A Kato type Theorem for the inviscid limit of the Navier-Stokes equations with a moving rigid body.

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Abstract

The issue of the inviscid limit for the incompressible Navier-Stokes equations when a no-slip condition is prescribed on the boundary is a famous open problem. A result by Kato [21] says that convergence to the Euler equations holds true in the energy space if and only if the energy dissipation rate of the viscous flow in a boundary layer of width proportional to the viscosity vanishes. Of course, if one considers the motion of a solid body in an incompressible fluid, with a no-slip condition at the interface, the issue of the inviscid limit is as least as difficult. However it is not clear if the additional difficulties linked to the body's dynamic make this issue more difficult or not. In this paper we consider the motion of a rigid body in an incompressible fluid occupying the complementary set in the space and we prove that a Kato type condition implies the convergence of the fluid velocity and of the body velocity as well, what seems to indicate that an answer in the case of a fixed boundary could also bring an answer to the case where there is a moving body in the fluid.

1 Introduction

In this paper we investigate the issue of the inviscid limit for a incompressible fluid, driven by the Navier-Stokes equations, in the case where there is a moving body in the fluid. When a no-slip condition is prescribed on a solid boundary this issue is still widely open, even if this boundary does not move (see for instance [4, 1, 9, 15]). However in this case a result by Kato [21] says that, in the inviscid limit, the convergence to the Euler equations holds true in the energy space if and only if the energy dissipation rate of the viscous flows in a boundary layer of width proportional to the viscosity vanishes. The main result in this paper is an extension of Kato's result in the case where there is a moving body in the fluid. In order to clarify the presentation of our result we first recall Kato's result in its original setting: the case of a fluid contained in a fixed bounded domain, along with a slight reformulation which will be natural in the case with a moving body.

1.1 A short review of Kato's result.

Let us first consider the case of a fluid alone, contained in a bounded domain $\Omega \subset \mathbb{R}^d$, with $d = 2$ or 3. We therefore consider the incompressible Navier-Stokes equations:

$$\frac{\partial U}{\partial t} + (U \cdot \nabla)U + \nabla P = \nu \Delta U \quad \text{for} \quad x \in \Omega, \tag{1}$$

$$\operatorname{div} U = 0 \quad \text{for} \quad x \in \Omega, \tag{2}$$

$$U = 0 \quad \text{for} \quad x \in \partial\Omega, \tag{3}$$

$$U|_{t=0} = U_0. \tag{4}$$

Here U and P denote respectively the velocity and pressure fields. The positive constant ν is the viscosity of the fluid. The condition (3) is the so-called no-slip condition.

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We are going to deal with weak solutions of (1)-(4). Let us recall the following result by Leray (cf. for instance [24]), where we denote

$$\begin{aligned} \mathcal{H}_\Omega &:= \{V \in L^2(\Omega)/ \ \operatorname{div} V = 0 \text{ in } \Omega \text{ and } V \cdot n = 0 \text{ on } \partial\Omega\}, \\ \mathcal{V}_\Omega &:= \{V \in H^1_0(\Omega)/ \ \operatorname{div} V = 0 \text{ in } \Omega\}. \end{aligned}$$

Let us warn here the reader that we use the following slight abuse of notations: if V denotes any scalar-valued function space and U is a function with its values in $\mathbb{R}^d$, we will say that $U \in V$ if its components are in V.

Theorem 1. *Let $U_0 \in \mathcal{H}_\Omega$ and $T > 0$. Then there exists a solution $U \in C_w([0,T];\mathcal{H}_\Omega) \cap L^2([0,T];\mathcal{V}_\Omega)$ of the equations* (1)-(4) *in the sense that for all $V \in H^1([0,T];\mathcal{H}_\Omega) \cap L^2([0,T];\mathcal{V}_\Omega)$, for all $t \in [0,T]$,*

$$\int_\Omega \Big(U(t,\cdot) \cdot V(t,\cdot) - U_0 \cdot V|_{t=0} \Big) dx = \int_0^t \int_\Omega \Big[U \cdot (\partial_t + U \cdot \nabla) V - 2\nu \nabla U : \nabla V \Big] dx ds. \tag{5}$$

Moreover this solution satisfies the following energy inequality: for any $t \in [0,T]$,

$$\frac{1}{2} \|U(t,\cdot)\|^2_{L^2(\Omega)} + \nu \int_{(0,t)\times\Omega} |\nabla U|^2 dx ds \leqslant \frac{1}{2} \|U_0\|^2_{L^2(\Omega)}. \tag{6}$$

Moreover when $d = 2$ this solution is unique, $U \in C([0,T];\mathcal{H}_\Omega)$ and there is equality in (6).

When the viscosity coefficient ν is set equal to 0 in the previous equations, it is expected that the system (1)-(4) degenerates into the following incompressible Euler equations:

$$\frac{\partial U^E}{\partial t} + (U^E \cdot \nabla) U^E + \nabla P^E = 0 \quad \text{for} \quad x \in \Omega, \tag{7}$$

$$\operatorname{div} U^E = 0 \quad \text{for} \quad x \in \Omega, \tag{8}$$

$$U^E \cdot n = 0 \quad \text{for} \quad x \in \partial\Omega, \tag{9}$$

$$U^E|_{t=0} = U^E_0. \tag{10}$$

Kato's result deals with classical solutions of the Euler equations (7)-(10), whose (local in time) existence and uniqueness are classical since the works of Lichtenstein, Günter and Wolibner. Let us also recall that in two dimensions they are global in time, cf. [35] in the case of a simply connected domain and [20] for multiply connected domains. More precisely we have the following result, where we make use of the notation $C^{1,\lambda}(\Omega)$ for the Hölder space, endowed with the norm:

$$\|V\|_{C^{1,\lambda}(\Omega)} := \|V\|_{L^\infty(\Omega)} + \sup_{x \neq y \in \Omega} \frac{|\nabla V(x) - \nabla V(y)|}{|x-y|^\lambda}.$$

Here $\lambda \in (0,1)$.

Theorem 2. *Let be given $U^E_0 \in \mathcal{H}_\Omega \cap C^{1,\lambda}(\Omega)$. Then there exists $T > 0$ and a unique solution U^E of* (7)-(10) *in $C([0,T];\mathcal{H}_\Omega) \cap C_{w*}([0,T];C^{1,\lambda}(\Omega))$. Moreover this solution satisfies the following energy equality: for any $t \in [0,T]$,*

$$\|U^E(t,\cdot)\|_{L^2(\Omega)} = \|U^E_0\|_{L^2(\Omega)}. \tag{11}$$

Moreover in two dimensions, T can be chosen arbitrarily.

We are now in position to recall Kato's result.

Theorem 3. *Let be given $c > 0$ and $T > 0$. Assume that $U^E_0 \in \mathcal{H}_\Omega \cap C^{1,\lambda}(\Omega)$ and that $U_0 \to U^E_0$ in $\mathcal{H}_\Omega$ when $\nu \to 0$. Let us denote by U a solution of* (1)-(4) *given by Theorem 1 and by U^E the solution of* (7)-(10) *given by Theorem 2. Let us denote*

$$\Gamma^\Omega_{c\nu} := \{x \in \Omega/ \ dist(x, \partial\Omega) < c\nu\},$$

which is well defined for $\nu > 0$ small enough.

Then the following conditions are equivalent, when $\nu \to 0$.

1. $\nu \int_{(0,T)\times\Gamma^{\Omega}_{c\nu}} |\nabla U|^2 dxdt \to 0$,

2. $U \to U^E$ *in* $C([0,T];\mathcal{H}_\Omega)$.

Comparing (6) and (11) we see that the quantity in the first condition in Theorem 3 can be interpreted as the energy dissipation rate of the viscous flows in a boundary layer of width proportional to the viscosity. This width is much smaller than the one given by Prandtl's theory, what seems to indicate that one has to go beyond Prandtl's description to understand the inviscid limit. Moreover some recent results [7, 16] show that Prandtl's equation is in general ill-posed.

Kato's result in [21] contains some extra considerations about source terms and weak convergence, but we will skip these considerations here for sake of simplicity. Furthermore there exists many variants of Kato's argument: see for instance [33, 32, 22, 25, 19]. In particular it is shown in [22] that another equivalent condition is[1]

$$\nu \int_{(0,T)\times\Gamma^{\Omega}_{c\nu}} |\operatorname{curl} U|^2 dxdt \to 0,$$

where $\operatorname{curl} U$ is the $d\times d$ skew symmetric matrix given by

$$\operatorname{curl} U := (\frac{1}{2}(\partial_j U_i - \partial_i U_j))_{1\leqslant i,j\leqslant d},$$

and a slight modification of the proof in [22] also yields that another equivalent condition is

$$\nu \int_{(0,T)\times\Gamma^{\Omega}_{c\nu}} |D(U)|^2 dxdt \to 0, \tag{12}$$

where $D(U)$ is the deformation tensor

$$D(U) := (\frac{1}{2}(\partial_j U_i + \partial_i U_j))_{1\leqslant i,j\leqslant d}. \tag{13}$$

Actually, the proof in [22] relies on the observations that

1. for any U, V in $H^1(\Omega)$ such that $\operatorname{div} V = 0$ and such that $(U\cdot\nabla V)\cdot n = 0$ on $\partial\Omega$,

$$\int_\Omega \nabla U : \nabla V = 2\int_\Omega \operatorname{curl} U : \operatorname{curl} V, \tag{14}$$

2. for any U in $H^1(\Omega)$ and V in $C^1(\Omega)$ such that $\operatorname{div} V = 0$ and such that $(n\cdot V)U = 0$ on $\partial\Omega$,

$$\int_\Omega V\cdot(U\cdot\nabla U) = 2\int_\Omega V\cdot\Big((\operatorname{curl} U)U\Big). \tag{15}$$

These properties also hold true when we substitute $D(U)$ to $\operatorname{curl} U$, that is

1. for any U, V in $H^1(\Omega)$ such that $\operatorname{div} V = 0$ and such that $(U\cdot\nabla V)\cdot n = 0$ on $\partial\Omega$,

$$\int_\Omega \nabla U : \nabla V = 2\int_\Omega D(U) : D(V), \tag{16}$$

2. for any U in $H^1(\Omega)$ and V in $C^1(\Omega)$ such that $\operatorname{div} V = 0$ and such that $(n\cdot V)U = 0$ on $\partial\Omega$,

$$\int_\Omega V\cdot(U\cdot\nabla U) = 2\int_\Omega V\cdot\Big(D(U)U\Big). \tag{17}$$

It is therefore sufficient to follow the proof in [22] with these subtitutions in order to add (12) to the list of the equivalent conditions in Theorem 3. We are going to use a condition similar to (12) in the case of a moving rigid body.

