepsilon>3^{2/3}\pi\gamma^{-1}}A_i^\varepsilon\tilde\delta_i^\varepsilon \\ +c_2\sum_{\beta\le A_i^\varepsilon\le\pi 3^{2/3}\gamma^{-1}}(A_i^\varepsilon-\pi\bar r_\varepsilon^2)^2\tilde\delta_i^\varepsilon+c_3\sum_{A_i^\varepsilon<\beta}A_i^\varepsilon\tilde\delta_i^\varepsilon\Bigg)dx\le\frac{C_R}{|\ln\varepsilon|},\quad(6.16)$$

where we recall that all the terms in the sums are nonnegative. It then easily follows that for all $i\in I_{K_R}$ the droplets with $A_i^\varepsilon>3^{2/3}\pi\gamma^{-1}$ do not exist when ε is small enough depending on R, and those with $A_i^\varepsilon<\beta$ satisfy $A_i^\varepsilon=C_R|\ln\varepsilon|^{-1}$ and $P_i^\varepsilon\le C_R|\ln\varepsilon|^{-1/2}$, for some $C_R>0$ independent of ε. This establishes item (i) of Part 1 of the theorem.

It remains to treat the case of $A_i^\varepsilon\in[\beta,3^{2/3}\pi\gamma^{-1}]$ when ε is small enough. It follows from (6.16) that

$$D(\Omega'_{i,\varepsilon})\le\frac{C_R}{|\ln\varepsilon|},\quad(6.17)$$

for some $C_R>0$ independent of ε, and since $\bar r_\varepsilon=3^{1/3}+o_\varepsilon(1)$, for all these droplets (or equivalently for all droplets with $A_i^\varepsilon\ge\beta$) we must have

$$A_i^\varepsilon\to3^{2/3}\pi\quad\text{and}\quad P_i^\varepsilon\to2\cdot3^{1/3}\pi\ \text{ as }\varepsilon\to0.\quad(6.18)$$

Using (6.3), (6.8) easily follows from (6.18) and (6.16).

- Step 2: Convergence results.
From boundedness of A_i^ε, (6.13) and (6.16) we know that $\# I_{\beta,K_R}$ and $\mu'_\varepsilon(K_R)$ are both bounded independently of ε as $\varepsilon \to 0$. We easily deduce from this, the previous step and the definition of μ'_ε that up to extraction, μ'_ε converges in each K_R to at most finitely many point masses which are integer multiples of $3^{2/3}\pi$ and, hence, to a measure of the form $\nu = 3^{2/3}\pi \sum_{a\in\Lambda} d_a \delta_a$, where $d_a \in \mathbb{N}$ and Λ is a discrete set in the whole of $\mathbb{R}^2$. In view of (6.15), we also have $h'_\varepsilon \rightharpoonup h \in \dot{W}^{1,p}_{loc}(\mathbb{R}^2)$ as $\varepsilon \to 0$, up to extraction (recall that we work with equivalence classes from (2.10)). Finally, from the definition of $\bar{g}_\varepsilon$ in (5.26) and the bound (6.7) we deduce that

$$\frac{\kappa^2}{|\ln\varepsilon|}\int_{K_R} |h'_\varepsilon|^2 \le C_R$$

from which it follows that $|\ln\varepsilon|^{-1} h'_\varepsilon$ tends to 0 in $L^2_{loc}(\mathbb{R}^2)$ as $\varepsilon \to 0$. Passing to the limit in the sense of distributions in (3.16), we then deduce from the above convergences that we must have

$$-\Delta h = 3^{2/3}\pi \sum_{a\in\Lambda} d_a \delta_a - \bar{\mu} \quad \text{ on } \mathbb{R}^2. \tag{6.19}$$

We will show below that $d_a = 1$ for every $a \in \Lambda$, and when this is done, this will complete the proof of the first item after recalling $\varphi^\varepsilon = 2\cdot 3^{-2/3} h'_\varepsilon$ and $m = 2\cdot 3^{-2/3}\bar{\mu}$.

- Step 3: There is only one droplet converging to any limit point a.
In order to prove this statement, we examine lower bounds for the energy. Fix $R > 1$ such that $\partial K_R \cap \Lambda = \varnothing$ and consider $a \in \Lambda \cap K_R$. From Step 1, (2.2) and Lemma 4.1, for any $\eta \in (0, \frac{1}{2})$ such that $\eta < \frac{1}{2}\min_{b\in\Lambda\cap K_R\setminus\{a\}} |a-b|$ and for all $r < \eta$, all the droplets converging to a are covered by $B(a,r)$, and $B(a,\eta)$ contains no other droplets with $A_i^\varepsilon \ge \beta$, for ε small enough. There are $d_a \ge 1$ droplets in $B(a,r)$ such that $A_i^\varepsilon \to 3^{2/3}\pi$ as $\varepsilon \to 0$, let us relabel them as $\Omega'_{1,\varepsilon}, \ldots, \Omega'_{d_a,\varepsilon}$.

Let $U = B(a,\eta)$. Arguing as in the proof of the first item of Proposition 4.2, by (6.18), we may construct a collection $\mathcal{B}_0$ of disjoint closed balls covering $\bigcup_{i\in I_{\beta,U}} \Omega'_{i,\varepsilon}$ and satisfying

$$r(\mathcal{B}_0) \le C d_a \rho_\varepsilon < \eta, \tag{6.20}$$

for some universal $C > 0$, provided ε is small enough, and a collection of disjoint balls $\mathcal{B}_r$ covering $\mathcal{B}_0$ of total radius $r \in [r(\mathcal{B}_0), \eta]$. Choosing $r = \eta^3$, which is always possible for small enough ε, it is clear that $\mathcal{B}_{\eta^3}$ consists of only a single ball contained in $B(a, \frac{3}{2}\eta^3)$ for ε small enough. Applying the second item of Proposition 4.2 to that ball, we then obtain

$$\int_{\mathcal{B}_{\eta^3}} \left(|\nabla h'_\varepsilon|^2 + \frac{\kappa^2}{4|\ln\varepsilon|}|h'_\varepsilon|^2 \right) dx' \ge \frac{1}{2\pi}\left(\ln \frac{\eta^3}{r(\mathcal{B}_0)} - c\eta^3 \right) \sum_{i=1}^{d_a} |\tilde{A}_i^\varepsilon|^2. \tag{6.21}$$

Therefore, we have

$$\int_{\mathcal{B}_{\eta^3}} \chi_R \left(|\nabla h'_\varepsilon|^2 + \frac{\kappa^2}{4|\ln\varepsilon|} |h'_\varepsilon|^2 \right) dx' \geq \frac{1}{2\pi} \left(\ln \frac{\eta^3}{r(\mathcal{B}_0)} - c\eta^3 \right) \left(\min_{B(a,\eta)} \chi_R \right) \sum_{i=1}^{d_a} |\tilde{A}_i^\varepsilon|^2. \quad (6.22)$$

On the other hand, we can estimate the contribution of the remaining part of $B(a,\eta)$ as

$$\begin{aligned}
&\int_{B(a,\eta)\setminus\mathcal{B}_{\eta^3}} \chi_R |\nabla h'_\varepsilon|^2 \, dx' + \frac{\kappa^2}{4|\ln\varepsilon|} \int_{B(a,\eta)} \chi_R |h'_\varepsilon|^2 \, dx' \\
&\qquad \geq \left(\min_{B(a,\eta)} \chi_R \right) \left(\int_{B(a,\eta)\setminus B(a,2\eta^3)} |\nabla h'_\varepsilon|^2 \, dx' + \frac{\kappa^2}{4|\ln\varepsilon|} \int_{B(a,\eta)} |h'_\varepsilon|^2 \, dx' \right) \\
&\quad \geq \left(\min_{B(a,\eta)} \chi_R \right) \int_{2\eta^3}^{\eta} \left(\int_{\partial B(a,r_B)} |\nabla h'_\varepsilon|^2 \, d\mathcal{H}^1(x') + \frac{\kappa^2}{4|\ln\varepsilon|} \int_{B(a,r_B)} |h'_\varepsilon|^2 \, dx' \right) dr_B. \quad (6.23)
\end{aligned}$$

Arguing as in (4.12) and using the fact that $\eta < \frac{1}{2}$, we obtain

$$\begin{aligned}
&\int_{B(a,\eta)\setminus\mathcal{B}_{\eta^3}} \chi_R |\nabla h'_\varepsilon|^2 dx' + \frac{\kappa^2}{4|\ln\varepsilon|} \int_{B(a,\eta)} \chi_R |h'_\varepsilon|^2 dx' \\
&\qquad\qquad \geq \frac{1}{2\pi} \left(\min_{B(a,\eta)} \chi_R \right) \ln \frac{1}{2\eta^2} \left(\sum_{i=1}^{d_a} A_i^\varepsilon \right)^2 (1 - C\eta), \quad (6.24)
\end{aligned}$$

where $C > 0$ is independent of η and ε, for small enough ε.

