THE DIFFUSION APPROXIMATION FOR THE LINEAR BOLTZMANN EQUATION WITH VANISHING ABSORPTION

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Abstract. The present paper discusses the diffusion approximation of the linear Boltzmann equation in cases where the collision frequency is not uniformly large in the spatial domain. Our result applies for instance to the case of radiative transfer in a composite medium with optically thin inclusions in an optically thick background medium. The equation governing the evolution of the approximate particle density coincides with the limit of the diffusion equation with infinite diffusion coefficient in the optically thin inclusions.

1. Presentation of the Problem

Consider the linear Boltzmann equation

$$(\partial_t + v\cdot\nabla_x)f(t,x,v) + \mathcal{L}_x f(t,x,v) = 0 \tag{1}$$

for the unknown $f \equiv f(t,x,v)$ that is the distribution function for a system of identical point particles interacting with some background material. In other words, $f(t,x,v)$ is the number density of particles located at the position $x\in\Omega$, where Ω is a domain of $\mathbf{R}^N$, with velocity $v\subset\mathbf{R}^N$ at time $t\ge 0$.

The notation $\mathcal{L}_x$ designates a linear integral operator acting on the v variable in f, i.e.

$$\mathcal{L}_x f(t,x,v) = \int_{\mathbf{R}^N} k(x,v,w)(f(t,x,v)-f(t,x,w))d\mu(w) \tag{2}$$

where μ is a Borel probability measure on $\mathbf{R}^N$, while k is a nonnegative function defined $\mu\otimes\mu$-a.e. on $\mathbf{R}^N\times\mathbf{R}^N$ that measures the probability of a transition from velocity v to velocity w for particles located at the position x.

Henceforth we denote

$$\langle\phi\rangle = \int_{\mathbf{R}^N}\phi(v)d\mu(v) \quad\text{and}\quad \langle\!\langle\Phi\rangle\!\rangle = \iint_{\mathbf{R}^N\times\mathbf{R}^N}\Phi(v,w)d\mu(v)d\mu(w) \tag{3}$$

for all $\phi\in L^1(\mathbf{R}^N,d\mu)$ and $\Phi\in L^1(\mathbf{R}^N_v\times\mathbf{R}^N_w;d\mu(v)d\mu(w))$.

We assume that k satisfies the semi-detailed balance condition

$$\int_{\mathbf{R}^N} k(x,v,w)d\mu(w) = \int_{\mathbf{R}^N} k(x,w,v)d\mu(w) \tag{4}$$

and introduce the notation

$$a(x,v) := \int_{\mathbf{R}^N} k(x,v,w)d\mu(w) \tag{5}$$

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for the absorption rate, so that

$$\mathcal{L}_x f(t,x,v) = a(x,v)f(t,x,v) - \mathcal{K}_x f(t,x,v)$$

where $\mathcal{K}_x$ designates the integral operator

$$\mathcal{K}_x f(t,x,v) := \int_{\mathbf{R}^N} k(x,v,w)f(t,x,w)d\mu(w)\,. \tag{6}$$

The semi-detailed balance assumption appears for instance in [11] — see formula (2.9) in §2.

The assumptions on the transition kernel other than (4) used in our discussion are introduced later.

We further assume that Ω is a bounded domain of $\mathbf{R}^N$ with C^1 boundary $\partial\Omega$ and denote by n_x the unit outward normal field at $x \in \partial\Omega$. Let

$$\begin{aligned}
\Gamma_+ &:= \{(x,v) \in \partial\Omega \times \mathbf{R}^N \,|\, v \cdot n_x > 0\}\,,\\
\Gamma_0 &:= \{(x,v) \in \partial\Omega \times \mathbf{R}^N \,|\, v \cdot n_x = 0\}\,,\\
\Gamma_- &:= \{(x,v) \in \partial\Omega \times \mathbf{R}^N \,|\, v \cdot n_x < 0\}\,.
\end{aligned}$$

The linear Boltzmann equation is supplemented with the absorption boundary condition

$$f(t,x,v) = 0\,, \quad (x,v) \in \Gamma_-\,,\ t > 0\,. \tag{7}$$

(In other words, it is assumed that there are no particles entering the domain Ω.) This choice is made for the sake of simplicity; other boundary conditions will be discussed later.

We are concerned with the diffusion approximation of the linear Boltzmann equation (1) — see [7] and the references therein for a general presentation of this approximation. We briefly recall its main features below.

Set L to be a length scale that measures the size of Ω while V is the average particle velocity; this defines a time scale $T := L/V$. The diffusion limit of (1) is based on the assumption that the dimensionless quantity $Ta(x,v)$ is large. Thus we introduce a scaling parameter $0 < \epsilon \ll 1$ and set

$$\hat{k}_\epsilon(x,v,w) := \epsilon k(x,v,w)$$

so that $\hat{k}_\epsilon(x,v,w)$ is of order unity. Accordingly, we define

$$\hat{a}_\epsilon(x,v) := \epsilon a_\epsilon(x,v)\,, \quad \hat{\mathcal{L}}_x = \epsilon\mathcal{L}_x\,, \quad \text{and } \hat{\mathcal{K}}_x = \epsilon\mathcal{K}_x\,.$$

(For notational simplicity, we do not mention explicitly the dependence of $\mathcal{L}_x$ and $\mathcal{K}_x$ in ϵ.) Assume further that variations of order unity of the boundary data driving the solution of (1) do not occur on time scales shorter than T/ϵ.

In that case, the solution f of (1) is sought in the form

$$f(t,x,v) = \hat{f}_\epsilon(\epsilon t,x,v)$$

with the notation $\hat{t} = \epsilon t$ for the rescaled time variable. Thus (1) takes the form

$$\epsilon\partial_{\hat{t}} f_\epsilon(\hat{t},x,v) + v\cdot\nabla_x f_\epsilon(\hat{t},x,v) + \frac{1}{\epsilon}\hat{\mathcal{L}}_x f_\epsilon(\hat{t},x,v) = 0\,.$$

Henceforth we drop hats on rescaled variables and consider the initial-boundary value problem for the scaled linear Boltzmann equation

$$(8)\quad \begin{cases} (\epsilon\partial_t + v\cdot\nabla_x)f_\epsilon(t,x,v) + \dfrac{1}{\epsilon}\mathcal{L}_x f_\epsilon(t,x,v) = 0\,, & x\in\Omega\,,\ v\in\mathbf{R}^N\,,\ t>0\,,\\ f_\epsilon(t,x,v) = 0\,, & (x,v)\in\Gamma_-\,,\ t>0\,,\\ f_\epsilon(0,x,v) = f^{in}(x,v) & x\in\Omega\,,\ v\in\mathbf{R}^N\,, \end{cases}$$

in the limit as $\epsilon \to 0$.

2. Existence, uniqueness and a priori estimates

Assume that k is a nonnegative measurable function on $\mathbf{R}^N\times\mathbf{R}^N$ satisfying the semi-detailed balance assumption (4) and the condition

$$(9)\qquad a_\epsilon \in L^\infty(\Omega\times\mathbf{R}^N, dxd\mu)\,.$$

Lemma 2.1. *Under assumption (9)*

a) the integral operators $\mathcal{L}_x$ and $\mathcal{K}_x$ are bounded on $L^2(\mathbf{R}^N,\mu)$ for a.e. $x\in\Omega$, with

$$\|\mathcal{K}_x\|_{\mathcal{L}(L^2(\mathbf{R}^N,\mu))} \le \|a_\epsilon(x,\cdot)\|_{L^\infty(\mathbf{R}^N,\mu)}\,;$$

b) the adjoints of $\mathcal{K}_x$ and $\mathcal{L}_x$ are given by the formulas

$$\mathcal{K}_x^*\phi(v) = \int_{\mathbf{R}^N} k_\epsilon(x,w,v)\phi(w)d\mu(w)$$

and

$$\mathcal{L}_x^*\phi(v) = \int_{\mathbf{R}^N} k_\epsilon(x,w,v)(\phi(v)-\phi(w))d\mu(w)$$

for a.e. $x\in\Omega$;

c) for a.e. $x\in\Omega$

$$\{\ \textit{functions a.e. constant on } \mathbf{R}^N\} = \mathbf{R} \subset \operatorname{Ker}(\mathcal{L}_x) \textit{ and } \operatorname{Ker}(\mathcal{L}_x^*)\,;$$

d) for each $\phi\in L^2(\mathbf{R}^N;d\mu)$,

$$\langle\phi\mathcal{L}_x\phi\rangle = \tfrac12\iint_{\mathbf{R}^N\times\mathbf{R}^N} k_\epsilon(x,v,w)(\phi(v)-\phi(w))^2 d\mu(v)d\mu(w)\,,$$

for a.e. $x\in\Omega$;

e) if in addition $k_\epsilon(x,v,w)>0$ for $d\mu(v)d\mu(w)$-a.e. $(v,w)\in\mathbf{R}^N\times\mathbf{R}^N$, then

$$\operatorname{Ker}(\mathcal{L}_x) = \operatorname{Ker}(\mathcal{L}_x^*) = \{\ \textit{functions a.e. constant on } \mathbf{R}^N\} = \mathbf{R}\,.$$

Remark. The discussion of the properties of the operator $\mathcal{L}_x$ differs from [3]. In [3], it is assumed that the measure μ is the uniform probability measure on the set V of admissible velocities, that can be a ball, or a sphere, or a spherical annulus centered at the origin in $\mathbf{R}^N$. The scattering kernel $k_\epsilon(x,v,w)$ is of the form

$$k_\epsilon(x,v,w) = \sigma_\epsilon(x)f(v,w)$$

where $f(v,w)=f(w,v)$ a.e. on $V\times V$ is positive and such that

$$\int_V f(v,w)dw = 1 \quad \text{for a.e. } v\in V\,.$$

Thus $\mathcal{L}_x = \sigma_\epsilon(x)(I-F)$, where F is the integral operator defined by

$$F\phi(v) := \int_V f(v,w)\phi(w)dw \quad \text{for a.e. } v\in V\,.$$

Then $\mathrm{Ker}(\mathcal{L}_x) = \mathrm{Ker}(I - F)$ whenever $\sigma_\epsilon(x) > 0$, and $\mathbf{R} \subset \mathrm{Ker}(I - F)$. On the other hand, it is assumed in [3] that f is chosen so that F is a compact operator on $L^2(V)$. Then 1 is the principal eigenvalue of F and $\mathrm{Ker}(I - F)$ is one-dimensional by the Krein-Rutman theorem, which implies that $\mathrm{Ker}(I - F) = \mathbf{R}$.

Proof. Statement a) follows from Schur's lemma (Lemma 18.1.12 in [10] or Lemma 1 in §2 of chapter XXI in [7]). The formula for $\mathcal{K}_x$ in statement b) and statement c) are obvious.

As for the formula for $\mathcal{L}_x^*$ in statement b), observe that

$$\begin{aligned}\mathcal{L}_x\phi(v) + \mathcal{K}_x\phi(v) &= \int_{\mathbf{R}^N} k_\epsilon(x, v, w)\phi(v)d\mu(w)\\ &= \int_{\mathbf{R}^N} k_\epsilon(x, w, v)\phi(v)d\mu(w) = a_\epsilon(x, v)\phi(v)\end{aligned}$$

for $dxd\mu$-a.e. $(x, v) \in \Omega \times \mathbf{R}^N$ by the semi-detailed balance assumption (4).