[1] Here we use the following notations: when A and B are two $d\times d$ matrices, we denote $A : B = \sum_{1\leqslant i,j\leqslant d} A_{ij}B_{ij}$ and $|A|^2 := A : A$.

1.2 The case of a fluid with a moving rigid body.

We now consider the case where there is a moving rigid body in a fluid. Let us focus here on the three dimensional case. We assume that the body initially occupies a closed, bounded, connected and simply connected subset $\mathcal{S}_0 \subset \mathbb{R}^3$ with smooth boundary. It rigidly moves so that at time t it occupies an isometric domain denoted by $\mathcal{S}(t)$. More precisely if we denote by $h(t)$ the position of the center of mass of the body at time t, then there exists a rotation matrix $Q(t) \in SO(3)$, such that the position $\eta(t,x) \in \mathcal{S}(t)$ at the time t of the point fixed to the body with an initial position x is

$$\eta(t,x) := h(t) + Q(t)(x - h(0)). \tag{18}$$

Of course this yields that $Q(0) = 0$. Since $Q^T Q'(t)$ is skew symmetric there exists (only one) $r(t)$ in $\mathbb{R}^3$ such that for any $x \in \mathbb{R}^3$,

$$Q^T Q'(t)x = r(t) \wedge x. \tag{19}$$

Accordingly, the solid velocity is given by

$$U_{\mathcal{S}}(t,x) := h'(t) + R(t) \wedge (x - h(t)) \text{ with } R(t) := Q(t)r(t).$$

Given a positive function ρ_{S_0}, say in $L^\infty(\mathcal{S}_0;\mathbb{R})$, describing the density in the solid, the solid mass $m > 0$, the center of mass $h(t)$ and the inertia matrix $\mathcal{J}(t)$ can be computed by it first moments. Let us recall that $\mathcal{J}(t)$ is symmetric positive definite and that $\mathcal{J}$ satisfies Sylvester's law:

$$\mathcal{J}(t) = Q(t)\mathcal{J}_0 Q^T(t), \tag{20}$$

where $\mathcal{J}_0$ is the initial value of $\mathcal{J}$.

In the rest of the plane, that is in the open set $\mathcal{F}(t) := \mathbb{R}^3 \setminus \mathcal{S}(t)$, evolves a planar ideal fluid driven by the incompressible Navier-Stokes equations. We denote correspondingly $\mathcal{F}_0 := \mathbb{R}^3 \setminus \mathcal{S}_0$ the initial fluid domain.

The complete system driving the dynamics reads

$$\frac{\partial U}{\partial t} + (U\cdot\nabla)U + \nabla P = \nu\Delta U + g \quad \text{for} \quad x \in \mathcal{F}(t), \tag{21}$$

$$\operatorname{div} U = 0 \quad \text{for} \quad x \in \mathcal{F}(t), \tag{22}$$

$$U = U_{\mathcal{S}} \quad \text{for} \quad x \in \partial\mathcal{S}(t), \tag{23}$$

$$mh''(t) = mg - \int_{\partial\mathcal{S}(t)} \Sigma n \, ds, \tag{24}$$

$$(\mathcal{J}R)'(t) = -\int_{\partial\mathcal{S}(t)} (x - h) \wedge \Sigma n \, ds, \tag{25}$$

$$U|_{t=0} = U_0, \tag{26}$$

$$h(0) = 0, \; h'(0) = \ell_0, \; R(0) = r_0. \tag{27}$$

Here U and P denote the fluid velocity and pressure, which are defined on $\mathcal{F}(t)$ for each t. The fluid is supposed to be homogeneous of density 1, to simplify the notations and without any loss of generality. The Cauchy stress tensor is defined by

$$\Sigma := -PId + 2\nu D(U),$$

where $D(U)$ is the deformation tensor defined in (13).

Above n denotes the unit outward normal on the boundary of the fluid domain, ds denotes the integration element on this boundary and g is the gravity force which is assumed to be a constant vector, we actually include it in our study as a physical example of source term.

Let us observe that the choice $h(0) = 0$ avoids to write an extra moment, the one due to the gravity force, in (25). Still this choice is only a matter of convention and does not decrease the generality.

The existence of a weak solution to the system (21)-(27) was given in [29]. Let us also refer here to the following subsequent works [5, 6, 2, 3, 30] and the references therein.

When the viscosity coefficient ν is set equal to 0 in the previous equations, formally, the system (21)-(27) degenerates into the following equations:

$$\frac{\partial U^E}{\partial t} + (U^E \cdot \nabla)U^E + \nabla P^E = g \quad \text{for} \quad x \in \mathcal{F}^E(t), \tag{28}$$

$$\operatorname{div} U^E = 0 \quad \text{for} \quad x \in \mathcal{F}^E(t), \tag{29}$$

$$U^E \cdot n = U^E_{\mathcal{S}} \cdot n \quad \text{for} \quad x \in \partial\mathcal{S}^E(t), \tag{30}$$

$$m(h^E)'' = mg + \int_{\partial\mathcal{S}^E(t)} P^E n\, ds, \tag{31}$$

$$(\mathcal{J}^E R^E)' = \int_{\partial\mathcal{S}^E(t)} P^E (x - h^E) \wedge n\, ds, \tag{32}$$

$$U^E|_{t=0} = U^E_0, \tag{33}$$

$$h^E(0) = 0, \; (h^E)'(0) = \ell^E_0, \; R^E(0) = r^E_0, \tag{34}$$

where

$$U^E_{\mathcal{S}}(t,x) := (h^E)'(t) + R^E(t) \wedge (x - h^E(t)),$$

and

$$\mathcal{S}^E(t) := \eta^E(t,\cdot)(\mathcal{S}_0), \text{ with } \eta^E(t,x) := h^E(t) + Q^E(t)x,$$

where the matrix Q^E solves the differential equation $(Q^E)' = R^E \wedge Q^E$ with $Q^E(0) = 0$. Finally $\mathcal{J}^E$ is given by $\mathcal{J}^E = Q^E \mathcal{J}_0 (Q^E)^T$.

Observe that we prescribe $h^E(0) = 0$ so that the initial position $\mathcal{S}^E(0)$ occupied by the solid also starts from $\mathcal{S}_0$ at $t = 0$. The mass m and the initial inertia matrix $\mathcal{J}_0$ are also the same than in the previous case of the Navier-Stokes equations.

The existence and uniqueness of classical solutions for short times to the equations (28)-(34) is now well understood thanks to the recent works [26, 27, 28, 18, 14, 13].

The aim of this paper is to show the following conditional result about the inviscid limit: if

$$\nu \int_{(0,T)} \int_{\Gamma_{c\nu}(t)} |D(U)|^2 dx dt \to 0, \tag{35}$$

when $\nu \to 0$, where, for some $c > 0$,

$$\Gamma_{c\nu}(t) := \{x \in \mathcal{F}(t) / \; dist(x, \mathcal{S}(t)) < c\nu\},$$

then the solution of (21)-(27) converges to the solution of (28)-(34).

A precise statement is given below. In particular we will see that the condition (35) is also necessary.

2 Change of variables

In order to write the equations of the fluid in a fixed domain, we are going to use some changes of variables.

2.1 Case of the Navier-Stokes equations

In the case of the Navier-Stokes equations we use the following change of variables:

$$\ell(t) := Q(t)^T h'(t), \; u(t,x) := Q(t)^T U(t, Q(t)x + h(t)), \; p(t,x) := P(t, Q(t)x + h(t)),$$

and we introduce

$$\sigma := -pId + 2\nu D(u), \text{ where } D(u) := (\frac{1}{2}(\partial_j u_i + \partial_i u_j))_{i,j}.$$

Therefore the system (21)-(27) now reads

$$\frac{\partial u}{\partial t} + (u - u_{\mathcal{S}}) \cdot \nabla u + r \wedge u + \nabla p = Q(t)^T g + \nu \Delta u \quad \text{for} \quad x \in \mathcal{F}_0, \tag{36}$$

$$\operatorname{div} u = 0 \quad \text{for} \quad x \in \mathcal{F}_0, \tag{37}$$

$$u = u_{\mathcal{S}} \quad \text{for} \quad x \in \partial \mathcal{S}_0, \tag{38}$$

$$m\ell' = mQ^T g - \int_{\partial \mathcal{S}_0} \sigma n \, ds + m\ell \wedge r, \tag{39}$$

$$\mathcal{J}_0 r' = - \int_{\partial \mathcal{S}_0} x \wedge \sigma n \, ds + (\mathcal{J}_0 r) \wedge r, \tag{40}$$

$$u|_{t=0} = u_0, \tag{41}$$

$$h(0) = 0, \; h'(0) = \ell_0, \; r(0) = r_0. \tag{42}$$

with

$$u_{\mathcal{S}}(t, x) := \ell(t) + r(t) \wedge x. \tag{43}$$

In order to write the weak formulation of the system (36)-(42) we introduce

$$\mathcal{H} := \{\phi \in L^2(\mathbb{R}^3)/ \; \operatorname{div} \phi = 0 \text{ in } \mathbb{R}^3 \text{ and } D(\phi) = 0 \text{ in } \mathcal{S}_0\}.$$

According to Lemma 1.1 in [31], p18, for all $\phi \in \mathcal{H}$, there exists $\ell_\phi \in \mathbb{R}^3$ and $r_\phi \in \mathbb{R}^3$ such that for any $x \in \mathcal{S}_0$, $\phi(x) = \ell_\phi + r_\phi \wedge x$. Therefore we extend the initial data u_0 (respectively u_0^E) by setting $u_0 := \ell_0 + r_0 \wedge x$ (resp. $u_0^E := \ell_0^E + r_0^E \wedge x$) for $x \in \mathcal{S}_0$.

We endow the space $L^2(\mathbb{R}^3)$ with the following inner product:

$$(\phi, \psi)_{\mathcal{H}} := \int_{\mathcal{F}_0} \phi \cdot \psi dx + \int_{\mathcal{S}_0} \rho_{S_0} \phi \cdot \psi dx.$$

When ϕ, ψ are in $\mathcal{H}$ then,

$$(\phi, \psi)_{\mathcal{H}} = \int_{\mathcal{F}_0} \phi \cdot \psi dx + ml_\phi \cdot l_\psi + \mathcal{J}_0 r_\phi \cdot r_\psi,$$

by definition of m and $\mathcal{J}_0$.

