Hybrid resonance of Maxwell's equations in slab geometry

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Abstract

Hybrid resonance is a physical mechanism for the heating of a magnetic plasma. In our context hybrid resonance is a solution of the time harmonic Maxwell's equations with smooth coefficients, where the dielectric tensor is a non diagonal hermitian matrix. The main part of this work is dedicated to the construction and analyzis of a mathematical solution of the hybrid resonance with the limit absorption principle. We prove that the limit solution is singular: it is constituted of a Dirac mass at the origin plus a principle value and a smooth square integrable function. The formula obtained for the plasma heating is directly related to the singularity.

1 Introduction

It is known in plasma physics that Maxwell's equation in the context of a strong background magnetic field may develop singular solutions even for smooth coefficients. This is related to what is called the hybrid resonance [11, 16, 7] for which we know no mathematical analysis. Hybrid resonance shows up in reflectometry experiments [14, 13] and heating devices in fusion plasma [15]. The energy deposit may exceed by far the energy exchange which occurs in Landau damping [16, 22]. In the physical literature it is somewhat considered as surprising: for example the 2×2 upper block of the dielectric tensor (1.2) generates what is called X-mode equations, that is eXtraordinary equations, as opposed to the classical O-mode equation (O for Ordinary) related to the right-bottom coefficient in (1.2). Hybrid resonance is a non damping dissipative phenomenon: a singularity of the solution makes it stronger in some sense than the Landau damping; for example Landau damping needs kinetic effects to be

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modeled, while hybrid resonance can be derived from the coupling of a fluid model with the non electrostatic part of the Maxwell's equation (derivation of the basic model is proposed in the appendix). Since the mathematical solution is not square integrable, hybrid resonance is also a paradoxal and non standard phenomenon in the context of the mathematical theory of Maxwell's equations for which we refer to [12, 10, 21, 31]. The situation can be compared with the mathematial theory of metametarials. In [32, 33] the electric permittivity and magnetic susceptibility tensors are degenerate -i.e. they have zero eigenvalues- in surfaces, but they remain positive definite. In this case, the solutions are singular, but the problem remains coercive. In [4, 5] the coefficient changes in a discontinuous way from being positive to negative. In this situation coerciveness is lost, but as the absolute value of the coefficient is bounded below by a positive constant, the solutions are regular. In our case we have both difficulties at the same time. As the coefficient α (see below) goes from being positive to negative in a continuous way, its absolute value is zero at a point, and, in consequence, our problem is not coercive and there are singular solutions.

Our purpose in work is to construct and analyze in Theorem 1.1 a mathematical solution with a hybrid resonance in slab (planar) geometry. We will use the limit absorption principle to construct the relevant solutions. We will show that an original singular integral equation is attached to the Fourier solution. Introduced in the seminal work of Hilbert [19] and Picard [24], this type of integral equation is referred to as integral equation of the third kind, by comparison with the more classical equations of the first and second kind. Some references about this type of equations may be found in [3, 27] for mathematical analysis, and [29, 9, 17] for relation with theory of particles or plasma physics. Our results are reminiscent of those of Bart and Warnock [3], even if our kernel does not satisfy exactly their hypothesis since it is less regular: that is the solution if the sum of a Dirac mass plus a principal value (plus a regular part). In their work it is stressed that non uniqueness is the rule for such equations. In our case, we are able to obtain uniqueness by means of the limit absorption principle which is a physically based selection principle. One originality of this work is the analysis of the properties of this singular equation for which we found no equivalent in the classical literature [1, 2, 6].

The model problem is based on the time-harmonic Maxwell's equations

$$\nabla \wedge \nabla \wedge E - \left(\frac{\omega}{c}\right)^2 \varepsilon E = 0, \tag{1.1}$$

where the unknown E is the electric field, ω is the frequency and c the velocity of light. Plasmas with strong background magnetic fields can be represented by the cold plasma model [16, 11] for which the dielectric tensor is defined by

$$\varepsilon(x) = P(x) \begin{pmatrix} 1 - \frac{\omega_p^2}{\omega^2 - \omega_c^2} & i\frac{\omega_c \omega_p^2}{\omega(\omega^2-\omega_c^2)} & 0 \\ -i\frac{\omega_c \omega_p^2}{\omega(\omega^2-\omega_c^2)} & 1 - \frac{\omega_p^2}{\omega^2 - \omega_c^2} & 0 \\ 0 & 0 & 1 - \frac{\omega_p^2}{\omega^2} \end{pmatrix} P(x)^T. \tag{1.2}$$

A derivation of this dielectric tensor is given for convenience in our appendix. The incident wave with frequency ω is typically generated by an antenna on the boundary of the plasma. Such devices are actually being studied for the purposes of reflectometry and heating of magnetic fusion plasmas in the context of the international ITER project: the ITER project is about the design of new Tokamak with enhanced fusion capabilities [20]. The parameters of the dielectric tensor are the cyclotron frequency

$$\omega_c = \frac{eB_0}{m_e}$$

which is taken as constant in our work, the plasma frequency

$$\omega_p = \sqrt{\frac{e^2 N_e}{\varepsilon_0 m_e}}$$

which depends on the electronic density N_e. The modulus of the background magnetic field $B_0 > 0$ will be assumed constant for simplicity in our work, $e > 0$ is the absolute value of the charge of electron, m_e is the mass of electrons and ε_0 is the permittivity of vacuum. In this work we consider that the orientation of the background magnetic field is fixed, namely P is the identity matrix and is discarded in the equations. Boundary conditions can be of usual types, that is metallic condition $n \wedge E = 0$, non homogeneous absorbing boundary condition like $curl E + i\lambda n \wedge E = g$ on some parts of the boundary or even natural absorbing boundary condition at infinity.

The equations for the transverse electric (TE) mode, $E = (E_x, E_y, 0)$, and E_x, E_y, independent of z, which correspond to the 2×2 upper-left block in (1.2) are

$$\left\{ \begin{array}{rrrl} W & +\partial_y E_x & -\partial_x E_y & = 0, \\ \partial_y W & -\alpha \frac{\omega^2}{c^2} E_x & -i\delta \frac{\omega^2}{c^2} E_y & = 0, \\ -\partial_x W & +i\delta \frac{\omega^2}{c^2} E_x & -\alpha \frac{\omega^2}{c^2} E_y & = 0, \end{array} \right. \tag{1.3}$$

where the coefficients are

$$\alpha = 1 - \frac{\omega_p^2}{\omega^2 - \omega_c^2} \qquad \delta = \frac{\omega_c \omega_p^2}{\omega\left(\omega^2 - \omega_c^2\right)},$$

and W is the vorticity.

In the plasma community this system is referred to as the X-mode equations, where the letter X stands for eXtraordinary mode or eXtraordinary waves. We suspect the reason is the non standard behavior of the solutions of this system. The case where $\omega = \omega_c$, i.e., when the frequency of the incident wave, ω, is equal to the cyclotron frequency, ω_c, will not be considered in this work. That is we consider that $\omega \neq \omega_c$. If $\omega < \omega_c$ it is called a low hybrid resonance. The other case $\omega > \omega_c$ is denoted as the upper hybrid resonance. On the other hand we will assume that the diagonal coefficient α is smooth and vanishes at $x = 0$. This configuration corresponds to the hybrid resonance.

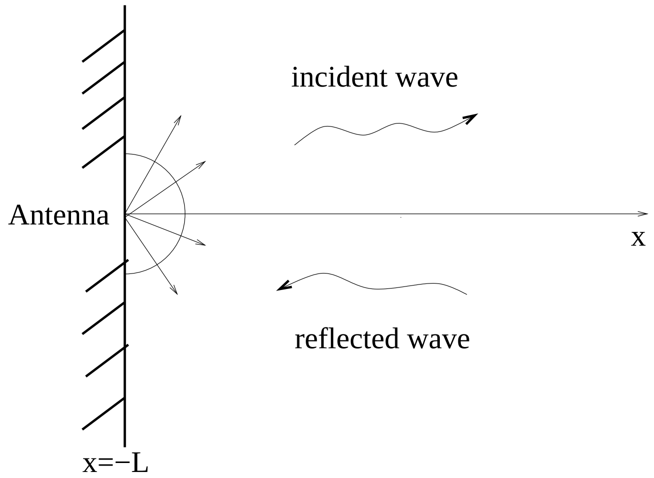

Figure 1: X-mode in slab geometry: the domain. In a real physical device an antenna is on the wall on the left and send an incident electromagnetic wave through a medium which is assumed infinite for simplicity. The incident wave generates a reflected wave. We will characterize the antenna by the knowledge of the non homogeneous boundary condition (1.4). The medium is filled with a plasma with dielectric tensor given by (1.2).

To be more specific we consider the simplified 2D domain

$$\Omega = \left\{(x,y) \in \mathbb{R}^2, \quad -L \le x, \quad y \in \mathbb{R}, \quad L > 0\right\}.$$

We supplement the X-mode equations (1.3) with a non homogeneous boundary condition

$$W + i\lambda n_x E_y = g \text{ on the left boundary } x = -L, \qquad \lambda > 0, \tag{1.4}$$

which models a given source, typically an radiating antenna. In real Tokamaks this antenna is used to heat or to probe the plasma. We will consider slab geometry also for the sake of simplicity, that is all coefficients α and δ are functions only of the variable x

$$\partial_y \alpha = \partial_y \delta = 0.$$

Other assumptions which correspond to the physical context of idealized reflectometry or heating devices are the following. We assume that

$$\delta \in \mathcal{C}^1[-L, \infty[, \qquad 0 < \delta_- \le \delta(x) \le \delta_+, \ \forall x. \tag{1.5}$$

The coefficient α vanishes at a finite number of points and $\alpha(x) = 0$ implies that $\alpha'(x) \neq 0$. For the sake of simplicity we will now suppose that α only vanishes at $x = 0$. More precisely

$$\alpha \in \mathcal{C}^2[-L, \infty[, \qquad \alpha(0) = 0, \quad \alpha'(0) < 0, \tag{H1}$$

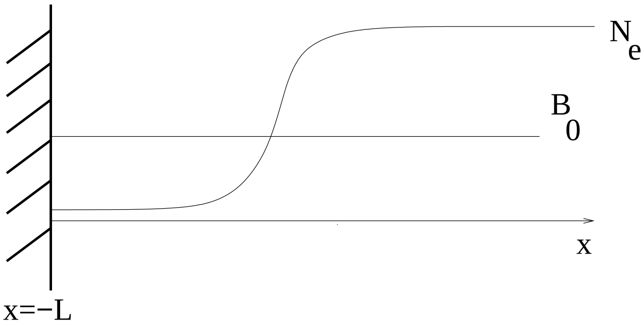

Figure 2: X-mode equations in slab geometry: the physical parameters. The electronic density $x \mapsto N_e(x)$ is low at the boundary, and increases towards a plateau. The background magnetic field B_0 is taken as constant for simplicity.

and

$$\alpha_- \leq \alpha(x) \leq \alpha_+, \ \forall x \in [-L, \infty[, \qquad \text{and} \quad 0 < r \leq \left| \frac{\alpha(x)}{x} \right|, \ \forall x \in [-L, H] \tag{H2}$$

where $H > 0$. We will also assume that the coefficients are constant at large scale: there exists δ_∞ and α_∞ so that

$$\delta(x) = \delta_\infty \text{ and } \alpha(x) = \alpha_\infty \qquad H \leq x < \infty. \tag{H3}$$

We assume that the problem is coercive at infinity

$$\alpha_\infty^2 - \delta_\infty^2 > 0. \tag{H4}$$

An additional condition is defined by

$$4\|\delta\|_\infty^2 H < r. \tag{H5}$$

It expresses the fact that the length of the transition zone between $x = 0$ and $x = H$ is small with respect to the other parameters of the problem. One can refer to figures 2 and 4 for a graphical representation. This hypothesis is physically very reasonnable. Assuming (H1)-(H4), it is known in the physical community that the problem is highly singular at the origin. Our main result can be summarized as follows. We denote by $\widehat{g}$ the Fourier transform of g,

$$\widehat{g}(\theta) := \int_{\mathbb{R}} g(y)\, e^{-i\theta y}\, dy.$$

We need the uniform transversality assumption H6 which is a generalization of assumption H5. See Section 7.

Theorem 1.1. *Assuming (H1-H6), for every $g \in L^2(\mathbb{R})$ with $\widehat{g}$ of compact support there exists a solution in the sense of distributions of (1.2) with boundary*

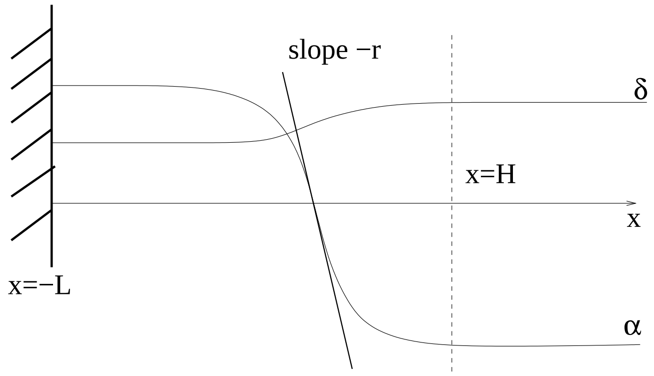

Figure 3: X-mode equations in slab geometry: parameters of the dielectric tensor deduced from the value of the physical parameters described in figure 2, assuming that $\omega > \omega_c$. The coefficient α decreases from positives to negative values. It crosses the axis with a slope bounded from below by r. The coefficient δ is positive and bounded. Since the electromagnetic wave is strongly absorbed for $x \geq H$, we simplify by taking all coefficients constant for $x \geq H$ because it does not change the physics of the problem.

condition (1.4) that goes to zero at infinity. Moreover, unless the source term g is identically zero, the electric field E_x does not belong to $L^1_{\rm loc}((-L,\infty)\times(-\infty,\infty))$. The other components E_y and W are more regular, they belong to $L^2((-L,\infty)\times(-\infty,\infty))$.

Remark 1. *The exact form of the solution is provided in (7.5), where we have taken $\omega = c = 1$. The energy lost by the electromagnetic wave, which is also the one transmitted to the plasma, is given in formula (7.6) which is the Fourier characterization of related physical formulas provided at the end of the appendix.*

The result will be obtained with the limit absorption principle combined with a specific original integral representation of the solution. The loss of regularity of the electric field is counter intuitive with respect to the standard theory of existence and uniqueness for solutions of time harmonic Maxwell's equations [12, 10, 21, 31]. The essential part of the proof consists in showing that the Fourier transform $\widehat{E}_x$ may be composed of three contributions: a Dirac mass at $x = 0$; a non integrable function proportional to $\frac{1}{\alpha(x)}$, that is interpreted as a distribution in the sense of principal value; and a regular part. The condition $\alpha'(0) < 0$ guarantees that the coefficient in front of the Dirac mass is finite. Moreover, the condition (H5) simplifies some parts of the mathematical analysis. The solution is a priori non unique since the limit absorption principle generates two solutions depending on the sign of the regularization. The heating of the plasma (7.6) is directly related to the singular part of the solution.

This work is organized as follows. Section 2 is devoted to basic considerations. In the next section we introduce a regularization parameter, and we propose a specific integral representation of the solution. After that we recall the Plemelj-Privalov theorem and explain why it cannot be used directly for our problem. Section 5 is where we prove the properties of the solutions of the regularized equations. In particular, we show that one basis function has a fundamental singularity. Next, in Section 6 we define the limit spaces. The main theorem is finally proved in section 7. Additional material is provided in the appendix.

2 Basic considerations

In this section we rederive the phase velocity, compute the analytic solutions of the simplified Budden problem and introduce the limit absorption principle.

2.1 Phase velocity

Recall that the phase velocity measures the velocity of individual Fourier modes.

2.1.1 Constant coefficients

Let us consider first that α and δ are constant at least locally. A plane wave $(E_x, E_y) = Re^{i(k_1x+k_2y)}$, $R \in \mathbb{C}^2$, is solution of X-mode equations (1.3) if and only if

$$\left[\begin{pmatrix} k_2^2 & -k_1\,k_2 \\ -k_1\,k_2 & k_1^2 \end{pmatrix} - \frac{\omega^2}{c^2}\begin{pmatrix} \alpha & i\delta \\ -i\delta & \alpha \end{pmatrix}\right] R = 0, \qquad k = (k_1, k_2) \in \mathbb{R}^2.$$

We assume that $c = 1$ for simplicity. We set $k = |k|d$ with $d = (\cos\theta, \sin\theta)$ the direction of the wave. The phase velocity $v_\varphi = \frac{\omega}{|k|}$ is solution of the eigenvalue problem

$$\begin{pmatrix} \sin\theta^2 - v_\varphi^2\alpha & -\cos\theta\sin\theta - iv_\varphi^2\delta \\ -\cos\theta\sin\theta + iv_\varphi^2\delta & \cos\theta^2 - v_\varphi^2\alpha \end{pmatrix} R = 0.$$

The determinant of the matrix is

$$D = v_\varphi^4\left(\alpha^2 - \delta^2\right) - v_\varphi^2\alpha.$$

Setting $D = 0$ we obtain the phase velocity,

$$v_\varphi^2 = \frac{\alpha}{\alpha^2 - \delta^2}.$$

2.1.2 Non constant coefficients

Let us assume for example that $\alpha = -x$ and that $\delta = 1$ which is locally compatible with the general assumptions. This represented in figure 4. We plot in figure 4 the phase velocity as a function of the horizontal space coordinate. When the phase velocity is real we are in a propagating region, and when the phase velocity is pure imaginary we are in a non-propagating region. One distinguishes two cutoffs where the local phase velocity is infinite

$$\text{Cutoff}: \quad \alpha(x) = \pm\delta(x)$$

and one resonance where the phase velocity is null

$$\text{Resonance}: \alpha(x) = 0.$$

This structure is characteristic of the hybrid resonance.