For each $\phi \in L^2(\mathbf{R}^N; d\mu)$ and a.e. in $x \in \Omega$

$$\begin{aligned}\langle \phi\mathcal{L}_x\phi\rangle &= \iint_{\mathbf{R}^N\times\mathbf{R}^N} k_\epsilon(x, v, w)(\phi(v)^2 - \phi(v)\phi(w))d\mu(v)d\mu(w)\\ &= \int_{\mathbf{R}^N} a_\epsilon(x, v)\phi(v)^2 d\mu(v) - \iint_{\mathbf{R}^N\times\mathbf{R}^N} k_\epsilon(x, v, w)\phi(v)\phi(w)d\mu(v)d\mu(w)\\ &= \tfrac{1}{2}\int_{\mathbf{R}^N} a_\epsilon(x, v)\phi(v)^2 d\mu(v) + \tfrac{1}{2}\int_{\mathbf{R}^N} a_\epsilon(x, w)\phi(w)^2 d\mu(v)\\ &- \iint_{\mathbf{R}^N\times\mathbf{R}^N} k_\epsilon(x, v, w)\phi(v)\phi(w)d\mu(v)d\mu(w)\\ &= \iint_{\mathbf{R}^N\times\mathbf{R}^N} k_\epsilon(x, v, w)\tfrac{1}{2}(\phi(v)^2 + \phi(w)^2)d\mu(v)d\mu(w)\\ &- \iint_{\mathbf{R}^N\times\mathbf{R}^N} k_\epsilon(x, v, w)\phi(v)\phi(w)d\mu(v)d\mu(w)\\ &= \tfrac{1}{2}\iint_{\mathbf{R}^N\times\mathbf{R}^N} k_\epsilon(x, v, w)(\phi(v) - \phi(w))^2 d\mu(v)d\mu(w)\end{aligned}$$

by Fubini's theorem and the semi-detailed balance assumption (4). This proves statement d).

By statement d), if $\phi \in L^2(\mathbf{R}^N; d\mu)$ satisfies $\mathcal{L}_x\phi = 0$, then

$$0 = \langle \phi\mathcal{L}_x\phi\rangle = \tfrac{1}{2}\iint_{\mathbf{R}^N\times\mathbf{R}^N} k_\epsilon(x, v, w)(\phi(v) - \phi(w))^2 d\mu(v)d\mu(w)\,.$$

Therefore

$$k_\epsilon(x, v, w)(\phi(v) - \phi(w)) = 0 \quad \text{for } d\mu(v)d\mu(w)- \text{ a.e. } (v, w) \in \mathbf{R}^N \times \mathbf{R}^N$$

so that

$$\phi(v) - \phi(w) = 0 \quad \text{for } d\mu(v)d\mu(w)- \text{ a.e. } (v, w) \in \mathbf{R}^N \times \mathbf{R}^N\,.$$

Averaging in w shows that

$$\phi(v) = \langle\phi\rangle \quad \text{for } d\mu(v)- \text{ a.e. } v \in \mathbf{R}^N\,,$$

so that

$$\mathrm{Ker}(\mathcal{L}_x) \subset \{ \text{ functions a.e. constant on } \mathbf{R}^N\} = \mathbf{R}\,.$$

With statement c), this shows that

$$\mathrm{Ker}(\mathcal{L}_x) = \{ \text{ functions a.e. constant on } \mathbf{R}^N \} = \mathbf{R}\,.$$

Since the function $(v, w) \mapsto k_\epsilon(x, w, v)$ satisfies the same properties as k_ϵ,

$$\mathrm{Ker}(\mathcal{L}_x) = \{ \text{ functions a.e. constant on } \mathbf{R}^N \} = \mathbf{R}\,.$$

□

Proposition 2.2. *Assume that k_ϵ is a nonnegative measurable function defined $dxd(\mu \otimes \mu)$-a.e. on $\Omega \times \mathbf{R}^N \times \mathbf{R}^N$ satisfying (4) and (9) with a_ϵ defined by (5). For each $\epsilon > 0$ and each $f^{in} \in L^2(\Omega \times \mathbf{R}^N; dxd\mu(v))$, there exists a unique weak solution of the initial-boundary value problem (8) in the space $C_b(\mathbf{R}_+; L^2(\Omega \times \mathbf{R}^N; dxd\mu(v)))$. This solution satisfies*

a) the continuity equation

$$\partial_t \langle f_\epsilon \rangle + \mathrm{div}_x \frac{1}{\epsilon} \langle v f_\epsilon \rangle = 0$$

in the sense of distributions on $\mathbf{R}_+^* \times \Omega$;

b) the "entropy inequality"

$$\int_\Omega \langle f_\epsilon(t, x, \cdot)^2 \rangle dx + \int_0^t \int_\Omega \langle\!\langle k_\epsilon(x, \cdot, \cdot) q_\epsilon(s, x, \cdot, \cdot)^2 \rangle\!\rangle dxds \le \int_\Omega \langle f^{in}(x, \cdot)^2 \rangle dx$$

for each $\epsilon > 0$ and each $t \ge 0$, where

$$q_\epsilon(t, x, v, w) = \frac{1}{\epsilon}(f_\epsilon(t, x, v) - f_\epsilon(t, x, w))\,.$$

Proof. The operator $\phi(x, v) \mapsto \mathcal{L}_x \phi(x, v)$ is a bounded perturbation of the advection operator $-v \cdot \nabla_x$ with absorbing boundary condition (7) that is the generator of a strongly continuous contraction semigroup on $L^2(\Omega \times \mathbf{R}^N; dxd\mu(v))$.

This implies the existence and uniqueness of the weak solution f_ϵ of the initial-boundary value problem (8) in the functional space $C_b(\mathbf{R}_+; L^2(\Omega \times \mathbf{R}^N; dxd\mu(v)))$.

Statement a) follows from the inclusion $\mathbf{R} \subset \mathrm{Ker}(\mathcal{L}_x^*)$ in Lemma 2.1. Statement b) follows from Lemma 2.1 d) and Lemma 2.3 below. □

Lemma 2.3. *Let $f^{in} \in L^2(\Omega \times \mathbf{R}^N; dxd\mu(v))$ and $S \in L^2([0, T] \times \Omega \times \mathbf{R}^N; dtdxd\mu(v))$. For each $\epsilon > 0$, let f_ϵ be the weak solution in $C_b(\mathbf{R}_+; L^2(\Omega \times \mathbf{R}^N; dxd\mu(v)))$ of*

$$\begin{cases} \epsilon \partial_t f_\epsilon + v \cdot \nabla_x f_\epsilon = S\,, \qquad x \in \Omega\,,\ v \in \mathbf{R}^N\,,\ t > 0\,, \\ f_\epsilon\big|_{\Gamma_-} = 0\,, \\ f_\epsilon\big|_{t=0} = f^{in}\,. \end{cases}$$

Then

$$\tfrac{1}{2} \int_\Omega \langle f(t, x, \cdot)^2 \rangle dx \le \frac{1}{\epsilon} \int_0^t \int_\Omega \langle S(s, x, \cdot) f_\epsilon(s, x, \cdot) \rangle dxds + \tfrac{1}{2} \int_\Omega \langle f^{in}(x, \cdot)^2 \rangle dx$$

The proof of this lemma is classical; we give it in the appendix for the sake of being self-contained.

3. Diffusion approximation with vanishing absorption rate: main results

Assume that the spatial domain $\Omega = A \cup B$, where A is open and B is closed in $\mathbf{R}^N$ (i.e. $B \cap \partial\Omega = \varnothing$), with finitely many connected components denoted B_l, for $l = 1, \ldots, m$. We further assume that B_l has piecewise C^1 boundary, that B_l is locally on one side of its boundary ∂B_l. Finally, we denote by n_x the unit normal field at $x \in \partial A$, oriented towards the exterior of A.

Henceforth, it is assumed that the measure μ satisfies

$$\mu(\{0\}) = 0\,. \tag{10}$$

We further assume that

$$\langle |v|^2 \rangle < \infty \quad \text{ and } \det(\langle v \otimes v \rangle) \neq 0\,. \tag{11}$$

For each $l = 1, \ldots, m$, denote by $\tau_l \equiv \tau_l(x, v)$ the forward exit time from B_l starting from the position x with the velocity v; in other words

$$\tau_l(x, v) := \inf\{t > 0 \text{ s.t. } x + tv \in \partial B_l\}\,. \tag{12}$$

We assume that, for each $l = 1, \ldots, m$ and for each $g \in L^2(\partial B_l)$,

$$\begin{aligned} &g(x + \tau_l(x, v)v) = g(x) \text{ for } d\sigma(x)d\mu(v) - \text{a.e. } (x, v) \in \partial B_l \times \mathbf{R}^N \\ &\qquad\qquad \Rightarrow g(x) = \frac{1}{|\partial B_l|} \int_{\partial B_l} g(y)d\sigma(y) \text{ for a.e. } x \in \partial B_l\,. \end{aligned} \tag{13}$$

We further assume that the scattering kernel k_ϵ in the linear Boltzmann equation is a $dxd\mu(v)d\mu(w)$-a.e. nonnegative measurable function on $\Omega \times \mathbf{R}^N \times \mathbf{R}^N$ satisfying the following assumptions, in addition to (4):

(a) the absorption rate a_ϵ is uniformly small on $B \times \mathbf{R}^N$ as $\epsilon \to 0$, i.e.

$$\|a_\epsilon\|_{L^\infty(B \times \mathbf{R}^N, dxd\mu)} \to 0 \quad \text{ as } \epsilon \to 0\,; \tag{14}$$

(b) the restriction of k_ϵ to $A \times \mathbf{R}^N \times \mathbf{R}^N$ is assumed to be independent of ϵ and denoted $k_A \equiv k_A(x, v, w)$; it satisfies

$$C_K := \operatorname*{supess}_{(x,v) \in A \times \mathbf{R}^N} \int_{\mathbf{R}^N} \left(k_A(x, v, w) + \frac{1}{k_A(x, v, w)} \right) d\mu(w) < \infty\,; \tag{15}$$

we henceforth denote

$$a_A(x, v) := \int_{\mathbf{R}^N} k_A(x, v, w)d\mu(w)\,, \quad \text{ for } dxd\mu(v) - \text{a.e. } (x, v) \in A \times \mathbf{R}^N\,; \tag{16}$$

(c) there exists a $\mathbf{R}^N$-valued vector field $b \equiv b(x, v)$ defined $dxd\mu$-a.e. on $A \times \mathbf{R}^N$ such that

$$b(x, \cdot) \in L^2(\mathbf{R}^N, d\mu)\,, \quad \langle b(x, \cdot) \rangle = 0 \quad \text{ and } \mathcal{L}_x b(x, \cdot) = \mathcal{L}_x^* b(x, \cdot) = v \tag{17}$$

for a.e. $x \in A$.

With the vector field b, we defined the $M_N(\mathbf{R})$- valued matrix field

$$M(x) = \langle b(x, \cdot) \otimes v \rangle\,, \quad \text{ for a.e. } x \in A\,. \tag{18}$$

3.1. **Coercivity properties of $\mathcal{L}_x$.** The coercivity properties of $\mathcal{L}_x$ on the orthogonal of its null-space for a.e. $x \in A$ are crucial in order to study the matrix field M. This matrix field is of fundamental importance in the sequel as it is the diffusion matrix that appears in the limit equation. Notice however that this diffusion matrix is (a.e.) defined on A only, and not in all of Ω.

Lemma 3.1. *Assume that k_ϵ satisfies the assumptions (4)-(14)-(15)-(17) while the probability measure μ satisfies (11). Then*

a) one has

$$\langle v \rangle = 0\,;$$

b) for a.e. $x \in A$ and each $\phi \in L^2(\mathbf{R}^N, d\mu)$

$$\|\phi - \langle\phi\rangle\|_{L^2(\mathbf{R}^N;d\mu)} \le 2C_K \|\mathcal{L}_x\phi\|_{L^2(\mathbf{R}^N,d\mu)}\,;$$

in particular

$$\|b(x,\cdot)\|_{L^2(\mathbf{R}^N,d\mu)} \le 2C_K\langle|v|^2\rangle^{1/2}\,;$$

c) the matrix field M satisfies

$$M(x) = M(x)^T \quad \textit{for a.e. } x \in A\,;$$

d) the matrix field M satisfies the bound

$$|M_{ij}(x)| \le 2C_K\|v_i\|_{L^2(\mathbf{R}^N,d\mu)}\|v_j\|_{L^2(\mathbf{R}^N,d\mu)} \quad \textit{for a.e. } x \in A\,;$$

e) denoting by $\beta > 0$ the smallest eigenvalue of the positive matrix $\langle v^{\otimes 2}\rangle$, one has

$$\xi \cdot M(x)\xi \ge \frac{\beta}{2C_K}|\xi|^2 \textit{ for all } \xi \in \mathbf{R}^N\,, \quad \textit{for a.e. } x \in A\,.$$

3.2. **The diffusion equation on Ω with infinite diffusivity in B.** In the classical diffusion approximation of the linear Boltzmann equation, the diffusion coefficient is proportional to the reciprocal absorption rate (see for instance formula (47) in [3]). In the situation considered above, the absorption rate vanishes as $\epsilon \to 0$ in the subregion B of the spatial domain Ω. This suggest that the limit equation should be a diffusion equation with infinite diffusion constant in B.