Proposition 1. *A smooth solution of* (36)-(42) *satisfies the following: for any* $v \in C^\infty([0,T]; \mathcal{H} \cap C_c^\infty(\mathbb{R}^3))$, *for all* $t \in [0,T]$,

$$(u, v)_{\mathcal{H}}(t) - (u_0, v|_{t=0})_{\mathcal{H}} = \int_0^t \Big[(u, \partial_t v)_{\mathcal{H}} + b(u, u, v) - 2\nu \int_{\mathcal{F}_0} D(u) : D(v) dx + f_s[u, v] \Big] ds, \tag{44}$$

with

$$f_t[u, v] := m_a Q(t)^T g \cdot \ell_v - Vol(\mathcal{S}_0) Q(t)^T g \cdot (r_v \wedge x_0),$$

where

$$m_a := m - Vol(\mathcal{S}_0) \text{ and } x_0 := (Vol(\mathcal{S}_0))^{-1} \int_{\mathcal{S}_0} x dx$$

are respectively the apparent mass and the centroïd of the solid, and

$$b(u, v, w) := m \det(r_u, \ell_v, \ell_w) + \det(\mathcal{J}_0 r_u, r_v, r_w) + \int_{\mathcal{F}_0} \Big([(u - u_{\mathcal{S}}) \cdot \nabla w] \cdot v - \det(r_u, v, w) \Big) dx$$

Let us stress that $f_t[u, v]$ is linear with respect to v, and depends on u via the rotation matrix $Q(t)$ which is obtained by solving the matrix differential equation

$$Q' = Q(r_u \wedge \cdot) \text{ with } Q(0) = Id. \tag{45}$$

We easily obtain that there exists a constant $C > 0$ such that, for any $t \in [0,T]$, for any $u, v \in C([0,T];\mathcal{H})$,

$$|f_t[u,v] - f_t[\tilde{u},v]| \leqslant C\|v\|_{\mathcal{H}}(t) \sup_{0\leqslant s\leqslant T} \|u-\tilde{u}\|_{\mathcal{H}}(s). \tag{46}$$

We postpone the proof of Proposition 1 to the Appendix. For the sequel we will need to enlarge the space of the test functions. Therefore we introduce the space

$$\mathcal{V} := \{\phi \in \mathcal{H} /\ \|\nabla\phi\|^2_{L^2(\mathbb{R}^3,(1+|y|^2)^{\frac{1}{2}}dy)} := \int_{\mathbb{R}^3} |\nabla\phi(y)|^2(1+|y|^2)dy < +\infty\} \text{ and } \underline{\mathcal{V}} := \mathcal{H}\cap H^1(\mathbb{R}^3),$$

respectively endowed with the norms

$$\|\phi\|_{\mathcal{V}} := \|\phi\|_{\mathcal{H}} + \|\nabla\phi\|_{L^2(\mathbb{R}^3,(1+|y|^2)^{\frac{1}{2}}dy)} \text{ and } \|\phi\|_{\underline{\mathcal{V}}} := \|\phi\|_{\mathcal{H}} + \|\phi\|_{H^1(\mathbb{R}^3)}.$$

It is worth to notice from now on that b is well-defined, trilinear and continuous on $\underline{\mathcal{V}}\times\underline{\mathcal{V}}\times\mathcal{V}$: there exists a constant $C > 0$ such that for any $(u,v,w) \in \underline{\mathcal{V}}\times\underline{\mathcal{V}}\times\mathcal{V}$,

$$|b(u,v,w)| \leqslant C\|u\|_{\underline{\mathcal{V}}}\,\|v\|_{\underline{\mathcal{V}}}\,\|w\|_{\mathcal{V}}. \tag{47}$$

This follows easily from Holder's inequality and the following interpolation inequality

$$\|v\|_{L^4(\mathcal{F}_0)} \leqslant \sqrt{2}\|v\|^{\frac{1}{4}}_{L^2(\mathcal{F}_0)}\|\nabla v\|^{\frac{3}{4}}_{L^2(\mathcal{F}_0)}. \tag{48}$$

Observe in particular that the weight in the definition of $\mathcal{V}$ allows to handle the rotation part of $u_{\mathcal{S}}$.

Moreover the trilinear form b satisfies the following crucial property

$$(u,v) \in \underline{\mathcal{V}}\times\mathcal{V} \text{ implies } b(u,v,v) = 0. \tag{49}$$

Definition 1. *We say that*

$$u \in C_w([0,T];\mathcal{H}) \cap L^2([0,T];H^1(\mathbb{R}^3))$$

is a weak solution of the system (36)-(42) *if for all* $v \in H^1([0,T];\mathcal{H})\cap L^2([0,T];\mathcal{V})$*, and for all* $t \in [0,T]$*,* (44) *holds true.*

As already said above the existence of weak solutions "à la Leray" for the system (36)-(42) is now well understood. Let us for instance refer to [29], Theorem 4.5.

Theorem 4. *Let be given* $u_0 \in \mathcal{H}$ *and* $T > 0$*. Then there exists a weak solution* u *of* (36)-(42) *in* $C_w([0,T];\mathcal{H}) \cap L^2([0,T];H^1(\mathbb{R}^3))$*. Moreover this solution satisfies the following energy inequality: for any* $t \in [0,T]$*,*

$$\frac{1}{2}\|u(t,\cdot)\|^2_{\mathcal{H}} + 2\nu\int_{(0,t)\times\mathbb{R}^3} |D(u)|^2\,dxdt \leqslant \frac{1}{2}\|u_0\|^2_{\mathcal{H}} + \int_0^t f_s[u,u]\,ds. \tag{50}$$

Let us stress that the integral in the right hand side above could innocuously be taken over $(0,t)\times\mathcal{F}_0$ since the deformation tensor $D(u)$ vanishes in the solid. Moreover using the identities (14) and (16) in the case where $\Omega = \mathbb{R}^3$ yields the following remarks.

Remark 1. *In the previous statement, it is possible to replace the weak formulation* (44) *by the following one: for any* $v \in H^1([0,T];\mathcal{H})\cap L^2([0,T];\mathcal{V})$*, for all* $t \in [0,T]$*,*

$$(u,v)_{\mathcal{H}}(t) - (u_0, v|_{t=0})_{\mathcal{H}} = \int_0^t \Big[(u,\partial_t v)_{\mathcal{H}} + b(u,u,v) - 2\nu\int_{\mathbb{R}^3} \operatorname{curl} u : \operatorname{curl} v\,dx + f_s[u,v]\Big]ds, \tag{51}$$

and the energy inequality (50) *by*

$$\frac{1}{2}\|u(t,\cdot)\|^2_{\mathcal{H}} + 2\nu\int_{(0,t)\times\mathbb{R}^3} |\operatorname{curl} u|^2\,dxdt \leqslant \frac{1}{2}\|u_0\|^2_{\mathcal{H}} + \int_0^t f_s[u,u]\,ds. \tag{52}$$

Remark 2. *In Theorem (4), it is also possible to replace (44) by: for any $v \in H^1([0,T];\mathcal{H}) \cap L^2([0,T];\mathcal{V})$, for all $t \in [0,T]$,*

$$(u,v)_{\mathcal{H}}(t) - (u_0, v|_{t=0})_{\mathcal{H}} = \int_0^t \Big[(u,\partial_t v)_{\mathcal{H}} + b(u,u,v) - \nu \int_{\mathbb{R}^3} \nabla u : \nabla v \, dx + f_s[u,v]\Big] ds, \tag{53}$$

and (50) *by*

$$\frac{1}{2}\|u(t,\cdot)\|^2_{\mathcal{H}} + \nu \int_{(0,t)\times\mathbb{R}^3} |\nabla u|^2 \, dxdt \leqslant \frac{1}{2}\|u_0\|^2_{\mathcal{H}} + \int_0^t f_s[u,u]\, ds. \tag{54}$$

2.2 Case of the Euler equations

Let us now see the case of the Euler equations. First performing the following change of variables:

$$\ell^E(t) := Q^E(t))^T\,(h^E)'(t),\; R^E(t) := Q^E(t) r^E(t),$$
$$u^E(t,x) := Q^E(t)^T\, U^E(t, Q^E(t)x + h^E(t)),\; \text{and } p^E(t,x) := P^E(t, Q^E(t)x + h^E(t)),$$

where $Q^E(t)$ is the rotation matrix associated to the motion of $\mathcal{S}^E(t)$, the system (28)-(34) now reads

$$\frac{\partial u^E}{\partial t} + (u^E - u^E_{\mathcal{S}}) \cdot \nabla u^E + r^E \wedge u^E + \nabla p^E = (Q^E)^T g \quad \text{for} \quad x \in \mathcal{F}_0, \tag{55}$$

$$\operatorname{div} u^E = 0 \quad \text{for} \quad x \in \mathcal{F}_0, \tag{56}$$

$$u^E(t,x)\cdot n = u^E_{\mathcal{S}} \cdot n \quad \text{for} \quad x \in \partial\mathcal{S}_0, \tag{57}$$

$$m(\ell^E)' = m(Q^E)^T g + \int_{\partial\mathcal{S}_0} p^E n \, ds + (m\ell^E) \wedge r^E, \tag{58}$$

$$\mathcal{J}_0 (r^E)' = \int_{\partial\mathcal{S}_0} p^E x \wedge n \, ds + (\mathcal{J}_0 r^E) \wedge r^E, \tag{59}$$

$$u^E|_{t=0} = u^E_0, \tag{60}$$

$$h^E(0) = 0,\; (h^E)'(0) = \ell^E_0,\; r^E(0) = r^E_0, \tag{61}$$

with

$$u^E_{\mathcal{S}}(t,x) := \ell^E(t) + r^E(t) \wedge x. \tag{62}$$

Here, in order to follow Kato's strategy we will need classical solutions. The existence and uniqueness of classical solutions to the equations (28)-(34) with finite energy is given by the following result.