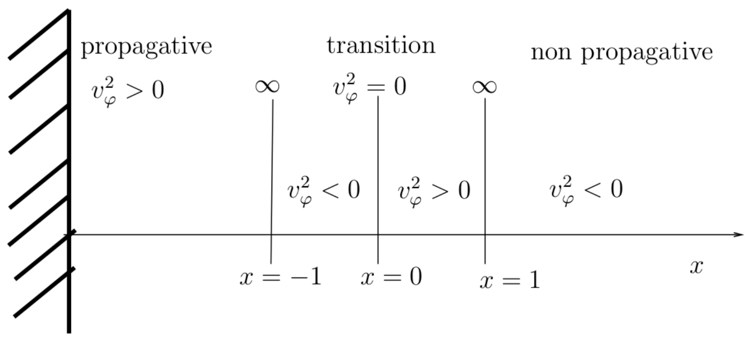

Figure 4: Sign of the square of the phase velocity. In this example $\alpha = -x$ and $\delta = 1$, so that $v_\varphi^2 = \frac{x}{1-x^2}$.

Remark 2. In what follows we always take $\omega = c = 1$.

2.2 The Budden problem

In the case where the solution is independent of y, what for the plane waves corresponds to normal incidence, that is $\theta = 0$, the system (1.3) is called the Budden problem [11]

$$\begin{cases} W - E_y' & = 0, \\ -\alpha E_x - i\delta E_y & = 0, \\ -W' + i\delta E_x - \alpha E_y & = 0. \end{cases}$$

After elimination of E_x and W we obtain that,

$$-E_y'' + \left(\frac{\delta^2}{\alpha} - \alpha\right) E_y = 0$$

This equation can be solved analytically in some cases which helps a lot to understand the singularity of the general problem. Let us consider that $\alpha = -x$ and δ is solution of $\frac{\delta^2}{x} - x = -\frac{1}{4} + \frac{1}{x}$. The positive solution is

$$\delta(x) = \sqrt{x^2 - \frac{x}{4} + 1} > 0.$$

The y-component of the electric field is solution of

$$E_y'' + \left(-\frac{1}{4} + \frac{1}{x}\right) E_y = 0. \tag{2.1}$$

This equation is of Whittaker type [1, 2]. It is a particular case of the confluent hypergeometric equation, and can also be rewritten under the Kummer form. The general theory shows that the first fundamental solution is regular

$$v(x) = e^{-\frac{x}{2}} x$$

Indeed $v'(x) = e^{-\frac{x}{2}} \left(1 - \frac{x}{2}\right)$ and $v''(x) = e^{-\frac{x}{2}} \left(-1 + \frac{x}{4}\right)$, so that

$$v'' + \left(-\frac{1}{4} + \frac{1}{x}\right) v = 0.$$

Let us consider a second solution w with linear independence with respect to the first one. The linear independence can be characterized by the normalized Wronskian relation

$$v(x)w'(x) - v'(x)w(x) = 1.$$

Seeking for a representation $w = vz$, one gets that

$$v^2 z' = 1 \Rightarrow z = \int \frac{dx}{v^2} \Rightarrow w = v \int \frac{dx}{v^2} = x\, e^{-x/2} \int \frac{e^x}{x^2}.$$

Moreover, [18],

$$\int \frac{e^x}{x^2} = -\frac{e^x}{x} + \int \frac{e^x}{x} = -\frac{e^x}{x} + E_i(x),$$

where $E_i(x)$ is the Exponential-integral function. It follows that,

$$\omega(x) = -e^{x/2} + x\, e^{-x/2}\, E_i(x).$$

Furthermore [18],

$$E_i(x) = \ln|x| + |\sum_{j=1}^{\infty} \frac{x^j}{j \cdot j!}.$$

It follows that,

$$\omega(x) = -1 + x \ln|x| + O(|x|), \quad |x| \to 0.$$

We notice that the second function w is bounded, but non regular at origin. It shows the subtleties associated with the singular Whittaker equation (2.1). Nevertheless we note that the general form of the y component of the electric field of the Budden problem is bounded

$$E_y = av + bw \Rightarrow E_y \in L^\infty] - \varepsilon, \varepsilon[.$$

The x component of the electric field is more singular. It is a linear combination of two functions, the first one which is regular and bounded

$$E_x^v(x) = i\frac{\sqrt{x^2 - \frac{x}{4} + 1}}{x} v(x) = ie^{-\frac{x}{2}} \sqrt{x^2 - \frac{x}{4} + 1},$$

and the second one singular which is singular at origin since $w(0) = 1$

$$E_x^w(x) = i\frac{\sqrt{x^2 - \frac{x}{4} + 1}}{x} w(x).$$

The general form of the x component of the electric field is a linear combination of these two functions. Since $E_x^w \notin L^2] - \varepsilon, \varepsilon[$, we notice that the electric field is not a square integrable function in general.

2.3 Limit absorption principle

We will develop a regularized approach to give a rigorous meaning to the solution at all incidences. This regularized approach is based on the limit absorption principle. One considers a parameter $\mu \neq 0$ (the precise sign will be justified later) and the regularized problem with unknown $(E_x^\mu, E_y^\mu, W^\mu)$

$$\left\{ \begin{array}{cccc} W^\mu & +\partial_y E_x^\mu & -\partial_x E_y^\mu & = 0, \\ \partial_y W^\mu & -(\alpha(x) + i\mu)E_x^\mu & -i\delta(x)E_y^\mu & = 0, \\ -\partial_x W^\mu & +i\delta(x)E_x^\mu & -(\alpha(x) + i\mu)E_y^\mu & = 0. \end{array} \right. \tag{2.2}$$

The regularization parameter μ can be interpreted as a small collision frequency. See Appendix B.

A further simplification consists in Fourier reduction. Since the coefficients do not depend on the y variable, one can perform the usual one dimension reduction. The system that will be studied in this article is obtained by applying the Fourier transformto the regularized system (2.2). Denoting the unknowns (U, V, W) it yields

$$\left\{ \begin{array}{cccc} W & +i\theta U & -V' & = 0, \\ i\theta W & -(\alpha(x) + i\mu)U & -i\delta(x)V & = 0, \\ -W' & +i\delta(x)U & -(\alpha(x) + i\mu)V & = 0. \end{array} \right. \tag{2.3}$$

Here the notation $'$ denotes the derivative with respect to the x variable.

3 A general integral representation

We begin by some notations. Let us denote by (A_μ, B_μ) the two fundamental solutions of the modified equation

$$-u'' - (\alpha(x) + i\mu)u = 0, \tag{3.1}$$

with the usual normalization

$$A_\mu(0) = 1, \quad A'_\mu(0) = 0 \qquad \text{and } B_\mu(0) = 0, \quad B'_\mu(0) = 1. \tag{3.2}$$

Various continuity estimates of A_μ and B_μ are derived in the appendix for the sake of the completeness of this work. Let us denote $\mathcal{D}_z^\theta$ the operator $i\theta\partial_z - i\delta(z)$ applied to any function h, that is

$$\mathcal{D}_z^\theta h = i\theta\partial_z h - i\delta(z)h. \tag{3.3}$$

Let us define the kernel

$$k^\mu(x,z) = B_\mu(z)A_\mu(x) - B_\mu(x)A_\mu(z). \tag{3.4}$$

Next we define

$$K_1^{\theta,\mu}(x,z;G) = \begin{cases} \dfrac{\mathcal{D}_x^\theta \mathcal{D}_z^\theta k^\mu(x,z)}{\alpha(x)+i\mu}, & \text{for } G \le z \le x \quad \text{or } x \le z \le G, \\ 0, & \text{in all other cases.} \end{cases} \tag{3.5}$$

Let us define the kernel sequence by

$$K_{n+1}^{\theta,\mu}(x,z;G) = \int_G^x \frac{\mathcal{D}_x^\theta \mathcal{D}_z^\theta k^\mu(x,t)}{\alpha(x)+i\mu} K_n^{\theta,\mu}(t,z;G)dt. \tag{3.6}$$

The sum is

$$\mathcal{K}^{\theta,\mu}(x,z;G) = \sum_{n=0}^{\infty} K_{n+1}^{\theta,\mu}(x,z;G). \tag{3.7}$$

The integration domain is centered on G, that is

$$\text{supp}\left(K_1^{\theta,\mu}(\cdot,\cdot;G)\right) \subset \left\{(x,y) \in \mathbb{R}^2; \quad G \le z \le x \text{ or } x \le z \le G\right\} \equiv \mathcal{D}_G, \tag{3.8}$$

which yields as well

$$\text{supp}\left(K^{\theta,\mu}(\cdot,\cdot;G)\right) \subset \mathcal{D}_G.$$

Proposition 3.1. *Any triplet (U,V,W) solution of the regularized system* (2.3) *admits the following integral representation.*

- *One first chooses an arbitrary reference point $G \in [-L,\infty[$.*
- *The x component of the electric field is solution of the integral equation*

$$U(x) - \int_G^x \frac{\mathcal{D}_x^\theta \mathcal{D}_z^\theta k^\mu(x,z)}{\alpha(x)+i\mu} U(z)dz = \frac{F^{\theta,\mu}(x)}{\alpha(x)+i\mu}, \tag{3.9}$$

where the right hand side is

$$F^{\theta,\mu}(x) = a_G \mathcal{D}_x^\theta A_\mu(x) + b_G \mathcal{D}_x^\theta B_\mu(x) \tag{3.10}$$

and the kernel is given in (3.3-3.4). The solution of this integral equation is naturally provided by the resolvent integral formula

$$U(x) = \frac{F^{\theta,\mu}(x)}{\alpha(x)+i\mu} + \int_G^x \mathcal{K}^{\theta,\mu}(x,z;G)\frac{F_G^{\theta,\mu}(z)}{\alpha(z)+i\mu}dz \tag{3.11}$$

where the resolvent kernel is constructed in (3.6).

- *The y component of the electric field is recovered as*

$$V(x) = a_G A_\mu(x) + b_G B_\mu(x) + \int_G^x \mathcal{D}_z^\theta k^\mu(x,z)U(z)dz, \tag{3.12}$$

and the vorticity is recovered as

$$W(x) = a_G A'_\mu(x) + b_G B'_\mu(x) + \int_G^x \partial_x \mathcal{D}_z^\theta k^\mu(x,z)U(z)dz. \tag{3.13}$$

- *The two complex numbers (a_G, b_G) solve the linear system*

$$\begin{cases} a_G A_\mu(G) + b_G B_\mu(G) = V(G), \\ a_G A'_\mu(G) + b_G B'_\mu(G) = W(G). \end{cases} \tag{3.14}$$

Proof. Eliminating W from the first and third equations of (2.3) gives

$$-V'' - (\alpha + i\mu)V = f \qquad \text{with } f = -i\theta U' - i\delta U.$$

Since the Wronskian is constant, it follows from the normalization (3.2) that $A_\mu B'_\mu - A'_\mu B_\mu = 1$. Then, from the variation of constants formula,

$$V(x) = a_f A_\mu(x) + b_f B_\mu(x) + \int_G^x f(z)k^\mu(x,z)dz, \quad \forall x. \tag{3.15}$$

where a_f and b_f are two integration constants. Now we replace f by the corresponding function of U and perform the integration by part

$$\int_G^x U'(z)k^\mu(x,z)dz = U(x)k^\mu(x,x) - U(G)k^\mu(x,G) - \int_G^x U(z)\partial_z k^\mu(x,z)dz.$$

Since $k^\mu(x,x) = 0$ there is a simplification. Therefore (3.15) yields (3.12) with $a_G = a_f + i\theta U(G)B_\mu(G)$ and $b_G = b_f - i\theta U(G)A_\mu(G)$. Next we eliminate W from the first and second equations of (2.3) and obtain

$$-i\theta V' - \theta^2 U + (\alpha + i\mu)U + i\delta V = 0. \tag{3.16}$$

The derivative of (3.12) yields

$$V'(x) = a_G A'_\mu(x) + b_G B'_\mu(x) + \int_G^x \partial_x \mathcal{D}_z^\theta k^\mu(x,z)U(z)dz + \mathcal{D}_z^\theta k^\mu(x,x)U(x).$$

Since $\mathcal{D}_z^\theta k^\mu(x,x) = i\theta\left(A_\mu B'_\mu - B_\mu A'_\mu\right) = i\theta$, one gets the identity

$$V'(x) = a_G A'_\mu(x) + b_G B'_\mu(x) + \int_G^x \partial_x \mathcal{D}_z^\theta k^\mu(x,z) U(z) dz + i\theta U(x).$$

Plugging this expression in (3.16) and performing all simplifications we obtain the integral equation (3.9). Finally, we get the last integral formula (3.13) from $W = -i\theta U + V'$. The linear system (3.14) is obvious from (3.12-3.13) at $x = G$. □

Following [24], the equation (3.9) is an integral equation of the third kind in the case $\mu = 0$. In this case the theory is rather incomplete regarding existence and uniqueness [3]. However as long as $\mu \neq 0$, the solution based on these integral equations is uniquely defined. Then, the question is to determine the behavior of these solutions when μ goes to 0. Moreover, different choices of G will give different kind of information. A strategy to study of the limit solution $\mu \to 0$ can be the following: *Choose an optimal G, so that a) the integration constants (a_G, b_G) are easy to determine, and b) the resolvent kernel $\mathcal{K}^{\theta,\mu}(\cdot,\cdot;G)$ admits a limit as $\mu \to 0$.* Considering the form of the right hand side in (3.11), a convenient tool is the Plemelj-Privalov Theorem [23, 26]. Unfortunately, we will see that a fundamental singularity of the kernel $\mathcal{K}^{\theta,\mu}(\cdot,\cdot;G)$ prevents any simple limit procedure. A more convenient technique will be proposed in Section 5.

4 Singularity of the kernels

A fundamental tool in order to pass to the limit in singular integrals is the Plemelj-Privalov theorem [23, 26]. However, to apply this theorem to pass to the limit $\mu \to 0$ in equation (3.11) it is necessary that the kernel $\mathcal{K}^{\theta,\mu}(x,z)$ be a Hölder continuous function of z for each fixed x. Unfortunately, this regularity is not available in our case. To illustrate this phenomenon, we study only the first term of the series (3.6) that defines $\mathcal{K}^{\theta,\mu}$, namely

$$\mathcal{K}_1^{\theta,\mu}(x,z) := \frac{\mathcal{D}_x^\theta \mathcal{D}_z^\theta k^\mu(x,z)}{\alpha(x) + i\mu}. \tag{4.1}$$

We consider two cases.

4.1 First case: $G \neq 0$

In this case there exists $(0,z) \in \mathcal{D}_G$ with $z \neq 0$. In the limit case $\mu = 0$ one has that $\mathcal{K}_1^{\theta,0}(x,z)$ admits the local expansion:

$$\mathcal{K}_1^{\theta,0}(x,z) \approx \frac{1}{x\ \alpha'(0)} \mathcal{D}_x^\theta \mathcal{D}_z^\theta k^0(x,z).$$

Therefore, $\mathcal{K}_1^{\theta,0}(x,z)$ blows up as $x \to 0$.

4.2 Second case: $G = 0$

We turn to the case $G = 0$. We begin with a preliminary result.

Proposition 4.1. *One has*

$$(\mathcal{D}_x^\theta \mathcal{D}_z^\theta k^\mu)(x,x) = 0 \quad \forall x \in \mathbb{R}. \tag{4.2}$$

Proof. Indeed by construction

$$(\mathcal{D}_x^\theta \mathcal{D}_z^\theta k^\mu)(x,x) = -\delta(x)\delta(x)k^\mu(x,x)$$

$$+\theta\,\delta(x)\left((\partial_x k^\mu)(x,x) + (\partial_z k^\mu)(x,x)\right) - \theta^2(\partial_x\partial_z k^\mu)(x,x).$$

We notice that by definition $k_\mu(x,x) = 0$ for all x so the first contribution vanishes in $(\mathcal{D}_x^\theta \mathcal{D}_z^\theta k^\mu)(x,x)$. One also has that

$$(\partial_x k^\mu)(x,x) + (\partial_z k^\mu)(x,x)$$

$$= B_\mu(x)A'_\mu(x) - B'_\mu(x)A_\mu(x) + B'_\mu(x)A_\mu(x) - B_\mu(x)A'_\mu(x) = 0,$$

so, the second contribution vanishes also. Furthermore,

$$(\partial_x\partial_z k^\mu)(x,x) = B'_\mu(x)\,A'_\mu(x) - B'_\mu(x)\,A'_\mu(x) = 0.$$

This completes the proof of equation (4.2). □

Proposition 4.2. *The limit kernel*

$$\frac{\mathcal{D}_x^\theta \mathcal{D}_z^\theta k^{\mu=0}(x,z)}{\alpha(x)}$$

belongs to $L^\infty(\mathcal{D}_0)$.