Before going further, we recall the precise statement of this problem and its variational formulation.

Define

$$\mathcal{H} := \left\{ u \in L^2(\Omega) \text{ s.t. } u(x) = \frac{1}{|B_l|}\int_{B_l} u(y)dy \text{ for a.e. } x \in B_l\,,\ l = 1,\dots,n \right\}\,,$$

and

$$\mathcal{V} := \mathcal{H} \cap H^1_0(\Omega) = \{u \in H^1_0(\Omega) \text{ s.t. } \nabla u(x) = 0 \text{ for a.e. } x \in B_l\,,\ l = 1,\dots,n\}\,.$$

For each $\rho^{in} \in \mathcal{H}$, consider the following variational problem

(19)
$$\begin{cases} \rho \in C_b(\mathbf{R}_+;\mathcal{H}) \cap L^2(\mathbf{R}_+;\mathcal{V})\,, \quad \partial_t\rho \in L^2(\mathbf{R}_+;\mathcal{V}')\,, & \text{and } \rho\big|_{t=0} = \rho^{in}\,, \\ \dfrac{d}{dt}\displaystyle\int_\Omega \rho(t,x)w(x)dx + \int_A \nabla w(x)\cdot M(x)\nabla_x\rho(t,x)dx = 0\,, & \text{for a.e. } t \ge 0\,, \\ & \text{for all } w \in \mathcal{V}\,. \end{cases}$$

Proposition 3.2. *Assume that $x \mapsto M(x)$ is an $M_N(\mathbf{R})$-valued measurable matrix field on A satisfying*

$$M_{ij} \in L^\infty(A) \text{ for all } i,j = 1,\ldots,N\,, \text{ and there exists } \alpha > 0 \text{ s.t.}$$
$$\xi \cdot M(x)\xi \geq \alpha|\xi|^2 \text{ for a.e. } x \in A \text{ and all } \xi \in \mathbf{R}^N\,.$$

For each $\rho^{in} \in \mathcal{H}$, the variational problem (19) has a unique solution. This solution satisfies the "energy" identity

$$\tfrac{1}{2}\int_\Omega \rho(t,x)^2 dx + \int_0^t \int_A \nabla_x \rho(s,x) \cdot M(x) \nabla_x \rho(s,x) dx ds = \tfrac{1}{2}\int_\Omega \rho^{in}(x)^2 dx$$

for each $t \geq 0$.

Next we state the PDE formulation equivalent to (19).

Proposition 3.3. *Assume that $x \mapsto M(x)$ is an $M_N(\mathbf{R})$-valued measurable matrix field on A such that $M_{ij} \in L^\infty(A)$ for all $i,j = 1,\ldots,N$. Let $\rho \in C([0,T];\mathcal{H}) \cap L^2([0,T];\mathcal{V})$ with $\partial_t \rho \in L^2([0,T];\mathcal{V}')$.*

Then ρ satisfies

$$\begin{cases} \dfrac{d}{dt}\displaystyle\int_\Omega \rho(t,x)w(x)dx + \int_A \nabla w(x) \cdot M(x)\nabla_x \rho(t,x)dx = 0\,, \quad \text{for all } w \in \mathcal{V}\,, \\ \\ \rho\big|_{t=0} = \rho^{in}\,. \end{cases}$$

if and only if

$$\begin{cases} \partial_t \rho - \operatorname{div}_x(M\nabla_x \rho) = 0 & \text{in } \mathcal{D}'(\mathbf{R}_+^* \times A)\,, \\ \rho(t,\cdot)\big|_{\partial\Omega} = 0 & \text{in } L^2([0,T];H^{1/2}(\partial\Omega))\,, \\ \dot\rho_l = \dfrac{1}{\beta_l}\left\langle \dfrac{\partial \rho}{\partial n_M}, 1 \right\rangle_{H^{-1/2}(\partial B_l),H^{1/2}(\partial B_l)} & \text{in } H^{-1}((0,T))\,,\ l = 1,\ldots,m \\ \rho\big|_{t=0} = \rho^{in}\,, \end{cases}$$

where

$$\beta_l = |B_l|\,, \quad l = 1,\ldots,m\,.$$

In the PDE formulation equivalent to (19), we have used the notation

$$\frac{\partial \rho}{\partial n_M}(t,x) := n_x \cdot M(x)\nabla_x \rho(t,x)\,.$$

Consider therefore the problem

$$\begin{cases} \partial_t \rho(t,x) = \operatorname{div}_x(M(x)\nabla_x \rho(t,x) & x \in A\,,\ t > 0\,, \\ \rho(t,x) = 0 & x \in \partial\Omega\,,\ t > 0\,, \\ \dot\rho_l(t) = \dfrac{1}{\beta_l}\displaystyle\int_{\partial B_l} \frac{\partial \rho}{\partial n_M}(t,x)d\sigma(x) & l = 1,\ldots,m\,,\ t > 0\,, \\ \rho(0,x) = \rho^{in}(x) & x \in \Omega\,. \end{cases} \tag{20}$$

The proposition above justifies the following definition.

Definition 3.4. *For $\rho^{in} \in \mathcal{H}$, a weak solution of the problem (20) is a function $\rho \equiv \rho(t,x)$ such that*

$$\rho \in C_b(\mathbf{R}_+;\mathcal{H}) \cap L^2(\mathbf{R}_+;\mathcal{V}) \quad \text{and } \partial_t \rho \in L^2(\mathbf{R}_+;\mathcal{V}')$$

which satisfies the variational formulation and the initial condition in (19).

Remark. The problem (20) is the limit of the diffusion equation in the case where the diffusion coefficient tends to $+\infty$ in B: see for instance Theorem 2.4 in [8] for a proof of this result.

3.3. **The diffusion approximation.** With the preparations above, we can formulate the diffusion approximation for the scaled linear Boltzmann equation with absorption rate vanishing in B.

Theorem 3.5. *Assume that μ satisfies (11) and that k_ϵ satisfies the assumptions (4)-(14)-(15)-(17). Let M be the matrix field defined on A by (18).*

Let $\rho^{in} \in \mathcal{H}$, and for each $\epsilon > 0$, let f_ϵ be the unique weak solution of the initial-boundary value problem (8) in the space $C_b(\mathbf{R}_+; L^2(\Omega \times \mathbf{R}^N; dxd\mu(v)))$. Then

$$f_\epsilon(t,\cdot,\cdot) \to \rho(t,\cdot) \text{ strongly in } L^2(\Omega \times \mathbf{R}^N; dxd\mu)$$

uniformly in $t \in [0,T]$ for all $T > 0$, where ρ is the unique weak solution of (20).

In addition

$$\frac{1}{\epsilon}(f_\epsilon(t,x,v) - f_\epsilon(t,x,w)) \to -(b(x,v) - b(x,w)) \cdot \nabla_x \rho(t,x)$$

in the strong topology of $L^2([0,T] \times A \times \mathbf{R}^N \times \mathbf{R}^N; k_A(x,v,w)dtdxd\mu(v)d\mu(w))$ for all $T > 0$ as $\epsilon \to 0$.

Remark. The last convergence statement above is equivalent to the following:

$$\frac{1}{\epsilon}(f_\epsilon - \langle f_\epsilon \rangle) \to -b \cdot \nabla_x \rho \quad \text{strongly in } L^2([0,T] \times A \times \mathbf{R}^N; dtdxd\mu) \tag{21}$$

as $\epsilon \to 0$. This statement is the analogue of formula (37) in [3] giving the $O(\epsilon)$ term in the Hilbert expansion of the solution f_ϵ as a formal power series in ϵ.

Proof of formula (21). Observe that the linear map $\Phi \mapsto \phi$ defined by

$$\phi(t,x,v) = \int_{\mathbf{R}^N} \Phi(t,x,v,w) d\mu(w)$$

is a bounded operator from $L^2([0,T] \times A \times \mathbf{R}^N \times \mathbf{R}^N; k_A(x,v,w)dtdxd\mu(v)d\mu(w))$ to $L^2([0,T] \times A \times \mathbf{R}^N; dtdxd\mu(v))$. Indeed

$$\phi(t,x,v)^2 \leq C_K \int_{\mathbf{R}^N} k_A(x,v,w)\Phi(t,x,v,w)^2 d\mu(w)$$

by the Cauchy-Schwarz inequality and assumption (15). The conclusion follows from integrating both sides of this inequality in t,x,v. Therefore

$$\begin{aligned}\frac{1}{\epsilon}(f_\epsilon(t,x,v) - \langle f_\epsilon \rangle(t,x)) &= \int_{\mathbf{R}^N} \frac{1}{\epsilon}(f_\epsilon(t,x,v) - f_\epsilon(t,x,w)) d\mu(w) \\ &\to -\int_{\mathbf{R}^N} (b(x,v) - b(x,w)) \cdot \nabla_x \rho(t,x) d\mu(w) \\ &= -b(x,v) \cdot \nabla_x \rho(t,x)\end{aligned}$$

in $L^2([0,T] \times A \times \mathbf{R}^N; dtdxd\mu(v))$ as $\epsilon \to 0$, since $\langle b(x,\cdot) \rangle = 0$ for a.e. $x \in A$. $\square$

3.4. **Remarks on the assumptions of Theorem 3.5.** The existence of the vector field b is obviously an important assumption as it enters the definition of the diffusion matrix M. In the present formulation, the existence of b is postulated in (17) and the condition $\langle v\rangle = 0$ in Lemma 3.1 deduced from the existence of b.

Conversely, one can assume that the transition kernel k_A is chosen so that $\mathcal{K}_x$ is a compact operator on $L^2(\mathbf{R}^N; d\mu)$ for a.e. $x \in A$. In that case, $\mathcal{L}_x$ is a Fredholm operator on $L^2(\mathbf{R}^N; d\mu)$ for a.e. $x \in A$, because the multiplication by $a_A(x, v)$ is an invertible operator on $L^2(\mathbf{R}^N; d\mu)$ for a.e. $x \in A$ — see Corollary 19.1.8 in [10] or chapter 6 in [5]. Indeed, by (15) and Jensen's inequality

$$|a_A(x, v)^{-1}| = \langle k_A(x, v, \cdot)\rangle^{-1} \leq \langle k_A(x, v, \cdot)^{-1}\rangle \leq C_K$$

for $dxd\mu(v)$-a.e. $(x, v) \in A \times \mathbf{R}^N$. Applying then Lemma 2.1 e) shows that

$$\mathrm{Im}(\mathcal{L}_x) = \mathrm{Ker}(\mathcal{L}_x^*)^{\perp} = \{\phi \in L^2(\mathbf{R}^N; d\mu(v)) \text{ s.t. } \langle\phi\rangle = 0\}\,.$$

Thus

$$\langle v\rangle = 0 \Leftrightarrow v \in \mathrm{Im}(\mathcal{L}_x)$$

for a.e. $x \in A$. If in addition k_A is chosen such that

$$k_A(x, v, w) = k_A(x, w, v) \quad \text{for } d\mu(v)d\mu(w) - \text{a.e. } (v, w) \in \mathbf{R}^N \times \mathbf{R}^N$$

for a.e. $x \in A$, then $\mathcal{L}_x^* = \mathcal{L}_x$ and assumption (17) holds.