Theorem 5. *Let be given $\lambda \in (0,1)$ and $u^E_0 \in \mathcal{H}$ such that $u^E_0|_{\mathcal{F}_0} \in H^1 \cap C^{1,\lambda}$ and $\operatorname{curl} u^E_0|_{\mathcal{F}_0}$ is compactly supported. Then there exists $T > 0$ and a unique solution u^E of (55)-(61) in $C^1([0,T];\mathcal{H})$ such that $(\nabla u^E)|_{[0,T]\times\mathcal{F}_0} \in C([0,T]; L^2(\mathcal{F}_0, (1+|x|^2)^{\frac{1}{2}} dx)) \cap C_{w*}([0,T]; C^{0,\lambda}(\mathcal{F}_0))$. Moreover for any $t \in [0,T]$,*

$$\frac{1}{2}\|u^E(t,\cdot)\|^2_{\mathcal{H}} = \frac{1}{2}\|u^E_0\|^2_{\mathcal{H}} + \int_0^t f_s[u^E, u^E] ds, \tag{63}$$

where we denote, for $s \in [0,T]$ and $v \in \mathcal{H}$,

$$f_s[u^E, v] := m_a Q^E(s)^T g \cdot \ell_v - Vol(\mathcal{S}_0) Q^E(s)^T g \cdot (r_v \wedge x_0).$$

where the rotation matrix $Q^E(t)$ is obtained by solving the matrix differential equation

$$(Q^E)' = Q^E(r^E \wedge \cdot) \text{ with } Q^E(0) = Id. \tag{64}$$

Theorem 5 extends Th. 4 in [14] to the case where the fluid-rigid body system occupies the whole space, instead of a bounded domain in [14]. Actually the proof of Theorem 5 is even simpler than the one of Th. 4 in [14], thanks to the change of variables performed at the beginning of this section, which leads here to a fluid domain independent of time. We provide a complete and independent proof in Appendix 8.

It is worth to point out here that solution u^E given by Theorem 5 satisfies the following property: for any $t \in [0,T]$, for any $v \in \mathcal{V}$,

$$(\partial_t u^E, v)_{\mathcal{H}} = -b(u^E, v, u^E) + f_t[u^E, v]. \tag{65}$$

To see that, multiply (55) by v and integrate by parts in space using (55)-(59). Actually this property even holds for a larger space of test functions, as there is no more the Laplace term to integrate by parts. More precisely (65) holds true for any $v \in \mathcal{H}$. Indeed, defining the space

$$\mathcal{W} := \{\phi \in \mathcal{H}/ \ (\nabla\phi)|_{\mathcal{F}_0} \in L^2(\mathcal{F}_0, (1+|y|^2)^{\frac{1}{2}} dy) \cap L^\infty(\mathcal{F}_0)\},$$

endowed with the norm

$$\|\phi\|_{\mathcal{W}} := \|\phi\|_{\mathcal{H}} + \|(\nabla\phi)|_{\mathcal{F}_0}\|_{L^2(\mathcal{F}_0,(1+|y|^2)^{\frac{1}{2}} dy)} + \|(\nabla\phi)|_{\mathcal{F}_0}\|_{L^\infty(\mathcal{F}_0)},$$

one easily sees that b is trilinear continuous on $\mathcal{H} \times \mathcal{H} \times \mathcal{W}$, that is there exists a constant $C > 0$ such that for any $(u, v, w) \in \mathcal{H} \times \mathcal{H} \times \mathcal{W}$,

$$|b(u, v, w)| \leqslant C\|u\|_{\mathcal{H}} \, \|v\|_{\mathcal{H}} \, \|w\|_{\mathcal{W}}. \tag{66}$$

Moreover the cancellation property (49) extends into

$$(u, v) \in \mathcal{H} \times \mathcal{W} \text{ implies } b(u, v, v) = 0. \tag{67}$$

3 Statement of the main result

Let us now state the main result of this paper.

Theorem 6. *Let be given $c > 0$, u_0^E and $T > 0$ as in Theorem 5. Assume that*

$$u_0 \to u_0^E \text{ in } \mathcal{H} \text{ when } \nu \to 0. \tag{68}$$

Let us denote u a solution of (36)-(42) given by Theorem 4 and by u^E the solution of (55)-(61) given by Theorem 5.

Let us introduce the strips

$$\Gamma_{c\nu} := \{x \in \mathcal{F}_0/ \ d(x) < c\nu\} \text{ with } d(x) := dist(x, \partial\mathcal{S}_0),$$

which are well-defined for ν small enough.

Then the following conditions are equivalent, when $\nu \to 0$:

$$u \to u^E \text{ in } C([0,T];\mathcal{H}), \tag{69}$$

$$\nu \int_{(0,T)\times\Gamma_{c\nu}} |D(u)|^2 dxdt \to 0, \tag{70}$$

$$\nu \int_{(0,T)\times\Gamma_{c\nu}} |\operatorname{curl} u|^2 dxdt \to 0, \tag{71}$$

$$\nu \int_{(0,T)\times\Gamma_{c\nu}} |\nabla u|^2 dxdt \to 0, \tag{72}$$

$$u(t,\cdot) \rightharpoonup u^E(t,\cdot) \text{ in } \mathcal{H} - w, \text{ for any } t \in [0,T]. \tag{73}$$

Before to start the proof of Theorem 6, let us give a few comments and open questions.

First as mentioned previously, a similar result can be obtained in two dimensions. The proof is even actually simpler. Still let us mention that in two dimensions the assumption that the energy is finite is rather restrictive, at least for what concerns the Euler equation, see [13] for a wider setting. Therefore it is natural to wonder whether or not the analysis performed here can be extended to this more general setting. In particular it could be that, even under Kato's condition, one misses some interesting dynamics of the Euler case, as for instance the one obtained in the particle limit in [12], by using the Navier-Stokes equations.

Another natural issue is to extend Theorem 6 to the case where there are several bodies, or to the case where the fluid-body system occupies a fixed bounded domain. This raises some extra technical difficulties as the change of variable performed in Section 2 does not lead to a time-independent domain. Let us also stress that the collision issues can be very different depending on whether one considers the Euler equations or the Navier-Stokes equations. Let us refer here to [8, 17] and to the references therein.

Also another interesting question raised by Theorem 6 is about the convergence of the time derivatives of the body's velocity. In particular it was shown in [14, 13] that in the Euler case, the body's velocity is actually analytic in time, if its boundary is analytic. It is therefore natural to wonder whether or not the time derivatives of the body's velocity for smooth solutions of the Navier-Stokes case also converge to the ones of the Euler case under a Kato type condition.

It is also probably possible to extend some of the variants of Kato's argument mentioned in the introduction in this setting of a moving body.

4 Beginning of the proof of Theorem 6

4.1 Easy part

As in Kato's original statement, the proof of the necessity of the condition (70) to get (69) is quite easy: if (69) holds true when $\nu \to 0$ then it suffices to combine (50), (63) and (68) to get that

$$\nu \int_{(0,T)\times\mathbb{R}^3} |D(u)|^2 dxdt \to 0, \tag{74}$$

when $\nu \to 0$. Of course (74) implies (70).

We obtain similarly that (69) implies (71) and (72) using Remark 1 and Remark 2.

Since it is straightforward that (69) implies (73), it remains now to see the converse statements.

Actually let us see that (73) implies (70) so that it will only remain to prove that either (70), or (71) or (72) implies (69).

Thanks to (50), we have for any $t \in [0,T]$, using (68),

$$\begin{aligned} 2\limsup \nu \int_{(0,t)\times\mathbb{R}^3} |D(u)|^2 dxdt &\leqslant \frac{1}{2}\|u_0^E\|^2_{\mathcal{H}} - \liminf \frac{1}{2}\|u(t,\cdot)\|^2_{\mathcal{H}} + \limsup \int_0^t f_s[u,u]ds \\ &\leqslant \frac{1}{2}\|u_0^E\|^2_{\mathcal{H}} - \frac{1}{2}\|u^E(t,\cdot)\|^2_{\mathcal{H}} + \int_0^t f_s[u^E,u^E]ds, \end{aligned}$$

using (73) and Fatou's lemma. It remains to use (63) to see that the right hand side above is 0, what yields (70).

We will detail how to prove that (70) implies (69) and then we will explain what modifications lead to the other cases. We first adapt the construction of a Kato type "fake" layer.

4.2 A Kato type "fake" layer

The goal of this section is to prove the following result, where we make use of the Landau notations $o(1)$ and $O(1)$ for quantities respectively converging to 0 and bounded with respect to the limit $\nu \to 0^+$.

Proposition 2. *Under the assumptions of Theorem 6 there exists $v_F \in C([0,T];\mathcal{H})$, supported in $\Gamma_{c\nu}$, such that*

$$v_F = O(1) \text{ in } C([0,T]\times\mathbb{R}^3), \tag{75}$$

$$v_F = O(\nu^{\frac{1}{2}}) \text{ in } C([0,T];\mathcal{H}), \tag{76}$$

$$\partial_t v_F = O(\nu^{\frac{1}{2}}) \text{ in } C([0,T];\mathcal{H}) \tag{77}$$

$$\|\nabla v_F\|_{L^\infty([0,T];L^2(\Gamma_{c\nu}))} = O(\nu^{-\frac{1}{2}}), \tag{78}$$

$$d(x)v_F = O(\nu) \text{ in } L^\infty([0,T]\times\mathbb{R}^3), \tag{79}$$

$$u^E - v_F \in C([0,T];\mathcal{H}) \cap L^2([0,T];\mathcal{V}) \tag{80}$$

Proof. According to [21], Lemma A1, we get that there exists an antisymmetric 2-tensor field $a_F(t,x)$ on $[0,T]\times\mathbb{R}^3$ such that,

$$\operatorname{div} a_F = u^E - u^E_{\mathcal{S}} \text{ and } a_F = 0 \text{ on } \partial\mathcal{S}_0. \tag{81}$$

Let us recall that for a smooth antisymmetric 2-tensor a, $\operatorname{div} a$ denotes the vector field $\operatorname{div} a := (\sum_k \partial_k a_{jk})_k$.