We will now use crucially the fact shown in Step 1 that all $A_i^\varepsilon \geq \beta$ approach the same limit as $\varepsilon \to 0$. We begin by adding (6.21) and (6.24) and subtracting $\frac{1}{2\pi} |\ln \rho_\varepsilon| \sum_{i=1}^{d_a} |\tilde{A}_i^\varepsilon|^2 \chi_R^i$ from both sides. With the help of (6.20) we can cancel out the leading order $O(|\ln \rho_\varepsilon|)$ term in the right-hand side of the obtained inequality. Replacing $\tilde{A}_i^\varepsilon$ and A_i^ε with $3^{2/3}\pi + o_\varepsilon(1)$ in the remaining terms and using the fact that $\min_{B(a,\eta)} \chi_R \geq \chi_R(a) - 2\eta \|\nabla \chi_R\|_\infty$ on $B(a,\eta)$, we then find

$$\begin{aligned}
&\int_{B(a,\eta)} \chi_R \left(|\nabla h'_\varepsilon|^2 + \frac{\kappa^2}{2|\ln\varepsilon|} |h'_\varepsilon|^2 - \frac{1}{2\pi} |\ln \rho_\varepsilon| \nu^\varepsilon \right) dx' \\
&\qquad\qquad \geq \frac{3^{4/3}\pi}{2} \chi_R(a) \left(d_a^2 \ln \frac{1}{2\eta^2} + d_a \ln \frac{\eta^3}{2} \right) - C, \quad (6.25)
\end{aligned}$$

where $C > 0$ is independent of ε or η.

Now, adding up the contributions of all $a \in \Lambda \cap K_R$ and recalling the definition of f_ε

in (5.2), we conclude that on the considered sequence

$$\limsup_{\varepsilon\to 0}\int_{K_R}\chi_R f_\varepsilon\,dx' \geq \limsup_{\varepsilon\to 0}\sum_{a\in\Lambda\cap K_R}\int_{B(a,\eta)}\chi_R f_\varepsilon\,dx'$$

$$\geq \frac{3^{4/3}\pi}{2}|\ln\eta|\sum_{a\in\Lambda\cap K_R}(2d_a^2-3d_a)\chi_R(a)-C, \quad (6.26)$$

for some $C>0$ is independent of ε or η. In particular, since $\chi_R(a)>0$ for all $a\in\Lambda\cap K_R$, the right-hand side of (6.26) goes to plus infinity as $\eta\to 0$, unless all $d_a=1$. But by the estimate (6.14) of Proposition 6.3, Corollary 5.3 and our assumption in (6.7) together with (5.26), the left-hand side of (6.26) is bounded independently of η, which yields the conclusion.

- *Step 4: Energy of each droplet.* Now that we know that for each $a_i\in\Lambda\cap K_R$ there exists exactly one droplet $\Omega'_{i,\varepsilon}$ such that $a_i^\varepsilon\to a_i$ and $A_i^\varepsilon\to 3^{2/3}\pi$, we can extract more precisely the part of energy that concentrates in a small ball around each such droplet. Let B_i be a ball that minimizes Fraenkel asymmetry defined in (6.4), i.e., let $B_i=B(a_i^\varepsilon,r_i^\varepsilon)$, and let B be a ball of radius r_B centered at a_i^ε. Arguing as in (4.12) in the proof of the second item of Proposition 4.2, we can write

$$\int_{\partial B}|\nabla h'_\varepsilon|^2 d\mathcal{H}^1(x') + \frac{\kappa^2}{4|\ln\varepsilon|}\int_B |h'_\varepsilon|^2 dx' \;\geq\; \frac{\varepsilon^{-4/3}|\ln\varepsilon|^{-2/3}|\Omega'_{i,\varepsilon}\cap B|^2}{2\pi r_B}(1-cr_i^\varepsilon)\,. \quad (6.27)$$

Observe that by the definition of Fraenkel asymmetry we have $|\Omega'_{i,\varepsilon}\cap B|\geq|B|-\frac{1}{2}\alpha(\Omega'_{i,\varepsilon})|B_i|$ for all $r_B<r_i^\varepsilon$. Hence, denoting by $\tilde r_i^\varepsilon$ the smallest value of r_B for which the right-hand side of this inequality is non-negative and integrating from $\tilde r_i^\varepsilon$ to r_i^ε, we find

$$\int_{B_i}\left(|\nabla h'_\varepsilon|^2+\frac{\kappa^2}{4|\ln\varepsilon|}|h'_\varepsilon|^2\right)dx'$$

$$\geq \frac{\pi}{2}(1+o_\varepsilon(1))\varepsilon^{-4/3}|\ln\varepsilon|^{-2/3}\int_{\tilde r_i^\varepsilon}^{r_i^\varepsilon} r_B^{-1}(r_B^2-|\tilde r_i^\varepsilon|^2)^2 dr_B. \quad (6.28)$$

Since by (6.8) and (6.18) we have $\tilde r_i^\varepsilon/r_i^\varepsilon\to 0$ and $\varepsilon^{-1/3}|\ln\varepsilon|^{-1/6}r_i^\varepsilon\to 3^{1/3}$ as $\varepsilon\to 0$, after an elementary computation we find

$$\int_{\Omega'_{i,\varepsilon}}\left(|\nabla h'_\varepsilon|^2+\frac{\kappa^2}{4|\ln\varepsilon|}|h'_\varepsilon|^2\right)dx' \geq \frac{3^{4/3}\pi}{8}+o_\varepsilon(1). \quad (6.29)$$

On the other hand, by (6.5) and (6.17) it is possible to choose a collection $\mathcal{B}_0\subset B(a_i,\eta)$, actually consisting of only a single ball $B(\tilde a_i^\varepsilon,R_i^\varepsilon)$ circumscribing $\Omega'_{i,\varepsilon}$, so that

$$r(\mathcal{B}_0)=R_i^\varepsilon\leq r_i^\varepsilon\left(1+C_R|\ln\varepsilon|^{-1/2}\right)=\rho_\varepsilon+o_\varepsilon(\rho_\varepsilon). \quad (6.30)$$

The corresponding ball construction $\mathcal{B}_r$ of the first item of Proposition 4.2, with $U = B(a_i, \eta)$ and η as in Step 3 of the proof (again, just a single ball $B(\tilde{a}_i^\varepsilon, r)$), exists and is contained in U for all $r \in [r(\mathcal{B}_0), \eta']$, for any $\eta' \in (r(\mathcal{B}_0), \eta)$, provided ε is sufficiently small depending on η'. In view of the fact that for small enough η' and small enough ε depending on η' we have $\chi_R(x) \geq \chi_R(\tilde{a}_i^\varepsilon) - c|x - \tilde{a}_i^\varepsilon| > 0$, with $c > 0$ independent of ε, η' or R, we obtain that

$$\begin{aligned}
&\int_{B(\tilde{a}_i^\varepsilon,\eta')\setminus\mathcal{B}_0} \chi_R|\nabla h'_\varepsilon|^2 dx' + \frac{\kappa^2}{4|\ln\varepsilon|}\int_{B(\tilde{a}_i^\varepsilon,\eta')} \chi_R |h'_\varepsilon|^2 dx' \\
&\quad \geq \int_{r(\mathcal{B}_0)}^{\eta'} (\chi_R(\tilde{a}_i^\varepsilon) - cr)\left(\int_{\partial\mathcal{B}_r} |\nabla h'_\varepsilon|^2 d\mathcal{H}^1(x)\right) dr + \frac{\kappa^2\chi_R(\tilde{a}_i^\varepsilon)}{8|\ln\varepsilon|}\int_{B(\tilde{a}_i^\varepsilon,\eta')} |h'_\varepsilon|^2 dx' \\
&\quad \geq \int_{r(\mathcal{B}_0)}^{\eta'} (\chi_R(\tilde{a}_i^\varepsilon) - cr)\left(\int_{\partial\mathcal{B}_r} |\nabla h'_\varepsilon|^2 d\mathcal{H}^1(x) + \frac{\kappa^2}{8\eta'|\ln\varepsilon|}\int_{B(\tilde{a}_i^\varepsilon,\eta')} |h'_\varepsilon|^2 dx'\right) dr \\
&\quad \geq \int_{r(\mathcal{B}_0)}^{\eta'} (\chi_R(\tilde{a}_i^\varepsilon) - cr)\left(\int_{\partial\mathcal{B}_r} |\nabla h'_\varepsilon|^2 d\mathcal{H}^1(x) + \frac{\kappa^2}{4|\ln\varepsilon|}\int_{\mathcal{B}_r} |h'_\varepsilon|^2 dx'\right) dr \\
&\quad \geq \frac{1}{2\pi}|\tilde{A}_i^\varepsilon|^2 \int_{r(\mathcal{B}_0)}^{\eta'} (\chi_R(\tilde{a}_i^\varepsilon) - cr)(1 - Cr)\frac{dr}{r}, \qquad (6.31)
\end{aligned}$$

for η' and ε sufficiently small, arguing as in (4.12) in the proof of Proposition 4.2 and taking into account Remark 4.4 in deducing the last line. Performing integration in (6.31) and using (6.30), we then conclude

$$\begin{aligned}
&\int_{B(\tilde{a}_i^\varepsilon,\eta')\setminus\mathcal{B}_0} \chi_R|\nabla h'_\varepsilon|^2 dx' + \frac{\kappa^2}{4|\ln\varepsilon|}\int_{B(\tilde{a}_i^\varepsilon,\eta')} \chi_R |h'_\varepsilon|^2 dx' \\
&\quad \geq \frac{1}{2\pi}|\tilde{A}_i^\varepsilon|^2 \chi_R(\tilde{a}_i^\varepsilon) \ln\left(\frac{\eta'}{\rho_\varepsilon}\right) - C\eta', \qquad (6.32)
\end{aligned}$$

for ε sufficiently small.