Another key assumption is (13).

If B_l is convex for $l = 1, \ldots, m$ and μ is of the form $d\mu(v) = r(|v|)dv$ or μ is the uniform probability measure on a sphere included in $\mathbf{R}^N$ centered at the origin, (13) is obviously satisfied. Indeed, for each $x, y \in \partial B_l$, the segment $[x, y]$ is included in B_l, so that $g(x) = g(y)$ for a.e. $x, y \in \partial B_l$.

But even when B_l is convex, the assumption (13) may fail to be satisfied for some measures μ. For instance, assume that $N = 2$, and take $B_l = \{x \in \mathbf{R}^2 \text{ s.t. } |x| \leq 1\}$. Denote by (e_1, e_2) the canonical basis of $\mathbf{R}^2$, and let

$$\mu = \tfrac{1}{4}(\delta_{e_1} + \delta_{-e_1} + \delta_{e_2} + \delta_{-e_2})\,.$$

For $x = (x_1, x_2) \in \partial B_l$, one has $\tau_l(x, \pm e_1) = 2|x_1|$ and $\tau_l(x, \pm e_2) = 2|x_2|$, so that

$$(-x_1, x_2) + \tau_l(x, e_1)e_1 = (x_1, x_2)\,, \qquad (x_1, x_2) - \tau_l(x, e_1)e_1 = (-x_1, x_2)\,,$$
$$(x_1, -x_2) + \tau_l(x, e_2)e_2 = (x_1, x_2)\,, \qquad (x_1, x_2) - \tau_l(x, e_2)e_2 = (x_1, -x_2)\,.$$

Thus $g(x) = |x_1|$ or $g(x) = |x_2|$ are not a.e. constant on ∂B_l and yet satisfy the condition

$$g(x + \tau_l(x, v)v) = g(x) \quad \text{for } dxd\mu(v) - \text{a.e. } (x, v) \in \partial B_l \times \mathbf{R}^2\,.$$

Assumption (14) is obviously satisfied if $k_\epsilon(x, v, w) = 0$ for $dxd\mu(v)d\mu(w)$-a.e. $(x, v, w) \in B \times \mathbf{R}^N \times \mathbf{R}^N$, or if $k_\epsilon(x, v, w) = O(\epsilon)$ on B. The assumption used in the present paper is obviously much more general. For instance, it is satisfied if one has $k_\epsilon(x, v, w) = O(|\ln\epsilon|^{-\gamma_l})$ on B_l with $\gamma_l > 0$ for each $l = 1, \ldots, m$. The Hilbert expansion method used in [3] does not apply to this situation, and therefore cannot be used on the problem considered here in its fullest generality.

Even in the nondegenerate case where $B = \varnothing$, observe that our assumptions on the transition kernel k_ϵ do not imply that the vector field b in (17) depends smoothly on x. This again excludes the possibility of using the Hilbert expansion as in [3] to establish the validity of the diffusion limit. Accordingly, the diffusion matrix field M obtained in Theorem 3.5 is in general not even continuous. In this regularity class, the classical interpretation of the diffusion equation with diffusion matrix M in terms of the associated stochastic differential equation fails (see for instance section 5.1 and Remark 5.1.6 in [14]). One should bear in mind that a diffusion equation with diffusion matrix that has a type I discontinuity across some smooth surface is equivalent to a transmission problem for two diffusion equations on each side of the discontinuity surface, with continuity of the solution and of the normal component of the current across the discontinuity surface. See for instance [12] on p. 107 or Lemma 1.1 in [8] for a discussion of this well known issue.

4. Proof of Lemma 3.1

For each $i = 1, \ldots, N$ and a.e. $x \in A$, one has

$$\langle v_i \rangle = \langle \mathcal{L}_x b_i(x, \cdot) \rangle = \langle (\mathcal{L}_x^* 1) b_i(x, \cdot) \rangle = 0$$

since $\mathcal{L}_x^* 1 = 0$ by Lemma 2.1 c), which proves statement a).

Set $\mathcal{L}_x \phi = \psi$; by statement d) in Lemma 2.1

$$\langle \phi\psi \rangle = \langle \phi \mathcal{L}_x \phi \rangle = \tfrac{1}{2} \iint_{\mathbf{R}^N \times \mathbf{R}^N} k_\epsilon(x, v, w)(\phi(v) - \phi(w))^2 d\mu(v) d\mu(w) \geq 0 \, .$$

By the Cauchy-Schwarz inequality, for a.e. $x \in A$,

$$\begin{aligned} |\phi(v) - \langle \phi \rangle|^2 &= \left(\int_{\mathbf{R}^N} (\phi(v) - \phi(w)) d\mu(w) \right)^2 \\ &\leq \int_{\mathbf{R}^N} \frac{d\mu(w)}{k_A(x, v, w)} \int_{\mathbf{R}^N} k_A(x, v, w)(\phi(v) - \phi(w))^2 d\mu(w) \end{aligned}$$

so that

$$\begin{aligned} \|\phi - \langle \phi \rangle\|^2_{L^2(\mathbf{R}^N; d\mu)} &\leq C_K \iint_{\mathbf{R}^N \times \mathbf{R}^N} k_A(x, v, w)(\phi(v) - \phi(w))^2 d\mu(v) d\mu(w) \\ &= 2C_K \langle \phi\psi \rangle \, . \end{aligned}$$

Next

$$\langle \psi \rangle = \langle \mathcal{L}_x \phi \rangle = \langle (\mathcal{L}_x^* 1) \phi \rangle = 0$$

since $\mathcal{L}_x^* 1 = 0$ by Lemma 2.1 c), so that

$$\langle \phi\psi \rangle = \langle (\phi - \langle \phi \rangle) \psi \rangle \leq \|\psi\|_{L^2(\mathbf{R}^N, d\mu)} \|\phi - \langle \phi \rangle\|_{L^2(\mathbf{R}^N, d\mu)}$$

by the Cauchy-Schwarz inequality. Putting together the last two inequalities, we obtain the bound

$$\|\phi - \langle \phi \rangle\|_{L^2(\mathbf{R}^N; d\mu)} \leq 2C_K \|\psi\|_{L^2(\mathbf{R}^N, d\mu)}$$

which is statement b).

Next

$$\begin{aligned} M_{ij}(x) = \langle b_i(x, \cdot) v_j \rangle &= \langle b_i(x, \cdot) \mathcal{L}_x b_j(x, \cdot) \rangle \\ &= \langle b_j(x, \cdot) \mathcal{L}_x^* b_i(x, \cdot) \rangle = \langle b_j(x, \cdot) v_i \rangle = M_{ji}(x) \end{aligned}$$

for all $i, j = 1, \ldots, N$ and a.e. $x \in A$. This proves statement c).

Statement d) follows from the identity

$$M_{ij}(x) = \langle b_i(x, \cdot) v_j \rangle \, ,$$

from the Cauchy-Schwartz inequality and statement b) with $\phi = b_i(x, \cdot)$.

Applying again the Cauchy-Schwarz inequality with $\phi(x,v) := \xi \cdot b(x,v)$ and $\psi(v) := \xi \cdot v = \mathcal{L}_x\phi(x,v)$, one has

$$\psi(v)^2 = \left(\int_{\mathbf{R}^N} k_A(x,v,w)(\phi(x,v)-\phi(x,w))d\mu(w)\right)^2$$
$$\leq a_A(x,v)\int_{\mathbf{R}^N} k_A(x,v,w)(\phi(x,v)-\phi(x,w))^2 d\mu(w)$$

for a.e. $x \in A$, so that, by Lemma 2.1 d)

$$\langle\psi^2\rangle \leq C_K \iint_{\mathbf{R}^N\times\mathbf{R}^N} k_A(x,v,w)(\phi(x,v)-\phi(x,w))^2 d\mu(v)d\mu(w)$$
$$= 2C_K\langle\psi\phi(x,\cdot)\rangle = 2C_K\xi\cdot M(x)\xi\,.$$

for a.e. $x \in A$. Obviously

$$\langle\psi^2\rangle = \xi\cdot\langle v^{\otimes 2}\rangle\xi \geq \beta|\xi|^2$$

and statement e) follows.

5. Proofs of Propositions 3.2 and 3.3

Proof of Proposition 3.2. The existence and uniqueness of the solution of the variational problem (19) is a straightforward consequence of the Lions-Magenes theorem, i.e. Theorem X.9 in [5], with the bilinear form

$$a(u,v) := \int_A \nabla u(x)\cdot M(x)\nabla v(x)dx\,, \quad u,v \in \mathcal{V}\,.$$

Indeed, this bilinear form satisfies the assumptions of the Lions-Magenes theorem since Lemma 3.1 d) implies that

$$|a(u,v)| \leq 2C_K\langle|v|^2\rangle\|\nabla u\|_{L^2(A)}\|\nabla v\|_{L^2(A)} \leq 2C_K\langle|v|^2\rangle\|u\|_{\mathcal{V}}\|v\|_{\mathcal{V}}\,,$$

while Lemma 3.1 e) implies that

$$a(u,u) \geq \frac{\beta}{2C_K}\|\nabla u\|^2_{L^2(A)} = \frac{\beta}{2C_K}\|\nabla u\|^2_{L^2(\Omega)} = \frac{\beta}{2C_K}(\|u\|^2_{\mathcal{V}} - \|u\|^2_{\mathcal{H}})$$

for each $u,v \in \mathcal{V}$.

Consider the linear functional

$$L(t):\ \mathcal{V} \ni w \mapsto \langle\partial_t\rho, w\rangle_{\mathcal{V}',\mathcal{V}} + a(\rho(t,\cdot),w)$$

defined for a.e. $t \geq 0$.

Since $L(t) = 0$ for a.e. $t \in \mathbf{R}$, one has

$$\langle L(t),\rho(t,\cdot)\rangle_{\mathcal{V}',\mathcal{V}} = 0 \quad \text{for a.e. } t \geq 0\,,$$

for each $w \in \mathcal{V}$. By Lemma B.2, one has

$$L(t) = 0 \text{ in } \mathcal{V}' \text{ for a.e. } t \in \mathbf{R}_+\,.$$

In particular, for a.e. $s \geq 0$, one has

$$0 = \langle L(s),\rho(s,\cdot)\rangle_{\mathcal{V}',\mathcal{V}} = \langle\partial_t\rho(s,\cdot),\rho(s,\cdot)\rangle_{\mathcal{V}',\mathcal{V}} + \int_A \nabla_x\rho(s,x)\cdot M(x)\nabla_x\rho(s,x)dx\,,$$

and one concludes by integrating in $s \in [0,t]$ and applying Lemma B.1 b). □

Proof of Proposition 3.3. Specializing (19) to the case where $w \in C_c^\infty(A)$ is equivalent to

$$\partial_t \rho - \operatorname{div}_x(M\nabla_x \rho) = 0 \quad \text{ in } \mathcal{D}'(\mathbf{R}_+^* \times A)\,.$$

In particular, the (a.e. defined) vector field

$$(0,\tau) \times A \ni (t,x) \mapsto (\rho(t,x), -M(x)\nabla_x \rho(t,x))$$

is divergence free in $(0,\tau) \times A$. Applying statement b) in Lemma B.3 shows that

$$\begin{aligned} 0 &= \frac{d}{dt}\int_\Omega \rho(t,x)w(x)dx + \int_A \nabla w(x) \cdot M(x)\nabla_x \rho(t,x)dx \\ &= \frac{d}{dt}\int_A \rho(t,x)w(x)dx + \sum_{l=1}^m \beta_l w_l \dot\rho_l(t) + \int_A \nabla w(x) \cdot M(x)\nabla_x \rho(t,x)dx \\ &= \sum_{l=1}^m w_l \left(\beta_l \dot\rho_l(t) - \left\langle \frac{\partial \rho}{\partial n_M}(t,\cdot)\Big|_{\partial B_l}, 1 \right\rangle_{H^{-1/2}(\partial B_l), H^{1/2}(\partial B_l)} \right) \end{aligned}$$

for each $w \in \mathcal{V}$, where

$$w_l := \frac{1}{|B_l|}\int_{B_l} w(y)dy\,, \quad l = 1,\ldots,m\,.$$

Since this is true for all $w \in \mathcal{V}$, and therefore for all $(w_1,\ldots,w_m) \in \mathbf{R}^m$, one concludes that

$$\beta_l \dot\rho_i - \left\langle \frac{\partial \rho}{\partial n_M}\Big|_{\partial B_l}, 1 \right\rangle_{H^{-1/2}(\partial B_l), H^{1/2}(\partial B_l)} = 0$$

in $H^{-1}((0,\tau))$ for all $l = 1,\ldots,m$, which is precisely the transmission condition on ∂B_l. Finally, the Dirichlet condition on $\partial\Omega$ comes from the condition $\rho \in L^2(\mathbf{R}_+;\mathcal{V})$ since $\mathcal{V} \subset H_0^1(\Omega)$.