Now we introduce a smooth cut-off function $\xi : [0,+\infty) \to [0,+\infty)$ such that $\xi(0) = 1$ and $\xi(r) = 0$ for $r \geqslant 1$. We define $z(x) := \xi(\frac{d(x)}{c\nu})$ and v_F by

$$v_F := \operatorname{div}(z a_F) \text{ in } \mathcal{F}_0 \text{ and } v_F := 0 \text{ in } \mathcal{S}_0. \tag{82}$$

In order to verify that v_F satisfies the desired properties, let us introduce $a^\flat_F(t,x) := \frac{1}{d(x)} a_F(t,x)$, $\tilde{\xi}(r) := r\xi'(r)$ and $\tilde{z}(x) := \tilde{\xi}(\frac{d(x)}{c\nu})$. Then, in $\mathcal{F}_0$,

$$v_F = z \operatorname{div} a_F + \tilde{z} a^\flat_F \nabla d. \tag{83}$$

First since z and $\tilde{z}$ are supported in $\Gamma_{c\nu}$ so is v_F. Furthermore, using (81) and that, for $x \in \partial\mathcal{S}_0$, $z(x) = 1$ and $\tilde{z}(x) = 0$, we get

$$v_F|_{\mathcal{F}_0} = u^E - u^E_{\mathcal{S}} \text{ on } \partial\mathcal{S}_0. \tag{84}$$

We observe that for any smooth antisymmetric 2-tensor a the vector field $\operatorname{div} a$ is divergence free, as $\operatorname{div}\operatorname{div} a = \sum_j \sum_k \partial_j a_{jk} = 0$. Therefore we obtain that $v_F \in C([0,T];\mathcal{H})$.

Moreover $u^E - v_F$ is H^1 in $\mathcal{F}_0$ and in $\mathcal{S}_0$. Using again (84) we get that $u^E - v_F$ is continuous across $\partial\mathcal{S}_0$. Therefore it belongs to $L^2([0,T];\mathcal{V})$.

The other estimates follow easily from (83) if one observes that the functions z and $\tilde{z}$ satisfy the required estimates and that, according to (83), v_F is a slow modulation (with respect to ν) of z and $\tilde{z}$ by some regular functions. □

5 Core of the proof of Theorem 6

In this section we prove that (70) implies (69). Let us give a few words of caution before entering in the proof:

1. We will use the same notation C for various constants (which may change from line to line).
2. For some functions ϕ and ψ depending on (t,x), such that for any t, $\phi(t,\cdot)$ and $\psi(t,\cdot)$ are in $\mathcal{H}$, we will denote $(\phi,\psi)_{\mathcal{H}}(t)$ for $(\phi(t,\cdot),\psi(t,\cdot))_{\mathcal{H}}$.
3. The identities (16) and (17) are also true for an unbounded domain, for instance if one substitutes the domain $\mathcal{F}_0$ to the domain Ω.

For any $t \in [0,T]$, we have, thanks to (50), (63), the Cauchy-Schwarz inequality, (76) and (68),

$$\begin{aligned}
\|u(t,\cdot) - u^E(t,\cdot)\|^2_{\mathcal{H}} &= \|u(t,\cdot)\|^2_{\mathcal{H}} + \|u^E(t,\cdot)\|^2_{\mathcal{H}} - 2(u,u^E)_{\mathcal{H}}(t) \\
&\leqslant \|u_0\|^2_{\mathcal{H}} + \|u^E_0\|^2_{\mathcal{H}} + 2\int_0^t (f_s[u^E,u^E] + f_s[u,u])ds - 2(u,u^E)_{\mathcal{H}}(t) \\
&\leqslant 2\|u^E_0\|^2_{\mathcal{H}} + 2\int_0^t (f_s[u^E,u^E] + f_s[u,u])ds - 2(u,u^E - v_F)_{\mathcal{H}}(t) + o(1).
\end{aligned} \tag{85}$$

We now apply (44) to $v = u^E - v_F$ (what is licit according to (80)) to get

$$\begin{aligned}
(u,u^E - v_F)_{\mathcal{H}}(t) - (u_0, u^E_0 - v_F|_{t=0})_{\mathcal{H}} = \int_0^t \Big[&(u,\partial_t(u^E - v_F))_{\mathcal{H}} + b(u,u,u^E - v_F) \\
&-2\nu \int_{\mathcal{F}_0} D(u) : D(u^E - v_F)dx + f_s[u,u^E]\Big] ds.
\end{aligned}$$

Let us stress that we used above that $f_s[u,v_F] = 0$. Now using (68), (50), (76), the Cauchy-Schwarz inequality and (77) we deduce that

$$-2(u,u^E - v_F)_{\mathcal{H}}(t) + 2\|u^E_0\|^2_{\mathcal{H}} = o(1) - 2\int_0^t \Big[R(s) + (u,\partial_t u^E)_{\mathcal{H}} + b(u,u,u^E) + f_s[u,u^E]\Big] ds, \tag{86}$$

where R denotes the time-dependent function:

$$\begin{aligned} R &:= -b(u,u,v_F) - 2\nu \int_{\mathcal{F}_0} D(u) : D(u^E - v_F)dx \\ &= -\int_{\mathcal{F}_0} \Big([(u-u_{\mathcal{S}}) \cdot \nabla v_F] \cdot u - \det(r_u, u, v_F) \Big) - 2\nu \int_{\mathcal{F}_0} D(u) : D(u^E - v_F)dx. \end{aligned}$$

On the other hand we have, for any $t \in [0,T]$,

$$(\partial_t u^E, u)_{\mathcal{H}} = -b(u^E, u, u^E) + f_t[u^E, u],$$

using (65) with $v = u$.

Combining with (86) we obtain

$$-2(u, u^E - v_F)_{\mathcal{H}}(t) + 2\|u_0^E\|^2_{\mathcal{H}} = o(1) - 2\int_0^t \Big[R(s) + b(u-u^E, u, u^E) + f_s[u,u^E] + f_s[u^E, u] \Big] ds \tag{87}$$

Using the property (67) we get

$$-2(u, u^E - v_F)_{\mathcal{H}}(t) + 2\|u_0^E\|^2_{\mathcal{H}} = o(1) - 2\int_0^t \Big[R(s) + b(u-u^E, u-u^E, u^E) + f_s[u,u^E] + f_s[u^E, u] \Big] ds,$$

and, then, using (66) with $(u-u^E, u-u^E, u^E)$ instead of (u,v,w) and that $u^E \in C([0,T];\mathcal{W})$, we get

$$-2(u, u^E - v_F)_{\mathcal{H}}(t) + 2\|u_0^E\|^2_{\mathcal{H}} \leqslant o(1) - 2\int_0^t R(s)ds + C\int_0^t \|u-u^E\|^2_{\mathcal{H}}(s)ds - 2\int_0^t \Big[f_s[u,u^E] + f_s[u^E, u] \Big] ds.$$

Now combining this with (85) yields

$$\|u(t,\cdot) - u^E(t,\cdot)\|^2_{\mathcal{H}} \leqslant o(1) - 2\int_0^t R(s)ds + C\int_0^t \|u-u^E\|^2_{\mathcal{H}}(s)ds + 2\int_0^t (f_s[u^E, u^E - u] + f_s[u, u-u^E])ds.$$

Moreover, combining (45) and (64), and using again the bounds given by (50) and (63), we obtain, for any $s \in [0,t]$,

$$|f_s[u^E, u^E - u] + f_s[u, u-u^E]| \leqslant C\|u-u^E\|_{\mathcal{H}}(s) \sup_{0\leqslant \tilde{s}\leqslant s} \|u-u^E\|_{\mathcal{H}}(\tilde{s})$$

As a consequence in order to achieve this part of the proof of Theorem 6 it only suffices to prove that

$$\int_0^t R(s)ds \to 0 \text{ when } \nu \to 0. \tag{88}$$

In order to prove (88) we first decompose $R(t)$ into

$$R(t) = R_1(t) + ... + R_5(t),$$

where

$$\begin{aligned} R_1 &:= -\int (u-u_{\mathcal{S}}) \cdot [(u-u_{\mathcal{S}}) \cdot \nabla v_F]dx, \\ R_2 &:= -\int u_{\mathcal{S}} \cdot [(u-u_{\mathcal{S}}) \cdot \nabla v_F]dx, \\ R_3 &:= -2\nu \int D(u) : D(u^E)dx, \\ R_4 &:= 2\nu \int D(u) : D(v_F)dx, \\ R_5 &:= \int_{\mathcal{F}_0} \det(r_u, u, v_F). \end{aligned}$$

Let us emphasize that the integrals in the expressions above, except the one corresponding to R_3, can be taken over $\Gamma_{c\nu}$, since the fake layer v_F is supported in $\Gamma_{c\nu}$. In particular we do not have to worry too much about the nondecreasing at infinity of the vector field $u_{\mathcal{S}}$. However let us gain in comfort by introducing a smooth cut-off function χ defined on $\mathcal{F}_0$ such that $\chi = 1$ in Γ_c and $\chi = 0$ in $\mathcal{F}_0 \setminus \Gamma_{2c}$. Let us denote

$$\psi_{\mathcal{S}}(t,x) := -\frac{1}{2}(\ell(t) \wedge x + \frac{1}{2} r(t)|x|^2) \text{ and } \tilde{u}_{\mathcal{S}} := \operatorname{curl}(\chi \psi_{\mathcal{S}}), \tag{89}$$

and observe that

$$R_1(t) = -\int (u - \tilde{u}_{\mathcal{S}}) \cdot [(u - \tilde{u}_{\mathcal{S}}) \cdot \nabla v_F] dx \text{ and } R_2(t) = -\int \tilde{u}_{\mathcal{S}} \cdot [(u - \tilde{u}_{\mathcal{S}}) \cdot \nabla v_F] dx,$$

since $\tilde{u}_{\mathcal{S}} = u_{\mathcal{S}}$ on the support of v_F for $\nu \leqslant 1$. Moreover, $\tilde{u}_{\mathcal{S}}$ is a H^1 divergence free vector field on $\mathcal{F}_0$ and, using (50), we have that

$$\|\tilde{u}_{\mathcal{S}}\|_{L^\infty([0,T];H^1(\mathcal{F}_0))} = O(1). \tag{90}$$

Regarding $R_1(t)$ we first integrate by parts to get

$$R_1(t) = \int v_F \cdot [(u - \tilde{u}_{\mathcal{S}}) \cdot \nabla (u - \tilde{u}_{\mathcal{S}})] dx.$$

Then we can use the equality (17) to obtain

$$R_1(t) = 2 \int v_F \cdot \Big(D(u - \tilde{u}_{\mathcal{S}})(u - \tilde{u}_{\mathcal{S}}) \Big) dx = 2 \int v_F \cdot \Big(D(u)(u - \tilde{u}_{\mathcal{S}}) \Big) dx,$$

since $\tilde{u}_{\mathcal{S}}$ is a rigid velocity on the support of v_F.