Proof. A first order Taylor expansion of $\mathcal{D}_x^\theta \mathcal{D}_z^\theta k^\mu$ around 0 yields

$$\mathcal{D}_x^\theta \mathcal{D}_z^\theta k^\mu(x,z) = \alpha_\mu x + \beta_\mu z + 0(|x|^2 + |z|^2).$$

Notice that (4.2) implies $\beta_\mu = -\alpha_\mu$. The coefficient α_μ is easily computed using

$$(\mathcal{D}_x^\theta \mathcal{D}_z^\theta k^\mu)(x,0) = \mathcal{D}_x^\theta A_\mu(x)\mathcal{D}_z^\theta B_\mu(0) - \mathcal{D}_x^\theta B_\mu(x)\mathcal{D}_z^\theta A_\mu(0)$$

and the definition (3.1-3.2). One gets that $\mathcal{D}_x^\theta A^\mu(x) = -i\delta(0) - i\delta'(0)x + \theta\mu x + O(x^2)$ and $\mathcal{D}_x^\theta B^\mu(x) = i\theta - i\delta(0)x + O(x^2)$. So

$$(\mathcal{D}_x^\theta \mathcal{D}_z^\theta k^\mu)(x,0) = \left(-i\delta(0) - i\delta'(0)x + \theta\mu x + O(x^2)\right)i\theta$$

$$-\left(i\theta - i\delta(0)x + O(x^2)\right)(-i\delta(0))$$

$$= \left(\delta(0)^2 + \theta\delta'(0) + i\theta^2\mu\right)x + O(x^2).$$

This coefficient α_μ being constant, one obtains that

$$\varphi_x(z) := \frac{\mathcal{D}_x^\theta \mathcal{D}_z^\theta k^{\mu=0}(x,z)}{\alpha(x)} = \frac{\left(\delta(0)^2 + \theta\delta'(0)\right)(x-z) + 0(|x|^2 + |z|^2)}{\alpha(x)}. \quad (4.3)$$

This expansion is valid for $(x,z) \in \mathcal{D}_0$ (the domain $\mathcal{D}_0$ is defined in (3.8)): in this case $|x-z| \leq |x|$ and $|z| \leq |x|$. Moreover, since $\alpha(x) = x(\alpha'(0)x + O(x))$ we obtain that,

$$|\varphi_x(z)| \leq \frac{\left|\delta(0)^2 + \theta\delta'(0)\right|}{\sqrt{\alpha'(0)^2}} + O(|x|).$$

Since there is no such difficulty for x away from 0, this inequality ends the proof of the proposition. □

Remark 3. *A similar property holds for* $\frac{\mathcal{D}_x^\theta \mathcal{D}_z^\theta k^\mu(x,z)}{\alpha(x)+i\mu}$ *which also belongs to* $L^\infty(\mathcal{D}_0)$ *for all* θ *and uniformly for* $\mu \in [-1,1] \setminus \{0\}$*, that is*

$$\left\| \frac{\mathcal{D}_x^\theta \mathcal{D}_z^\theta k^\mu(x,z)}{\alpha(x)+i\mu} \right\|_{L^\infty(\mathcal{D}_0)} \leq C^\theta, \quad \mu \in [-1,1] \setminus \{0\}. \quad (4.4)$$

Such estimate is sufficient to control some L^∞ bounds of the series that defines the iterated kernel $\mathcal{K}^{\theta,\mu}(x,z;0)$. However, L^∞ bounds are not sufficient to show that $\mathcal{K}^{\theta,\mu}(x,z;0)$ is of Hölder class in z in the vicinity of $x=0$: That is, one cannot pass to the limit using the Plemelj-Privalov theorem for all values of the parameters involved in (3.6, 3.11). This is why we will develop another approach to give a meaning to the limit value.

5 The space $\mathbb{X}^{\theta,\mu}$ $(\mu \neq 0)$

The solutions of the integral equations evidently belong to a vectorial space of dimension two: see also (5.2). In a first stage we will design a particular basis in this space, in a second stage we will study the properties of the two basis functions. A careful analysis of this singularity will allow to show that one basis function (more precisely the x component of electric field) is the sum of a singular part $\frac{1}{\alpha(x)+i\mu}$ plus a term which is bounded in L^p $(1 \leq p < \infty)$ uniformly with respect to μ. It will be the central result of this part.

For the simplicity of notations, we restrict the parameter to $0 < \mu \leq 1$ without loss of generality. The extension to negative μ will be considered in section (6.2). We define the vectorial space of all solutions of the X-mode equations

$$\mathbb{X}^{\theta,\mu} = \{x \mapsto (U(x), V(x), W(x)), \text{ for all solutions of the system (2.3)}\}. \quad (5.1)$$

One may also use the notation

$$\mathbf{U}^{\theta,\mu} = (U^{\theta,\mu}, V^{\theta,\mu}, W^{\theta,\mu}) \in \mathbb{X}^{\theta,\mu}.$$

This section is devoted to the analysis of this space.

Remark 4. *The property that* $\dim \mathbb{X}^{\theta,\mu} = 2$ *is also evident considering the right hand side of the integral equation (3.9).*

By elimination $U^{\theta,\mu}$ in (2.3), one gets a system of two coupled ordinary differential equations

$$\frac{d}{dx}\begin{pmatrix} V^{\theta,\mu} \\ W^{\theta,\mu} \end{pmatrix} = A^{\theta,\mu}(x)\begin{pmatrix} V^{\theta,\mu} \\ W^{\theta,\mu} \end{pmatrix} \tag{5.2}$$

with

$$A^{\theta,\mu}(x) = \begin{pmatrix} \frac{\theta\delta(x)}{\alpha(x)+i\mu} & 1 - \frac{\theta^2}{\alpha(x)+i\mu} \\ \frac{\delta(x)^2}{\alpha(x)+i\mu} - \alpha(x) - i\mu & -\frac{\theta\delta(x)}{\alpha(x)+i\mu} \end{pmatrix}. \tag{5.3}$$

In the case $\mu \neq 0$ the matrix is non singular for all x, which gives a meaning to the regularized problem. One notices the matrix is singular for $\mu = 0$.

Lemma 5.1. *Take two solutions* $\left(V^{\theta,\mu}, W^{\theta,\mu}\right)$ *and* $\left(\widetilde{V}^{\theta,\mu}, \widetilde{W}^{\theta,\mu}\right)$ *of (5.2). Define the Wronskian*

$$\mathcal{W}(x) = V^{\theta,\mu}(x)\widetilde{W}^{\theta,\mu}(x) - W^{\theta,\mu}(x)\widetilde{V}^{\theta,\mu}(x). \tag{5.4}$$

Then the Wronskian is constant

$$\mathcal{W}(x) = \mathcal{W}(0), \qquad \forall x.$$

Proof. The system (5.2) main be rewritten as

$$\frac{d}{dx}\begin{pmatrix} V \\ W \end{pmatrix} = \begin{pmatrix} a & b \\ c & -a \end{pmatrix}\begin{pmatrix} V \\ W \end{pmatrix}.$$

Therefore

$$\frac{d}{dx}\mathcal{W} = \frac{d}{dx}\left(V(x)\widetilde{W}(x) - W(x)\widetilde{V}(x)\right)$$

$$= (aV + bW)\,\widetilde{W} + V\left(c\widetilde{V} - a\widetilde{W}\right) - (cV - aW)\,\widetilde{V} - W\left(a\widetilde{V} + b\widetilde{W}\right) = 0$$

since all terms cancel each other. □

5.1 The first basis function

Next we desire to particularize a convenient basis in this space. The first basis function

$$\mathbf{U}_1^{\theta,\mu} = \left(U_1^{\theta,\mu}, V_1^{\theta,\mu}, W_1^{\theta,\mu}\right) \in \mathbb{X}^{\theta,\mu} \tag{5.5}$$

is the natural one which is smooth at the origin: $U_1^{\theta,\mu}(0) = 0$. For that reason G is chosen to be the origin in this subsection, so that the corresponding integral equation has a bounded right-hand side and a bounded kernel. It is naturally characterized by

$$V_1^{\theta,\mu}(0) = i\theta, \qquad \text{and } W_1^{\theta,\mu}(0) = i\delta(0) \qquad (\neq 0). \tag{5.6}$$

Proposition 5.1. *The basis function (5.5) is uniformly bounded with respect to μ: for any interval $\theta \in [\theta_-, \theta_+]$ and any $H \in]L, \infty[$, there exists a constant independent of μ such that*

$$\left\| U_1^{\theta,\mu} \right\|_{L^\infty(-L,H)} + \left\| V_1^{\theta,\mu} \right\|_{L^\infty(-L,H)} + \left\| W_1^{\theta,\mu} \right\|_{L^\infty(-L,H)} \leq C. \tag{5.7}$$

Proof. The right hand side in the integral equation (3.9) is

$$g^\mu(x) = \frac{h^\mu(x)}{\alpha(x) + i\mu} \text{ with } h^\mu(x) = i\theta \mathcal{D}_x^\theta A_\mu(x) + i\delta(0)\mathcal{D}_x^\theta B_\mu(x).$$

With the choice (5.5) one has

$$h^\mu(0) = i\theta(-i\delta(0)) + i\delta(0)(i\theta) = 0 \qquad \forall \mu.$$

Therefore the right hand side of the integral equation

$$g^\mu(x) = \frac{h^\mu(x) - h^\mu(0)}{\alpha(x) + i\mu}$$

is bounded in $L^\infty(-L, H))$ uniformly with respect to μ. The solution $U_1^{\theta,\mu}$ (3.11) is also bounded, since by the results of Subsection 4.2 the kernel $\mathcal{K}^{\theta,\mu}(x, z, 0)$ is also uniformly bounded. These bounds are uniform with respect to μ. The integral representation (3.12) of the $V_1^{\theta,\mu}$ yields that $V_1^{\theta,\mu}$ is also bounded. It is similar concerning the integral representation (3.13) of the $W_1^{\theta,\mu}$, so $W_1^{\theta,\mu}$ is also bounded. □

5.2 Behavior at infinity

Hypothesis (H3) allows to study a simplified model with constant coefficients for $x \geq H$. In fact, it corresponds to a system as in (5.2) with constant coefficients, which matrix will be denoted $A_\infty^{\theta,\mu}$.

Proposition 5.2. *The matrix $A_\infty^{\theta,\mu}$ has two distinct eigenvalues. The first eigenvalue $\lambda^{\theta,\mu}$ has a positive real part. The second eigenvalue is $-\lambda^{\theta,\mu}$.*

Proof. The eigenvalues are solution to the characteristic equation

$$\lambda^2 - \text{tr}(A_\infty^{\theta,\mu})\lambda + \det(A_\infty^{\theta,\mu}) = 0$$

where $\text{tr}(A_\infty^{\theta,\mu}) = 0$ and

$$\det(A_\infty^{\theta,\mu}) = \alpha_\infty + i\mu - \theta^2 - \frac{\delta_\infty^2}{\alpha_\infty + i\mu}.$$

The real part is

$$\text{real}\left(\det(A_\infty^{\theta,\mu})\right) = \alpha_\infty - \theta^2 - \frac{\delta_\infty \alpha_\infty}{\alpha_\infty^2 + \mu^2} = \alpha_\infty \left(1 - \frac{\delta_\infty^2}{\alpha_\infty^2 + \mu^2}\right) - \theta^2$$

and is therefore negative due to the coercivity assumption (H4). So the usual square root $\lambda^{\theta,\mu} = \sqrt{-\det(A_\infty^{\theta,\mu})}$ has a positive real part. The other one has a negative real part. □

As a consequence any $\mathbf{U} \in \mathbb{X}^{\theta,\mu}$ is at large scale a linear combination of the exponential increasing function and a exponential decreasing function

$$\mathbf{U}(x) = c_+ R_+ e^{\lambda^{\theta,\mu} x} + c_- R_- e^{-\lambda^{\theta,\mu} x} \qquad H \le x \tag{5.8}$$

where $R_+ \in \mathbb{C}^3$ and $R_- \in \mathbb{C}^3$ are constant vectors and $(c_+, c_-) \in \mathbb{C}^2$ are arbitrary complex numbers. Regarding the structure of the matrix and using the second equation of the system (2.3), one gets that $R_+ = (r_+^1, r_+^2, r_+^3)$ with

$$r_+^1 = \frac{i\theta r_+^3 - i\delta(H) r_+^2}{\alpha(H) + i\mu}, \; r_+^2 = 1 - \frac{\theta^2}{\alpha(H) + i\mu}, \; r_+^3 = \sqrt{-\det(A_\infty^{\theta,\mu})} - \frac{\theta\delta(H)}{\alpha(H) + i\mu}.$$

The other vector $R_- = (r_-^1, r_-^2, r_-^3)$ is characterized by

$$r_-^1 = \frac{i\theta r_-^3 - i\delta(H) r_-^2}{\alpha(H) + i\mu}, \; r_-^2 = 1 - \frac{\theta^2}{\alpha(H) + i\mu}, \; r_-^3 = -\sqrt{-\det(A_\infty^{\theta,\mu})} - \frac{\theta\delta(H)}{\alpha(H) + i\mu}.$$

One notices that R_+ and R_- are well defined for all $\mu \in \mathbb{R}$, in particular even for $\mu = 0$.

Proposition 5.3. *The first basis function (5.5) is exponentially growing at large scale ($\mu \neq 0$).*

Proof. For the sake of simplicity, denote $\mathbf{U}_1^{\theta,\mu} = (U_1, V_1, W_1)$, dropping the θs and μs. Then from system (2.3) one gets

$$\begin{cases} W_1 & +i\theta U_1 & -V_1' & = 0, \\ i\theta W_1 & -(\alpha + i\mu) U_1 & -i\delta V_1 & = 0, \\ -W_1' & +i\delta U_1 & -(\alpha + i\mu) V_1 & = 0. \end{cases}$$

Multiplying the second equation by $\overline{U_1}$ and the third one by $\overline{V_1}$, the sum writes

$$i\theta W_1 \overline{U_1} - W_1' \overline{V_1} - \left(\alpha |U_1|^2 + \alpha |V_1|^2 + i\delta V_1 \overline{U_1} - i\delta U_1 \overline{V_1}\right) - i\mu \left(|U_1|^2 + |V_1|^2\right) = 0.$$

On the other hand an integration in the interval $]M, N[$ yields

$$\int_M^N \left(i\theta W_1 \overline{U_1} - W_1' \overline{V_1}\right) dx$$

$$= \int_M^N \left(i\theta W_1 \overline{U_1} + W_1 \overline{V_1}'\right) dx - W_1(N) \overline{V_1}(N) + W_1(M) \overline{V_1}(M)$$

$$= \int_M^N |W_1|^2 dx - W_1(N) \overline{V_1}(N) + W_1(M) \overline{V_1}(M),$$

where we used the first equation. We obtain the identity,

$$\int_M^N \left(|W_1|^2 - \alpha |U_1|^2 - \alpha |V_1|^2 - i\delta V_1 \overline{U_1} + i\delta U_1 \overline{V_1}\right) dx - i\mu \int_M^N \left(|U_1|^2 + |V_1|^2\right) dx \tag{5.9}$$

$$= W_1(N)\overline{V_1}(N) - W_1(M)\overline{V_1}(M).$$

Splitting between the real and imaginary parts, one gets the important relation

$$\mu \int_M^N \left(|U_1|^2 + |V_1|^2\right) dx = \operatorname{Im}\left(W_1(M)\overline{V_1}(M)\right) - \operatorname{Im}\left(W_1(N)\overline{V_1}(N)\right) \qquad (5.10)$$

which is true in fact for any element in $\mathbb{X}^{\theta,\mu}$ and for any $M < N$.

Let us take $M = 0$: so $V_1(0) = \frac{\theta}{\delta(0)} W_1(0)$ and $\operatorname{Im}\left(W_1(M)\overline{V_1}(M)\right) = 0$. Therefore

$$\mu \int_M^N \left(|U_1|^2 + |V_1|^2\right) dx = -\operatorname{Im}\left(W_1(N)\overline{V_1}(N)\right).$$

It shows that $W_1(N)\overline{V_1}(N) \not\to 0$ for $N \to \infty$. In other words the first basis function does not decrease exponentially at infinity. Considering (5.8) it means that this function is exponentially increasing at infinity. □

5.3 The second basis function

The second basis function

$$\mathbf{U}_2^{\theta,\mu} = (U_2^{\theta,\mu}, V_2^{\theta,\mu}, W_2^{\theta,\mu}) \in \mathbb{X}^{\theta,\mu}$$

is built with two requirements.

- It is exponentially decreasing at infinity, that is

$$\mathbf{U}_2^{\theta,\mu}(x) = c_- R_- e^{-\lambda^{\theta,\mu} x}, \qquad H \le x, \qquad (5.11)$$

  for some $c_- \in \mathbb{C}$.

- Its value at the origin is normalized with the requirement

$$i\mu U_2^{\theta,\mu}(0) = 1. \qquad (5.12)$$

To ensure that these conditions can be satisfied, consider the third function

$$\mathbf{U}_3^{\theta,\mu} = (U_3^{\theta,\mu}, V_3^{\theta,\mu}, W_3^{\theta,\mu})(x) = R_- e^{-\lambda^{\theta,\mu} x} \qquad H \le x, \qquad (5.13)$$

where R_- and λ_- are defined in section 5.2, smoothly extended so that $\mathbf{U}_3^{\theta,\mu} \in \mathbb{X}^{\theta,\mu}$. The identity

$$\mu \int_M^N \left(|U_3^{\theta,\mu}|^2 + |V_3^{\theta,\mu}|^2\right) dx$$
$$= \operatorname{Im}\left(W_3^{\theta,\mu}(M)\overline{V_3^{\theta,\mu}}(M)\right) - \operatorname{Im}\left(W_3^{\theta,\mu}(N)\overline{V_3^{\theta,\mu}}(N)\right)$$

with $N \to \infty$ and $M = 0$ shows that

$$\mu \int_0^\infty \left(|U_3^{\theta,\mu}|^2 + |V_3^{\theta,\mu}|^2\right) dx = \operatorname{Im}\left(W_3^{\theta,\mu}(0)\overline{V_3^{\theta,\mu}}(0)\right).$$

However, from (2.3), $V_3^{\theta,\mu}(0) = \frac{\theta}{\delta(0)} W_3(0) - \frac{\mu}{\delta(0)} U_3^{\theta,\mu}(0)$, so one gets

$$\mu \int_0^\infty \left(|U_3^{\theta,\mu}|^2 + |V_3^{\theta,\mu}|^2 \right) dx = -\frac{\mu}{\delta(0)} \mathrm{Im} \left(W_3^{\theta,\mu}(0) \overline{U_3^{\theta,\mu}}(0) \right).$$

It shows that $U_3^{\theta,\mu}(0) \neq 0$. This is why it is always possible to renormalize with a parameter

$$\mathbf{U}_2^{\theta,\mu} = c_- \mathbf{U}_3^{\theta,\mu}, \qquad c_- = \frac{1}{i\mu U_3^{\theta,\mu}(0)} \tag{5.14}$$

so as to enforce (5.12).