Conversely, if $\rho \in C_b(\mathbf{R}_+,\mathcal{H}) \cap L^2(\mathbf{R}_+,\mathcal{V})$ s.t. $\partial_t\rho \in L^2(\mathbf{R}_+,\mathcal{V}')$ satisfies the initial condition and the diffusion equation in (20) in the sense of distributions on $\mathbf{R}_+^* \times A$, together with the transmission condition on ∂B_l for each $l = 1,\ldots,m$, it follows from the identity above that ρ must satisfy (19). □

6. Proof of Theorem 3.5

The proof is split in several steps.

Step 1: uniform bounds and weak compactness.

By the entropy inequality (statement b) in Proposition 2.2), one has the bounds

$$\left\{ \begin{aligned} &\|f_\epsilon(t,\cdot,\cdot)\|_{L^2(\Omega\times\mathbf{R}^N;dxd\mu)} \le \|\rho^{in}\|_{L^2(\Omega)} \quad \text{and} \\ &\|\sqrt{k_\epsilon} q_\epsilon\|_{L^2(\mathbf{R}_+\times\Omega\times\mathbf{R}^N\times\mathbf{R}^N;dtdxd(\mu\otimes\mu)} \le \|\rho^{in}\|_{L^2(\Omega)} \end{aligned} \right. \tag{22}$$

By the Banach-Alaoglu theorem, the families f_ϵ and $\sqrt{k_\epsilon} q_\epsilon$ are relatively compact in $L^\infty(\mathbf{R}_+;L^2(\Omega\times\mathbf{R}^N;dxd\mu))$ weak-* and $L^2(\mathbf{R}_+\times\Omega\times\mathbf{R}^N\times\mathbf{R}^N;dtdxd(\mu\otimes\mu))$ weak respectively. Extracting subsequences if needed, one has

$$f_\epsilon \to f \text{ in } L^\infty(\mathbf{R}_+;L^2(\Omega\times\mathbf{R}^N;dxd\mu)) \text{ weak-*} \tag{23}$$

while

$$\sqrt{k_\epsilon} q_\epsilon \to r \text{ in } L^2(\mathbf{R}_+\times\Omega\times\mathbf{R}^N\times\mathbf{R}^N;dtdxd(\mu\otimes\mu)) \text{ weak.} \tag{24}$$

In particular

$$q_\epsilon \to q \text{ in } L^2(\mathbf{R}_+ \times A \times \mathbf{R}^N \times \mathbf{R}^N; k_A(x,v,w)dtdxd(\mu\otimes\mu)) \text{ weak,} \tag{25}$$

where

$$q(t,x,v,w) := r(t,x,v,w)/\sqrt{k_A(x,v,w)}\,, \tag{26}$$

for $dtdxd\mu(vd\mu(w)$-a.e. $(t,x,v,w) \in \mathbf{R}_+ \times A \times \mathbf{R}^N \times \mathbf{R}^N$.

Step 2: asymptotic form of the linear Boltzmann equation

One has

$$\begin{aligned}\frac{1}{\epsilon}\mathcal{L}_x f_\epsilon(t,x,v) = &\int_{\mathbf{R}^N} \mathbf{1}_A(x)k_A(x,v,w)q_\epsilon(t,x,v,w)d\mu(w)\\ &+\int_{\mathbf{R}^N} \mathbf{1}_B(x)k_\epsilon(x,v,w)q_\epsilon(t,x,v,w)d\mu(w)\end{aligned}$$

Since $(x,v,w) \mapsto \mathbf{1}_A(x)$ belongs to $L^2(A\times\mathbf{R}^N\times\mathbf{R}^N; k(x,v,w)dxd(\mu\otimes\mu))$ by (15)

$$\int_{\mathbf{R}^N} \mathbf{1}_A(x)k_A(x,v,w)q_\epsilon(t,x,v,w)d\mu(w) \to \int_{\mathbf{R}^N} \mathbf{1}_A(x)k_A(x,v,w)q(t,x,v,w)d\mu(w)$$

in the weak topology of $L^2(\mathbf{R}_+\times A\times\mathbf{R}^N; dtdxd\mu)$ as $\epsilon\to 0$. On the other hand, the Cauchy-Schwarz inequality and (15) imply that

$$\begin{aligned}&\left\|\int_{\mathbf{R}^N} k_\epsilon(\cdot,\cdot,w)q_\epsilon(\cdot,\cdot,\cdot,w)d\mu(w)\right\|^2_{L^2(\mathbf{R}_+\times B\times\mathbf{R}^N;dtdxd\mu)}\\ &\le \|a_\epsilon\|_{L^\infty(B\times\mathbf{R}^N)}\iint_{\mathbf{R}_+\times\Omega}\langle\!\langle k_\epsilon(x,\cdot,\cdot)q_\epsilon(t,x,\cdot,\cdot)^2\rangle\!\rangle dtdx\\ &\le \|a_\epsilon\|_{L^\infty(B\times\mathbf{R}^N)}\|\rho^{in}\|^2_{L^2(\Omega)} \to 0\end{aligned}$$

as $\epsilon\to 0$, by (14) and the entropy inequality in Proposition 2.2. Thus

$$\frac{1}{\epsilon}\mathcal{L}_x f_\epsilon(t,x,v) \to \int_{\mathbf{R}^N} \mathbf{1}_A(x)k_A(x,v,w)q(t,x,v,w)d\mu(w) \tag{27}$$

in the weak topology of $L^2(\mathbf{R}_+\times\Omega\times\mathbf{R}^N; dxd\mu)$ as $\epsilon\to 0$. Passing to the limit in the scaled Boltzmann equation (8) we see that

$$\begin{aligned}&v\cdot\nabla_x f \in L^2(\mathbf{R}_+\times\Omega\times\mathbf{R}^N, dtdxd\mu) \quad \text{and}\\ &\int_{\mathbf{R}^N} k_A(\cdot,\cdot,w)q(\cdot,\cdot,\cdot,w)d\mu(w) \in L^2(\mathbf{R}_+\times A\times\mathbf{R}^N, dtdxd\mu)\,,\end{aligned} \tag{28}$$

while

$$v\cdot\nabla_x f(t,x,v) + \mathbf{1}_A(x)\int_{\mathbf{R}^N} k_A(x,v,w)q(t,x,v,w)d\mu(w) = 0\,, \tag{29}$$

for $dtdxd\mu$-a.e. $(t,x,v)\in\mathbf{R}_+\times\Omega\times\mathbf{R}^N$.

Step 3: asymptotic form of f_ϵ.

Multiplying both sides of the scaled linear Boltzmann equation (8) by ϵ and passing to the limit in the sense of distributions as $\epsilon\to 0$, one finds that

$$\mathcal{L}_x f(t,x,v) = 0 \quad \text{for a.e. } (t,x,v)\in\mathbf{R}_+^*\times\Omega\times\mathbf{R}^N\,.$$

By Lemma 2.1 e), this implies that $f(t,x,v)$ is independent of v for a.e. $x\in A$, i.e. is of the form

$$f(t,x,v) = \rho(t,x) \quad \text{for a.e. } (t,x,v)\in\mathbf{R}_+^*\times A\times\mathbf{R}^N\,. \tag{30}$$

By (23) and (28)

$$\rho \in L^\infty(\mathbf{R}_+; L^2(A)) \quad \text{and } \nabla_x \rho \in L^2(\mathbf{R}_+ \times A)\,, \tag{31}$$

since

$$(v \cdot \nabla_x f)v = (v \otimes v) \cdot \nabla_x \rho \in L^2(\mathbf{R}_+ \times A; L^1(\mathbf{R}^N, d\mu))$$

so that

$$\langle v \otimes v \rangle \cdot \nabla_x \rho \in L^2(\mathbf{R}_+ \times A)\,;$$

one concludes since $\det(\langle v \otimes v \rangle) \neq 0$ by assumption (11).

In particular

$$\rho\big|_{\partial B_i} \in L^2([0,T]; H^{1/2}(\partial B_i))$$

for each $T > 0$ and each $i = 1, \ldots, n$.

In particular, the first condition in (28) and (10) imply that $s \mapsto f(t, x+sv, v)$ is continuous in s for $dtdxd\mu$-a.e. $(t,x,v) \in \mathbf{R}_+ \times \Omega \times \mathbf{R}^N$. Therefore, we deduce from (29) and (30) that, for each $l = 1, \ldots, m$

$$\begin{cases} v \cdot \nabla_x f(t,x,v) = 0\,, & x \in B_l\,,\ v \in \mathbf{R}^N\,,\ t > 0\,, \\ f(t,x,v) = \rho(t,x)\,, & x \in \partial B_l\,,\ v \in \mathbf{R}^N\,,\ t > 0\,. \end{cases}$$

Hence

$$\frac{d}{ds} f(t, x+sv, v) = 0 \quad \text{for all } s \text{ s.t. } x + sv \in B_l$$

for $dtdxd\mu(v)$-a.e. $(t,x,v) \in \mathbf{R}_+ \times B_l \times \mathbf{R}^N$. By assumption (10), one concludes that

$$\rho(t, x + \tau_l(x,v)v) = \rho(t,x) \text{ for } d\sigma(x)d\mu(v) - \text{ a.e. } (x,v) \in \partial B_l \times \mathbf{R}^N$$

by solving the boundary value problem above by the method of characteristics. By assumption (13)

$$\rho(t,x) = \frac{1}{|\partial B_l|} \int_{\partial B_l} \rho(t,y) d\sigma(y) =: \rho_l(t) \quad \text{for a.e. } x \in \partial B_l\,,$$

for a.e. $t \geq 0$. In other words, $\rho(t,\cdot)$ is a.e. equal to a constant on ∂B_l.

Solving for f along characteristics, this implies that $f(t,\cdot,\cdot)$ itself is a.e. equal to a constant on $\partial B_l \times \mathbf{R}^N$, i.e.

$$f(t,x,v) = \frac{1}{|B_l|} \int_{B_l} \langle f(t,x,\cdot) \rangle dx =: \rho_l(t)$$

for $dtdxd\mu$-a.e. $(t,x,v) \in \mathbf{R}_+ \times B_l \times \mathbf{R}^N$, for $l = 1, \ldots, m$.