- *Step 5: Convergence.* Using the fact, seen in Step 2, that $h'_\varepsilon \rightharpoonup h$ in $\dot{W}^{1,p}_{loc}(\mathbb{R}^2)$, we have, by lower semi-continuity,

$$\liminf_{\varepsilon\to 0} \int_{\mathbb{R}^2\setminus\cup_{a\in\Lambda}B(a,\eta)} \chi_R|\nabla h'_\varepsilon|^2\, dx' \geq \int_{\mathbb{R}^2\setminus\cup_{a\in\Lambda}B(a,\eta)} \chi_R|\nabla h|^2 dx'. \qquad (6.33)$$

On the other hand, in view of $\chi_R(\tilde{a}_i^\varepsilon) = \chi_R^i + O(\rho_\varepsilon)$ by (6.30), from (6.32) we obtain

$$\begin{aligned}
\liminf_{\varepsilon\to 0} \int_{B(a_i^\varepsilon,\eta)\setminus\mathcal{B}_0} \chi_R|\nabla h'_\varepsilon|^2 dx' + \int_{B(a_i^\varepsilon,\eta)} \chi_R\left(\frac{\kappa^2}{4|\ln\varepsilon|}|h'_\varepsilon|^2 - \frac{1}{2\pi}|\ln\rho_\varepsilon|\nu^\varepsilon\right) dx' \\
\geq \liminf_{\varepsilon\to 0} \int_{B(\tilde{a}_i^\varepsilon,\eta')\setminus\mathcal{B}_0} \chi_R|\nabla h'_\varepsilon|^2 dx' + \int_{B(\tilde{a}_i^\varepsilon,\eta')} \chi_R\left(\frac{\kappa^2}{4|\ln\varepsilon|}|h'_\varepsilon|^2 - \frac{1}{2\pi}|\ln\rho_\varepsilon|\nu^\varepsilon\right) dx' \\
\geq \frac{3^{4/3}\pi}{2}\chi_R(a_i)\ln\eta' - C\eta', \quad (6.34)
\end{aligned}$$

where we also used that $\chi_R^i \to \chi_R(a)$ as $\varepsilon \to 0$.

We now convert the estimate in (6.29) to one over $\mathcal{B}_0$ and involving χ_R as well. Observing that $\Omega'_{i,\varepsilon} \subseteq \mathcal{B}_0$ and that $\chi_R(x') \geq \chi_i^R - 4\rho_\varepsilon\|\nabla\chi_R\|_\infty$ for all $x' \in \Omega'_{i,\varepsilon}$ and ε small enough by (6.30), from (6.29) and (3.1) we obtain

$$\liminf_{\varepsilon\to 0} \int_{\mathcal{B}_0} \chi_R\left(|\nabla h'_\varepsilon|^2 + \frac{\kappa^2}{4|\ln\varepsilon|}|h'_\varepsilon|^2\right) dx' \geq \frac{3^{4/3}\pi}{8}\chi_R(a_i), \quad (6.35)$$

where we used the fact that by (3.2), (3.12) and (4.5) the integral in the left-hand side of (6.29) may be bounded by $C|\ln\varepsilon|$, for some $C > 0$ independent of ε and R. Adding up (6.33) with (6.34) and (6.35) summed over all $a_i \in K_R$, in view of the arbitrariness of $\eta' < \eta$ we then obtain

$$\begin{aligned}
\liminf_{\varepsilon\to 0} \int_{\mathbb{R}^2} \chi_R\left(|\nabla h'_\varepsilon|^2 + \frac{\kappa^2}{2|\ln\varepsilon|}|h'_\varepsilon|^2 - \frac{1}{2\pi}|\ln\rho_\varepsilon|\nu^\varepsilon\right) dx' \\
\geq \int_{\mathbb{R}^2\setminus\cup_{a\in\Lambda}B(a,\eta)} \chi_R|\nabla h|^2 dx' + \frac{3^{4/3}\pi}{2}\sum_{a\in\Lambda}\chi_R(a)\left(\ln\eta + \frac{1}{4}\right) - C\eta. \quad (6.36)
\end{aligned}$$

Letting now $\eta \to 0$ in (6.36), and recalling that $\varphi = 2\cdot 3^{-2/3}h$ and that the definition of $W(\varphi,\chi)$ is given by Definition 6.1, we obtain

$$\begin{aligned}
\liminf_{\varepsilon\to 0} \int_{\mathbb{R}^2} \chi_R\left(|\nabla h'_\varepsilon|^2 + \frac{\kappa^2}{2|\ln\varepsilon|}|h'_\varepsilon|^2 - \frac{1}{2\pi}|\ln\rho_\varepsilon|\nu^\varepsilon\right) dx' \\
\geq \frac{3^{4/3}}{2}W(\varphi,\chi_R) + \frac{3^{4/3}\pi}{8}\sum_{a\in\Lambda}\chi_R(a). \quad (6.37)
\end{aligned}$$

From (6.14) we may replace $f_\varepsilon = |\nabla h'_\varepsilon|^2 + \frac{\kappa^2}{2|\ln\varepsilon|}|h'_\varepsilon|^2 - \frac{1}{2\pi}|\ln\rho_\varepsilon|\nu^\varepsilon$ by g_ε in (6.37) with an additional error term:

$$\liminf_{\varepsilon\to 0} \int_{\mathbb{R}^2} \chi_R g_\varepsilon dx' \geq \frac{3^{4/3}}{2}W(\varphi,\chi_R) + \frac{3^{4/3}\pi}{8}\sum_{a\in\Lambda}\chi_R(a) - c\Delta(R), \quad (6.38)$$

where
$$\Delta(R) = \limsup_{\varepsilon\to 0} \sum_{\alpha | K_{R-C} \subset U_\alpha \subset K_{R+C}} (n_\alpha + 1)\ln(n_\alpha + 2),$$
for some $c, C > 0$ independent of R. Under hypothesis (6.9), from (6.12) we have
$$\limsup_{\varepsilon\to 0} \sum_{\alpha | U_\alpha \subset K_R} n_\alpha^2 \le CR^2,$$
and thus, using Hölder inequality and bounding the number of α's involved in the sum by CR we find
$$\Delta(R) \le C \limsup_{\varepsilon\to 0} \sum_{\alpha | U_\alpha \subset K_{R+C} \setminus K_{R-C}} (n_\alpha^{3/2} + 1)$$
$$\le C' R^{1/4} \limsup_{\varepsilon\to 0} \left(\sum_{\alpha | U_\alpha \subset K_{R+C}} n_\alpha^2 \right)^{3/4} + CR \le C'' R^{7/4},$$
for some $C, C', C'' > 0$ independent of R. Hence
$$\limsup_{R\to\infty} \limsup_{\varepsilon\to 0} \frac{\Delta(R)}{R^2} = 0,$$
which together with (6.38) and the fact that $\bar{g}_\varepsilon \ge g_\varepsilon$ establishes (6.11). □

6.3 Local to Global bounds via the Ergodic Theorem: proof of Theorem 1, item i.

The proof follows the procedure outlined in [35]. We refer the reader to Sections 4 and 6 of [35] for the proof adapted to the case of the magnetic Ginzburg-Landau energy, which is essentially identical to the present one, with some simplifications due to the fact that we work on the torus. As in [35], we say that $\mu \in \mathcal{M}_0(\mathbb{R}^2)$, if the measure $d\mu + C dx$ is a positive locally bounded measure on $\mathbb{R}^2$, where C is the constant appearing in Corollary 5.3. The measures $d\bar{g}_\varepsilon$ and the functions φ_ε will be alternatively seen as functions on $\mathbb{T}^2_{\ell^\varepsilon}$ or as periodically extended to the whole of $\mathbb{R}^2$, which will be clear from the context. We let χ be a smooth non-negative function on $\mathbb{R}^2$ with support in $B(0,1)$ and with $\int_{\mathbb{R}^2} \chi(x)dx = 1$. We set $X = \dot{W}^{1,p}_{loc}(\mathbb{R}^2) \times \mathcal{M}_0(\mathbb{R}^2)$, and define for every $\mathbf{x} = (\varphi, g) \in X$ the following functional
$$\mathbf{f}(\mathbf{x}) := 2 \int_{\mathbb{R}^2} \chi(y) dg(y). \tag{6.39}$$
We note that from (5.27) we have for ε sufficiently small
$$F^\varepsilon[u^\varepsilon] + o_\varepsilon(1) \ge \frac{2}{\ell^2 |\ln \varepsilon|} \int_{\mathbb{T}^2_{\ell^\varepsilon}} d\bar{g}_\varepsilon = \fint_{\mathbb{T}^2_{\ell^\varepsilon}} \mathbf{f}(\theta_\lambda \mathbf{x}_\varepsilon) d\lambda, \tag{6.40}$$

where $\mathbf{x}_\varepsilon := (\varphi^\varepsilon, \bar{g}_\varepsilon)$, θ_λ denotes the translation operator by $\lambda \in \mathbb{R}^2$, i.e., $\theta_\lambda f(x) := f(x+\lambda)$, and $\fint$ stands for the average. Here the last equality follows by an application of Fubini's theorem and the fact that $\int_{\mathbb{R}^2} \chi(x)dx = 1$.