Proposition 5.4. *With the normalizations (5.6) and (5.11-5.12), the Wronskian relation takes the form*

$$V_1^{\theta,\mu}(x) W_2^{\theta,\mu}(x) - W_1^{\theta,\mu}(x) V_2^{\theta,\mu}(x) = 1 \qquad \forall x. \tag{5.15}$$

Proof. It is sufficient to compute it at the origin

$$\begin{aligned} V_1^{\theta,\mu}(0) W_2^{\theta,\mu}(0) - W_1^{\theta,\mu}(0) V_2^{\theta,\mu}(0) &= i\theta W_2^{\theta,\mu}(0) - i\delta(0) V_2^{\theta,\mu}(0) \\ &= (\alpha(0) + i\mu) U_2^{\theta,\mu}(0) = i\mu U_2^{\theta,\mu}(0) = 1 \end{aligned}$$

using (2.3) and thanks to (5.12). □

Remark 5. *The value of the Wronskian (5.15) is independent of μ. It will be of major interest in the limit regime $\mu \to 0$.*

The non zero Wronskian shows (5.15) shows that the two basis function are linearly independent. So they span the whole space

$$\mathbb{X}^{\theta,\mu} = \mathrm{Span} \left\{ \mathbf{U}_1^{\theta,\mu}, \mathbf{U}_2^{\theta,\mu} \right\}, \qquad \mu > 0.$$

5.4 Passing to the limit $\mu \to 0$

We now study the limit $\mu \to 0$. An important result is that the first basis function admits a limit which is defined as a continuous function in $\mathcal{C}^0[-L, \infty[$ and is independent of the sign of μ. On the other hand the second basis function admits a limit which is singular at $x = 0$. Moreover the limit is different for $\mu \to 0^+$ and for $\mu \to 0^-$. The linear independence of these limits will be establish with a transversality condition.

5.4.1 The first basis function

There is no difficulty for this case which is easily treated passing to the limit in the integral equation (3.11), choosing $G = 0$. The limit basis function is referred to as

$$\mathbf{U}_1^\theta = (U_1^\theta, V_1^\theta, W_1^\theta)$$

$\mathbf{U}_1^\theta$ is and will be called the regular solution by analogy with the terminology in scattering on the half-line.

Proposition 5.5. *The first basis function satisfies for $\mu \to 0$*

$$\left\| \mathbf{U}_1^{\theta,\mu} - \mathbf{U}_1^{\theta} \right\|_{(L^\infty(-L,H))^3} \to 0.$$

Proof. Setting $G = 0$ both the resolvent kernel and the function $\frac{F^{\theta,\mu}(x)}{\alpha(x)+i\mu}$ are bounded for all $\mu \neq 0$, and admit a continuous limit with respect to $\mu \to 0$. The limit basis function is defined by

$$V_1^\theta(0) = i\theta, \qquad \text{and } W_1^\theta(0) = i\delta(0) \qquad (\neq 0). \tag{5.16}$$

Then U_1^θ is the solution of the non singular integral equation (3.9)

$$U_1^\theta(x) - \int_0^x \frac{\mathcal{D}_x^\theta \mathcal{D}_z^\theta k(x,z)}{\alpha(x)} U_1^\theta(z)dz = \frac{i\theta \mathcal{D}_x^\theta A(x) + i\delta(0)\mathcal{D}_x^\theta B(x))}{\alpha(x)},$$

with $k(x,z) = A(x)B(z) - B(x)A(z)$. This equation is well posed since the kernel is locally bounded and the right-hand side is a locally bounded function. The other components are

$$V_1^\theta(x) = i\theta A(x) + i\delta(0)B(x) + \int_0^x \mathcal{D}_z^\theta k(x,z) U_1^\theta(z)dz$$

and

$$W_1^\theta(x) = i\theta A'(x) + i\delta(0)B'(x) + \int_0^x \partial_x \mathcal{D}_z^\theta k(x,z) U_1^\theta(z)dz.$$

Therefore the convergence is granted by classical dominated convergence theorems. □

The next result establishes that $\mathbf{U}_1^\theta$ is still exponentially increasing at infinity with a technical condition.

Proposition 5.6. *Assume hypothesis* (H5). *Then $\mathbf{U}_1^{\theta=0}$ increases exponentially at infinity.*

Remark 6. *The constant 4 in the condition* (H5) *is probably non optimal.*

Proof. We drop the super-index $\cdot^{\theta=0}$ to simplify: that is (U_1, V_1, W_1) stands for (U_1^0, V_1^0, W_1^0). Let us consider the identity (5.9) which holds true at the limit $\mu = 0$

$$\int_0^N \left(|W_1|^2 - \alpha|U_1|^2 - \alpha|V_1|^2 - i\delta V_1 \overline{U_1} + i\delta U_1 \overline{V_1} \right) dx$$
$$= W_1(N)\overline{V_1}(N) - W_1(0)\overline{V_1}(0), \quad 0 < N < \infty.$$

Since we consider the case $\theta = 0$, $V_1(0) = 0$. Notice also that $W_1 = V_1'$, so the relation is rewritten as

$$\int_0^N \left(|V_1'|^2 - \alpha|U_1|^2 - \alpha|V_1|^2 - i\delta V_1 \overline{U_1} + i\delta U_1 \overline{V_1} \right) dx = W_1(N)\overline{V_1}(N).$$

Let us proceed by contradiction: we assume that the function is exponentially decreasing at infinity. It yields

$$\int_0^\infty \left(|V_1'|^2 - \alpha|U_1|^2 - \alpha|V_1|^2 - i\delta V_1\overline{U_1} + i\delta U_1\overline{V_1}\right) dx = 0.$$

Notice that $-\alpha|U_1|^2 - \alpha|V_1|^2 - i\delta V_1\overline{U_1} + i\delta U_1\overline{V_1} \geq 0$ for $x \geq H$ due to the coercivity property (H4). Therefore it implies that

$$\int_0^H \left(|V_1'|^2 - \alpha|U_1|^2 - \alpha|V_1|^2 - i\delta V_1\overline{U_1} + i\delta U_1\overline{V_1}\right) dx \leq 0.$$

Next observe that $U_1 = -i\frac{\delta}{\alpha}V_1$, so that

$$\int_0^H \left(|V_1'|^2 + \frac{\delta^2}{\alpha}|V_1|^2 - \alpha|V_1|^2\right) dx \leq 0.$$

Since $V_1(0) = 0$ and $\alpha(x) \approx \alpha'(0)x$ with $\alpha'(0) < 0$ (see hypothesis H1), it is convenient to notice the proximity with the famous Hardy inequality that we recall,

$$\int_0^H \frac{u(x)^2}{x^2} < 4\int_0^H u'(x)^2, \qquad u \in H^1(0,H), \quad u(0) = 0, \quad u \neq 0. \tag{5.17}$$

Since, thanks to hypothesis (H2),

$$\int_0^H \frac{\delta^2}{|\alpha|}|V_1|^2 = \int_0^H \delta^2 x \frac{x}{|\alpha|}\frac{|V_1|^2}{x^2} \leq \frac{\|\delta\|_\infty^2 H}{r}\int_0^H \frac{|V_1|^2}{x^2},$$

it yields the inequality

$$0 \leq \left(1 - 4\frac{\|\delta\|_\infty^2 H}{r}\right)\int_0^H |V_1'|^2 dx \leq \int_0^H \left(|V_1'|^2 + \frac{\delta}{\alpha}|V_1|^2 - \alpha|V_1|^2\right) dx \leq 0,$$

where we used (H5). Therefore V_1 vanishes on the interval $[0,H]$. So U_1 vanishes and W_1 also vanishes on the interval which is not compatible with $W_1(0) = i\delta(0) \neq 0$.

□

Proposition 5.7. *There exists a maximal value $\theta_{\text{thresh}} > 0$ such that: If $\|\delta\|_\infty^2 H < 4r$ and $|\theta| < \theta_{\text{thresh}}$, then $\mathbf{U}_1^\theta$ increases exponentially at infinity.*

Let us denote by $(U_3^\theta(H), V_3^\theta(H), W_3^\theta(H))$ the solution to (2.3) for $x > 0$ that satisfies (5.13) with $\mu = 0$.

Proof. Let us consider the function

$$\sigma(\theta) = V_1^\theta(H)W_3^\theta(H) - W_1^\theta(H)V_3^\theta(H) \tag{5.18}$$

By definition

$$(V_3(H), W_3(H)) = \left(1 - \frac{\theta^2}{\alpha_\infty}, -\frac{\theta\delta_\infty}{\alpha_\infty} - \sqrt{-\alpha_\infty + \theta^2 + \frac{\delta_\infty^2}{\alpha_\infty}}\right) e^{-\sqrt{-\det A_\infty^{\theta,\mu=0}}H}.$$

This vector is real and always non zero. Therefore the function $\theta \mapsto f(\theta)$ is well defined. This function naturally satisfies two properties

- $\sigma(0) \neq 0$ since (V_1^0, W_1^0) is exponentially increasing by virtue of the previous property. Indeed $\sigma(0) = 0$ if and only if the functions $x \mapsto (V_1^0(x), W_1^0(x))$ and $x \mapsto (V_3^0(x), W_3^0(x))$ are linearly dependent, which is not true.

- the function σ is continuous since the first basis function is continuous wit respect to θ.

Therefore there exists an interval around 0 in which $\sigma(\theta)$ is non zero, which in turn yields the fact that $\mathbf{U}_1^\theta$ is linearly independent of $\mathbf{U}_3^\theta$. Therefore $\mathbf{U}_1^\theta$ is exponentially increasing. □

5.4.2 The transversality condition

Passing to the limit in the second basis function near the origin is involved. Indeed we expect that the limit U_2^θ is such that $U_2^\theta \approx \frac{C}{x}$ for some local constant C. Therefore the limit is singular and special care has to be provided to avoid any artifacts in the analysis.

Let us define the special Wronskian between the first and third basis functions

$$\sigma(\theta, \mu) = V_1^{\theta,\mu}(H)W_3^{\theta,\mu}(H) - W_1^{\theta,\mu}(H)V_3^{\theta,\mu}(H).$$

It is the natural continuous extension with respect to μ of the function $\theta \mapsto \sigma(\theta)$. We rewrite (5.14) as

$$\mathbf{U}_2^{\theta,\mu} = \xi^{\theta,\mu}\mathbf{U}_3^{\theta,\mu}.$$

Plugging this relation in the Wronskian (5.15) one gets that

$$1 = \xi^{\theta,\mu}\sigma(\theta, \mu)$$

This function is continuous with respect to μ. Moreover the function defined in (5.18) satisfies $\sigma(\theta) = \sigma(\theta, 0)$. The transversality condition is defined as the condition

$$\sigma(\theta) \neq 0. \qquad (5.19)$$

If the transversality condition is not satisfied, that is $\sigma(\theta) = 0$, then by continuity $|\xi^{\theta,\mu}| \to \infty$ for $\mu \to 0$. If $\sigma(\theta) = 0$, then the first basis function and the third function are linearly dependent at the limit $\mu = 0$. It is of course possible to develop the theory in this direction, but it seems to us less interesting. Therefore we will always assume the transversality condition[1] from now on. We postpone to the appendix some aspects of the case where is not satisfied.

[1]The "transversality condition" is a sufficient condition of linear independence.

Proposition 5.8. *Assume the transversality condition (5.19). Then for all* $\varepsilon > 0$ *one has the limit*

$$\left\| \mathbf{U}_2^{\theta,\mu} - \frac{1}{\sigma(\theta)} \mathbf{U}_3^{\theta} \right\|_{(L^\infty[\varepsilon,\infty[)^3} \to 0.$$

Proof. Evident. □

In order to show that the second basis function admits a continuous limit for $x < 0$, the strategy is to solve the integral equation (3.9) from $G = H$ backward, and to show that fine estimates on the solution give knowledge of the limit even for $x < 0$.

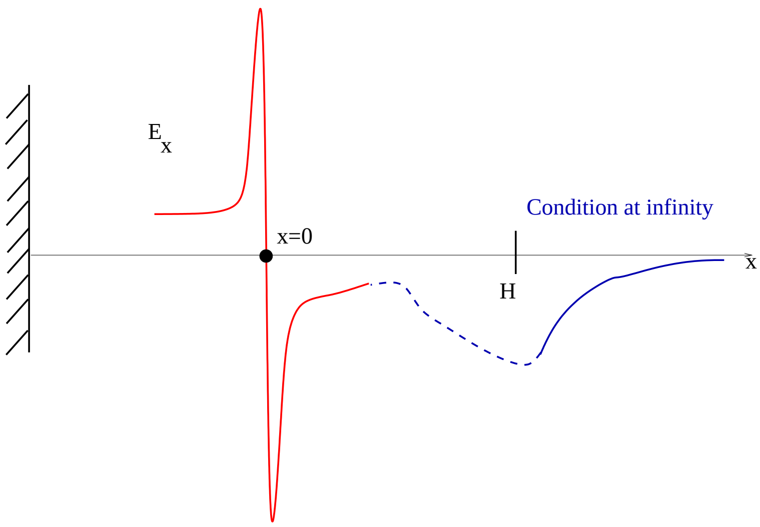

Figure 5: Schematic representation of the real part of the limit electric field of the second basis function $U_2^{\theta,\mu}$, $\mu > 0$. Here the transversality condition $\sigma(\theta) \neq 0$ is satisfied, which turns into a singular behavior at the limit $\mu \to 0$.

5.4.3 Continuity estimates

The integral equation (3.9) is singular at the limit. The whole problem comes form the singularity at $x = 0$. By comparison with the standard literature [28, 23, 30, 12, 24, 3, 27] we found no convenient mathematical tool to analyze its properties. That is why we develop in the following new continuity estimates with respect to the parameters of the problem. On this basis we will manage to pass to to the limit $\mu \to 0$.

Let us consider a general solution $\mathbf{U} = (U, V, W) \in \mathbb{X}^{\theta,\mu}$ of the integral equation (3.9) with prescribed data in H under the form

$$V(H) = a_H \text{ and } W(H) = b_H.$$

Let us introduce the compact notation

$$\|H\| = |a_H| + |b_H| .$$

Our goal is to obtain some sharp continuity estimates on the solution $\mathbf{U}$ with respect to $\|H\|$. The main point is to bound the constants uniformly with respect to $0 < \mu \le 1$ which is hereafter taken positive for the simplicity of notation. The reference point can be different from H as well, but non equal to zero. Once these continuity estimates are proved, they will provide enough information to define the limit $\mu \to 0$ of the second basis function.

Proposition 5.9. *There exists a constant C_θ with continuous dependence with respect to θ such that*

$$|U(x)| \le \frac{C_\theta}{\sqrt{r^2x^2+\mu^2}}\|H\|, \qquad 0 < x \le H. \tag{5.20}$$

Proof. Let us consider

$$\gamma_\theta = \left(\sup_{0\le\mu\le1} \|A_\mu\|_{W^{1,\infty}(0,H)} + \sup_{0\le\mu\le1} \|B_\mu\|_{W^{1,\infty}(0,H)} \right) (\|\delta\|_\infty + |\theta|).$$

The integral equation (3.9) with $G = H$ implies that

$$|U(x)| \le \frac{\gamma_\theta\|H\|}{\sqrt{r^2x^2+\mu^2}} + \int_x^H \frac{|\mathcal{D}_x^\theta\mathcal{D}_z^\theta k(x,z)|}{\sqrt{r^2x^2+\mu^2}}|U(z)|dz,$$

where we used (H2). Since $\mathcal{D}_x^\theta\mathcal{D}_z^\theta k(x,x) = 0$ for all x, there exists a constant β_θ such that

$$\left\|\mathcal{D}_x^\theta\mathcal{D}_z^\theta k(x,z)\right\|_{L^\infty]0,H[} \le \beta_\theta|x-z| \le \beta_\theta z \quad \text{ for } 0 \le x \le z.$$

So

$$\sqrt{r^2x^2+\mu^2}|U(x)| \le \gamma_\theta\|H\| + \beta_\theta \int_x^H z|U(z)|dz$$

and

$$rx|U(x)| \le \gamma_\theta\|H\| + \beta_\theta \int_x^H z|U(z)|dz, \qquad 0 \le x \le H.$$

The Gronwall lemma is useful to study this inequality. Indeed let us set $g(x) = \int_x^H |zU(z)|dz$, so that the previous inequality is rewritten as

$$-rg'(x) \le \gamma_\theta\|H\| + \beta_\theta g(x).$$

Therefore $0 \le \gamma_\theta\|H\| + rg'(x) + \beta_\theta g(x)$, that is

$$0 \le \gamma_\theta\|H\|e^{\frac{\beta_\theta}{r}x} + r\left(e^{\frac{\beta_\theta}{r}x}g(x)\right)'.$$

Next we integrate on the interval $[x, H]$ and use the fact that $g(H) = 0$ by definition. It yields

$$0 \leq \gamma_\theta \|H\| \frac{e^{\frac{\beta_\theta}{r}H} - e^{\frac{\beta_\theta}{r}x}}{\frac{\beta_\theta}{r}} - re^{\frac{\beta_\theta}{r}x} g(x),$$

that is

$$g(x) \leq \frac{e^{\frac{\beta_\theta}{r}(H-x)} - 1}{\beta_\theta} \gamma_\theta \|H\|. \tag{5.21}$$

Finally one checks that

$$\sqrt{r^2x^2 + \mu^2}|U(x)| \leq \gamma_\theta \|H\| + \beta_\theta g(x) \leq e^{\frac{\beta_\theta}{r}(H-x)} \gamma_\theta \|H\|$$

which proves (5.20). □

Next define $\|0\| = |V(0)| + |W(0)|$.