Summarizing, we have proved that

$$\begin{aligned} &f(t,x,v) = \rho(t,x) \text{ for } dtdxd\mu - \text{a.e. } (t,x,v) \in \Omega \\ &\text{with } \rho \in L^\infty(\mathbf{R}_+; \mathcal{H}) \quad \text{and } \nabla_x \rho \in L^2(\mathbf{R}_+ \times \Omega)\,. \end{aligned} \tag{32}$$

Step 4: Fourier's law and continuity equation

Observe that the flux satisfies

$$\begin{aligned} \frac{1}{\epsilon} \langle v f_\epsilon(t,x,\cdot) \rangle &= \frac{1}{\epsilon} \langle (\mathcal{L}_x^* b(x,\cdot)) f_\epsilon(t,x,\cdot) \rangle = \left\langle b(x,\cdot) \frac{1}{\epsilon} \mathcal{L}_x f_\epsilon(t,x,\cdot) \right\rangle \\ &= \iint_{\mathbf{R}^N \times \mathbf{R}^N} b(x,v) k(x,v,w) q_\epsilon(t,x,v,w) d\mu(v) d\mu(w) \end{aligned} \tag{33}$$

for a.e. $(t,x) \in \mathbf{R}_+ \times A$ and for all $\epsilon > 0$.

Since $b \in L^\infty(A; L^2(\mathbf{R}^N; d\mu))$ by statement b) in Lemma 3.1 , the function $(x, v, w) \mapsto \sqrt{k_A(x, v, w)}b(x, v)$ belongs to $L^\infty(A; L^2(\mathbf{R}^N \times \mathbf{R}^N; d\mu(v)\mu(w)))$. Thus

$$(34)\qquad \begin{aligned} \frac{1}{\epsilon}\langle v f_\epsilon(t, x, \cdot)\rangle &= \iint_{\mathbf{R}^N\times\mathbf{R}^N} b(x, v)k(x, v, w)q_\epsilon(t, x, v, w)d\mu(v)d\mu(w) \\ &\to \iint_{\mathbf{R}^N\times\mathbf{R}^N} b(x, v)k(x, v, w)q(t, x, v, w)d\mu(v)d\mu(w) \\ &= \langle b(x, \cdot)v \cdot \nabla_x\rho(t, x)\rangle = M(x)\nabla_x\rho(t, x) \end{aligned}$$

in for the weak topology of $L^2(\mathbf{R}_+ \times A)$ as $\epsilon \to 0$, on account of (29).

Therefore, for each $w \in \mathcal{V}$, one has

$$\frac{d}{dt}\int_\Omega \langle f_\epsilon(t, x, \cdot)\rangle w(x)dx + \int_A \frac{1}{\epsilon}\langle v f_\epsilon(t, x, \cdot)\rangle \cdot \nabla w(x)dx = 0\,,$$

(since $\nabla w = 0$ on $\mathring{B}$) and passing to the limit in each side of this identity as $\epsilon \to 0$ shows that

$$(35)\qquad \frac{d}{dt}\int_\Omega \rho(t, x)w(x)dx + \int_A \nabla w(x) \cdot M(x)\nabla_x\rho(t, x)dx = 0$$

in the sense of distributions on $\mathbf{R}_+^*$.

Step 5: limiting entropy production

By definition of q_ϵ, one has

$$q_\epsilon(t, x, v, w) = -q_\epsilon(t, x, w, v)$$

for $dtdxd\mu(vd\mu(w)$-a.e. $(t, x, v, w) \in \mathbf{R}_+ \times A \times \mathbf{R}^N \times \mathbf{R}^N$ and each $\epsilon > 0$; by passing to the limit as $\epsilon \to 0$

$$q(t, x, v, w) = -q(t, x, w, v)$$

for $dtdxd\mu(vd\mu(w)$-a.e. $(t, x, v, w) \in \mathbf{R}_+ \times A \times \mathbf{R}^N \times \mathbf{R}^N$. Defining

$$k_A^s(t, x, v, w) = \tfrac{1}{2}(k_A(t, x, v, w) + k_A(t, x, w, v))$$

one has

$$\langle\!\langle k_A(x, \cdot, \cdot)q(t, x, \cdot, \cdot)^2\rangle\!\rangle = \langle\!\langle k_A^s(x, \cdot, \cdot)q(t, x, \cdot, \cdot)^2\rangle\!\rangle$$

for a.e. $(t, x) \in \mathbf{R}_+ \times A$. Likewise

$$\begin{aligned} &\iint_{\mathbf{R}^N\times\mathbf{R}^N} k_A(x, v, w)(\phi(v) - \phi(w))^2 d\mu(v)d\mu(w) \\ &= \iint_{\mathbf{R}^N\times\mathbf{R}^N} k_A^s(x, v, w)(\phi(v) - \phi(w))^2 d\mu(v)d\mu(w) \end{aligned}$$

and

$$\begin{aligned} &\iint_{\mathbf{R}^N\times\mathbf{R}^N} k_A(x, v, w)(\phi(v) - \phi(w))q(t, x, v, w)d\mu(v)d\mu(w) \\ &= \iint_{\mathbf{R}^N\times\mathbf{R}^N} k_A^s(x, v, w)(\phi(v) - \phi(w))q(t, x, v, w)d\mu(v)d\mu(w) \end{aligned}$$

for a.e. $(t,x)\in\mathbf{R}_+\times\Omega$. With $\phi(v)=\xi\cdot b(x,v)$ for some $\xi\in\mathbf{R}^N$ to be chosen later, and applying the Cauchy-Schwarz inequality, one finds that

(36)
$$\left(\iint_{\mathbf{R}^N\times\mathbf{R}^N}k_A^s(x,v,w)\xi\cdot(b(x,v)-b(x,w))q(t,x,v,w)d\mu(v)d\mu(w)\right)^2$$
$$\le\iint_{\mathbf{R}^N\times\mathbf{R}^N}k_A(x,v,w)(\xi\cdot(b(x,v)-b(x,w)))^2d\mu(v)d\mu(w)\langle\!\langle k_A(x,\cdot,\cdot)q(t,x,\cdot,\cdot)^2\rangle\!\rangle\,.$$

On the other hand, by definition of k_A^s

$$\begin{aligned}&\iint_{\mathbf{R}^N\times\mathbf{R}^N}k_A^s(x,v,w)\xi\cdot(b(x,v)-b(x,w))q_\epsilon(t,x,v,w)d\mu(v)d\mu(w)\\&=\frac{2}{\epsilon}\iint_{\mathbf{R}^N\times\mathbf{R}^N}k_A^s(x,v,w)\xi\cdot(b(x,v)-b(x,w))f_\epsilon(t,x,v)d\mu(v)d\mu(w)\\&\qquad=\frac{1}{\epsilon}\langle f_\epsilon(t,x,\cdot)(\mathcal{L}_x+\mathcal{L}_x^*)\xi\cdot b(x,\cdot)\rangle=\frac{2}{\epsilon}\langle\xi\cdot vf_\epsilon(t,x,\cdot)\rangle\end{aligned}$$

for a.e. $(t,x)\in\mathbf{R}_+\times A$ where the last equality follows from (17). Passing to the limit as $\epsilon\to0$, one finds that

$$\begin{aligned}\iint_{\mathbf{R}^N\times\mathbf{R}^N}k_A^s(x,v,w)\xi\cdot(b(x,v)-b(x,w))q(t,x,v,w)d\mu(v)d\mu(w)\\=-2\xi\cdot M(x)\nabla_x\rho(t,x)\end{aligned}$$

for a.e. $(t,x)\in\mathbf{R}_+\times A$. On the other hand

$$\begin{aligned}&\iint_{\mathbf{R}^N\times\mathbf{R}^N}k_A(x,v,w)(\xi\cdot(b(x,v)-b(x,w)))^2d\mu(v)d\mu(w)\\&\qquad=2\langle\xi\cdot b(x,\cdot)\mathcal{L}_x(\xi\cdot b(x,\cdot))\rangle=2\langle\xi\cdot b(x,\cdot)\xi\cdot v\rangle=2\xi\cdot M(x)\xi\end{aligned}$$

for a.e. $x\in A$, by Lemma 2.1 b). Hence

$$2(\xi\cdot M(x)\nabla_x\rho(t,x))^2\le\xi\cdot M(x)\xi\langle\!\langle k_A(x,\cdot,\cdot)q(t,x,\cdot,\cdot)^2\rangle\!\rangle$$

and choosing $\xi=\nabla_x\rho(t,x)$, we find that

$$2\nabla_x\rho(t,x)\cdot M(x)\nabla_x\rho(t,x)\le\langle\!\langle k_A(x,\cdot,\cdot)q(t,x,\cdot,\cdot)^2\rangle\!\rangle \tag{37}$$

for a.e. $(t,x)\in\mathbf{R}_+\times A$. By convexity and weak convergence

(38)
$$\int_0^\infty\int_A\langle\!\langle k_A(x,\cdot,\cdot)q(t,x,\cdot,\cdot)^2\rangle\!\rangle dxdt\le\varliminf_{\epsilon\to0}\int_0^\infty\int_A\langle\!\langle k_A(x,\cdot,\cdot)q_\epsilon(t,x,\cdot,\cdot)^2\rangle\!\rangle dxdt\,.$$

Using Lemma 3.1 e) and the entropy inequality

(39)
$$\frac{\beta}{C_K}\int_0^\infty\|\nabla_x\rho(t,\cdot)\|^2_{L^2(A)}dt\le2\int_0^\infty\int_A\nabla_x\rho(t,x)\cdot M(x)\nabla_x\rho(t,x)dxdt\le\|\rho^{in}\|^2_{L^2(\Omega)}\,.$$

Step 6: limiting initial condition

By (33) and the Cauchy-Schwarz inequality

$$\begin{aligned}\left\|\frac{1}{\epsilon}\langle vf_\epsilon\rangle\right\|^2_{L^2([0,T]\times A)} &\le \int_{\mathbf{R}_+}\int_A \langle\!\langle k_A(x,\cdot,\cdot)q_\epsilon(t,x,\cdot,\cdot)^2\rangle\!\rangle dxdt\\ &\times \operatorname{supess}_{x\in A}\iint_{\mathbf{R}^N\times\mathbf{R}^N} k_A(x,v,w)|b(x,v)|^2 d\mu(v)d\mu(w)\\ &\le 8C_K^3\langle|v|^2\rangle\|\rho^{in}\|^2_{L^2(\Omega)}\end{aligned}$$

using the entropy inequality in Proposition 2.2 and Lemma 3.1 b) and d). Since

$$\frac{d}{dt}\int_\Omega \langle f_\epsilon(t,x,\cdot)\rangle w(x)dx = \int_A \frac{1}{\epsilon}\langle vf_\epsilon(t,x,\cdot)\rangle\cdot\nabla w(x)dx$$

for each $w\in\mathcal{V}$, one has

$$\left\|\frac{d}{dt}\int_\Omega\langle f_\epsilon(\cdot,x,\cdot)\rangle w(x)dx\right\| \le (2C_K)^{3/2}\langle|v|^2\rangle^{1/2}\|\rho^{in}\|_{L^2(\Omega)}\|\nabla w\|_{L^2(\Omega)}\,. \tag{40}$$

Applying the Ascoli-Arzela theorem shows that, for each $w\in\mathcal{V}$

$$\int_\Omega(\langle f_\epsilon(t,x,\cdot)\rangle-\rho(t,x))w(x)dx\to 0 \text{ uniformly in } t\in[0,T] \tag{41}$$

for all $T>0$. In particular

$$\int_\Omega \rho^{in}(x)w(x)dx = \int_\Omega\langle f_\epsilon(0,x,\cdot)\rangle w(x)dx \to \int_\Omega \rho(0,x))w(x)dx$$

so that

$$\int_\Omega \rho(0,x)w(x)dx = \int_\Omega \rho^{in}(x)w(x)dx \quad \text{for each } w\in\mathcal{V}\,. \tag{42}$$