It can be easily shown as in [35] that $\mathbf{f}_\varepsilon = \mathbf{f}$ satisfies the coercivity and Γ-liminf properties required for the application of Theorem 3 in [35] on sequences consisting of $\mathbf{x}_\varepsilon = (\varphi^\varepsilon, \bar{g}_\varepsilon)$ obtained from (u^ε) obeying (2.19). This is done by starting with a sequence $\{\mathbf{x}_\varepsilon\}_\varepsilon$ in X such that

$$\limsup_{\varepsilon\to 0} \int_{K_R} \mathbf{f}(\theta_\lambda \mathbf{x}_\varepsilon) d\lambda < +\infty, \tag{6.41}$$

for every $R > 0$, which implies that the integral is finite whenever ε is small enough. Consequently $\mathbf{f}_\varepsilon(\theta_\lambda \mathbf{x}_\varepsilon) < +\infty$ for almost every $\lambda \in K_R$. Applying Fubini's theorem again, (6.41) becomes

$$\limsup_{\varepsilon\to 0} \int_{\mathbb{R}^2} \chi_R(y) d\bar{g}_\varepsilon(y) < +\infty,$$

where $\chi_R = \chi * \mathbf{1}_{K_R}$, and "$*$" denotes convolution. Then since $\chi_R = 1$ in K_{R-1} and $\bar{g}_\varepsilon$ is bounded below by a constant, the assumption (6.7) in Part 1 of Theorem 4 is satisfied, and we deduce from that theorem that φ^ε and $\bar{g}_\varepsilon$ converge, upon extraction of a subsequence, weakly in $\dot{W}^{1,p}_{loc}(\mathbb{R}^2)$ and weakly in the sense of measures, respectively. Furthermore, if $\mathbf{x}_\varepsilon \to \mathbf{x} = (\varphi, g)$ on this subsequence, we have $2\int_{\mathbb{R}^2} \chi(y) d\bar{g}_\varepsilon(y) = \mathbf{f}(\mathbf{x}_\varepsilon) \to \mathbf{f}(\mathbf{x}) = 2\int_{\mathbb{R}^2} \chi(y) d\bar{g}(y)$.

We may then apply Theorem 3 of [35] to $\mathbf{f}$ on $\mathbb{T}^2_{\ell^\varepsilon}$ and conclude that the measure $\{\widetilde{P}^\varepsilon\}_\varepsilon$ defined as the push-forward of the normalized uniform measure on $\mathbb{T}^2_{\ell^\varepsilon}$ by

$$\lambda \mapsto (\theta_\lambda \varphi^\varepsilon, \theta_\lambda \bar{g}_\varepsilon),$$

converges to a translation-invariant probability measure $\widetilde{P}$ on X with

$$\liminf_{\varepsilon\to 0} F^\varepsilon[u^\varepsilon] \geq \int \mathbf{f}(\mathbf{x}) d\widetilde{P}(\mathbf{x}) = \int \mathbf{f}^*(\mathbf{x}) d\widetilde{P}(\mathbf{x}), \tag{6.42}$$

where

$$\mathbf{f}^*(\varphi, g) = \lim_{R\to\infty} \fint_{K_R} \mathbf{f}(\theta_\lambda \mathbf{x}) d\lambda = \lim_{R\to+\infty} \left(\frac{2}{|K_R|} \int_{\mathbb{R}^2} \chi_R(y) dg(y) \right), \tag{6.43}$$

provided that $\mathbf{x}$ is in the support of $\widetilde{P}$.

The next step is to show that for $\widetilde{P}$-a.e. $\mathbf{x}$ we have $\varphi \in \mathcal{A}_m$ with $m = 3^{-2/3}(\bar{\delta} - \bar{\delta}_c)$, and $\mathbf{f}^*$ can be computed. By [35, Remark 1.6], we have that for $\widetilde{P}$-a.e $\mathbf{x}$, there exists a sequence $\{\lambda_\varepsilon\}_\varepsilon$ such that $\mathbf{x}_\varepsilon = (\theta_{\lambda_\varepsilon} \varphi^\varepsilon, \theta_{\lambda_\varepsilon} \bar{g}_\varepsilon)$ converges to $\mathbf{x}$ in X. In addition, from (6.42)–(6.43), for $\widetilde{P}$-a.e. $\mathbf{x}$, we have

$$\lim_{R\to+\infty} \fint_{K_R} \mathbf{f}(\theta_\lambda \mathbf{x}) d\lambda \; < +\infty,$$

for $\widetilde{P}$-almost every $\mathbf{x}$. Using Fubini's theorem again, together with the definition of $\mathbf{f}$, we then find

$$\lim_{R\to+\infty}\left(\frac{1}{|K_R|}\int_{\mathbb{R}^2}\chi_R(y)dg(y)\right)<+\infty.$$

Therefore, since

$$\int_{\mathbb{R}^2}\chi_R(y)d\bar{g}_\varepsilon(y)\to\int_{\mathbb{R}^2}\chi_R(y)dg(y)\qquad\text{as }\varepsilon\to 0,\tag{6.44}$$

a bound of the type (6.9) holds, and the results of Part 2 of Theorem 4 hold for $\mathbf{x}_\varepsilon$. In particular, we find that

$$-\Delta\varphi=2\pi\sum_{a\in\Lambda}\delta_a-m,\tag{6.45}$$

with $m=3^{-2/3}(\bar{\delta}-\bar{\delta}_c)$, and that

$$\mathbf{f}^*(\varphi,g)=\lim_{R\to\infty}\left(\frac{2}{|K_R|}\lim_{\varepsilon\to 0}\int_{\mathbb{R}^2}\chi_R\bar{g}_\varepsilon dx\right)\geq 3^{4/3}W(\varphi)+\frac{3^{4/3}}{8}m.\tag{6.46}$$

The result in (6.46) follows from the definition of $\mathbf{f}^*$, (6.44), (6.11), the definition of W, provided we can show that

$$\lim_{R\to+\infty}\frac{1}{|K_R|}\sum_{a\in\Lambda}\chi_R(a)=\lim_{R\to+\infty}\frac{\nu(K_R)}{2\pi|K_R|}=\frac{m}{2\pi}.\tag{6.47}$$

The latter can be obtained from (6.15), exactly as in Lemma 4.11 of [35], so we omit the proof. Note that with (6.45), it proves that $\varphi\in\mathcal{A}_m$, and we thus have the claimed result. Combining (6.42) and (6.46), we obtain

$$\liminf_{\varepsilon\to 0}F^\varepsilon[u^\varepsilon]\geq\int\left(3^{4/3}W(\varphi)+\frac{3^{2/3}}{8}(\bar{\delta}-\bar{\delta}_c)\right)d\widetilde{P}(\varphi,g).$$

Letting now P^ε and P be the first marginals of $\widetilde{P}^\varepsilon$ and $\widetilde{P}$ respectively, this proves (2.20) and the fact that P-almost every φ is in $\mathcal{A}_m$ with $m=3^{-2/3}(\bar{\delta}-\bar{\delta}_c)$. □

7 Upper bound construction: proof of Part ii) of Theorem 1

We follow closely the construction performed for the magnetic Ginzburg-Landau energy in [35], but our situation is somewhat simpler, since we work on a torus (instead of a domain bounded by a free boundary). The construction given in [35] relies on a result stated as Corollary 4.5 in [35], which we repeat below with slight modifications to adapt it to our setting. These results imply, in particular, that the minimum of W may be approximated by sequences of periodic configurations of larger and larger period. Below for any discrete set of points Λ, $|\Lambda|$ will denote its cardinal.

Proposition 7.1 (Corollary 4.5 in [35]). *Let $p \in (1,2)$ and let P be a probability measure on $\dot{W}^{1,p}_{loc}(\mathbb{R}^2)$ which is invariant under the action of translations and concentrated on $\mathcal{A}_1$. Let Q be the push-forward of P under $-\Delta$. Then there exists a sequence $R \to \infty$ with $R^2 \in 2\pi\mathbb{N}$ and a sequence $\{b_R\}_R$ of $2R$-periodic vector fields such that:*

- *There exists a finite subset Λ_R of the interior of K_R such that*

$$\begin{cases} -\mathrm{div}\ b_R = 2\pi \sum_{a\in\Lambda_R} \delta_a - 1 & \text{in } K_R \\ b_R \cdot \nu = 0 & \text{on } \partial K_R. \end{cases}$$

- *Letting Q_R be the probability measure on $W^{-1,p}_{loc}(\mathbb{R}^2)$, which is defined as the image of the normalized Lebesgue measure on K_R by $x \mapsto -\mathrm{div}\ b_R(x+\cdot)$, we have $Q_R \to Q$ weakly as $R \to \infty$.*

- $\displaystyle\limsup_{R\to\infty} \frac{1}{|K_R|} \lim_{\eta\to 0} \left(\frac{1}{2} \int_{K_R \setminus \cup_{a\in\Lambda_R} B(a,\eta)} |b_R|^2 dx + \pi |\Lambda_R| \ln \eta \right) \le \int W(\varphi)\, dP(\varphi).$