Then

$$R_1(t) = 2 \int d(x) v_F \cdot \Big(D(u) \tau \Big) dx,$$

where

$$\tau(t,x) := d(x)^{-1}(u(t,x) - \tilde{u}_{\mathcal{S}}(t,x)). \tag{91}$$

Since the vector field $u - \tilde{u}_{\mathcal{S}}$ is vanishing on $\partial \mathcal{S}_0$, according to Hardy's inequality we have, uniformly in t,

$$\|\tau\|_{L^2(\Gamma_{c\nu})} \leqslant C \|\nabla (u - \tilde{u}_{\mathcal{S}})\|_{L^2(\mathcal{F}_0)}. \tag{92}$$

Thus

$$|R_1(t)| \leqslant C\nu \|D(u)\|_{L^2(\Gamma_{c\nu})} \|\nabla (u - \tilde{u}_{\mathcal{S}})\|_{L^2(\mathcal{F}_0)},$$

thanks to the Cauchy-Schwarz inequality, (92) and (79). Using again the Cauchy-Schwarz inequality with respect to the time integration, that

$$\|\nabla (u - \tilde{u}_{\mathcal{S}})\|_{L^2(\mathcal{F}_0)} = 2 \|D(u - \tilde{u}_{\mathcal{S}})\|_{L^2(\mathcal{F}_0)} \tag{93}$$

according to the identity (16), (50), (90) and (70), we obtain

$$\int_0^t |R_1(t)| ds \to 0 \text{ when } \nu \to 0. \tag{94}$$

Similarly, we integrate by parts $R_2(t)$ to get

$$R_2(t) = \int v_F \cdot [(u - \tilde{u}_{\mathcal{S}}) \cdot \nabla \tilde{u}_{\mathcal{S}}] dx = \int v_F \cdot [r(t) \wedge (u - \tilde{u}_{\mathcal{S}})] dx,$$

using that, on the support of v_F, $\tilde{u}_{\mathcal{S}}$ is given by the formula (43). Then

$$\int_0^t R_2(s) ds = O(\nu^{1/2}), \tag{95}$$

thanks to (50) and (76).

It remains to deal with R_3 and R_4. Using the Cauchy-Schwarz inequality and that $u^E \in L^\infty((0,T); H^1(\mathcal{F}_0))$, we get

$$|\int_0^t R_3(s)ds| \leqslant C \int_0^t \nu \|D(u)(s,\cdot)\|_{L^2(\mathcal{F}_0)} ds \leqslant C t^{\frac{1}{2}} \nu \|D(u)\|_{L^2((0,t)\times\mathcal{F}_0)}$$

by using again the Cauchy-Schwarz inequality. Thanks to (50) we obtain

$$|\int_0^t R_3(s)ds| \leqslant C(t\nu)^{\frac{1}{2}}. \tag{96}$$

Regarding $R_4(t)$, we have, using (78), that

$$|\int_0^t R_4(s)ds| \leqslant C\nu^{\frac{1}{2}} \|D(u)\|_{L^2((0,t)\times\Gamma_{c\nu})} = o(1), \tag{97}$$

thanks to (70).

Finally, thanks to (50) and (76) we obtain

$$\int_0^t R_5(s)ds = O(\nu^{\frac{1}{2}}). \tag{98}$$

Gathering (94)-(98) we obtain (88) and the proof is over.

6 End of the proof of Theorem 6

In this section we explain how to modify the proof of the previous section in order to obtain that either (71) or (72) implies (69). Of course the idea is to use the weak formulations (51) and (53) instead of (44) and the energy inequalities (52) and (54) instead of (50). Thus we proceed as previously so that, in order to prove that (71) (respectively (72)) implies (69), it suffices to prove (88) where R now denotes the time-dependent function:

$$R = -b(u,u,v_F) - 2\nu \int_{\mathbb{R}^3} \operatorname{curl} u : \operatorname{curl}(u^E - v_F)dx) \text{ (respectively } R = -b(u,u,v_F) - \nu \int_{\mathbb{R}^3} \nabla u : \nabla(u^E - v_F)dx).$$

6.1 Proof of "(71) implies (69)"

In the first case, we decompose $R(t)$ into $R(t) = R_1(t) + R_2(t) + R_3(t)$, where

$$R_1 := -\int_{\mathcal{F}_0} u \cdot [(u-u_{\mathcal{S}})\cdot\nabla v_F]dx, \quad R_2 := -2\nu \int_{\mathbb{R}^3} \operatorname{curl} u : \operatorname{curl}(u^E - v_F)dx \text{ and } R_3 := \int_{\mathcal{F}_0} \det(r_u, u, v_F).$$

We start as in the previous section by substituting the function $\tilde{u}_{\mathcal{S}}$ defined in (89) to $u_{\mathcal{S}}$ and then we integrate by parts to obtain:

$$R_1 = \int_{\mathcal{F}_0} v_F \cdot [(u-\tilde{u}_{\mathcal{S}})\cdot\nabla u]dx = \int_{\mathcal{F}_0} v_F \cdot [(u-\tilde{u}_{\mathcal{S}})\cdot\nabla(u-\tilde{u}_{\mathcal{S}})]dx + \int_{\mathcal{F}_0} v_F \cdot [(u-\tilde{u}_{\mathcal{S}})\cdot\nabla\tilde{u}_{\mathcal{S}}]dx.$$

Then, using (15) and that, on the support of v_F, $\tilde{u}_{\mathcal{S}}$ is given by the formula (43), we get

$$R_1 = 2\int_{\mathcal{F}_0} v_F \cdot [\big(\operatorname{curl}(u-\tilde{u}_{\mathcal{S}})\big)(u-\tilde{u}_{\mathcal{S}})]dx + \int_{\mathcal{F}_0} v_F \cdot [r \wedge (u-\tilde{u}_{\mathcal{S}})]dx = 2\int_{\mathcal{F}_0} d(x) v_F \cdot [\big(\operatorname{curl} u\big)\tau]dx,$$

where τ is the function defined in (91).

Thus, using the Cauchy-Schwarz inequality, (92) and (79), we obtain:

$$|R_1(t)| \leqslant C\nu \|\operatorname{curl} u\|_{L^2(\Gamma_{c\nu})} \|\nabla(u-\tilde{u}_{\mathcal{S}})\|_{L^2(\mathcal{F}_0)},$$

Using again the Cauchy-Schwarz inequality, the identity (14), (52), (90) and (71), we obtain

$$\int_0^t |R_1(t)| ds \to 0 \text{ when } \nu \to 0.$$

Regarding R_2 we first use (80) and that v_F is supported in $\Gamma_{c\nu}$ to decompose R_2 into:

$$-R_2 = 2\nu \int_{\mathcal{F}_0} \operatorname{curl} u : \operatorname{curl} u^E dx + 2\nu \int_{\mathcal{S}_0} \operatorname{curl} u : \operatorname{curl} u^E dx - 2\nu \int_{\Gamma_{c\nu}} \operatorname{curl} u : \operatorname{curl} v_F dx =: R_{2,a} + R_{2,b} + R_{2,c}.$$

Then, using the Cauchy-Schwarz inequality and that the restrictions of u^E to $\mathcal{F}_0$ and to $\mathcal{S}_0$ are H^1, uniformly in t, we get

$$|\int_0^t R_{2,a}(s) ds| + |\int_0^t R_{2,b}(s) ds| \leqslant C t^{\frac{1}{2}} \nu \| \operatorname{curl} u \|_{L^2((0,t)\times\mathbb{R}^3)} \leqslant C(t\nu)^{\frac{1}{2}}$$

thanks to (52).

Regarding $R_{2,c}(t)$, we have, using (78), that

$$|\int_0^t R_{2,c}(s) ds| \leqslant C \nu^{\frac{1}{2}} \| \operatorname{curl} u \|_{L^2((0,t)\times\Gamma_{c\nu})} = o(1),$$

thanks to (71).

Finally the term R_3 can be bounded as in the previous section (where it was denoted by R_5).

We therefore obtain (88) in the first case and therefore that (71) implies (69).

6.2 Proof of "(72) implies (69)"

In the second case, we decompose $R(t)$ into $R(t) = R_1(t) + R_2(t) + R_3(t)$, where

$$R_1 := -\int_{\mathcal{F}_0} u \cdot [(u - u_{\mathcal{S}}) \cdot \nabla v_F] dx, \quad R_2 := -\nu \int_{\mathbb{R}^3} \nabla u : \nabla(u^E - v_F) dx \text{ and } R_3 := \int_{\mathcal{F}_0} \det(r_u, u, v_F).$$

Using an integration by parts, we obtain:

$$R_1(t) = -\int u \cdot [(u - \tilde{u}_{\mathcal{S}}) \cdot \nabla v_F] dx = \int v_F \cdot [(u - \tilde{u}_{\mathcal{S}}) \cdot \nabla u] dx = \int d(x) v_F \cdot [\tau \cdot \nabla u] dx,$$

so that using Hardy's and Cauchy-Schwarz's inequalities, (92), (79), (54), (90) and (72), we obtain

$$\int_0^t |R_1(t)| ds \to 0 \text{ when } \nu \to 0.$$

Regarding R_2 we proceed as previously decomposing R_2 into:

$$-R_2 = \nu \int_{\mathcal{F}_0} \nabla u : \nabla u^E dx + \nu \int_{\mathcal{S}_0} \nabla u : \nabla u^E dx - \nu \int_{\Gamma_{c\nu}} \nabla u : \nabla v_F dx =: R_{2,a} + R_{2,b} + R_{2,c},$$

so that

$$|\int_0^t R_{2,a}(s) ds| + |\int_0^t R_{2,b}(s) ds| \leqslant C t^{\frac{1}{2}} \nu \|\nabla u\|_{L^2((0,t)\times\mathbb{R}^3)} \leqslant C(t\nu)^{\frac{1}{2}},$$

thanks to (54), and that

$$|\int_0^t R_{2,c}(s) ds| \leqslant C \nu^{\frac{1}{2}} \|\nabla u\|_{L^2((0,t)\times\Gamma_{c\nu})} = o(1),$$

thanks to (78) and (72).

Finally the term R_3 is treated as previously.

We thus obtain (88) in the second case and therefore that (72) implies (69).