Proposition 5.10. *There exists a constant C_θ with continuous dependence with respect to θ such that*

$$\|0\| \leq C_\theta (1 + |\ln \mu|)\|H\|. \tag{5.22}$$

Proof. We adopt the same notations as above. The integral expression of V (3.12) with $G = H$ yields the inequality

$$|V(0)| \leq \gamma_\theta \, \|H\| + \left| \int_0^H \mathcal{D}_z^\theta k(0, z).U(0) \right| dz$$

We notice that $\mathcal{D}_z^\theta k(0, z) = \left(\mathcal{D}_z^\theta k(0, z) - \mathcal{D}_z^\theta k(0, 0)\right) + \mathcal{D}_z^\theta k(0, 0)$. Since

$$\mathcal{D}_z^\theta k(0, 0) = i\theta \partial_z k(0, 0) - i\delta k(0, 0) = i\theta$$

one gets

$$\left|\mathcal{D}_z^\theta k(0, z) - i\theta\right| \leq \eta_\theta |z|$$

for some constant $\eta_\theta > 0$. Which gives

$$|V(0)| \leq \underbrace{\gamma_\theta \, \|H\| + \eta_\theta \int_0^H z|U(z)|dz}_{Q} + |\theta| \underbrace{\left| \int_0^H U(z)dz \right|}_{R}.$$

By (5.20) $Q \leq C_\theta \|H\|$, and moreover,

$$R := \left| \int_0^H U(z)dz \right| \leq C_\theta \|H\| \left| \int_0^H \frac{1}{r|x| + \mu} dz \right| \leq C_\theta \|H\| |\ln \mu|.$$

This completes the proof for $|V_2(0)|$. The term $|W(0)|$ is bounded with the same method starting from the integral (3.13) and using the identity $\partial_x \mathcal{D}_z^\theta k_\mu(x, x) = i\delta(z)A_\mu(z)$. □

An interesting question is the following. Let us consider the integral equation (3.9) with $G = 0$. That is the starting point of the integral is the singularity. One may wonder if a direct use of the Gronwall lemma may yield valuable estimates, or not. It appears that a pollution with $\log \mu$ terms render the result of little interest.

Consider firstly for simplicity $0 \leq x$. Then (3.9) with $G = 0$ turns into

$$|U(x)| \leq C_\theta \frac{\|0\|}{\sqrt{r^2x^2 + \mu^2}} + C \int_0^x |U(z)|\, dz \tag{5.23}$$

where we used (4.4) to bound the kernel. The constant $C_\theta > 0$ is chosen large enough. Set $h(x) = \int_0^x |U(z)|\, dz$ so that

$$h'(x) \leq C_\theta \frac{\|0\|}{\sqrt{r^2x^2 + \mu^2}} + C_\theta h(x).$$

Since $h(0) = 0$ the Gronwall lemma yields the inequality

$$h(x) \leq C'_\theta \int_0^x \frac{\|0\|}{|z| + |\mu|} dz$$

that is after integration ($0 \leq x \leq H$) $|h(x)| \leq C''_\theta \|0\| \left(1 + |\ln \mu|\right)$, for some constant $C''_\theta > 0$ with continuous dependance with respect to θ. Considering the bound (5.22) and the symetry between $0 < x$ and $x < 0$ in the integral (3.9) (with $G = 0$) one obtains the estimate

$$\left| \int_0^x U(z) dz \right| \leq C'''_\theta \|H\| \left(1 + |\ln \mu|\right)^2, \quad -L \leq x \leq H. \tag{5.24}$$

Going back to (5.23) which is easily generalized to $x < 0$, one gets

$$|U(x)| \leq C_\theta \left(\frac{1}{\sqrt{r^2x^2 + \mu^2}} + 1 + |\ln \mu| \right) \left(1 + |\ln \mu|\right) \|H\|, \quad -L \leq x \leq H. \tag{5.25}$$

By comparison of (5.20) and (5.25), it is clear that this technique generates spurious terms of order $\log \mu$ for positive x. It spoils the possibility of having sharp estimates also for negative x. With this respect, the rest of this section is devoted to the obtention of various sharp inequalites which are free of such spurious terms.

Let us define

$$Q(\mathbf{U}) = V_1^{\theta,\mu}(H)W(H) - W_1^{\theta,\mu}(H)V(H). \tag{5.26}$$

This quantity is the Wronskian of the current solution $\mathbf{U}$ against the first basis function. It is therefore independent of the position H which is used to evaluate $Q(\mathbf{U})$.

Proposition 5.11. *There exists a constant C_θ with continuous dependence with respect to θ and a continuous function $\mu \mapsto \varepsilon(\mu)$ with $\varepsilon(0) = 0$ such that*

$$\left| |\mu| \, \|U\|^2_{L^2(-L,H)} - \left| \frac{\pi Q(\mathbf{U})^2}{\alpha'(0)} \right| \right| \le C_\theta \varepsilon(\mu) \|H\|^2. \tag{5.27}$$

Proof. We consider positive μ to simplify the notations. The proof is easily adapted for negative μ.

Consider the integral equation (3.9) with $G = 0$. One gets

$$U(x) = \frac{a_0 \mathcal{D}^\theta_x A(x) + b_0 \mathcal{D}^\theta_x B(x)}{\alpha(x) + i\mu} + \int_0^x \frac{\mathcal{D}^\theta_x \mathcal{D}^\theta_z k(x,z)}{\alpha(x) + i\mu} U(z) dz.$$

Here (a_0, b_0) are a priori different from (a_H, b_H). Due to (5.26), the normalization of $\mathbf{U}_1$ and thanks to Lemma 5.1 one has that

$$a_0 \mathcal{D}^\theta_x A(0) + b_0 \mathcal{D}^\theta_x B(0) = Q(\mathbf{U}).$$

So the integral equation can be written as

$$U(x) = \underbrace{\frac{Q(\mathbf{U})}{\alpha(x) + i\mu}}_{S_1} \tag{5.28}$$

$$\underbrace{+\, a_0 \frac{\mathcal{D}^\theta_x A(x) - \mathcal{D}^\theta_x A(0)}{\alpha(x) + i\mu} + b_0 \frac{\mathcal{D}^\theta_x B(x) - \mathcal{D}^\theta_x B(0)}{\alpha(x) + i\mu}}_{S_2} + \underbrace{\int_0^x \frac{\mathcal{D}^\theta_x \mathcal{D}^\theta_z k(x,z)}{\alpha(x) + i\mu} U(z) dz}_{S_3}.$$

- The L^2 norm of the first term S_1 depends upon the value of

  $$D_\mu = \int_{-L}^H \frac{\mu}{\alpha(x)^2 + \mu^2} dx.$$

  Make the change of variable $x = \mu w$ so that $D_\mu = \int_{-\frac{L}{\mu}}^{\frac{H}{\mu}} \frac{1}{b_\mu(w)^2 + 1} dw$ and $b_\mu(w) = \frac{\alpha(\mu w)}{\mu}$. Using the hypothesis (H2) one has that $|b_\mu(w)| \ge rw$, $r > 0$. Since $\int_{\mathbb{R}} \frac{dw}{r^2 w^2 + 1} = \frac{\pi}{r} < \infty$ and the point-wise limit of $b_\mu(x)$ is $\alpha'(0)$, the Lebesgue dominated convergence theorem states that $\lim_{0^+} D_\mu = \frac{\pi}{|\alpha'(0)|}$. Considering that

  $$|Q(\mathbf{U})| \le C^1_\theta \|H\| \tag{5.29}$$

  using (5.26), there exists a continuous function $\mu \mapsto \varepsilon^1(\mu)$ with $\varepsilon^1(0) = 0$ such that

  $$\left| \mu \|S_1\|^2_{L^2(-L-,H)} - \left| \frac{\pi Q(\mathbf{U})^2}{\alpha'(0)} \right| \right| \le C^1_\theta \varepsilon^1(\mu) \|H\|^2. \tag{5.30}$$

- The functions $\frac{\mathcal{D}_x^\theta A_\mu(x)-\mathcal{D}_x^\theta A_\mu(0)}{\alpha(x)+i\mu}$ and $\frac{\mathcal{D}_x^\theta B_\mu(x)-\mathcal{D}_x^\theta B_\mu(0)}{\alpha(x)+i\mu}$ can bounded in L^∞ uniformly with respect to μ. So

  $$\int_{-L}^{H} |S_2(z)|^2 dz \leq c_\theta^2 \|0\|^2.$$

  Estimate (5.22) yields

  $$\mu\|S_2\|_{L^2(-L-,H)}^2 \leq C_\theta^2 \mu(1+|\ln\mu|)^2\|H\|^2$$

  for some constant $C_\theta^2 > 0$.

- The last term S_3 is

  $$|S_3(x)| = \left|\int_0^x \frac{\mathcal{D}_x^\theta \mathcal{D}_z^\theta k^\mu(x,z)}{\alpha(x)+i\mu} U(z)dz\right| \leq c_3^\theta \left|\int_0^x |U(z)|dz\right|$$

  since the kernel is bounded (4.4) with respect to θ and uniformly for $\mu \in [0,1]$. Inequality (5.24) implies that $|S_3(x)| \leq c_\theta^3 (1+|\ln\mu|)^2 \|H\|$. Therefore this term is bounded like

  $$\mu\|S_3\|_{L^2(-L-,H)}^2 \leq c_\theta^4 \mu (1+|\ln\mu|)^4 \|H\|^2$$

  for some constant c_θ^4.

We complete the proof adding these three inequalities. □

To pursue the analysis, we begin by rewriting the general form of the integral equation (3.9), showing that the various singularities of the equation can be recombined under a more convenient form. This intermediate result is essential to obtain all following results. Indeed the integral equation for U (3.9) choosing $G = 0$ writes

$$(\alpha(x)+i\mu)U(x) = a_0\mathcal{D}_x^\theta A_\mu(x) + b_0\mathcal{D}_x^\theta B_\mu(x) + \int_0^x \mathcal{D}_x^\theta \mathcal{D}_z^\theta k^\mu(x,z)U(z)dz.$$

Since by construction $a_0\mathcal{D}_x^\theta A_\mu(0) + b_0\mathcal{D}_x^\theta B_\mu(0) = Q(\mathbf{U})$ one also has

$$(\alpha(x)+i\mu)U(x) = a_0\left(\mathcal{D}_x^\theta A_\mu(x) - \mathcal{D}_x^\theta A_\mu(0)\right) + b_0\left(\mathcal{D}_x^\theta B_\mu(x) - \mathcal{D}_x^\theta B_\mu(0)\right)$$
$$+Q(\mathbf{U}) + \int_0^x \mathcal{D}_x^\theta \mathcal{D}_z^\theta k^\mu(x,z)U(z)dz.$$

But one also has due to the integral equation for V (3.12) choosing $G = H$

$$V(0) = a_0 = a_H A_\mu(0) + b_H B_\mu(0) - \int_0^H \mathcal{D}_z^\theta k^\mu(0,z)U(z)dz.$$

Basic manipulations yield

$$a_0 = a_H - \int_0^H \left(\mathcal{D}_z^\theta k^\mu(0,z) - \mathcal{D}_z^\theta k^\mu(0,0)\right)U(z)dz - i\theta\int_0^H U(z)dz$$

because $\mathcal{D}_z^\theta k^\mu(0,0) = i\theta$. Since the function $\mathcal{D}_z^\theta k^\mu$ is continuous, there exists a constant C_4^θ independent of μ such that

$$\left|\mathcal{D}_z^\theta k^\mu(x,z) - \mathcal{D}_z^\theta k^\mu(x,x)\right| \le C_4^\theta(z-x) \le C_4^\theta z \quad \text{ for } 0 \le x \le z \le H.$$

Therefore the integral

$$\int_0^H \left|\mathcal{D}_z^\theta k^\mu(0,z) - \mathcal{D}_z^\theta k^\mu(0,0)\right| |U(z)|dz \le C_4^\theta \int_0^H z|U(z)|dz$$

is bounded uniformly with respect to μ thanks to the bound given in (5.20). We summarize this as

$$a_0 = \tilde{a} - i\theta \int_0^H U(z)dz \tag{5.31}$$

where $|\tilde{a}| \le C_5^\theta \|H\|$ is bounded uniformly with respect to μ. Similarly

$$b_0 = b_H - \int_0^H \partial_x \mathcal{D}_z^\theta k^\mu(0,z)U(z)dz \tag{5.32}$$

and since the function $\partial_x \mathcal{D}_z^\theta k^\mu$ is continuous and $\partial_x \mathcal{D}_z^\theta(0,0) = i\delta(0)$

$$b_0 = \tilde{b} - i\int_0^H \delta(0)U^{\theta,\mu}(z)dz \tag{5.33}$$

where $\tilde{b}$ is also bounded uniformly with respect to μ: $|\tilde{b}| \le C_6^\theta \|H\|$. The integral equation then gives

$$(\alpha(x)+i\mu)U(x) = \tilde{a}\left(\mathcal{D}_x^\theta A_\mu(x) - \mathcal{D}_x^\theta A_\mu(0)\right) + \tilde{b}\left(\mathcal{D}_x^\theta B_\mu(x) - \mathcal{D}_x^\theta B_\mu(0)\right)$$
$$+Q(\mathbf{U}) - \int_0^H Q(x,z)U(z)dz + \int_0^x \mathcal{D}_x^\theta \mathcal{D}_z^\theta k^\mu(x,z)U(z)dz$$

where the new kernel is

$$Q(x,z) = \left(\mathcal{D}_x^\theta A_\mu(x) - \mathcal{D}_x^\theta A_\mu(0)\right) i\theta + \left(\mathcal{D}_x^\theta B_\mu(x) - \mathcal{D}_x^\theta B_\mu(0)\right) i\delta(z)$$
$$= \mathcal{D}_x^\theta A^\mu(x)\mathcal{D}_z^\theta B^\mu(0) - \mathcal{D}_x^\theta B^\mu(x)\mathcal{D}_z^\theta A^\mu(0) = \mathcal{D}_x^\theta \mathcal{D}_z^\theta k^\mu(x,0)$$

after evident simplifications. It is convenient to introduce two bounded functions $m^{\theta,\mu} = \frac{\mathcal{D}_x^\theta A^\mu(x) - \mathcal{D}_x^\theta A^\mu(0)}{x}$ and $n^{\theta,\mu} = \frac{\mathcal{D}_x^\theta B^\mu(x) - \mathcal{D}_x^\theta B^\mu(0)}{x}$ so that (3.9) is rewritten as

$$(\alpha(x)+i\mu)U(x) = \tilde{a}m^{\theta,\mu}(x)x + \tilde{b}n^{\theta,\mu}(x)x + Q(\mathbf{U}) - \int_x^H \mathcal{D}_x^\theta \mathcal{D}_z^\theta k^\mu(x,0)U(z)dz$$
$$+\int_0^x \left(\mathcal{D}_x^\theta \mathcal{D}_z^\theta k^\mu(x,z) - \mathcal{D}_x^\theta \mathcal{D}_z^\theta k^\mu(x,0)\right) U(z)dz, \quad \forall x \in [-L,\infty[. \tag{5.34}$$

A first property which shows that (5.34) is less singular that its initial form (3.9) is the following lemma which uses the pointwise estimate (5.20) on U (so an important restriction is nevertheless that $x > 0$).

Lemma 5.2. *The first component U of any element $\mathbf{U} \in \mathbb{X}^{\theta,\mu}$ satisfies*

$$(\alpha(x) + i\mu)U(x) = p^{\theta,\mu}(x)x + Q(\mathbf{U}) - \int_x^H \mathcal{D}_x^\theta \mathcal{D}_z^\theta k^\mu(x,0)U(z)dz \tag{5.35}$$

where

$$\|p^{\theta,\mu}\|_{L^\infty(0,H)} \le C^\theta \|H\|, \qquad \forall \mu \in [0,1]. \tag{5.36}$$

Proof. Let us focus on the second integral in (5.34). Continuity properties with respect to the second variable z imply that there exists a constant C_7^θ independent of μ such that

$$\left|\mathcal{D}_x^\theta \mathcal{D}_z^\theta k^\mu(x,z) - \mathcal{D}_x^\theta \mathcal{D}_z^\theta k^\mu(x,0)\right| \le C_7^\theta z. \tag{5.37}$$

So, for $x \ge 0$,

$$\left|\int_0^x \left(\mathcal{D}_x^\theta \mathcal{D}_z^\theta k^\mu(x,z) - \mathcal{D}_x^\theta \mathcal{D}_z^\theta k^\mu(x,0)\right) U(z)dz\right| \le C_7^\theta \int_0^x z|U(z)|dz \le C_7^\theta C_\theta \|H\| x$$

using estimate (5.20). Set

$$p^{\theta,\mu}(x) = \tilde{a} m^{\theta,\mu}(x) + \tilde{b} n^{\theta,\mu}(x) + \frac{1}{x}\int_0^x \left(\mathcal{D}_x^\theta \mathcal{D}_z^\theta k^\mu(x,z) - \mathcal{D}_x^\theta \mathcal{D}_z^\theta k^\mu(x,0)\right) U(z)dz \tag{5.38}$$

which satisfies by construction (5.36). □

As a consequence one has

Proposition 5.12. *For all $1 \le p < \infty$, there exists a constant C_p^θ independent of μ and which depends continuously on θ such that*

$$\left\| U - \frac{Q(\mathbf{U})}{\alpha(\cdot) + i\mu} \right\|_{L^p(0,H)} \le C_p^\theta \|H\|. \tag{5.39}$$

Proof. From lemma 5.2 one has that

$$U(x) - \frac{Q(\mathbf{U})}{\alpha(x) + i\mu} = \frac{x}{\alpha(x) + i\mu} p^{\theta,\mu}(x) - \frac{\mathcal{D}_x^\theta \mathcal{D}_z^\theta k^\mu(x,0)}{\alpha(x) + i\mu} \int_x^H U(z)dz,$$

which turns into

$$\left(U(x) - \frac{Q(\mathbf{U})}{\alpha(x) + i\mu}\right) + \frac{\mathcal{D}_x^\theta \mathcal{D}_z^\theta k^\mu(x,0)}{\alpha(x) + i\mu} \int_x^H \left(U(z) - \frac{Q(\mathbf{U})}{\alpha(z) + i\mu}\right) dz$$

$$= \frac{x}{\alpha(x) + i\mu} p^{\theta,\mu}(x) - Q(\mathbf{U}) \frac{\mathcal{D}_x^\theta \mathcal{D}_z^\theta k^\mu(x,0)}{\alpha(x) + i\mu} \int_x^H \frac{1}{\alpha(z) + i\mu} dz \tag{5.40}$$

By virtue of (H2) we notice that

$$\left|\int_x^H \frac{1}{\alpha(z) + i\mu} dz\right| \le \int_x^H \frac{1}{|\alpha(z)|} dz \le \frac{1}{r} \log(H/x).$$

Since all powers of the function $x \mapsto \ln|x|$ are integrable, the right-hand side (5.40) is naturally bounded in any L^p, $1 \le p < \infty$. Therefore the function $Z(x) = U(x) - \frac{Q(\mathbf{U})}{\alpha(x)+i\mu}$ is solution of an integral equation with a bounded kernel and a right hand side in L^p. The form of this integral equation is

$$Z(x) + \widetilde{K}^{\theta,\mu}(x) \int_x^H Z(z)dz = b^{\theta,\mu}(x)$$

with $\left\|\widetilde{K}^{\theta,\mu}(x)\right\|_{L^\infty(0,H)} \le C_8^\theta$ independently of μ. One also uses $\left\|b^{\theta,\mu}\right\|_{L^p(0,H)} \le c_p^\theta \|H\|$ for $0 \le \mu \le 1$: the key estimate is (5.36) which explains why the result is restricted to $x > 0$. Since this is a standard non-singular integral equation, see [28], the claim is proved. $\square$

The previous result (5.39) shows that some singularities of the integral equation can be recombined in a less singular formulation, so that the dominant part of U is $\frac{1}{\alpha(\cdot)+i\mu}$. An important restriction of this technique, for the moment, is that it needs the a priori estimate (5.20) on U. This explains why inequality (5.39) is restricted to $x > 0$. By inspection of the structure of the algebra, it appears that one has the same kind of inequalities on the entire interval by replacing U directly by the function $\frac{1}{\alpha(\cdot)+i\mu}$ in the integrals. A preliminary and fundamental result in this direction concerns the function

$$D^{\theta,\mu}(x) = -\frac{\mathcal{D}_x^\theta \mathcal{D}_z^\theta k^\mu(x,0)}{\alpha(x)+i\mu} \int_0^H \frac{1}{\alpha(z)+i\mu} dz + \int_0^x \frac{\mathcal{D}_x^\theta \mathcal{D}_z^\theta k^\mu(x,z)}{\alpha(x)+i\mu} \frac{1}{\alpha(z)+i\mu} dz$$

which is nothing than the integral part of (5.34) where U is replaced by the function $\frac{1}{\alpha(\cdot)+i\mu}$.