Remark 7.2. *We would like to make the following observations concerning the vector field b_R constructed in Proposition 7.1.*

1. *By construction, the vector fields b_R has no distributional divergence concentrating on ∂K_R and its translated copied since $b_R \cdot \nu$ is continuous across ∂K_R. However, $b_R \cdot \tau$ may not be, and this may create a singular part of the distributional $\mathrm{curl}\, b_R$. This is the difficulty that prevents us from stating the convergence result for P directly in Theorem 1, Part ii).*

2. *We also note that an inspection of the construction in [35] shows that b_R is curl-free in a neighborhood of each point $a \in \Lambda_R$ and that $\mathrm{curl}\, b_R$ belongs to $W^{-1,p}_{\mathrm{loc}}(\mathbb{R}^2)$ for $p < \infty$.*

7.1 Definition of the test configuration

We take R the sequence given by Proposition 7.1. The first thing to do is to change the density 1 into a suitably chosen density $m_{\varepsilon,R}$, in order to ensure the compatibility of the functions with the torus volume. Recalling that $\bar{\mu}^\varepsilon > 0$ for $\bar{\delta} > \bar{\delta}_c$ and ε small enough, we set

$$m_{\varepsilon,R} = \frac{4R^2}{|\ell^\varepsilon|^2} \left\lfloor \frac{\ell^\varepsilon \sqrt{2\bar{\mu}^\varepsilon}}{2R\bar{r}_\varepsilon} \right\rfloor^2 \tag{7.1}$$

where, as usual, $\lfloor x \rfloor$ denotes the integer part of a x. We note for later that

$$\left| m_{\varepsilon,R} - \frac{2\bar{\mu}^\varepsilon}{\bar{r}_\varepsilon^2} \right| \le \frac{CR}{\ell^\varepsilon} = o_\varepsilon(1). \tag{7.2}$$

Recalling also that $\bar{r}_\varepsilon = 3^{1/3} + O\left(\frac{\ln|\ln\varepsilon|}{|\ln\varepsilon|}\right)$ and $\bar{\mu}^\varepsilon - \bar{\mu} = O\left(\frac{\ln|\ln\varepsilon|}{|\ln\varepsilon|}\right)$, we deduce that $m_{\varepsilon,R} \to m$, where $m := 2\cdot 3^{-2/3}\bar{\mu}$, as $\varepsilon \to 0$, for each R. In particular, $m_{\varepsilon,R}$ is bounded above and below by constants independent of ε and R. The choice of $m_{\varepsilon,R}$ ensures that we can split the torus into an integer number of translates of the square $K_{R'}$ with $R' := \frac{R}{\sqrt{m_{\varepsilon,R}}}$, each of which containing an identical configuration of $\frac{2R^2}{\pi}$ points.

Let $P \in \mathcal{P}$ be given as in the assumption of Part 2 of Theorem 1, i.e., let P be a probability measure concentrated on $\mathcal{A}_m$. Letting $\bar{P}$ be the push-forward of P by $\varphi \mapsto \varphi(\frac{\cdot}{\sqrt{m}})$, it is clear that $\bar{P}$ is concentrated on $\mathcal{A}_1$, and by the change of scales formula (2.16) we have

$$\int W(\varphi)\, d\bar{P}(\varphi) = \frac{1}{m}\int W(\varphi)\, dP(\varphi) + \frac{1}{4}\ln m. \tag{7.3}$$

We may then apply Proposition 7.1 to $\bar{P}$. It yields a vector field $\bar{b}_R$. We may then rescale it by setting

$$b_{\varepsilon,R}(x) = \sqrt{m_{\varepsilon,R}}\,\bar{b}_R(\sqrt{m_{\varepsilon,R}}x).$$

We note that $b_{\varepsilon,R}$ is a well-defined periodic vector-field on $\mathbb{T}^2_{\ell^\varepsilon}$ because $\frac{\ell^\varepsilon\sqrt{m_{\varepsilon,R}}}{2R}$ is an integer. This new vector field satisfies

$$-\mathrm{div}\ b_{\varepsilon,R} = 2\pi\sum_{a\in\Lambda_{\varepsilon,R}}\delta_a - m_{\varepsilon,R} \quad \text{in } \mathbb{T}^2_{\ell^\varepsilon} \tag{7.4}$$

for some set of points that we denote $\Lambda_{\varepsilon,R}$, and

$$\frac{1}{|K_R|}\lim_{\eta\to 0}\left(\frac{1}{2}\int_{K_{\frac{R}{\sqrt{m_{\varepsilon,R}}}}\setminus\cup_{a\in\Lambda_{\varepsilon,R}}B(a,\eta)}|b_{\varepsilon,R}|^2 dx + \pi|\Lambda_{\varepsilon,R}\cap K_{R/\sqrt{m_{\varepsilon,R}}}|\ln(\eta\sqrt{m_{\varepsilon,R}})\right)$$
$$\le \int W(\varphi)\, d\bar{P}(\varphi) + o_R(1) \quad \text{as } R\to\infty.$$

Using (7.3) and $|\Lambda_{\varepsilon,R}\cap K_{R/\sqrt{m_{\varepsilon,R}}}| = \frac{2R^2}{\pi}$, this can be rewritten as

$$\frac{m_{\varepsilon,R}}{|K_R|}\lim_{\eta\to 0}\left(\frac{1}{2}\int_{K_{\frac{R}{\sqrt{m_{\varepsilon,R}}}}\setminus\cup_{a\in\Lambda_{\varepsilon,R}}B(a,\eta)}|b_{\varepsilon,R}|^2 dx + \pi|\Lambda_{\varepsilon,R}\cap K_{R/\sqrt{m_{\varepsilon,R}}}|\ln\eta\right) + \frac{m_{\varepsilon,R}}{4}\ln\frac{m_{\varepsilon,R}}{m}$$
$$\le \frac{m_{\varepsilon,R}}{m}\int W(\varphi)\, dP(\varphi) + o_R(1).$$

But we saw that $m_{\varepsilon,R} \to m$ as $\varepsilon \to 0$ hence $\ln\left(\frac{m_{\varepsilon,R}}{m}\right) \to 0$. Therefore, recalling the definition of R' we have

$$\frac{1}{|K_{R'}|}\lim_{\eta\to 0}\left(\frac{1}{2}\int_{K_{R'}\setminus\cup_{a\in\Lambda_{\varepsilon,R}}B(a,\eta)}|b_{\varepsilon,R}|^2dx+\pi|\Lambda_{\varepsilon,R}\cap K_{R/\sqrt{m_{\varepsilon,R}}}|\ln\eta\right)$$
$$\leq\int W(\varphi)\,dP(\varphi)+o_R(1)+o_\varepsilon(1). \quad (7.5)$$

It thus follows that

$$\frac{1}{(\ell^\varepsilon)^2}\lim_{\eta\to 0}\left(\frac{1}{2}\int_{\mathbb{T}^2_{\ell^\varepsilon}\setminus\cup_{a\in\Lambda_{\varepsilon,R}}B(a,\eta)}|b_{\varepsilon,R}|^2dx'+\pi|\Lambda_{\varepsilon,R}|\ln\eta\right)\leq\int W(\varphi)\,dP(\varphi)+o_R(1)+o_\varepsilon(1). \quad (7.6)$$

Note that $\Lambda_{\varepsilon,R}$ is a dilation by the factor $1/\sqrt{m_{\varepsilon,R}}$, uniformly bounded above and below, of the set of points Λ_R, hence the minimal distance between the points in $\Lambda_{\varepsilon,R}$ is bounded below by a constant which may depend on R but does not depend on ε. For the same reason, estimates on $b_{R,\varepsilon}$ are uniform with respect to ε.

In addition, we have that $\bar{Q}_{\varepsilon,R}$, the push-forward of the normalized Lebesgue measure on $\mathbb{T}^2_{\ell^\varepsilon}$ by $x\mapsto -\mathrm{div}\ b_{\varepsilon,R}(x+\cdot)$ converges to Q, the push-forward of P by $-\Delta$, as $\varepsilon\to 0$ and $R\to\infty$. The final step is to replace the Dirac masses appearing above by their non-singular approximations:

$$\tilde{\delta}_a := \frac{\chi_{B(a,r'_\varepsilon)}}{\pi|r'_\varepsilon|^2} \qquad r'_\varepsilon := \varepsilon^{1/3}|\ln\varepsilon|^{1/6}\bar{r}_\varepsilon, \quad (7.7)$$

where $\bar{r}_\varepsilon$ was defined in (3.1). Note also that in view of the discussion of Section 3 it is crucial to use droplets with the corrected radius $\varepsilon^{1/3}|\ln\varepsilon|^{1/6}\bar{r}_\varepsilon$ instead of its leading order value $\rho_\varepsilon = 3^{1/3}\varepsilon^{1/3}|\ln\varepsilon|^{1/6}$.