Returning to (40), we have proved that $\partial_t\langle f_\epsilon\rangle$ is bounded in $L^2(\mathbf{R}_+,\mathcal{V}')$ for each $T>0$, so that

$$\partial_t\rho\in L^2(\mathbf{R}_+;\mathcal{V}')\,. \tag{43}$$

Step 7: Dirichlet condition

Next we establish the Dirichlet condition on $\partial\Omega$ for the diffusion equation. The scaled linear Boltzmann equation implies that, for each $\chi\in C^1_c(\mathbf{R}^*_+)$,

$$v\cdot\nabla_x\int_0^\infty \chi(t)f_\epsilon(t,x,v)dt = -\int_0^\infty\chi(t)\frac{1}{\epsilon}\mathcal{L}_x f_\epsilon(t,x,v)dt + \epsilon\int_0^\infty \chi'(t)f_\epsilon(t,x,v)dt$$

is bounded in $L^2(\Omega\times\mathbf{R}^N;dxd\mu)$ by (27) and the uniform boundedness principle (Banach-Steinhaus' theorem) and the entropy inequality in Proposition 2.2, while

$$\int_0^\infty \chi(t)f_\epsilon(t,x,v)dt$$

is bounded in $L^2(\Omega\times\mathbf{R}^N;dxd\mu)$ by the same entropy inequality. Hence

$$0=\int_0^\infty\chi(t)f_\epsilon(t,\cdot,\cdot)dt\Big|_{\Gamma^-}\to\int_0^\infty\chi(t)\rho(t,\cdot)dt\Big|_{\Gamma^-}$$

in $L^2(\Gamma^-;|v\cdot n_x|\tau(x,v)\wedge 1d\sigma(x)dv)$ by Cessenat's trace theorem [6], where the notation $\tau(x,v)$ designates the forward exit time from Ω starting from x with velocity v, i.e.

$$\tau(x,v):=\inf\{t>0 \text{ s.t. } x+tv\in\partial\Omega\}\,,\quad x\in\Omega\,,\ v\in\mathbf{R}^N\,.$$

In particular

$$\int_0^\infty \chi(t)\rho(t,\cdot)dt\bigg|_{\partial\Omega} = 0\,.$$

By (32), we already know that the limiting density $\rho \in L^2([0,T];H^1(\Omega))$. Therefore

$$\rho(t,\cdot)\big|_{\partial\Omega} = 0 \quad \text{in } L^2([0,T];H^{1/2}(\partial\Omega)) \tag{44}$$

for each $T > 0$.

Step 8: convergence to the diffusion equation

Summarizing, we have proved that

$$f_\epsilon \text{ is relatively compact in } L^\infty(\mathbf{R}_+;L^2(\Omega\times\mathbf{R}^N,dxd\mu(v))) \text{ weak-*}$$

and that, if f is a limit point of f_ϵ as $\epsilon \to 0$, it is of the form

$$f(t,x,v) = \rho(t,x) \quad dtdxd\mu(v) - \text{a.e. in } (t,x,v) \in \mathbf{R}_+\times\Omega\times\mathbf{R}^N$$

where

$$\begin{aligned}&\rho \in L^\infty(\mathbf{R}_+;\mathcal{H})\cap L^2(\mathbf{R}_+;H^1_0(\Omega)) = L^\infty(\mathbf{R}_+;\mathcal{H})\cap L^2(\mathbf{R}_+;\mathcal{V})\\ &\text{and } \partial_t\rho \in L^2(\mathbf{R}_+;\mathcal{V}')\end{aligned} \tag{45}$$

since ρ satisfies the Dirichlet boundary condition (44) and $\nabla_x\rho \in L^2(\mathbf{R}_+\times\Omega)$ by (32), together with (43). In particular, this implies that

$$\rho \in C_b(\mathbf{R}_+;\mathcal{H})\,. \tag{46}$$

Besides ρ satisfies (35) for each test function $w \in \mathcal{V}$, together with the initial condition (42). Therefore ρ is the unique solution of the Dirichlet problem for the diffusion equation with diffusion matrix $M(x)$ defined in (18) with infinite diffusivity in B, with initial data ρ^{in}. By compactness and uniqueness of the limit point, we conclude that

$$f_\epsilon \to \rho \quad \text{in } L^\infty(\mathbf{R}_+;L^2(\Omega\times\mathbf{R}^N,dxd\mu(v))) \text{ weak-*}$$

as $\epsilon \to 0$.

Step 9: strong convergence

The weak solution ρ of the initial-boundary value problem for the diffusion equation with infinite diffusivity in B is known to satisfy the identity

$$\tfrac{1}{2}\int_\Omega \rho(t,x)^2dx + \int_0^t\int_A \nabla_x\rho(s,x)\cdot M(x)\nabla_x\rho(s,x)dxds = \tfrac{1}{2}\int_\Omega \rho^{in}(x)^2dx$$

for each $t \geq 0$.

By Jensen's inequality

$$\int_\Omega \langle f_\epsilon(t,x,\cdot)^2\rangle dx \geq \int_\Omega \langle f_\epsilon(t,x,\cdot)\rangle^2 dx$$

while, by convexity and weak convergence,

$$\varliminf_{\epsilon\to 0}\int_\Omega \langle f_\epsilon(t,x,\cdot)\rangle^2 dx \geq \int_\Omega \rho(t,x)^2dx$$

uniformly in $t \in [0,T]$ for each $T > 0$ by (41).

With the entropy identity satisfied by f_ϵ (see Proposition 2.2) and the inequality (38), the two inequalities above imply that

$$\int_\Omega \langle f_\epsilon(t,x,\cdot)^2\rangle dx \to \int_\Omega \rho(t,x)^2dx$$

uniformly in $t \in [0,T]$ for each $T > 0$, so that

$$f_\epsilon(t,\cdot,\cdot) \to \rho(t,\cdot) \quad \text{strongly in } L^2(\Omega \times \mathbf{R}^N, dxd\mu)$$

uniformly in $t \in [0,T]$ for each $T > 0$.

By the same token

$$\begin{aligned}\tfrac{1}{2}\int_0^t \int_A \langle\!\langle k_A(x,\cdot,\cdot) q_\epsilon(s,x,\cdot,\cdot)^2 \rangle\!\rangle dxds &\to \tfrac{1}{2}\int_0^t \int_A \langle\!\langle k_A(x,\cdot,\cdot) q_\epsilon(s,x,\cdot,\cdot)^2 \rangle\!\rangle dxds \\ &= \int_0^t \int_A \nabla_x \rho(s,x) \cdot M(x) \nabla_x \rho(s,x) dxds\,.\end{aligned}$$

Therefore

$$q_\epsilon \to q \quad \text{strongly in } L^2([0,t] \times A \times \mathbf{R}^N \times \mathbf{R}^N; k_A(x,v,w)dsdxd\mu(v)d\mu(w)))$$

as $\epsilon \to 0$. Besides q satisfies the equality in the Cauchy-Schwarz inequality (36) so that q is of the form

$$q(t,x,v,w) = \lambda(t,x)(b(x,v) - b(x,w)) \cdot \nabla_x \rho(t,x)\,.$$

for some measurable function λ defined a.e. on $\mathbf{R}_+ \times A$. Using (29) shows that $\lambda(t,x) = -1$ for a.e. $(t,x) \in \mathbf{R}_+ \times A$, which concludes the proof.

Remark. The proof of Theorem 3.5 is inspired from the discussion of the Stokes and of the Navier-Stokes limit of the Boltzmann equation initiated in [1]. The procedure in step 9 for obtaining strong convergence is a simplified analogue of the proof of Theorem 6.2 in [1] using the notion of "entropic convergence" (see formula (4.32) in [1]). The discussion bearing on the limiting entropy production in step 5 is a simplified version of Lemma 4.7 in [1]. Notice also the role of the limiting linearized Boltzmann equation (29) and of the limiting collision integrand q, that is reminiscent of the analogous objects considered in [1] (see formula (4.3) and Proposition 4.1 there) in the case of the Boltzmann equation of the kinetic theory of gases. Notice in particular that the next to leading order in the Hilbert expansion, i.e. the second convergence statement in Theorem 3.5 is stated in complete analogy with formula (6.18) in [1] for the case of the Stokes limit of the Boltzmann equation.

7. Conclusions

The main result presented above (Theorem 3.5) can obviously be generalized in several directions.

First, our method obviously applies to a scaled linear Boltzmann equation of the form

$$(\epsilon\partial_t + v \cdot \nabla_x) f_\epsilon(t,x,v) + \frac{1}{\epsilon}\mathcal{L}_x f_\epsilon(t,x,v) + \epsilon \mathcal{B} f_\epsilon(t,x,v) = \epsilon S(t,x,v)$$

where $\mathcal{B}$ is a bounded operator on $L^2(\Omega \times \mathbf{R}^N; dxd\mu)$ and $S \in L^1(\mathbf{R}_+; L^2(\Omega \times \mathbf{R}^N, dxd\mu))$ is a source term. For instance $\mathcal{B}$ could be the multiplication by an amplifying or damping coefficient, i.e. $\mathcal{B} f_\epsilon(t,x,v) = \gamma(x) f_\epsilon(t,x,v)$ as in [3]. In other words, problems where the collision process is nearly, but not exactly conservative can be treated as above.

More general boundary conditions than the absorbing condition on $\partial\Omega$ can also be considered. For instance, imposing a specular or diffuse reflection condition at the boundary, or a convex combination thereof, i.e. assuming that

$$f_\epsilon(t,x,v) = (1-\theta(x))f_\epsilon(t,x,v-2v\cdot n_x n_x) + \frac{\theta(x)}{\langle (w\cdot n_x)_+\rangle}\int_{\mathbf{R}^N} f_\epsilon(t,x,w)(w\cdot n_x)_+ d\mu(w)$$

with $\theta \in C(\partial\Omega)$ satisfying $0 \leq \theta(x) \leq 1$ for all $x \in \partial\Omega$ and with a measure μ invariant under all transformations of the form $v \mapsto Qv$ for $Q \in O_N(\mathbf{R})$ leads to the same result as in Theorem 3.5, except that the homogeneous Dirichlet condition on $\partial\Omega$ should be replaced with the homogeneous Neuman condition.

The methods presented in this paper should also apply to some nonlinear problems, such as the radiative transfer equations. In fact the compactness method used in the proof of Theorem 3.5 finds its origin in [2].

Comparing Theorem 3.5 with the result in [9] is a more delicate issue. We recall that the problem considered in [9] involves the juxtaposition of a medium where the collision cross-section is of order 1 and a highly collisional medium, where the collision cross section is of order $1/\epsilon$. The setting is one dimensional, but extensions to higher dimensions are possible and discussed in [9]. The main result in [9] is a proof of the validity of a domain decomposition strategy where the highly collisional medium is treated by the diffusion equation, with a boundary layer term that is the solution of a Milne problem (see [3]) to accurately describe the interface. The interested reader is referred to [9] for a more accurate description of this domain decomposition algorithm.

At first sight, the situation considered in the present paper is of the same type, as the case of a transition kernel k_ϵ such that $k_\epsilon(x,v,w) = O(1)$ for a.e. $x \in B$ is covered by our assumptions. Yet the result in Theorem 3.5 obviously does not involve any sophisticated treatment of the interface between A and B that would require solving a Milne problem. The difference between both results comes from the type of boundary data considered in [9] and here. In the situation considered in Theorem 3.5, the distribution function of particles entering each connected component of B, i.e. of the region where the collision cross-section is of order 1, is independent of the variable v. For such boundary data, one easily verifies that the boundary layer matching the kinetic and the diffusion domain in [9] is trivial to leading order.