Proposition 5.13. *Let $1 \le p < \infty$. One has $\left\|D^{\theta,\mu}\right\|_{L^p(-L,H)} \le C_p^\theta$ where the constant depends continuously on θ and does not depend on μ.*

Proof. Two cases occur.

- **Assume** $0 \le x \le H$. The analysis is similar to the one of proposition 5.12. One has the same kind of rearrangement (5.34), that is

$$D^{\theta,\mu}(x) = -\frac{\mathcal{D}_x^\theta \mathcal{D}_z^\theta k^\mu(x,0)}{\alpha(x)+i\mu} \int_x^H \frac{1}{\alpha(z)+i\mu} dz$$
$$+ \int_0^x \frac{\mathcal{D}_x^\theta \mathcal{D}_z^\theta k^\mu(x,z) - \mathcal{D}_x^\theta \mathcal{D}_z^\theta k^\mu(x,0)}{\alpha(x)+i\mu} \frac{1}{\alpha(z)+i\mu} dz.$$

  The first term is bounded like $C^\theta \frac{|\log x|}{r}$ which is in all L^p, $p < \infty$. The second term is immediately bounded using (5.37): indeed

$$\left| \int_0^x \frac{\mathcal{D}_x^\theta \mathcal{D}_z^\theta k^\mu(x,z) - \mathcal{D}_x^\theta \mathcal{D}_z^\theta k^\mu(x,0)}{\alpha(x)+i\mu} \frac{1}{\alpha(z)+i\mu} dz \right|$$
$$\le C_7^\theta \frac{1}{\sqrt{\alpha(x)^2+\mu^2}} \int_0^x \frac{z}{\sqrt{\alpha(z)^2+\mu^2}} dz \le C_7^\theta \frac{1}{r^2}.$$

- **Assume** $-L \le x \le 0$. The decomposition is slightly different and uses some cancellations permitted by the symmetry properties of the kernels. One has

$$D^{\theta,\mu}(x) = -\frac{\mathcal{D}_x^\theta \mathcal{D}_z^\theta k^\mu(x,0)}{\alpha(x)+i\mu}\int_{-x}^{H}\frac{1}{\alpha(z)+i\mu}dz$$

$$+\int_0^x \frac{\mathcal{D}_x^\theta \mathcal{D}_z^\theta k^\mu(x,z)}{\alpha(x)+i\mu}\frac{1}{\alpha(z)+i\mu}dz - \frac{\mathcal{D}_x^\theta \mathcal{D}_z^\theta k^\mu(x,0)}{\alpha(x)+i\mu}\int_0^{-x}\frac{1}{\alpha(z)+i\mu}dz,$$

which emphasizes the importance of some symmetry properties of the kernels. Indeed

$$\int_0^{-x}\frac{1}{\alpha(z)+i\mu}dz = -\int_0^x \frac{1}{\alpha(-w)+i\mu}dw$$

$$= \int_0^x \frac{1}{\alpha(w)+i\mu}dw + \int_0^x \left(\frac{1}{-\alpha(-w)-i\mu} - \frac{1}{\alpha(w)+i\mu}\right)dw.$$

Notice that

$$\frac{1}{-\alpha(-w)-i\mu} - \frac{1}{\alpha(w)+i\mu} = \frac{\alpha(w)+\alpha(-w)+2i\mu}{(\alpha(w)+i\mu)(-\alpha(-w)-i\mu)}.$$

So, since $\alpha(0)=0$,

$$\left|\frac{1}{-\alpha(-w)-i\mu} - \frac{1}{\alpha(w)+i\mu}\right| \le \frac{2\,\|\alpha\|_{W^{2,\infty}(-L,H)}\,w^2+2\mu}{r^2w^2+\mu^2},$$

since $\alpha \in W^{2,\infty}(-L,H)$. One can bound

$$\left|\int_0^x \frac{1}{-\alpha(-w)-i\mu}dw - \int_0^x \frac{1}{\alpha(w)+i\mu}dw\right|$$

$$\le \frac{\|\alpha\|_{W^{2,\infty}(-L,H)}}{r^2}x + \int_0^x \frac{2\mu}{r^2z^2+\mu^2}dz$$

$$\le \frac{\|\alpha\|_{W^{2,\infty}(-L,H)}}{r^2}H + \int_0^\infty \frac{2\mu}{r^2z^2+\mu^2}dz \le \frac{\|\alpha\|_{W^{2,\infty}(-L,H)}}{r^2}H + \frac{\pi}{r}.$$

As a consequence $D^{\theta,\mu}$ can be expressed as

$$D^{\theta,\mu}(x) = -\frac{\mathcal{D}_x^\theta \mathcal{D}_z^\theta k^\mu(x,0)}{\alpha(x)+i\mu}\int_{-x}^{H}\frac{1}{\alpha(z)+i\mu}dz$$

$$+\int_0^x \frac{\mathcal{D}_x^\theta \mathcal{D}_z^\theta k^\mu(x,z) - \mathcal{D}_x^\theta \mathcal{D}_z^\theta k^\mu(x,0)}{\alpha(x)+i\mu}\frac{1}{\alpha(z)+i\mu}dz + R(x)$$

with $\|R\|_\infty(-L,H) \le C_{10}^\theta$. The two integrals have the same structure as for the first case, in particular the interval of integration is $[-x,H]$ with $0 \le -x$. So the same result holds.

□

Proposition 5.14. *For all $1 \le p < \infty$, there exists a constant C_p^θ independent of μ such that*

$$\left\| U - \frac{Q(\mathbf{U})}{\alpha(\cdot) + i\mu} \right\|_{L^p(-L,H)} \le C_p^\theta \|H\|. \tag{5.41}$$

Proof. We start from (5.34) written as

$$U(x) = \frac{Q(\mathbf{U})}{\alpha(x) + i\mu} + \frac{x}{\alpha(x) + i\mu} \widetilde{p}^{\theta,\mu}(x)$$

$$- \int_0^H \frac{\mathcal{D}_x^\theta \mathcal{D}_z^\theta k^\mu(x,0)}{\alpha(x) + i\mu} U(z) dz + \int_0^x \frac{\mathcal{D}_x^\theta \mathcal{D}_z^\theta k^\mu(x,z)}{\alpha(x) + i\mu} U(z) dz.$$

Here $\widetilde{p}^{\theta,\mu}(x) = \tilde{a} m^{\theta,\mu}(x) + \tilde{b} n^{\theta,\mu}(x)$, so that $\|\widetilde{p}^{\theta,\mu}\|_{L^\infty(-L,H)} \le C^\theta \|H\|$ over the whole interval $(-L, H)$. Notice that $\widetilde{p}^{\theta,\mu}$ is the first part of $p^{\theta,\mu}$ defined in (5.38). Setting $u(x) = U(x) - \frac{Q(\mathbf{U})}{\alpha(x)+i\mu}$ one gets

$$u(x) - \int_0^x \frac{\mathcal{D}_x^\theta \mathcal{D}_z^\theta k^\mu(x,z)}{\alpha(x) + i\mu} u(z) dz$$

$$= \frac{x}{\alpha(x) + i\mu} \widetilde{p}^{\theta,\mu}(x) - Q(\mathbf{U}) D^{\theta,\mu}(x) - \int_0^H \frac{\mathcal{D}_x^\theta \mathcal{D}_z^\theta k^\mu(x,0)}{\alpha(x) + i\mu} u(z) dz.$$

The left-hand side is an non singular integral operator of the second kind with and with a bounded kernel thanks to the fundamental property (4.4). The right-hand side is bounded in L^p with a continuous dependence with respect to $\|H\|$, see Lemma 5.2, estimation (5.29) and estimation (5.39).

□

5.4.4 The second basis function

We apply the above material to the second basis function for which $Q(\mathbf{U}_2^{\theta,\mu}) = 1$. The inequality (5.41) writes

$$\left\| U_2^{\theta,\mu} - \frac{1}{\alpha(\cdot) + i\mu} \right\|_{L^p(-L,H)} \le C_p^\theta \left(\left| V_2^{\theta,\mu}(H) \right| + \left| W_2^{\theta,\mu}(H) \right| \right), \tag{5.42}$$

for $1 \le p < \infty$.

Proposition 5.15. *Assume the transversality condition (5.19). There exists a constant D^θ independent of μ and continuous with respect to θ such that*

$$\left| V_2^{\theta,\mu}(H) \right| + \left| W_2^{\theta,\mu}(H) \right| \le C^\theta. \tag{5.43}$$

Proof. Indeed, regarding relation (5.14), (5.15) the pair $(v,w) = (V_2^{\theta,\mu}(H), W_2^{\theta,\mu}(H))$ is solution of the linear system

$$\left\{ \begin{array}{l} -vW_1^{\theta,\mu}(H) + wV_1^{\theta,\mu}(H) = 1, \\ vW_3^{\theta,\mu}(H) - wV_3^{\theta,\mu}(H) = 0. \end{array} \right. \tag{5.44}$$

The determinant of this linear system is equal to the value of the function $-\sigma(\theta,\mu)$. So the transversality condition establishes that

$$\det \begin{pmatrix} -W_1^{\theta,\mu}(H) & V_1^{\theta,\mu}(H) \\ W_3^{\theta,\mu}(H) & -V_3^{\theta,\mu}(H) \end{pmatrix} = -\sigma(\theta,\mu) \neq 0.$$

Therefore the solution of the linear system

$$v = -\frac{V_3^{\theta,\mu}(H)}{\sigma(\theta,\mu)}, \quad w = -\frac{W_3^{\theta,\mu}(H)}{\sigma(\theta,\mu)} \tag{5.45}$$

is bounded uniformly with respect to μ. □

Theorem 5.1. *Assume the same transversality condition. The second basis function satisfies the following estimates for some C_p^θ and C^θ which are continuous with respect to θ*

$$\left\| U_2^{\theta,\mu} - \frac{1}{\alpha(\cdot) + i\mu} \right\|_{L^p(-L,H)} \leq C_p^\theta, \quad 1 \leq p < \infty, \tag{5.46}$$

$$\left\| \mathbf{U}_2^{\theta,\mu} \right\|_{H^1_{\mathrm{loc}}[-L,0)\cup(0,H]} \leq C^\theta. \tag{5.47}$$

Proof. The first estimate is a straightforward consequence of (5.42), (5.43) . The use of the integral representations (3.12-3.13) shows that,

$$\left\| V_2^{\theta,\mu} \right\|_{L^\infty_{\mathrm{loc}}[-L,0)\cup(0,H]} + \left\| W_2^{\theta,\mu} \right\|_{L^\infty_{\mathrm{loc}}[-L,0)\cup(0,H]} \leq C^\theta \tag{5.48}$$

for some C^θ. Then the second equation of (2.3) shows that one has the same bound for $U_2^{\theta,\mu}$

$$\left\| U_2^{\theta,\mu} \right\|_{L^\infty_{\mathrm{loc}}[-L,0)\cup(0,H]} \leq C^\theta. \tag{5.49}$$

The bound on the derivatives follows from (2.3) □

Remark 7. *Let us set $H' = -L$. From (5.48) one gets that $\|H'\|$ is bounded uniformly also, therefore (5.20) can be generalized for $x < 0$ (resp. H') instead of $x > 0$ (resp. H). In summary one has for a constant K^θ that can be further specified*

$$\left| U_2^{\theta,\mu}(x) \right| \leq \frac{K^\theta}{r^2 x^2 + \mu^2}, \qquad x \in (-L,H).$$

We now pass to the limit $\mu \to 0^+$.

Proposition 5.16. *Assume the transversality condition. The second basis function admits a limit in the sense of distribution for $\mu = 0^{\pm}$ as follows:*

$$\mathbf{U}_2^{\theta,\mu} \to \mathbf{U}_2^{\theta,\pm} = \left(P.V. \frac{1}{\alpha(x)} \pm \frac{i\pi}{\alpha'(0)} \delta_D + u_2^{\theta,\pm},\ v_2^{\theta,\pm},\ w_2^{\theta,\pm} \right)$$

where $u_2^{\theta,\pm}, v_2^{\theta,\pm}, w_2^{\theta,\pm} \in L^2(-L,\infty)$ and δ_D is the Dirac mass at the origin.

Remark 8. *The limits $\mathbf{U}_2^{\theta,\pm}$ are solutions of (2.3) in the sense of distribution. they will be called the singular solutions.*

Proof. We consider the case $\mu \downarrow 0$. Some parts of the proof are already evident, essentially for quantities which are regular enough ($V_2^{\theta,\mu}$ and $W_2^{\theta,\mu}$) or for regions where all functions are regular (typically $x > 0$). Therefore the whole point is to pass to the limit in the singular part of the solution $U_2^{\theta,\mu}$. We will make wide use of the equivalence between the integral formulation of proposition 3.1 and the differential formulation (2.3).

• **Passing to the weak limit:** By continuity of the first basis function with respect to μ, one can pass to the limit concerning $(V_2^{\theta,\mu}(H), W_2^{\theta,\mu}(H))$. One gets that $(v,w) = (V_2^{\theta,0^+}(H), W_2^{\theta,0^+}(H))$ is the unique solution of the linear system

$$\begin{cases} -vW_1^\theta(H) + wV_1^\theta(H) = 1, \\ vW_3^\theta(H) - wV_3^\theta(H) = 0, \end{cases} \tag{5.50}$$

where the coefficients are defined in terms of the first basis function for $\mu = 0$. By continuity away from the singularity at $x = 0$, one has that $\mathbf{U}_2^{\theta,\mu} \to \mathbf{U}_2^\theta$ in $L^\infty(\varepsilon, H)$ for all $\varepsilon > 0$. Using (5.46) it is clear that $U^{\theta,\mu} - \frac{1}{\alpha(\cdot)+i\mu}$ is bounded in $L^2(-L,H)$ uniformly with respect to μ. Therefore there exists a limit function denoted as u_2^{θ,O^+} such that for a subsequence

$$U_2^{\theta,\mu} - \frac{1}{\alpha(\cdot)+i\mu} \to_{weak} u_2^{\theta,O^+} \text{ in } L^2(-L,H).$$

Moreover the first derivative of $U_2^{\theta,\mu}$ is bounded in $L^2(-L,-\varepsilon)$ by virtue of (5.47). Therefore

$$U_2^{\theta,\mu} \to_{strong} \frac{1}{\alpha(\cdot)} + u_2^{\theta,O^+} \text{ in } L^2(-L,-\varepsilon)$$

at least for a subsequence. Considering the integral relations (3.12-3.13), these subsequences are such that

$$V_2^{\theta,\mu}(x) \to v_2^{\theta,O^+}(x), \tag{5.51}$$

and

$$W_2^{\theta,\mu}(x) \to w_2^{\theta,O^+}(x), \tag{5.52}$$

with the convergence uniform in compact sets of $(-L,H)\setminus\{0\}$ and where,

$$v_2^{\theta,0^+} := a_H\,A_0(x)+b_H\,B_0(x)+\int_H^x \mathcal{D}_x^\theta(k^0(x,z)-k^0(x,0))\left(\frac{1}{\alpha(z)}+u^{\theta,+}(z)\right)+$$

$$\int_H^x \mathcal{D}_x^\theta k^0(x,0)u^{\theta,+}(z)+\tilde{v}(x),$$

with

$$\tilde{v}(x) := \int_H^x \mathcal{D}_x^\theta k^0(x,0)\,\frac{1}{\alpha(z)}\,dz,\quad \text{for}\, x>0,$$

$$\tilde{v}(x) := \mathcal{D}_x^\theta k^0(x,0)\left[\int_x^{-\varepsilon}\frac{1}{\alpha(z)}\,dz+\ln\alpha(\varepsilon)\,\beta(\varepsilon)-\right.$$

$$\left.\ln\alpha(-\varepsilon)\,\beta(-\varepsilon)+\int_{\alpha(\varepsilon)}^{\alpha(-\varepsilon)}\ln(z)\,\beta'(z)\,dz\right]+\int_\varepsilon^H \mathcal{D}_x^\theta k^0(x,0)\,\frac{1}{\alpha(z)}\,dz\quad \text{for,}\, x<0,$$

$$w_2^{\theta,0^+} := a_H\,A_0'(x)+b_H\,B_0'(x)+\int_H^x \partial_x\mathcal{D}_x^\theta(k^0(x,z)-k^0(x,0))\left(\frac{1}{\alpha(z)}+u^{\theta,+}(z)\right)$$

$$+\int_H^x \partial_x\mathcal{D}_x^\theta k^0(x,0)u^{\theta,+}(z)+\tilde{w}(x),$$

with

$$\tilde{w}(x) := \int_H^x \partial_x\mathcal{D}_x^\theta k^0(x,0)\,\frac{1}{\alpha(z)}\,dz,\quad \text{for}\, x>0,$$

$$\tilde{w}(x) := \partial_x\mathcal{D}_x^\theta k^0(x,0)\left[\int_x^{-\varepsilon}\frac{1}{\alpha(z)}\,dz+\ln\alpha(\varepsilon)\,\beta(\varepsilon)-\right.$$

$$\left.\ln\alpha(-\varepsilon)\,\beta(-\varepsilon)+\int_{\alpha(\varepsilon)}^{\alpha(-\varepsilon)}\ln(z)\,\beta'(z)\,dz\right]+\int_\varepsilon^H \partial_x\mathcal{D}_x^\theta k^0(x,0)\,\frac{1}{\alpha(z)}\,dz\quad \text{for,}\, x<0,$$

where $0<\varepsilon$ is so small that $\alpha(x)$ is invertible for $x\in[-\varepsilon,\varepsilon]$ and $\beta(z) := 1/\alpha'(\alpha^{-1}(z))$ with α^{-1} the inverse function of α. The limits in (5.51), (5.52) also hold in the strong topology of $L^2(-L,H)$.