Once the set $\Lambda_{\varepsilon,R}$ has been defined, the definition of the test function $u^\varepsilon\in\mathcal{A}$ follows: it suffices to take

$$u^\varepsilon(x) = -1+2\sum_{a\in\Lambda_{\varepsilon,R}}\chi_{B(a,r'_\varepsilon)}\left(x|\ln\varepsilon|^{1/2}\right),$$

which means (after blow up) that all droplets are round of identical radii r'_ε and centered at the points of $\Lambda_{\varepsilon,R}$. We now need to compute $F^\varepsilon[u^\varepsilon]$ and check that all the desired properties are satisfied. This is done by working with the associated function h'_ε defined in (3.16), i.e. the solution in $\mathbb{T}^2_{\ell^\varepsilon}$ to

$$-\Delta h'_\varepsilon + \frac{\kappa^2}{|\ln\varepsilon|}h'_\varepsilon = \pi\bar{r}_\varepsilon^2\sum_{a\in\Lambda_{\varepsilon,R}}\tilde{\delta}_a - \bar{\mu}^\varepsilon, \quad (7.8)$$

obtained from (3.16) by explicitly setting all $A_i^\varepsilon = \pi\bar{r}_\varepsilon^2$.

7.2 Reduction to auxiliary functions

Let us introduce ϕ_ε, which is the solution with mean zero of

$$-\Delta\phi_\varepsilon = 2\pi \sum_{a\in\Lambda_{\varepsilon,R}} \tilde{\delta}_a - m_{\varepsilon,R} \quad \text{in} \quad \mathbb{T}^2_{\ell^\varepsilon}, \tag{7.9}$$

where $m_{\varepsilon,R}$ is as in (7.1), and f_ε the solution with mean zero of

$$-\Delta f_\varepsilon = 2\pi \sum_{a\in\Lambda_{\varepsilon,R}} \delta_a - m_{\varepsilon,R} \quad \text{in} \quad \mathbb{T}^2_{\ell^\varepsilon}. \tag{7.10}$$

We note that f_ε is a rescaling by the factor $m_{\varepsilon,R} \to m$ of a function independent of ε, so all estimates on f_ε can be made uniform with respect to ε.

Lemma 7.3. *Let h'_ε and ϕ_ε be as above. We have as $\varepsilon \to 0$*

$$\int_{\mathbb{T}^2_{\ell^\varepsilon}} |h'_\varepsilon|^2 dx' \le C_R |\ln\varepsilon| \tag{7.11}$$

and for any $1 \le q < \infty$

$$\left\| \nabla \left(h'_\varepsilon - \frac{\bar{r}_\varepsilon^2}{2}\phi_\varepsilon \right) \right\|_{L^q(\mathbb{T}^2_{\ell^\varepsilon})} \le C_{R,q}, \tag{7.12}$$

for some constant $C_{R,q} > 0$ independent of ε.

Proof. Since $\Lambda_{\varepsilon,R}$ is $2R'$-periodic, h'_ε is too, and thus

$$\int_{\mathbb{T}^2_{\ell^\varepsilon}} |h'_\varepsilon|^2 dx' = \ell^2 |\ln\varepsilon| \fint_{K_{R'}} |h'_\varepsilon|^2 dx' \le C_R |\ln\varepsilon|.$$

For the second assertion, let

$$h_\varepsilon(x) = h'_\varepsilon(x\sqrt{|\ln\varepsilon|}) \qquad \hat{\phi}_\varepsilon(x) = \phi_\varepsilon(x\sqrt{|\ln\varepsilon|})$$

be the rescalings of h'_ε and ϕ_ε onto the torus $\mathbb{T}^2_\ell$. Rescaling (7.11) gives

$$\|h_\varepsilon\|_{L^2(\mathbb{T}^2_\ell)} \le C_R. \tag{7.13}$$

Furthermore, the function $w_\varepsilon := h_\varepsilon - \frac{1}{2}\bar{r}_\varepsilon^2 \hat{\phi}_\varepsilon$ is easily seen to solve

$$-\Delta w_\varepsilon = -\kappa^2 \left(h_\varepsilon - \fint_{\mathbb{T}^2_\ell} h_\varepsilon dx \right) \quad \text{in } \mathbb{T}^2_\ell.$$

But from elliptic regularity, Cauchy-Schwarz inequality and (7.13), we must have

$$\|\nabla w_\varepsilon\|_{L^q(\mathbb{T}^2_\ell)} \le C \left\| h_\varepsilon - \fint_{\mathbb{T}^2_\ell} h_\varepsilon \right\|_{L^2(\mathbb{T}^2_\ell)} \le C_{R,q},$$

which yields (7.12). □

The next lemma consists in comparing ϕ_ε and f_ε.

Lemma 7.4. *We have*

$$\|\nabla(f_\varepsilon - \phi_\varepsilon)\|_{L^\infty\left(\mathbb{T}^2_{\ell^\varepsilon}\setminus\cup_a B(a,r'_\varepsilon)\right)} \le C_R \varepsilon^{1/4}.$$

Proof. We observe that f_ε and ϕ_ε are both $2R'$-periodic. We may thus write

$$\phi_\varepsilon(x) - f(x) = 2\pi \int_{\mathbb{T}^2_{2R'}} G_{2R'}(x-y) \sum_{a\in\Lambda_{\varepsilon,R}} d(\tilde{\delta}_a - \delta_a)(y),$$

where $G_{2R'}$ is the zero mean Green's function for the Laplace's operator on the square torus of size $2R'$ with periodic boundary conditions, i.e. the solution to

$$-\Delta G_{2R'} = \delta_0 - \frac{1}{|\mathbb{T}^2_{2R'}|} \quad \text{in } \mathbb{T}^2_{2R'} \tag{7.14}$$

which we may be split as $G_{2R'}(x) = -\frac{1}{2\pi}\log|x| + S_{2R'}(x)$ with $S_{2R'}$ a smooth function. By Newton's theorem (or equivalently by the mean value theorem for harmonic functions applied to the function $\log|\cdot|$ away from the origin), the contribution due to the logarithmic part is zero outside of $\cup_{a\in\Lambda_{\varepsilon,R}} B(a,r'_\varepsilon)$. Differentiating the above we may thus write that for all $x \notin \cup_{a\in\Lambda_{\varepsilon,R}} B(a,r'_\varepsilon)$,

$$\nabla(\phi_\varepsilon - f)(x) = 2\pi \int_{\mathbb{T}^2_{2R'}} \nabla S_{2R'}(x-y) \sum_{a\in\Lambda_{\varepsilon,R}} d(\tilde{\delta}_a - \delta_a)(y). \tag{7.15}$$

Using the C^2 character of $S_{2R'}$ we deduce that

$$\|\nabla(f_\varepsilon - \phi_\varepsilon)\|_{L^\infty\left(\mathbb{T}^2_{\ell^\varepsilon}\setminus\cup_a B(a,r'_\varepsilon)\right)} \le C_{R'}|\Lambda_{\varepsilon,R}\cap K_{R'}|r'_\varepsilon$$

and the result follows in view of (7.7). □

The next step involves a comparison of the energy of ϕ_ε and that of $b_{\varepsilon,R}$ and leads to the following conclusion.

Lemma 7.5. *Given $\Lambda_{\varepsilon,R}$ as constructed above, and h'_ε the solution to* (7.8)*, we have*

$$\frac{1}{(\ell^\varepsilon)^2}\lim_{\eta\to 0}\left(\int_{\mathbb{T}^2_{\ell^\varepsilon}\setminus\bigcup_{a\in\Lambda_{\varepsilon,R}} B(a,\eta)} \frac{2}{\bar{r}_\varepsilon^4}|\nabla h'_\varepsilon|^2 dx' + \pi|\Lambda_{\varepsilon,R}|\ln\eta\right) \le \int W(\varphi)\,dP(\varphi) + o_\varepsilon(1) + o_R(1).$$

Proof. In view of Lemmas 7.3 and 7.4, it suffices to show the corresponding result for $\int_{\mathbb{T}^2_{\ell^\varepsilon}\setminus\bigcup_{a\in\Lambda_{\varepsilon,R}} B(a,\eta)} \frac{1}{2}|\nabla f_\varepsilon|^2\,dx'$ instead of the one for h'_ε. From (7.10) and (7.4), we have $\mathrm{div}\,(b_{\varepsilon,R} - \nabla f_\varepsilon) = 0$ hence by Poincaré's lemma we may write $\nabla f_\varepsilon = b_{\varepsilon,R} + \nabla^\perp \xi_\varepsilon$. We note that $-\Delta\xi_\varepsilon = \operatorname{curl} b_{\varepsilon,R}$, which is in $W^{-1,p}_{loc}$ for any $p < +\infty$ as mentioned in Remark 7.2. By elliptic regularity we find that $\nabla\xi_\varepsilon \in L^p_{loc}(\mathbb{R}^2)$ for all $1 \le p < +\infty$, uniformly with respect to ε. We may thus write

$$\int_{\mathbb{T}^2_{\ell^\varepsilon}\setminus\bigcup_{a\in\Lambda_{\varepsilon,R}} B(a,\eta)} \frac{1}{2}|b_{\varepsilon,R}|^2 dx' \\ = \int_{\mathbb{T}^2_{\ell^\varepsilon}\setminus\bigcup_{a\in\Lambda_{\varepsilon,R}} B(a,\eta)} \left(\frac{1}{2}|\nabla f_\varepsilon|^2 + \frac{1}{2}|\nabla\xi_\varepsilon|^2 - \nabla f_\varepsilon\cdot\nabla^\perp\xi_\varepsilon\right) dx', \quad (7.16)$$