7 Appendix. Proof of Proposition 1

First observe that the result of Proposition 1 will follow, by an integration by parts in time, from the following claim: for any $v \in \mathcal{H} \cap C_c^\infty(\mathbb{R}^3)$, for any $t \in [0,T]$,

$$(\partial_t u, v)_{\mathcal{H}} = b(u,u,v) - 2\nu \int_{\mathcal{F}_0} D(u) : D(v) dx + f_t[u,v]. \tag{99}$$

Then we multiply the equation (36) by v and integrate over $\mathcal{F}_0$:

$$\int_{\mathcal{F}_0} \frac{\partial u}{\partial t} \cdot v + \int_{\mathcal{F}_0} [\Big((u - u_{\mathcal{S}}) \cdot \nabla\Big) u] \cdot v + \int_{\mathcal{F}_0} (r(t) \wedge u) \cdot v + \int_{\mathcal{F}_0} \nabla p \cdot v = \int_{\mathcal{F}_0} \nu \Delta u \cdot v + \int_{\mathcal{F}_0} Q(t)^T g \cdot v.$$

We then use some integrations by parts, taking into account (37) and (38), to get

$$\begin{aligned}
\int_{\mathcal{F}_0} [\Big((u - u_{\mathcal{S}}) \cdot \nabla\Big) u] \cdot v &= -\int_{\mathcal{F}_0} u \cdot \Big((u - u_{\mathcal{S}}) \cdot \nabla\Big) v, \\
\int_{\mathcal{F}_0} (r(t) \wedge u) \cdot v &= \int_{\mathcal{F}_0} \det(r, u, v), \\
\int_{\mathcal{F}_0} \nabla p \cdot v &= \int_{\partial \mathcal{S}_0} pn \cdot v, \\
\int_{\mathcal{F}_0} \nu \Delta u \cdot v &= 2\nu \int_{\partial \mathcal{S}_0} (D(u)v) \cdot n - 2\nu \int_{\mathcal{F}_0} D(u) : D(v), \\
&= 2\nu \int_{\partial \mathcal{S}_0} (D(u)n) \cdot v - 2\nu \int_{\mathcal{F}_0} D(u) : D(v),
\end{aligned}$$

since $D(u)$ is symmetric. Then we observe that

$$\begin{aligned}
\int_{\partial \mathcal{S}_0} pn \cdot v - 2\nu \int_{\partial \mathcal{S}_0} (D(u)n) \cdot v &= -\ell_v \cdot \int_{\partial \mathcal{S}_0} \sigma n \, ds - r_v \cdot \int_{\partial \mathcal{S}_0} x \wedge \sigma n \, ds \\
&= m\ell_v \cdot \ell' + \mathcal{J}_0 r_v \cdot r' - \det(ml, r, \ell_v) - \det(\mathcal{J}_0 r, r, r_v) - m\ell_v \cdot Q(t)^T g,
\end{aligned}$$

thanks to (39)-(40).

Finally we have the following simplification of the gravity contribution, what corresponds to the Archimedes' principle,

$$\begin{aligned}
\int_{\mathcal{F}_0} Q(t)^T g \cdot v &= \int_{\mathbb{R}^3} Q(t)^T g \cdot v - \int_{\mathcal{S}_0} Q(t)^T g \cdot v \\
&= \int_{\mathbb{R}^3} \nabla(Q(t)^T g \cdot x) \cdot v - \int_{\mathcal{S}_0} Q(t)^T g \cdot v \\
&= -\int_{\mathcal{S}_0} Q(t)^T g \cdot v,
\end{aligned}$$

since v is divergence free. Moreover in $\mathcal{S}_0$, $v = \ell_v + r_v \wedge x$ so that

$$\int_{\mathcal{F}_0} Q(t)^T g \cdot v = -Vol(\mathcal{S}_0) Q(t)^T g \cdot (\ell_v + r_v \wedge x_0),$$

by definition of x_0.

Gathering all these equalities yields (99).

8 Appendix. Proof of Theorem 5

This section is devoted to the proof of Theorem 5. We will show the existence part by applying the Schauder fixed point Theorem and the uniqueness part by an energy estimate. To simplify the notation we will drop the index E.

8.1 Existence

We are going to use the vorticity formulation of the problem. We denote by $\omega := \nabla \wedge u$ the vorticity vector in $\mathbb{R}^3$ associated to u. From (55)-(56)-(62) we infer that the vorticity ω satisfies the equation:

$$\partial_t \omega + (u - u_{\mathcal{S}}) \cdot \nabla \omega = (\omega \cdot \nabla)(u - u_{\mathcal{S}}). \tag{100}$$

At least formally this means that the vorticity is transported with stretching, according to the formula:

$$\omega(t,x) = (\nabla \eta)(t, \eta^{-1}(t,x)) \cdot \omega_0(\eta^{-1}(t,x)),$$

where η is the flow associated to $u - u_{\mathcal{S}}$, that is the solution of

$$\partial_t \eta(t,x) = (u - u_{\mathcal{S}})(t, \eta(t,x)) \ \text{ and } \ \eta(0,x) = x \ \text{ for } (t,x) \in [0,T] \times \mathcal{F}_0, \tag{101}$$

and $\eta^{-1}(t,x)$ denotes the inverse at time t of the diffeomorphism $x \mapsto \eta(t,x)$.

On the other hand we are also going to reformulate the solid equation as follows. Let us introduce the functions Φ_i, usually referred to as the Kirchhoff potentials, as the solutions of the following problems:

$$-\Delta \Phi_i = 0 \quad \text{for } x \in \mathcal{F}_0, \tag{102}$$

$$\Phi_i \to 0 \quad \text{for } x \to \infty, \tag{103}$$

$$\frac{\partial \Phi_i}{\partial n} = K_i \quad \text{for } x \in \partial \mathcal{S}_0, \tag{104}$$

where

$$K_i := \begin{cases} n_i & \text{if } i = 1,2,3, \\ [x \wedge n]_{i-3} & \text{if } i = 4,5,6. \end{cases} \tag{105}$$

Thanks to some integrations by parts the equations (58)-(59) can now be recast as:

$$\mathcal{M}_1 \begin{bmatrix} \ell \\ \\ r \end{bmatrix}' = \begin{bmatrix} \int_{\partial \mathcal{S}_0} \big((Q^T g) \cdot x\big) n \, ds \\ \\ \int_{\partial \mathcal{S}_0} \big((Q^T g) \cdot x\big) x \wedge n \, ds \end{bmatrix} + \left(\int_{\mathcal{F}_0} \nabla \tilde{p} \cdot \nabla \Phi_i \, dx \right)_{i \in \{1,\dots,6\}} + \begin{bmatrix} mQ^T g + (m\ell) \wedge \ell \\ \\ (\mathcal{J}_0 r) \wedge r \end{bmatrix}, \tag{106}$$

where

$$\tilde{p} := p - (Q^T g) \cdot x$$

and where the 6×6 matrix $\mathcal{M}_1$ is given by

$$\mathcal{M}_1 := \begin{bmatrix} m \, \mathrm{Id}_3 & 0 \\ 0 & \mathcal{J}_0 \end{bmatrix}.$$

Let us observe that the first term in the right hand side of (106) can be written

$$\begin{bmatrix} \int_{\partial \mathcal{S}_0} \big((Q^T g) \cdot x\big) n \, ds \\ \\ \int_{\partial \mathcal{S}_0} \big((Q^T g) \cdot x\big) x \wedge n \, ds \end{bmatrix} = \begin{bmatrix} -Vol(\mathcal{S}_0) Q^T g \\ \\ -Vol(\mathcal{S}_0)(Q^T g) \wedge x_0 \end{bmatrix}. \tag{107}$$

Let us recall that x_0 denotes the centroïd of the solid, cf. Proposition 1. We now use (55) to get

$$\int_{\mathcal{F}_0} \nabla \tilde{p} \cdot \nabla \Phi_i \, dx = - \int_{\mathcal{F}_0} \partial_t u \cdot \nabla \Phi_i \, dx - \int_{\mathcal{F}_0} \big((u - u_{\mathcal{S}}) \cdot \nabla u\big) \cdot \nabla \Phi_i \, dx - \int_{\mathcal{F}_0} \big(r \wedge u\big) \cdot \nabla \Phi_i \, dx.$$

Integrating by parts and using (56), (57) and (62) we get

$$\left(\int_{\mathcal{F}_0} \partial_t u \cdot \nabla \Phi_i \, dx \right)_{i \in \{1,\dots,6\}} = \mathcal{M}_2 \begin{bmatrix} \ell \\ \\ r \end{bmatrix}', \quad - \int_{\mathcal{F}_0} \big((u - u_{\mathcal{S}}) \cdot \nabla u\big) \cdot \nabla \Phi_i \, dx = \int_{\mathcal{F}_0} \big((u - u_{\mathcal{S}}) \cdot \nabla \nabla \Phi_i\big) \cdot u \, dx,$$

where

$$\mathcal{M}_2 := \left[\int_{\mathcal{F}_0} \nabla \Phi_i \cdot \nabla \Phi_j \; dx \right]_{i,j \in \{1,\dots,6\}}$$

is usually referred to as the added mass of the solid. It is now elementary to see that the equation of the solid reads

$$\mathcal{M} \begin{bmatrix} \ell \\ \\ r \end{bmatrix}' = \begin{bmatrix} m_a Q^T g \\ \\ -Vol(\mathcal{S}_0)(Q^T g) \wedge x_0 \end{bmatrix} + \int_{\mathcal{F}_0} \big((u - u_{\mathcal{S}}) \cdot \nabla \nabla \Phi_i \big) \cdot u \, dx - \int_{\mathcal{F}_0} \big(r \wedge u \big) \cdot \nabla \Phi_i \, dx + \begin{bmatrix} (m\ell) \wedge \ell \\ \\ (\mathcal{J}_0 r) \wedge r \end{bmatrix}, \quad (108)$$

where m_a is the apparent mass introduced in Proposition 1 and

$$\mathcal{M} := \mathcal{M}_1 + \mathcal{M}_2.$$

Let us now introduce an operator, whose fixed points give local in time solutions to the system. We denote

$$\overline{\rho} := \min\{\rho > 0 \; / \; \mathrm{Supp}\,(w_0) \subset \overline{B}(0,\rho)\}.$$

For $T > 0$, we let

$$\begin{aligned} \mathcal{C} := \Big\{ (\omega, \ell, r) \in C^0([0,T]; C^{0,\lambda}(\mathcal{F}_0; \mathbb{R}^3)) \times W^{1,1}([0,T]; \mathbb{R}^6) \; / \\ i. \; \|\omega\|_{L^\infty(0,T; C^{0,\lambda}(\mathcal{F}_0))} \leqslant 4 \|w_0\|_{C^{0,\lambda}(\mathcal{F}_0)}, \\ ii. \; \mathrm{Supp}\,(\omega(t)) \subset \overline{B}(0, \overline{\rho} + 1), \\ iii. \; \partial_t \omega \in L^1(0,T; C^{-1,\lambda}(\mathcal{F}_0))) \;\; \text{and} \;\; \|\partial_t \omega\|_{L^1(0,T; C^{-1,\lambda}(\mathcal{F}_0)))} \leqslant 1, \\ iv. \; \|\ell - \ell_0\|_{W^{1,1}(0,T)} \leqslant 1, \; \|r - r_0\|_{W^{1,1}(0,T)} \leqslant 1 \Big\}. \end{aligned}$$

Now we define, for T small enough, the operator $\mathcal{V} : \mathcal{C} \to \mathcal{C}$, mapping (ω, ℓ, r) to $(\tilde{\omega}, \tilde{\ell}, \tilde{r})$ as follows.