Appendix A. Proof of Lemma 2.3

Let ϕ^{in} and Σ be the extensions of f^{in} and S respectively by 0 for $x \notin \Omega$. Let ϕ be the solution of the Cauchy problem

$$\begin{cases} \epsilon\partial_t\phi_\epsilon + v\cdot\nabla_x\phi_\epsilon = \Sigma\,, & x,v \in \mathbf{R}^N\,,\ t>0\,,\\ \phi_\epsilon\big|_{t=0} = \phi^{in}\,. \end{cases}$$

Then, for all $t \in [0,T]$

$$f_\epsilon(t,x,v) = \phi_\epsilon(t,x,v) \quad \text{for } dxd\mu(v) \text{ a.e. } (x,v) \in \Omega\times\mathbf{R}^N\,. \tag{47}$$

Denoting by $\hat\phi_\epsilon$, $\hat\phi^{in}$ and $\hat\Sigma$ the partial Fourier transforms of ϕ_ϵ, ϕ^{in} and Σ_ϵ respectively, one finds that

$$\epsilon\frac{d}{dt}\left(e^{itv\cdot\xi/\epsilon}\hat\phi_\epsilon(t,\xi,v)\right) = e^{itv\cdot\xi/\epsilon}\hat\Sigma(t,\xi,v)$$

so that the function

$$t \mapsto \hat\psi_\epsilon(t,\xi,v) := e^{itv\cdot\xi/\epsilon}\hat\phi_\epsilon$$

belongs to $C^1([0,T];L^2(\mathbf{R}^N\times\mathbf{R}^N;d\xi d\mu(v)))$ and one has

$$\epsilon\tfrac{1}{2}\frac{d}{dt}|\psi_\epsilon(t,\xi,v)|^2=\Re\left(e^{itv\cdot\xi/\epsilon}\hat{\Sigma}(t,\xi,v)\overline{\hat{\psi}_\epsilon(t,\xi,v)}\right)$$

or equivalently

$$\epsilon\tfrac{1}{2}\frac{d}{dt}|\phi_\epsilon(t,\xi,v)|^2=\Re\left(\hat{\Sigma}(t,\xi,v)\overline{\hat{\phi}_\epsilon(t,\xi,v)}\right)$$

Integrating in $t\in[0,T]$ and $\xi,v\in\mathbf{R}^N$ and applying Plancherel's theorem shows that

$$\begin{aligned}\tfrac{1}{2}\iint_{\Omega\times\mathbf{R}^N}\phi_\epsilon(T,x,v)^2dxdv&\le\tfrac{1}{2}\iint_{\mathbf{R}^N\times\mathbf{R}^N}\phi_\epsilon(T,x,v)^2dxdv\\ =\frac{1}{\epsilon}\int_0^T\iint_{\mathbf{R}^N\times\mathbf{R}^N}\Sigma(t,x,v)\phi_\epsilon(t,x,v)^2dxdvdt&+\tfrac{1}{2}\iint_{\mathbf{R}^N\times\mathbf{R}^N}\phi(0,x,v)^2dxdv\\ =\frac{1}{\epsilon}\int_0^T\iint_{\Omega\times\mathbf{R}^N}S(t,x,v)\phi_\epsilon(t,x,v)^2dxdvdt&+\tfrac{1}{2}\iint_{\Omega\times\mathbf{R}^N}f^{in}(x,v)^2dxdv\end{aligned}$$

and the result follows from (47).

Appendix B. Auxiliary Lemmas on Evolution Equations

Let $\mathcal{V}$ and $\mathcal{H}$ be two separable Hilbert spaces such that $\mathcal{V}\subset\mathcal{H}$ with continuous inclusion and $\mathcal{V}$ is dense in $\mathcal{H}$. The Hilbert space $\mathcal{H}$ is identified with its dual and the map

$$\mathcal{H}\ni u\mapsto L_u\in\mathcal{V}',$$

where L_u is the linear functional

$$L_u:\mathcal{V}\ni v\mapsto(u|v)_{\mathcal{H}}\in\mathbf{R},$$

identifies $\mathcal{H}$ with a dense subspace of $\mathcal{V}'$.

Lemma B.1. *Assume that*

$$v\in L^2(0,T;\mathcal{V})\quad\text{ and }\quad\frac{dL_v}{dt}\in L^2(0,T;\mathcal{V}').$$

Then

a) the function v is a.e. equal to a unique element of $C([0,T],\mathcal{H})$ still denoted v;

b) this function $v\in C([0,T],\mathcal{H})$ satisfies

$$\tfrac{1}{2}|v(t_2)|^2_{\mathcal{H}}-\tfrac{1}{2}|v(t_1)|^2_{\mathcal{H}}=\int_{t_1}^{t_2}\left\langle\frac{dL_v}{dt}(t),v(t)\right\rangle_{\mathcal{V}',\mathcal{V}}dt$$

for all $t_1,t_2\in[0,T]$

Statement a) follows from Proposition 2.1 and Theorem 3.1 in chapter 1 of [13], and statement b) from Theorem II.5.12 of [4].

Lemma B.2. *Let $L\in L^2(0,T;\mathcal{V}')$ satisfy*

$$\langle L(t),w\rangle_{\mathcal{V}',\mathcal{V}}=0\text{ for a.e. }t\in[0,T]$$

for all $w\in\mathcal{V}$. Then

$$L(t)=0\text{ for a.e. }t\in[0,T].$$

Proof. Pick $\mathcal{N}_w \subset [0,T]$ negligible such that L is defined on $[0,T]\setminus\mathcal{N}_w$ and

$$\langle L(t), w\rangle_{\mathcal{V}',\mathcal{V}} = 0 \text{ for all } t \in [0,T]\setminus\mathcal{N}_w\,.$$

Let $\mathcal{D}$ be a dense countable subset of $\mathcal{V}$ and let

$$\bar{\mathcal{N}} := \bigcup_{w\in\mathcal{D}} \mathcal{N}_w\,.$$

For all $t\in[0,T]\setminus\bar{\mathcal{N}}$, one has

$$\langle L(t), w\rangle_{\mathcal{V}',\mathcal{V}} = 0 \text{ for all } w\in\mathcal{D} \quad \text{ so that } L(t)=0$$

because $L(t)$ is a continuous linear functional on $\mathcal{V}$ and $\mathcal{D}$ is dense in $\mathcal{V}$. □

The next lemma recalls the functional background for Green's formula in the context of evolution equations.

Lemma B.3. *Let Ω be an open subset of $\mathbf{R}^N$ with smooth boundary, and let $T>0$. Denote by n the unit outward normal field on $\partial\Omega$. Let $\rho\in C([0,T];L^2(\Omega))$ and $m\in L^2((0,T)\times\Omega,\mathbf{R}^N)$. Assume that*

$$\partial_t\rho + \operatorname{div}_x m = 0 \quad \textit{in the sense of distributions in } (0,T)\times\Omega\,.$$

Then

a) the vector field m has a normal trace $m\cdot n\big|_{(0,T)\times\partial\Omega}\in H^{1/2}_{00}((0,T)\times\partial\Omega)'$;

b) for each $\psi\in H^1(\Omega)$

$$\frac{d}{dt}\int_\Omega \rho(\cdot,x)\psi(x)dx - \int_\Omega m(\cdot,x)\cdot\nabla_x\psi(x)dx = -\langle m\cdot n\big|_{\partial\Omega}, \psi\big|_{\partial\Omega}\rangle_{H^{-1/2}(\partial\Omega),H^{1/2}(\partial\Omega)}$$

in $H^{-1}(0,T)$.

Proof. Let $\chi\in C^\infty_c(\mathbf{R})$ be such that

$$\chi(t)=1 \text{ for } t\in[-1,T+1] \quad \text{ and } \operatorname{supp}(\chi)\subset[-2,T+2]\,.$$

Define

$$\bar\rho(t,x) := \begin{cases} \rho(t,x) & \text{if } 0\le t\le T\\ \chi(t)\rho(0,x) & \text{if } t<0\\ \chi(t)\rho(T,x) & \text{if } t>T\end{cases}$$

and

$$\bar m(t,x) := \begin{cases} m(t,x) & \text{if } 0\le t\le T\\ 0 & \text{if } t\notin[0,T]\end{cases}$$

so that the vector field $X:=(\bar\rho,\bar m)$ is an extension of (ρ,m) to $\mathbf{R}\times\Omega$ satisfying

$$X\in L^2(\mathbf{R}\times\Omega;\mathbf{R}^{N+1})\,.$$

Besides

$$(\partial_t\bar\rho + \operatorname{div}_x\bar m)(t,x) = \chi'(t)(\mathbf{1}_{t<0}\rho(0,x)+\mathbf{1}_{t>T}\rho(T,x)) =: S(t,x)$$

with $S\in L^2(\mathbf{R}\times\Omega)$ so that

$$\operatorname{div}_{t,x} X = S\in L^2(\mathbf{R}\times\Omega)\,.$$

Therefore X has a normal trace on the boundary $\partial(\mathbf{R}\times\Omega)=\mathbf{R}\times\partial\Omega$, denoted $X\cdot n\big|_{\mathbf{R}\times\partial\Omega}\in H^{-1/2}(\mathbf{R}\times\partial\Omega)$.

Let $\phi \in H^{1/2}_{00}((0,T)\times\partial\Omega)$; denote by $\bar\phi$ its extension by 0 to $\mathbf{R}\times\partial\Omega$. Thus $\bar\phi \in H^{1/2}(\mathbf{R}\times\partial\Omega)$ and there exists $\bar\Phi \in H^1(\mathbf{R}\times\Omega)$ such that $\bar\phi = \bar\Phi\big|_{\mathbf{R}\times\partial\Omega}$. The normal trace of m is then defined as follows: by Green's formula

$$\begin{aligned}
&\langle m\cdot n\big|_{\mathbf{R}\times\partial\Omega}, \phi\rangle_{H^{1/2}_{00}((0,T)\times\partial\Omega)', H^{1/2}_{00}((0,T)\times\partial\Omega)}\\
:=\ &\langle X\cdot n\big|_{\mathbf{R}\times\partial\Omega}, \bar\phi\rangle_{H^{1/2}((0,T)\times\partial\Omega)', H^{1/2}((0,T)\times\partial\Omega)}\\
=\ &\iint_{\mathbf{R}\times\Omega}(\bar\rho\partial_t\bar\Phi + \bar m\cdot\nabla_x\bar\Phi + S\bar\Phi)(t,x)\,dxdt\,.
\end{aligned}$$

Applying Green's formula on $(0,T)\times\Omega$ shows that two different extensions of the vector field (ρ,m) define the same distribution $m\cdot n\big|_{(0,T)}\times\partial\Omega)$ on $(0,T)\times\partial\Omega$. This completes the proof of statement a).

As for statement b), let $\kappa\in H^1_0(0,T)$ and $\psi\in H^1(\Omega)$, define $\Phi(t,x) := \kappa(t)\psi(x)$ and let $\bar\Phi$ be the extension of Φ by 0 to $\mathbf{R}\times\Omega$, so that $\bar\Phi\in H^1(\mathbf{R}\times\Omega)$. Thus $\phi = \Phi\big|_{(0,T)\times\partial\Omega} \in H^{1/2}_{00}((0,T)\times\partial\Omega)$ and

$$\begin{aligned}
&\langle\langle m\cdot n\big|_{\partial\Omega}, \psi\big|_{\partial\Omega}\rangle_{H^{-1/2}(\partial\Omega),H^{1/2}(\partial\Omega)}, \kappa\rangle_{H^{-1}(0,T),H^1_0(0,T)}\\
&\quad := \langle m\cdot n\big|_{\mathbf{R}\times\partial\Omega}, \phi\rangle_{H^{1/2}_{00}((0,T)\times\partial\Omega)', H^{1/2}_{00}((0,T)\times\partial\Omega)}\\
&\quad = \iint_{\mathbf{R}\times\Omega}(\bar\rho\partial_t\bar\Phi + \bar m\cdot\nabla_x\bar\Phi + S\bar\Phi)(t,x)\,dxdt\\
&\quad = \int_0^T\int_\Omega(\rho(t,x)\kappa'(t)\psi(x) + m(t,x)\cdot\nabla\psi(x)\kappa(t))\,dxdt\\
&\quad = -\left\langle \frac{d}{dt}\int_\Omega \rho(t,x)\psi(x)dx, \kappa\right\rangle_{H^{-1}(0,T),H^1_0(0,T)}\\
&\qquad + \int_0^T\int_\Omega m(t,x)\cdot\nabla\psi(x)\kappa(t))\,dxdt
\end{aligned}$$

which is precisely the identity in statement b). □

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