where $\nabla f_\varepsilon \cdot \nabla^\perp \xi_\varepsilon$ makes sense in the duality $\nabla\xi_\varepsilon \in L^p$, $p > 2$, $\nabla f_\varepsilon \in L^q$, $q < 2$. In addition, by the same duality, we have for any $a \in \Lambda_{\varepsilon,R}$,

$$\lim_{\eta\to 0}\int_{B(a,\eta)} \nabla f_\varepsilon \cdot \nabla^\perp\xi_\varepsilon = 0$$

uniformly with respect to ε. Therefore, we may extend the domain of integration in the last integral in (7.16) to the whole of $\mathbb{T}^2_{\ell^\varepsilon}$ at the expense of an error $o_\eta(1)$ multiplied by the number of points, and obtain

$$\int_{\mathbb{T}^2_{\ell^\varepsilon}\setminus\bigcup_{a\in\Lambda_{\varepsilon,R}} B(a,\eta)} \frac{1}{2}|\nabla f_\varepsilon|^2 dx' \le \int_{\mathbb{T}^2_{\ell^\varepsilon}\setminus\bigcup_{a\in\Lambda_{\varepsilon,R}} B(a,\eta)} \frac{1}{2}|b_{\varepsilon,R}|^2 dx' \\ + \int_{\mathbb{T}^2_{\ell^\varepsilon}} \nabla f_\varepsilon \cdot \nabla^\perp \xi dx' + o_\eta(|\ln\varepsilon|). \quad (7.17)$$

Noting that the last integral on the right-hand side vanishes by Stokes' theorem (and by approximating ∇f_ε and $\nabla^\perp\xi_\varepsilon$ by smooth functions), adding $\pi|\Lambda_{\varepsilon,R}|\ln\eta$ to both sides, and combining with (7.6) we obtain the result. □

In view of (7.4) and (7.9) we have that $-\text{div } b_{\varepsilon,R} + \Delta\phi_\varepsilon = 2\pi\sum_{a\in\Lambda_{\varepsilon,R}}(\delta_a - \tilde{\delta}_a) \to 0$ in $W^{-1,p}_{loc}(\mathbb{R}^2)$, so we deduce, since the push-forward of the normalized Lebesgue measure on $\mathbb{T}^2_{\ell^\varepsilon}$ by $x \mapsto -\text{div } b_{\varepsilon,R}(x+\cdot)$ converges to Q, that the push-forward of it by $x \mapsto -\Delta\varphi^\varepsilon(x+\cdot)$ also converges to Q. Thus, part ii) of Theorem 1 is established modulo (2.21), which remains to be proved.

7.3 Calculating the energy

We begin by calculating the exact amount of energy contained in a ball of radius η.

Lemma 7.6. *Let h'_ε be as above. Then we have for any $a \in \Lambda_{\varepsilon,R}$,*

$$\int_{B(a,r'_\varepsilon)} |\nabla h'_\varepsilon|^2 dx' = \frac{3^{4/3}\pi}{8} + o_\varepsilon(1) \quad (7.18)$$

and

$$\int_{B(a,\eta)\setminus B(a,r'_\varepsilon)} |\nabla h'_\varepsilon|^2 dx' \le \frac{\pi}{2}\bar r_\varepsilon^4 \ln\frac{\eta}{\rho_\varepsilon} + o_\varepsilon(1) + o_\eta(1). \tag{7.19}$$

Proof. In view of (7.12) applied with $q > 2$ and using Hölder's inequality, we have that for all $a \in \Lambda_{\varepsilon,R}$,

$$\int_{B(a,\eta)} \left|\nabla(h'_\varepsilon - \frac{\bar r_\varepsilon^2}{2}\phi_\varepsilon)\right|^2 dx' \le o_\eta(1). \tag{7.20}$$

Thus it suffices to compute the corresponding integrals for ϕ_ε. Using again the $2R'$-periodicity of ϕ_ε, we may write, with the same notation as in the proof of Lemma 7.4

$$\phi_\varepsilon(x) = \int_{\mathbb{T}^2_{2R'}} G_{2R'}(x-y)\left(2\pi\sum_{a\in\bar\Lambda_{\varepsilon,R}}\tilde\delta_a(y) - m_{\varepsilon,R}\right) dy.$$

Since the distances between the points in $\bar\Lambda_{\varepsilon,R}$ are bounded below independently of ε, and the number of points is bounded as well, we may write ϕ_ε in $B(a,\eta)$ as

$$\phi_\varepsilon(x) = \psi_\varepsilon(x) - \int_{\mathbb{T}^2_{2R'}} \ln|x-y|\,\tilde\delta_a(y)\,dy \tag{7.21}$$

where $\psi_\varepsilon(x)$ is smooth and its derivative is bounded independently of ε (but depending on R).

Thus the contribution of ψ_ε to the integrals $\int_{B(a,\eta)}|\nabla\phi_\varepsilon|^2$ is $o_\eta(1)$, and its contribution to $\int_{B(a,r'_\varepsilon)}|\nabla\phi_\varepsilon|^2$ is $o_\varepsilon(1)$. There remains to compute the contribution of the logarithmic term in (7.21). But this is almost exactly the same computation as in (6.27)–(6.29), and with (7.20) it yields (7.18), while it yields as well that

$$\int_{B(a,\eta)\setminus B(a,r'_\varepsilon)} |\nabla\phi_\varepsilon|^2 dx' \le 2\pi\ln\frac{\eta}{r'_\varepsilon} + o_\eta(1). \tag{7.22}$$

Now

$$\frac{r'_\varepsilon}{\rho_\varepsilon} = \frac{1}{3^{1/3}}\left(\frac{|\ln\varepsilon|}{|\ln\rho_\varepsilon|}\right)^{1/3} = \left(1 + O\left(\frac{\ln|\ln\varepsilon|}{|\ln\varepsilon|}\right)\right)^{1/3}.$$

Consequently $\ln\frac{r'_\varepsilon}{\rho_\varepsilon} = o_\varepsilon(1)$, and so we may replace r'_ε with ρ_ε at an extra cost of $o_\varepsilon(1)$ in (7.22), and the result follows with (7.20). □

We can now combine all the previous results to compute the energy of the test-function u^ε. By following the lower bounds of Proposition 3.1, it is easy to see that in our case (all the droplets being balls of radius r'_ε) all the inequalities in that proof become equalities, and thus recalling (3.1):

$$F^\varepsilon[u^\varepsilon] = \frac{1}{|\ell^\varepsilon|^2}\left(2\int_{\mathbb{T}^2_{\ell^\varepsilon}}\left(|\nabla h'_\varepsilon|^2 + \frac{\kappa^2}{|\ln\varepsilon|}|h'_\varepsilon|^2\right)dx' + \pi\bar r_\varepsilon^4|\Lambda_{\varepsilon,R}|\ln\rho_\varepsilon\right) + o_\varepsilon(1),$$

with the help of Lemma 7.6 we have for every R

$$F^\varepsilon[u^\varepsilon] \le \frac{1}{|\ell^\varepsilon|^2}\left(2\int_{\mathbb{T}^2_{\ell^\varepsilon}\setminus\cup_{a\in\Lambda_{\varepsilon,R}}B(a,\eta)} |\nabla h'_\varepsilon|^2\,dx' + \pi\bar{r}_\varepsilon^4|\Lambda_{\varepsilon,R}|\ln\eta + \frac{3^{4/3}\pi}{4}|\Lambda_{\varepsilon,R}|\right) + o_\varepsilon(1) + o_\eta(1).$$

In view of Lemma 7.5, letting $\eta\to 0$, we obtain

$$F^\varepsilon[u^\varepsilon] \le \bar{r}_\varepsilon^4\left(\int W(\varphi)\,dP(\varphi) + o_\varepsilon(1) + o_R(1)\right) + \frac{3^{4/3}\pi}{4|\ell^\varepsilon|^2}|\Lambda_{\varepsilon,R}| + o_\varepsilon(1).$$

Letting $\varepsilon\to 0$, using that $\bar{r}_\varepsilon \to 3^{1/3}$ and the fact that $|\Lambda_{\varepsilon,R}| = \frac{1}{2\pi}m_{\varepsilon,R}|\ell^\varepsilon|^2$ with $m_{\varepsilon,R}\to m$, and then finally letting $R\to\infty$, we conclude that

$$\limsup_{\varepsilon\to 0} F^\varepsilon[u^\varepsilon] \le 3^{4/3}\int W(\varphi)\,dP(\varphi) + \frac{3^{4/3}m}{8}.$$

Since $\frac{1}{8}3^{2/3}m = \frac{1}{8}(\bar\delta - \bar\delta_c)$, this completes the proof of part ii) of Theorem 1. $\square$

7.4 Proof of Theorem 2

In order to prove Theorem 2, it suffices to show that

$$\min_{P\in\mathcal{P}} F^0[P] = 3^{4/3}\min_{\mathcal{A}_m} W + \frac{3^{2/3}(\bar\delta - \bar\delta_c)}{8}. \tag{7.23}$$

For the proof, we use the following result, adapted from Corollary 4.4 in [35].