Given $(\omega, \ell, r) \in \mathcal{C}$, we introduce u as the unique solution in $L^\infty(0,T; C^{1,\lambda}(\mathcal{F}_0))$ of the following system

$$\begin{cases} \mathrm{curl}\, u = \omega \text{ in } [0,T] \times \mathcal{F}_0, \\ \mathrm{div}\, u = 0 \text{ in } [0,T] \times \mathcal{F}_0, \\ u \cdot n = u_{\mathcal{S}} \cdot n \text{ on } [0,T] \times \partial \mathcal{S}_0, \\ u \to 0 \quad \text{for } x \to \infty, \end{cases} \quad (109)$$

with $u_{\mathcal{S}}(t,x) := \ell(t) + r(t) \wedge x$. Moreover there exists a constant $C > 0$, depending only on $\mathcal{S}_0$ and on $\overline{\rho}$, such that

$$\|u\|_{L^\infty(0,T; C^{1,\lambda}(\mathcal{F}_0))} \leqslant C(\|\omega\|_{L^\infty(0,T; C^{0,\lambda}(\mathcal{F}_0))} + \|\ell\|_{L^\infty(0,T)} + \|r\|_{L^\infty(0,T)}). \quad (110)$$

This follows easily from classical elliptic theory, see e.g. [11].

As a consequence there exists a unique flow η in $L^\infty(0,T; C^{1,\lambda}(\mathcal{F}_0))$ associated to $u - u_{\mathcal{S}}$, that is η solution of (101). Moreover $\eta^{-1}(t,x)$ satisfies

$$\eta^{-1}(t,x) = x - \int_0^t (u - u_{\mathcal{S}})(s, \eta^{-1}(s,x)) ds,$$

so that

$$|\eta^{-1}(t,x) - x| \leqslant 1, \quad (111)$$

for $t \in [0,T]$ and T small enough. Then we defined

$$\tilde{\omega}(t,x) := (\nabla \eta)(t, \eta^{-1}(t,x)) \cdot \omega_0(\eta^{-1}(t,x)).$$

Thanks to (111) we obtain that

$$\mathrm{Supp}\,(\tilde{\omega}(t)) \subset \overline{B}(0, \overline{\rho} + 1). \quad (112)$$

Moreover $\tilde{\omega}$ satisfies

$$\partial_t \tilde{\omega} + (u - u_{\mathcal{S}}) \cdot \nabla \tilde{\omega} = (\tilde{\omega} \cdot \nabla)(u - u_{\mathcal{S}}). \tag{113}$$

One infers, see e.g. [10, Corollary 2.4], that there exists a constant $c > 0$, depending only on λ, such that

$$\begin{aligned} \|\tilde{\omega}\|_{L^\infty(0,T;C^{0,\lambda}(\mathcal{F}_0))} &\leqslant \|\omega_0\|_{C^{0,\lambda}(\mathcal{F}_0)} \exp\Big(c \int_0^t (\|u\|_{C^{1,\lambda}(\mathcal{F}_0))}(s) + |r|(s)) ds\Big) \\ &\leqslant 4\|w_0\|_{C^{0,\lambda}(\mathcal{F}_0)}, \end{aligned} \tag{114}$$

for T small enough.

Using that (ω, ℓ, r) is in $\mathcal{C}$, (110), (112) and (114), and that

$$(u - u_{\mathcal{S}}) \cdot \nabla \tilde{\omega} = \operatorname{div}\Big(\tilde{\omega}(u - u_{\mathcal{S}})\Big),$$

since $u - u_{\mathcal{S}}$ is divergence free, we obtain that $\partial_t \tilde{\omega} \in L^1(0,T;C^{-1,\lambda}(\mathcal{F}_0))$ with $\|\partial_t \tilde{\omega}\|_{L^1(0,T;C^{-1,\lambda}(\mathcal{F}_0))} \leqslant 1$ for T small enough.

Next we can define $(\tilde{\ell}, \tilde{r})$ by

$$\mathcal{M} \begin{bmatrix} \tilde{\ell} \\ \\ \tilde{r} \end{bmatrix} = \mathcal{M} \begin{bmatrix} \ell_0 \\ \\ r_0 \end{bmatrix} + \int_0^t \Big(\begin{bmatrix} m_a Q^T g \\ \\ -Vol(\mathcal{S}_0)(Q^T g) \wedge x_0 \end{bmatrix} + \int_{\mathcal{F}_0} \big((u - u_{\mathcal{S}}) \cdot \nabla\nabla\Phi_i\big) \cdot u \, dx - \int_{\mathcal{F}_0} \big(r \wedge u\big) \cdot \nabla\Phi_i \, dx\Big) \, ds, \tag{115}$$

and we verify easily that $\|\tilde{\ell} - \ell_0\|_{W^{1,1}(0,T)} \leqslant 1$ and $\|\tilde{r} - r_0\|_{W^{1,1}(0,T)} \leqslant 1$ for T small enough, so that the operator $\mathcal{V}$ defined by $\mathcal{V}(\omega, \ell, r) = (\tilde{\omega}, \tilde{\ell}, \tilde{r})$ maps $\mathcal{C}$ into $\mathcal{C}$.

Comparing (100) to (113) and (108) to (115) one sees that fixed points give local in time solutions to the system.

Let us now prove that for $T > 0$ suitably small, $\mathcal{V}$ admits a fixed point in $\mathcal{C}$. We endow $\mathcal{C}$ with the $L^\infty(0,T;C^{0,\tilde{\lambda}}(\mathcal{F}_0) \times \mathbb{R}^6)$ topology for some $\tilde{\lambda} \in (0, \lambda)$ and use Schauder's fixed point theorem. It is clear that $\mathcal{C}$ is a closed convex subset of $L^\infty(0,T;C^{0,\tilde{\lambda}}(\mathcal{F}_0) \times \mathbb{R}^6)$. Hence it remains to prove that $\mathcal{V}(\mathcal{C})$ is relatively compact and that $\mathcal{V}$ is continuous.

In order to prove that $\mathcal{V}(\mathcal{C})$ is relatively compact it is sufficient to prove that $\mathcal{C}$ is compact. The compactness for (ℓ, r) follows from Ascoli's theorem and the compactness for ω follows from the Aubin-Lions lemma.

We finally prove the continuity of $\mathcal{V}$. Suppose that (ω_n, ℓ_n, r_n) converges to (ω, ℓ, r) in $L^\infty(0,T;C^{0,\tilde{\lambda}}(\mathcal{F}_0) \times \mathbb{R}^6)$ and let us prove that $(\tilde{\omega}_n, \tilde{\ell}_n, \tilde{r}_n) := \mathcal{V}(\omega_n, \ell_n, r_n)$ converges to $(\tilde{\omega}, \tilde{\ell}, \tilde{r}) := \mathcal{V}(\omega, \ell, r)$ in $L^\infty(0,T;C^{0,\tilde{\lambda}}(\mathcal{F}_0) \times \mathbb{R}^6)$. First using again elliptic regularity as in (109)-(110) we infer that the fluid velocity u_n associated to (ω_n, ℓ_n, r_n) converges in $L^\infty(0,T;C^{1,\tilde{\lambda}}(\mathcal{F}_0))$ to the fluid velocity u associated to (ω, ℓ, r). Then we can deduce that the corresponding flow η_n converges in $L^\infty(0,T;C^{1,\tilde{\lambda}}(\mathcal{F}_0))$ to the limit fluid flow η, and next, that $\tilde{\omega}_n$ converges in $L^\infty(0,T;C^{0,\tilde{\lambda}}(\mathcal{F}_0))$ to $\tilde{\omega}$. Finally we deduce from the definition (115) that (ℓ_n, r_n) converges to $(\tilde{\ell}, \tilde{r})$ in $L^\infty(0,T;\mathbb{R}^6)$.

This achieves the existence part of the proof.

The decay at infinity of the fluid velocity and of its gradient follows from the fact that the vorticity is compactly supported uniformly in time and that the fluid velocity u can be recovered from the vorticity and the solid vorticity by solving the elliptic problem (109).

8.2 Uniqueness

Let us assume that u and $\tilde{u}$ are two solutions of (55)-(61). Then using, for any $t \in [0,T]$, (65) with u instead of u^E (respectively with and $\tilde{u}$ instead of u^E) and $u - \tilde{u}$ instead of v, and substracting the resulting identities, we obtain

$$(\partial_t(u - \tilde{u}), u - \tilde{u})_{\mathcal{H}} = -b(u, u - \tilde{u}, u) + b(\tilde{u}, u - \tilde{u}, \tilde{u}) + f_t[u, u - \tilde{u}] - f_t[\tilde{u}, u - \tilde{u}],$$

that is, using that b is trilinear and property (67),

$$\frac{1}{2}\partial_t \|u - \tilde{u}\|_{\mathcal{H}}^2 = -b(u - \tilde{u}, u - \tilde{u}, u) + f_t[u, u - \tilde{u}] - f_t[\tilde{u}, u - \tilde{u}], \tag{116}$$

Using (46) with $v = u - \tilde{u}$ and (66) with $(u - \tilde{u}, u - \tilde{u}, u)$ instead of (u, v, w), we get there exists a constant $C > 0$ such that for any $t \in [0, T]$,

$$|f_t[u, u - \tilde{u}] - f_t[\tilde{u}, u - \tilde{u}]| \leqslant C \sup_{0 \leqslant s \leqslant T} \|u - \tilde{u}\|^2_{\mathcal{H}}(s) \text{ and } |b(u - \tilde{u}, u - \tilde{u}, u)| \leqslant C \|u - \tilde{u}\|^2_{\mathcal{H}}.$$

Thus a Gronwall lemma yields that $u = \tilde{u}$ on $[0, T]$.

Remark 3. *After completion of this paper we learned that the paper [34] establishes a variant of Theorem 5 with a different method.*

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