These weak or strong limits are naturally weak solution of the initial system (2.3): denoting for simplicity

$$(u_2,v_2,w_2)=(u_2^{\theta,0+},v_2^{\theta,0+},w_2^{\theta,0+}),$$

these functions are solutions of

$$\begin{cases} \displaystyle\int w_2\varphi_1 dx+i\theta\; P.V.\int\left(\frac{1}{\alpha(x)}+u_2\right)\varphi_1 dx-\frac{\theta\pi}{\alpha'(0)}\varphi_1(0)+\int v_2\varphi_1' dx=0,\\ \displaystyle i\theta\int w_2\varphi_2 dx-\int(\alpha u_2+1)\varphi_2 dx-i\int\delta(x)v_2\varphi_2 dx=0,\\ \displaystyle\int w_2\varphi_3' dx+i\; P.V.\int\delta\left(\frac{1}{\alpha(x)}+u_2\right)\varphi_3 dx-\frac{\delta(0)\pi}{\alpha'(0)}\varphi_3(0)\\ \qquad\qquad\qquad\qquad -\int\alpha(x)v_2\varphi_3 dx=0, \end{cases} \tag{5.53}$$

for any sufficiently smooth test functions with compact support, for example $(\varphi_1, \varphi_2, \varphi_3) \in \mathcal{C}_0^1(-L, H)$. To pass to the limit we have used that in distribution sense, $\lim_{u\to 0+} \frac{1}{\alpha(x)+i\mu} = P.V \frac{1}{\alpha(x)} + i\pi \frac{1}{\alpha'(0)}\delta_D$. The signs of $-\frac{\theta\pi}{\alpha'(0)}\varphi_1(0)$ and $-\frac{\delta(0)\pi}{\alpha'(0)}\varphi_3(0)$ are compatible[2] with the fact the limit is for positive μ. The principal value is defined as:

$$P.V. \int \frac{1}{\alpha(x)}\,\varphi(x)\,dx := \lim_{\varepsilon\downarrow 0}\left(\int_{-L}^{\rho(-\varepsilon)} \frac{1}{\alpha(x)}\,\varphi(x) + \int_{\rho(\varepsilon)}^{H} \frac{1}{\alpha(x)}\,\varphi(x)\right) dx,$$

where $\alpha(\rho(\mp\varepsilon)) = \pm\varepsilon$.

• **Uniqueness of the weak limit:** If there is another triplet $(\widetilde{u_2}, \widetilde{v_2}, \widetilde{w_2})$ solution of the same weak formulation (5.53), then the difference

$$(\widehat{u_2}, \widehat{v_2}, \widehat{w_2}) = (\widetilde{u_2} - u_2, \widetilde{v_2} - v_2, \widetilde{w_2} - w_2)$$

satisfies

$$\begin{cases} \int \widehat{w_2}\varphi_1 dx + i\theta \int \widehat{u_2}\varphi_1 dx + \int \widehat{v_2}\varphi_1' dx = 0, \\ i\theta \int \widehat{w_2}\varphi_2 dx - \int \alpha\widehat{u_2}\varphi_2 dx - i\int \delta(x)\widehat{v_2}\varphi_2 dx = 0, \\ \int \widehat{w_2}\varphi_3' dx + i\int \delta\widehat{u_2}\varphi_3 dx - \int \alpha(x)\widehat{v_2}\varphi_3\, dx = 0, \end{cases} \tag{5.54}$$

By construction $(\widehat{u_2}, \widehat{v_2}, \widehat{w_2}) = (0, 0, 0)$ for $x > 0$. For $x < 0$, we deduce from (5.54) that $(\widehat{u_2}, \widehat{v_2}, \widehat{w_2})$ is a solution of the X-mode equations. Therefore these functions can be expressed as a linear combination of the first and second basis functions for $x < 0$. Since $\widehat{u_2} \in L^2(-L, 0)$ is non singular, only the first basis function is involved that is

$$(\widehat{u_2}, \widehat{v_2}, \widehat{w_2}) = \lambda\left(U_1^\theta, V_1^\theta, W_1^\theta\right) \qquad x < 0.$$

From (5.54) we get for example

$$\int_{-L}^{0} \widehat{w_2}\varphi_3' dx + i\int_{-L}^{0} \delta\widehat{u_2}\varphi_3 dx - \int_{-L}^{0} \alpha(x)\widehat{v_2}\varphi_3\, dx = 0$$

where $\varphi_3(-L) = 0$ and $\varphi_3(0)$ is arbitrary. We integrate by parts

$$\int_{-L}^{0} \left(-\widehat{w_2}' + i\delta\widehat{u_2} - \alpha\widehat{v_2}\right)\varphi_3 dx + \widehat{\omega_2}(0)\varphi_3(0) = 0.$$

Since $(\widehat{u_2}, \widehat{v_2}, \widehat{w_2})$ is a non singular solution of the X-mode equations, one has that $-\widehat{w_2}' + i\delta\widehat{u_2} - \alpha\widehat{v_2} = 0$. Finally

$$\widehat{\omega_2}(0)\varphi_3(0) = 0.$$

Since we can take $\varphi_3(0) \neq 0$, it follows that $0 = \widehat{\omega_2}(0) = \lambda W_1^\theta(0)$. Considering the normalization (5.6) one gets that $\lambda = 0$. Therefore $(\widehat{u_2}, \widehat{v_2}, \widehat{w_2}) = (0, 0, 0)$.

[2]If one takes the limit $\mu \uparrow 0$, the signs of these terms are changed

It means that the weak limit is unique: all the sequence tends to the same weak limit.

• **Regularity:** By Theorem 5.1 the limit belongs to $H^1\left([-L,-\varepsilon]\cup[\varepsilon,\infty)\right)^3$.
□

6 The limit spaces $\mathbb{X}^{\theta,\pm}$

We can now define the limit spaces in which the limit basis functions live.

6.1 The space $\mathbb{X}^{\theta,+}$

Passing to the limit $\mu \to 0^+$, the limit space $\mathbb{X}^{\theta,+}$ is

$$\mathbb{X}_\varepsilon^{\theta,+} = \mathrm{Span}\left\{\mathbf{U}_1^\theta, \mathbf{U}_2^{\theta,+}\right\} \subset H^1_{loc}\left((-L,\infty)\setminus\{0\}\right). \tag{6.1}$$

6.2 The space $\mathbb{X}^{\theta,-}$

It is of course possible do all the analysis with negative $\mu < 0$ and to study the limit $\mu \to 0^-$. The first basis function is exactly the same. The second basis function is chosen exponentially decreasing at infinity and such that

$$i\mu U_2^{\theta,\mu} = 1 \qquad \mu < 0.$$

The generalization of the preliminary result (5.46) is straightforward

$$\left\| U_2^{\theta,\mu} - \frac{1}{\alpha(\cdot)+i\mu} \right\|_{L^p(-L,H)} \le C_p^\theta, \qquad -1 \le \mu < 0. \tag{6.2}$$

Passing to the limit $\mu \to 0^-$, it defines the limit space $\mathbb{X}^{\theta,-}$

$$\mathbb{X}_\varepsilon^{\theta,-} = \mathrm{Span}\left\{\mathbf{U}_1^\theta, \mathbf{U}_2^{\theta,-}\right\} \subset H^1_{loc}\left((-L,\infty)\setminus\{0\}\right). \tag{6.3}$$

We observe of course that the first basis function belongs to $\mathbb{X}_\varepsilon^{\theta,+} \cap \mathbb{X}_\varepsilon^{\theta,-}$. Since the limit equation is the same, and the normalization at $x = H$ is also the same we readily observe that the second basis functions are identical for $0 < x$

$$U_2^{\theta,+}(x) = U_2^{\theta,-}(x) \qquad 0 < x. \tag{6.4}$$

The main point is to determine the difference between the limit of the two singular functions for $x < 0$.

Proposition 6.1. *One has*

$$U_2^{\theta,+}(x) - U_2^{\theta,-}(x) = \frac{-2i\pi}{\alpha'(0)} U_1^\theta(x) \qquad x < 0. \tag{6.5}$$

Proof. We notice that the Wronskian relations (5.15) are the same at the limit $\mu = 0^{\pm}$. By subtraction

$$V_1^\theta(x)\left(W_2^{\theta,+}(x) - W_2^{\theta,-}(x)\right) - W_1^\theta(x)\left(V_2^{\theta,\mu}(x) - V_2^{\theta,-}(x)\right) = 0.$$

It show that the difference is proportional to the first basis function

$$U_2^{\theta,+}(x) - U_2^{\theta,-}(x) = \gamma U_1^\theta(x) \qquad x < 0. \tag{6.6}$$

It remains to determine γ. We already now that the limit $\mu \to 0^+$ can be characterized by (5.53). The third equation writes

$$\int w_2^+ \varphi_3' dx + i\; P.V. \int \delta \left(\frac{1}{\alpha(x)} + u_2^+\right)\varphi_3 dx - \frac{\delta(0)\pi}{\alpha'(0)}\varphi_3(0) - \int \alpha(x) v_2^+ \varphi_3 dx = 0$$

where (u_2^+, v_2^+, w_2^+) refers to the non singular part of the limit $\mu \to 0^+$. The equivalent equation for the non singular part (u_2^-, v_2^-, w_2^-) of the limit $\mu \to 0^-$ is

$$\int w_2^- \varphi_3' dx + i\; P.V. \int \delta \left(\frac{1}{\alpha(x)} + u_2^-\right)\varphi_3 dx + \frac{\delta(0)\pi}{\alpha'(0)}\varphi_3(0) - \int \alpha(x) v_2^- \varphi_3 dx = 0.$$

By subtraction, one gets

$$\begin{aligned}&\int (w_2^+ - w_2^-)\varphi_3' dx + i \int \delta(u_2^+ - u_2^-)\varphi_3 dx \\ &- \frac{2\delta(0)\pi}{\alpha'(0)}\varphi_3(0) - \int \alpha(x)(v_2^+ - v_2^-)\varphi_3 dx = 0.\end{aligned}$$

Due to (6.4) these differences vanishes for $x > 0$. We get

$$\begin{aligned}&\int_{-L}^0 (w_2^+ - w_2^-)\varphi_3' dx + i \int_{-L}^0 \delta(u_2^+ - u_2^-)\varphi_3 dx \\ &- \frac{2\delta(0)\pi}{\alpha'(0)}\varphi_3(0) - \int_{-L}^0 \alpha(x)(v_2^+ - v_2^-)\varphi_3 dx = 0\end{aligned}$$

where φ_3 is a smooth test function that vanishes at $-L$. Integration by part yields

$$\begin{aligned}&\int_{-L}^0 \left(-(w_2^+ - w_2^-)' + i\delta(u_2^+ - u_2^-) - \alpha(x)(v_2^+ - v_2^-)\right)\varphi_3 dx \\ &\qquad - \frac{2\delta(0)\pi}{\alpha'(0)}\varphi_3(0) + (w_2^+ - w_2^-)(0)\varphi_3(0).\end{aligned}$$

Due to (6.6) one has that

$$-(w_2^+ - w_2^-)' + i\delta(u_2^+ - u_2^-) - \alpha(x)(v_2^+ - v_2^-) = 0 \qquad x < 0.$$

Since $\varphi_3(0)$ is arbitrary, it means that $w_2^+(0) - w_2^-(0) = \frac{2\delta(0)\pi}{\alpha'(0)}$. We obtain $\gamma W_1^\theta(0) = \frac{2\delta(0)\pi}{\alpha'(0)}$, that is $i\delta(0)\gamma = \frac{2\delta(0)\pi}{\alpha'(0)} = 0$. Therefore $\gamma = \frac{-2i\pi}{\alpha'(0)}$. The claim is proved. □

7 Proof of the main theorem

All the information about the first and second basis functions is now used to construct the solution of the system (2.3) with the boundary condition (1.4). The function g depends only of the vertical variable y. Under convenient condition g admits the Fourier representation

$$g(y) = \frac{1}{2\pi} \int_{\mathbb{R}} \widehat{g}(\theta) e^{i\theta y} d\theta. \tag{7.1}$$

We first consider a small but non zero regularization parameter $\mu > 0$. For the sake of simplicity we will assume that the transversality condition is satisfied for all θ in the support of $\widehat{g}$

$$|\sigma(\theta)| \geq c > 0 \quad \forall \theta \in \text{supp}\,(\widehat{g})\,. \tag{H6}$$

It is just a convenient uniform version of the point-wise transversality condition (5.19). Additional comments are in section 7.3.

7.1 One Fourier mode

For one Fourier mode, one needs to consider the solution of (2.3) with boundary condition

$$\widehat{W}^{\mu}(-L) + i\lambda \widehat{V}^{\mu}(-L) = \widehat{g}.$$

Since we add of course that the solution must decrease (exponentially) at $x \approx \infty$ to guarantee that no energy comes from infinity, the solution is proportional to the second basis function. That is there is a coefficient $\gamma^{\theta,\mu}$ such that $\widehat{\mathbf{U}}^{\mu} = \gamma^{\theta,\mu}\mathbf{U}^{\theta,\mu}$. The coefficient satisfies the equation

$$\gamma^{\theta,\mu} \left(W_2^{\theta,\mu}(-L) + i\lambda V_2^{\theta,\mu}(-L) \right) = \widehat{g}(\theta)$$

that is

$$\gamma^{\theta,\mu} = \frac{\widehat{g}(\theta)}{\tau^{\theta,\mu}}$$

from which it is clear that we must study the coefficient/function

$$\tau^{\theta,\mu} = W_2^{\theta,\mu}(-L) + i\lambda V_2^{\theta,\mu}(-L). \tag{7.2}$$

Proposition 7.1. *Assume (H6). For every compact set $S \subset \mathbb{R}$, there exists $\varepsilon > 0$, τ^+ and $\tau_- > 0$ such that $\tau^- \leq \left|\tau^{\theta,\mu}\right| \leq \tau^+$ for $0 < \mu \leq \varepsilon$ and $\theta \in S$.*

Proof. The upper bound is a direct consequence of (5.48). To prove the lower bound, a useful result is the formula which comes from (5.10)

$$\text{Im}\left(W_2^{\theta,\mu}(-L)\overline{V_2^{\theta,\nu}(-L)} \right) \geq \mu \int_{-L}^{\infty} \left| U_2^{\theta,\mu}(x) \right|^2 dx$$

Combining with (5.27) and $Q\left(\mathbf{U}^{\theta,\mu}\right) = 1$ (by construction), it yields

$$\operatorname{Im}\left(W_2^{\theta,\mu}(-L)\overline{V_2^{\theta,\nu}(-L)}\right) \geq \tau_- > 0.$$

Plugging the definition of $\tau^{\theta,\mu}$ inside this inequality, one gets

$$\operatorname{Im}\left(\tau^{\theta,\mu}\overline{V_2^{\theta,\nu}(-L)}\right) \geq \tau_- + \lambda\left|V^{\theta,\mu}(-L)\right|^2 \geq \tau_- > 0.$$

Therefore $\left|V_2^{\theta,\mu}(-L)\right| \times |\tau(\theta,\mu)| \geq \tau_-$. The L^∞ bounds (5.48) shows that there exists $C > 0$ such that $C\,|\tau(\theta,\mu)| \geq \tau_-$. □

By (5.51), (5.52),

$$\tau^{\theta,+} := W_2^{\theta,0^+}(-L) + i\lambda V_2^{\theta,0^+}(-L) = \lim_{\mu\to 0^+} \tau^{\theta,\mu}. \tag{7.3}$$

Proposition 7.2. *For every compact set $S \subset \mathbb{R}$, there exists $\varepsilon > 0$, τ^+ and $\tau_- > 0$ such that $\tau^- \leq \left|\tau^{\theta,+}\right| \leq \tau^+$ for $0 < \mu \leq \varepsilon$ and $\theta \in S$.*

Proof. This is immediate from Proposition 7.1 and (7.3). □

7.2 Fourier representation of the solution

The solution of (2.2) with the boundary condition (1.4) is given by the inverse Fourier formula

$$\begin{pmatrix} E_x^\mu \\ E_y^\mu \\ W^\mu \end{pmatrix}(x,y) = \frac{1}{2\pi}\int_{\mathbb{R}} \frac{\widehat{g}(\theta)}{\tau^{\theta,\mu}}\mathbf{U}^{\theta,\mu}(x)e^{i\theta y}d\theta \tag{7.4}$$

where we assume that $g \in L^2(\mathbb{R})$ and that $\widehat{g}$ has compact support. Passing to the limit in (7.4) one gets

$$\begin{pmatrix} E_x^+ \\ E_y^+ \\ W^+ \end{pmatrix}(x,y) = \frac{1}{2\pi}\int_{\mathbb{R}} \frac{\widehat{g}(\theta)}{\tau^{\theta,+}}\begin{pmatrix} P.V.\frac{1}{\alpha(x)} + \frac{i\pi}{\alpha'(0)}\delta_D + u_2^{\theta,+} \\ v_2^{\theta,+} \\ w_2^{\theta,+} \end{pmatrix} e^{i\theta y}d\theta \tag{7.5}$$

This formula has to be understood in the sense of distribution. Since by Theorem 5.1 $\left\|u^{\theta,+}\right\| \leq C_2^\theta$, $\left\|v^{\theta,+}\right\| \leq C_2^\theta$,$\left\|w^{\theta,+}\right\| \leq C_2^\theta$ with C_2^θ a continuous function of θ, and considering that $\tau^{\theta,\mu}$ converges to $\tau^{\theta,+}$there is sufficient regularity to pass to the limit. The value of the heating is

$$\mathcal{Q} = \lim_{\mu\to 0^+} \mu\int |E_x^\mu(x,y)|^2 dxdy = \frac{1}{2}\int_{\mathbb{R}} \frac{|\widehat{g}(\theta)|^2}{|\alpha'(0)|\left|\tau^{\theta,+}\right|^2}d\theta. \tag{7.6}$$

To our knowledge this is the first time that such a formula is written where all terms are explicitly given. A similar but much less precise formula can be found in [11] derived by means of analogies, see also [25]. All these integrals are convergent provided $\widehat{g}$ decays sufficiently fast at infinity. Provided the transversality condition is satisfied, and it is always the case in a neighborhood of $\theta = 0$ under hypothesis (H5), the heating is generically positive. This is of course related to the presence of the strong Dirac singularity in the solution (7.5).