Proposition 7.7 (Corollary 4.4 in [35])**.** *Let $\varphi\in\mathcal{A}_1$ be given, such that $W(\varphi)<\infty$. For any R such that $R^2\in 2\pi\mathbb{N}$, there exists a $2R$-periodic φ_R such that*

$$\begin{cases} -\Delta\varphi_R = 2\pi\sum_{a\in\Lambda_R}\delta_a - 1 & \text{in } K_R,\\ \dfrac{\partial\varphi_R}{\partial\nu} = 0 & \text{on } \partial K_R,\end{cases}$$

where Λ_R is a finite subset of the interior of K_R, and such that

$$\limsup_{R\to\infty}\frac{W(\varphi_R,\mathbf{1}_{K_R})}{|K_R|} \le W(\varphi).$$

Let us take φ to be a minimizer of W over $\mathcal{A}_m$ (which exists from [35]). We may rescale it to be an element of $\mathcal{A}_1$. Then Proposition 7.7 yields a φ_R, which can be extended periodically. We can then repeat the same construction as in the beginning of this section, starting from $\nabla\varphi_R$ instead of b_R, and in the end it yields a u^ε with

$$\limsup_{\varepsilon\to 0} F^\varepsilon[u^\varepsilon] \le 3^{4/3}\min_{\mathcal{A}_m} W + \frac{3^{2/3}(\bar\delta - \bar\delta_c)}{8}.$$

It follows that

$$\limsup_{\varepsilon\to 0} \min_{\mathcal{A}} F^{\varepsilon} \leq 3^{4/3} \min_{\mathcal{A}_m} W + \frac{3^{2/3}(\bar{\delta} - \bar{\delta}_c)}{8}.$$

But by part i) of Theorem 1 applied to a sequence of minimizers of F^{ε}, we also have

$$\liminf_{\varepsilon\to 0} \min_{\mathcal{A}} F^{\varepsilon} \geq \inf_{\mathcal{P}} F^{0} \geq 3^{4/3} \min_{\mathcal{A}_m} W + \frac{3^{2/3}(\bar{\delta} - \bar{\delta}_c)}{8}$$

where the last inequality is an immediate consequence of the definition of F^0. Comparing the inequalities yields that there must be equality and (7.23) is proved, which completes the proof of Theorem 2. □

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References

[1] E. Acerbi, N. Fusco, M. Morini. Minimality via second variation for a nonlocal isoperimetric problem, Preprint: http://cvgmt.sns.it/paper/540/

[2] G. Alberti, R. Choksi, F. Otto. Uniform Energy Distribution for an Isoperimetric Problem With Long-range Interactions. *Journal Amer. Math. Soc.*, 2:569-605, 2010.

[3] R. Alicandro, M. Cicalese, M. Ponsiglione. Variational equivalence between Ginzburg-Landau, XY spin systems and screw dislocations energies, to appear in *Indiana Univ. Math. J.*

[4] L. Ambrosio, V. Caselles, S. Masnou, and J.M Morel. Connected components of sets of finite perimeter with applications to image processing. *J. Eur. Math. Soc.*, 3:39-92, 2001

[5] F. Bethuel, H. Brézis, F. Hélein. *Ginzburg-Landau Vortices*, Birkhauser Progress in Non. Partial Diff. Eqns and Their Appns. **70**, (1994)

[6] T. Bonnesen. Über das isoperimetrische Defizit ebener Figuren. *Math. Ann.*, 91:252–268, 1924.

[7] A. Braides. *Gamma-Convergence for Beginners*, Oxford Lecture Series in Math., (2002).

[8] A. Braides and L. Truskinovsky. Asymptotic expansions by Γ-convergence. *Continuum Mech. Thermodyn.*, 20:21–62, 2008.

[9] H. Brézis and F. Browder. A property of Sobolev spaces. *Comm. Partial Differential Equations*, 4:1077–1083, 1979.

[10] X. Chen, Y. Oshita. An Application of the Modular Function in Nonlocal Variational Problems, *Arch. Rational Mech. Anal.* **186** (2007) 109–132.

[11] R. Choksi. Mathematical Aspects of Microphase Separation in Diblock Copolymers. *Ann. Sci. de l'ENS*, 33:4, 2000

[12] R. Choksi and M. A. Peletier. Small volume fraction limit of the diblock copolymer problem: I. Sharp interface functional. *SIAM J. Math. Anal.*, 42:1334–1370, 2010.

[13] R. Choksi and M. A. Peletier. Small volume fraction limit of the diblock copolymer problem: II. Diffuse interface functional. *SIAM J. Math. Anal.*, 43:739–763, 2011.

[14] R. Choksi, M. Peletier, J. F. Williams. On the Phase Diagram for Microphase Separation of Diblock Copolymers: An Approach via a Nonlocal Cahn–Hilliard Functional. *SIAM Journal of Applied Mathematics,* 69:1712–1738, 2009.

[15] M. Cicalese, E. Spadaro. Droplet Minimizers of an Isoperimetric Problem with long-range interactions, Preprint: http://arxiv.org/abs/1110.0031

[16] P. G. de Gennes. Effect of cross-links on a mixture of polymers. *J. de Physique – Lett.*, 40:69–72, 1979.

[17] N. Fusco, F. Maggi, and A. Pratelli. The sharp quantitative isoperimetric inequality. *Ann. of Math.*, 168:941–980, 2008.

[18] S. Glotzer, E. A. Di Marzio, and M. Muthukumar. Reaction-controlled morphology of phase-separating mixtures. *Phys. Rev. Lett.*, 74:2034–2037, 1995.

[19] D. Goldman, C. B. Muratov, and S. Serfaty. The Γ-limit of the two-dimensional Ohta-Kawasaki energy. I. Droplet density. (submitted to Arch. Rat. Mech. Anal.).

[20] R. Jerrard. Lower bounds for generalized Ginzburg-Landau functionals *SIAM J. Math. Anal*, 30:721–746, 1999.

[21] E. H. Lieb and M. Loss. *Analysis.* Amer. Math. Soc., 2001.

[22] S. Lundqvist and N. H. March, editors. *Theory of inhomogeneous electron gas.* Plenum Press, New York, 1983.

[23] S. Müller. Singular perturbations as a selection criterion for periodic minimizing sequences. *Calc. Var. Part. Dif.*, 1:169–204, 1993.

[24] C. B. Muratov. *Theory of domain patterns in systems with long-range interactions of Coulombic type*. Ph. D. Thesis, Boston University, 1998.

[25] C. B. Muratov. Theory of domain patterns in systems with long-range interactions of Coulomb type. *Phys. Rev. E*, 66:066108 pp. 1–25, 2002.

[26] C. B. Muratov. Droplet phases in non-local Ginzburg-Landau models with Coulomb repulsion in two dimensions. *Comm. Math. Phys.*, 299:45–87, 2010.

[27] I. A. Nyrkova, A. R. Khokhlov, and M. Doi. Microdomain structures in polyelectrolyte systems: calculation of the phase diagrams by direct minimization of the free energy. *Macromolecules*, 27:4220–4230, 1994.

[28] T. Ohta and K. Kawasaki. Equilibrium morphologies of block copolymer melts. *Macromolecules*, 19:2621–2632, 1986.

[29] C. Ortner and E. Süli. A note on linear elliptic systems on $\mathbb{R}^d$. arXiv:1202.3970v3, 2012.

[30] R. Osserman. Bonnesen-style isoperimetric inequalities. *Amer. Math. Monthly*, 86:1–29, 1979.

[31] X. Ren and L. Truskinovsky. Finite scale microstructures in nonlocal elasticity. *J. Elasticity*, 59:319–355, 2000.

[32] E. Sandier, S. Serfaty. Improved Lower Bounds for Ginzburg-Landau Energies via Mass Displacement, Analysis & PDE, 4-5:757–795, 2011.

[33] E. Sandier. Lower bounds for the energy of unit vector fields and applications, J. Funct. Anal., 152:379–403, 1998

[34] E. Sandier, S. Serfaty. *Vortices in the Magnetic Ginzburg-Landau Model*, Birkhauser Progress in Non. Partial Diff. Eqns and Their Appns. **70**, (2007)

[35] E. Sandier, S. Serfaty. From Ginzburg Landau to Vortex Lattice Problems, *Comm. Math. Phys.*, 313:635-743 (2012).

[36] E. Sandier, S. Serfaty. A rigorous derivation of a free boundary problem arising in superconductivity, *Ann. Sci. de l'ENS,* 33:561–592, 2000.

[37] E. Sandier, S. Serfaty. 2D Coulomb gases and the Renormalized Energy `arXiv:1201.3503`.

[38] E. Spadaro. Uniform energy and density distribution: diblock copolymers' functional. *Interfaces Free Bound.*, 11:447–474, 2009.

[39] P. Sternberg and I. Topaloglu. A note on the global minimizers of the nonlocal isoperimetric problem in two dimensions. *Interfaces Free Bound.*, 13:155–169, 2010.

[40] F. H. Stillinger. Variational model for micelle structure. *J. Chem. Phys.*, 78:4654–4661, 1983.

[41] M. Tinkham. *Introduction to superconductivity. Second edition.* McGraw-Hill, New York, 1996.

[42] N. K. Yip. Structure of stable solutions of a one-dimensional variational problem. *ESAIM Control Optim. Calc. Var.*, 12:721–751, 2006.

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