Remark 9. *An essential consequence of this analysis is the physical heating* $\mathcal{Q}$ *which is related to the singularity* $P.V.\frac{1}{\alpha(x)} \pm \frac{i\pi}{\alpha'(0)}\delta_D$ *of the mathematical solution. The singularity is not an artifact of the model. It is on the contrary a direct way to measure the amount of heating provided to the plasma by the electromagnetic wave.*

Remark 10. *Observe that the singular solutions* $\mathbf{U}_2^{\theta,\pm}$ *are the unique solutions of the following initial value problem.*

Find a triplet $(u_2^{\theta,\pm}, v_2^{\theta,\pm}, w_2^{\theta,\pm}) \in L^2(-L,\infty)^3$ *which satisfies the constraints* $v_2^{\theta,\pm}(H) = V_3^{\theta,0}(H)$, $w_2^{\theta,\pm}(H) = W_3^{\theta,0}(H)$, *and*

$$w_2^{\theta,\pm} - \frac{d}{dx}v_2^{\theta,\pm} + i\theta u_2^{\theta,\pm} = -i\theta P.V.\frac{1}{\alpha(x)} \pm \frac{\theta\pi}{\alpha'(0)}\delta_D,$$

$$i\theta w_2^{\theta,\pm} - \alpha u_2^{\theta,\pm} - i\delta(x)v_2^{\theta,\pm} = 1,$$

$$-\frac{d}{dx}w_2^{\theta,\pm} + iu_2^{\theta,\pm} - \alpha(x)v_2^{\theta,\pm} = -iP.V.\frac{\delta(x)}{\alpha(x)} \pm \frac{\delta(0)\pi}{\alpha'(0)}\delta_D.$$

We prove that this problem has an unique solution by the argument given to prove the uniqueness of the weak limits.. For this purpose observe that

$$\left(\frac{1}{\alpha(x)} + u_2^{\theta,\pm}, v_2^{\theta,\pm}, w_2^{\theta,\pm}\right)(x) = \left(U_3^{\theta,0}, V_3^{\theta,0}, W_3^{\theta,0}\right)(x) \text{ for } x > 0.$$

We observe the similarity with the standard limiting absorption principle in scattering theory. In scattering theory the solutions obtained by the limiting absorption principle are characterized as the unique solutions that satisfy the radiation condition, i.e., they are uniquely determined by the behavior at infinity. Here, the singular solutions are uniquely determined by their behavior at $+\infty$ *and by their singular part* $P.V.\frac{1}{\alpha(x)} \pm \frac{i\pi}{\alpha'(0)}\delta_D$ *Note that it is natural that we have to specify the singularity at* $x = 0$ *because our equations are degenerate at* $x = 0$. *We think this principle could be used for practical computations. It is however a little more subtle since a boundary condition at finite distance* $x = -L$ *must be prescribed. That is the singular part is itself dependant on the boundary condition where the energy comes in the system. Mathematically it corresponds to the coefficient* $\tau^{\theta,+}$ *in the representation formula (7.5).*

7.3 What happens if the transversality condition is not satisfied

An interesting question is to determine what happens if the transversality condition is not satisfied. We begin with some simple remarks. Firstly the point-wise transversality condition (5.19) or the uniform one (H6) greatly simplify the analysis. They are satisfied at least for θ close to zero provided the transition zone is small enough (H5). Secondly some technical intermediate results may be wrong if these conditions are not satisfied: for example it is not clear whether σ is still regular at points θ such that $\sigma(\theta) = 0$.

Our purpose is not to answer to the questions raised by this possibilities, but only to give understanding of the physical situation hidden behind and to explain what is the limit value of the heating (7.6). The analysis is as follows.

Physical picture: If $\sigma(\theta) = 0$ then the first basis function $\mathbf{U}_1^\theta$ is proportional to $\mathbf{U}_3^\theta$ (at least for $x \geq H$). It means that it is also exponentially decaying at infinity. That is the excitation provided by the boundary condition "captures" this non singular first basis function which is also the physical one. We remark that no heating is provided by $\mathbf{U}_1$ because $Q(\mathbf{U}_1) = 0$ by definition. Therefore a shortcut is: if $\sigma(\theta) = 0$, then physical heating vanishes.

Mathematical picture: On the other hand it is also clear that the function $\tau^{\theta,+}$ is upper bounded by virtue of the analysis provided in section 7.1. A consequence of the transversality condition is the fact that $V_2^{\theta,\mu}$ is uniformly bounded, see (5.48). So if $\sigma(\theta) = 0$, it is possible that $\lim_{\mu \to 0} \left| V_2^{\theta,\mu} \right| = \infty$. In this case $|\tau(\theta)| = \infty$ which yields in turn once again that the associated heating vanishes in (7.6).

In summary it is possible to conjecture that (7.5) is still valid even if the transversality condition is wrong: in this case the heating associated to the Fourier mode vanishes. The limit of (7.4) may a priori be more singular. More research is nevertheless needed to provide a rigorous basis to this analysis.

A Appendix

We provide additional material to obtain a self contained paper.

A.1 Approximation of Airy functions

Suppose that A and B are the two fundamental solutions of the equation $-u'' - \alpha u = 0$ satisfying the normalization conditions

$$\begin{cases} A(0) = 1, \ A'(0) = 0, \\ B(0) = 0, \ B'(0) = 1, \end{cases} \tag{A.1}$$

such that the corresponding Wronskian is equal to 1. Let A_μ be an approximation of A in the following sense

$$\begin{cases} -A_\mu'' - \alpha A_\mu = f_{A_\mu}, \qquad \text{with } f_{A_\mu} := i\mu A_\mu, \\ \\ A_\mu(0) = 1, \ A_\mu'(0) = 0. \end{cases} \tag{A.2}$$

From the variation of constants one gets

$$A_\mu(x) = A(x) \left(c_A + \int_0^x f_{A_\mu}(t) B(t) \, dt \right) + B(x) \left(c_B - \int_0^x f_{A_\mu}(t) A(t) \, dt \right).$$

The initial values (A.2) yield $A_\mu(x) = A(x) + \int_0^x f_{A_\mu}(t)k(x,t)\,dt$, where $k(x,y) = A(x)B(y) - A(y)B(x)$. So A_μ satisfies a classical Volterra integral equation

$$A_\mu(x) - i\mu \int_0^x A_\mu(t)k(x,t)\,dt = A(x). \tag{A.3}$$

Define the series of integral kernels

$$\begin{cases} K_1(x,y) = k(x,y), \\ K_{n+1}(x,y) = \int_0^x k(x,x_n)K_n(x_n,y)\,dx_n. \end{cases}$$

The solution of the integral equation (A.3) is

$$A_\mu(x) = A(x) + \int_0^x \left(\sum_{n=0}^{\infty} (i\mu)^{n+1} K_{n+1}(x,y) \right) A(y)\,dy.$$

For $n > 1$

$$K_{n+1}(x,y) = \int_{0<x_1<\cdots<x_n<x} k(x,x_n) \prod_{1\le i\le n} k(x_{i+1},x_i)\,dx_{i+1} k(x_1,y)\,dx_1,$$

and

$$\begin{aligned} I_n(x) &= \int_{(x_1,\dots,x_n)\in\{0<x_1<\cdots<x_n<x\}} \prod_{1\le i\le n} dx_i, \\ &= \int_0^x I_{n-1}(x_n)\,dx_n, \\ &= \tfrac{x^n}{n!} \text{ since } I_1(x) = x. \end{aligned}$$

So the iterated kernels satisfy $\forall n \ge 0$

$$|K_{n+1}(x,y)| \le (2\|A\|_\infty \|B\|_\infty)^{n+1} x^n/n!.$$

On the compact interval $]0, L_+[$ the sum and integral symbols can be inverted, which gives with a shift of the index n

$$A_\mu(x) = A(x) + \sum_{n=1}^{\infty} \left(\int_0^x K_n(x,y)A(y)\,dy \right) (i\mu)^n.$$

Assuming μ is bounded positive for the simplicity of notations, A_μ is indeed bounded independently for $0 < \mu \le 1$

$$|A_\mu(x)| \le \|A\|_\infty \left(1 + \sum_{n=1}^{\infty} \frac{(\mu C_0)^n}{n!} \right) = \|A\|_\infty \left(1 + e^{\mu C_0} - 1\right) = \|A\|_\infty e^{\mu C_0},$$

with $C_0 = 2L_+\|A\|_\infty\|B\|_\infty$. From (A.3) it yields

$$|A_\mu(x) - A(x)| \le \mu C_0 \|A\|_\infty e^{\mu C_0}.$$

Similarly if B_μ approximates B in the following sense

$$\begin{cases} -B_\mu'' - \alpha B_\mu = i\mu B_\mu, \\ \\ B_\mu(0) = 0, \; B_\mu'(0) = 1, \end{cases}$$

then one has the inequality $|B_\mu(x)| \le \|B\|_\infty e^{\mu C_0}$ together with

$$|B_\mu(x) - B(x)| \le \mu C_0 \|B\|_\infty e^{\mu C_0}.$$

Both $A_\mu(x)$ and $B_\mu(x)$ are $\mathcal{C}^\infty$ functions with respect to μ and x. Since any H^1 function f is 1/2 Hölder thanks to the inequality $|f(x)-f(y)| \le \|f'\|_{L^2}|x-y|^{1/2}$, then $A_\mu(x)$, $B_\mu(x)$ as well as all their derivatives with respect to μ and x are also 1/2 Hölder, with constants bounded independently of μ as far as μ is bounded.

B Derivation of the cold plasma dielectric tensor

Our aim is at the derivation of the cold plasma dielectric tensor used in this work. Much more material can be found in classical physical textbooks [16, 7]. Let us consider Maxwell's equations with a linear current

$$\begin{cases} -\frac{1}{c^2}\partial_t \mathbf{E} + \nabla \wedge \mathbf{B} = \mu_0 \mathbf{J}, & \mathbf{J} = -eN_e \mathbf{u}_e, \\ \partial_t \mathbf{B} + \nabla \wedge \mathbf{E} = 0, \\ m_e \partial_t \mathbf{u}_e = -e\left(\mathbf{E} + \mathbf{u}_e \wedge \mathbf{B}_0\right) - m_e \nu \mathbf{u}_e. \end{cases} \tag{B.1}$$

Here the linearized equation for the electrons corresponds to moving electrons (velocity $\mathbf{u}_e$) with a given electronic density N_e which is given but may be non constant in space. One implicitly assumes that an ion bath, which is the reason of the friction between the electrons and the ions with collision frequency ν. Shifting to the frequency domain (that $\partial_t = -i\omega$) yields

$$\begin{cases} \frac{1}{c^2} i\omega \mathbf{E} + \nabla \wedge \mathbf{B} = -\mu_0 e N_e \mathbf{u}_e, \\ -i\omega \mathbf{B} + \nabla \wedge \mathbf{E} = 0, \\ -i m_e \omega \mathbf{u}_e = -e\left(\mathbf{E} + \mathbf{u}_e \wedge \mathbf{B}_0\right) - m_e \nu \mathbf{u}_e. \end{cases} \tag{B.2}$$

One can compute the velocity using the third equation

$$\widetilde{\omega}\mathbf{u}_e + \omega_c i \mathbf{u}_e \wedge \mathbf{b}_0 = -\frac{e}{m_e} i\mathbf{E} \tag{B.3}$$

where the cyclotron frequency is $\omega_c = \frac{e|\mathbf{B}_0|}{m_e}$, $\mathbf{b}_0 = \frac{\mathbf{B}_0}{|\mathbf{B}_0|}$ is the normalized magnetic field and $\widetilde{\omega} = \omega + i\nu$ is the equivalent a priori complex pulsation. This is a linear equation. Assuming that $\mathbf{b}_0 = (0, 0, 1)$ one gets

$$\mathbf{u}_e = -\frac{e}{m_e} i \begin{pmatrix} \frac{\widetilde{\omega}}{\widetilde{\omega}^2 - \omega_c^2} & -i\frac{\omega_c}{\widetilde{\omega}^2 - \omega_c^2} & 0 \\ i\frac{\omega_c}{\widetilde{\omega}^2 - \omega_c^2} & \frac{\widetilde{\omega}}{\widetilde{\omega}^2 - \omega_c^2} & 0 \\ 0 & 0 & \frac{1}{\widetilde{\omega}} \end{pmatrix} \mathbf{E}. \tag{B.4}$$

It is then easy to eliminate $\mathbf{u}_e$ from the first equation of the system (B.2) and to obtain the Maxwell's equation (1.1) with the dielectric tensor

$$\varepsilon = \begin{pmatrix} 1 - \frac{\widetilde{\omega}\omega_p^2}{\omega(\widetilde{\omega}^2 - \omega_c^2)} & i\frac{\omega_c \omega_p^2}{\omega(\widetilde{\omega}^2 - \omega_c^2)} & 0 \\ -i\frac{\omega_c \omega_p^2}{\omega(\widetilde{\omega}^2 - \omega_c^2)} & 1 - \frac{\widetilde{\omega}\omega_p^2}{\omega(\widetilde{\omega}^2 - \omega_c^2)} & 0 \\ 0 & 0 & 1 - \frac{\omega_p^2}{\omega\widetilde{\omega}} \end{pmatrix}. \tag{B.5}$$

The dielectric tensor is slightly more general than (1.2) in the sense that $\widetilde{\omega} \neq \omega$ a priori. However physical values in fusion plasmas are such that the collision frequency is much smaller than the frequency ($\nu << \omega$) which means that some simplifications can be done in the dielectric tensor. We refer for example to [16], page 197. In this case one recovers exactly the limit tensor (1.2). This derivation is the explanation of the non zero extra-diagonal terms in the dielectric tensor. Notice that we consider in this work $\omega \neq \omega_c$, that is the frequency is away of the cyclotron frequency. Furthermore, when $\nu \neq 0$ the diagonal components of the dielectric tensor, $1 - \frac{\widetilde{\omega}\omega_p^2}{\omega(\omega^2-\omega_c^2)}$ have an imaginary part. This shows that our regularization parameter μ can be interpreted as a small collision frequency.

The loss of energy in domain Ω can easily be computed in the time domain starting from (B.1). One obtains

$$\frac{d}{dt}\int_\Omega \left(\frac{\varepsilon_0 |\mathbf{E}|^2}{2} + \frac{|\mathbf{B}|^2}{2\mu_0} + \frac{m_e N_e |\mathbf{u}_e|^2}{2} \right) = -\int_\Omega \nu m_e N_e |\mathbf{u}_e|^2 + \text{ boundary terms.}$$

Therefore

$$\mathcal{Q} = \int_\Omega \nu m_e N_e |\mathbf{u}_e|^2$$

represents the total loss of energy of the electromagnetic field plus the electrons. This loss of energy is necessarily equal to what is gained by the ions. This expression can be evaluated either in the time domain or also in the frequency domain. Considering (B.3) another equivalent formula in the frequency domain is

$$\mathcal{Q} = -\,\mathrm{Re}\left(\int_\Omega e N_e (\mathbf{E}, \mathbf{u}_e) \right)$$

Finally one can eliminate the electron velocity in function of the electric field using (B.4). Therefore a third formula is

$$\mathcal{Q} = -\omega\varepsilon_0\ \mathrm{Im}\left(\int_\Omega (\mathbf{E}, \varepsilon\mathbf{E}) \right)$$

where ε is the dielectric tensor (B.5). These formulas are the physical analoguous to the mathematical formula (7.6).

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