THE EULER EQUATIONS IN PLANAR NONSMOOTH CONVEX DOMAINS

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ABSTRACT. Let $\Omega \subset \mathbb{R}^2$ be a bounded convex domain. We consider the Euler system

$$\text{(P)} \qquad \begin{cases} \partial_t \mathbf{u}(x,t) + (\mathbf{u}\cdot\nabla)\mathbf{u}(x,t) + \nabla\pi(x,t) = \mathbf{f}(x,t), & x\in\Omega, t\in(0,T), \\ \nabla\cdot\mathbf{u}(x,t) = 0 & x\in\Omega, t\in(0,T), \end{cases}$$

with impermeability boundary conditions. For $\frac{4}{3} \le p \le 2$, divergence-free initial data $\mathbf{u}_0 = \mathbf{u}(t=0) \in \mathbf{W}^{1,p}(\Omega)$ with zero normal component on $\partial\Omega$, and forcing term $\mathbf{f} \in L^p(0,T;\mathbf{W}^{1,p}(\Omega))$, we show the existence of a pair

$$\mathbf{u} = \mathbf{u}(x,t) \in L^\infty\big(0,T;\mathbf{W}^{1,p}(\Omega)\big) \cap \mathscr{C}\big([0,T];L^2(\Omega)\big), \qquad \pi \in L^p\big(0,T;W^{1,s(p)}(\Omega)\big),$$

which solves the Euler system (P) almost everywhere on $\Omega\times(0,T)$. This enriches and extends the results of Taylor [28].

In the physically interesting case of a rectangle $\Omega = [0,L_1]\times[0,L_2]$, a similar result holds for all $2 < p < \infty$ as well. Moreover, if

$$\operatorname{curl}\mathbf{u}_0 \in L^\infty(\Omega), \quad \mathbf{f} \in L^\infty(0,T;\mathbf{W}^{1,\infty}(\Omega))$$

the solution $\mathbf{u}$ is unique and $\nabla\mathbf{u} \in L^\infty\big(0,T;\mathrm{bmo}_z(\Omega)^{2\times2}\big)$. This is a consequence of a new $\mathrm{bmo}_z(\Omega)$ regularity estimate for the Dirichlet problem on a rectangle.

1. INTRODUCTION

Let $\Omega \subset \mathbb{R}^2$ be a bounded open set. We are concerned with the Euler equations, describing the motion of a perfect inviscid fluid inside Ω,

$$\text{(P)} \qquad \begin{cases} \partial_t \mathbf{u}(x,t) + (\mathbf{u}\cdot\nabla)\mathbf{u}(x,t) + \nabla\pi(x,t) = \mathbf{f}(x,t), & x\in\Omega, t\in(0,T), \\ \nabla\cdot\mathbf{u}(x,t) = 0, & x\in\Omega, t\in(0,T), \end{cases}$$

where $\mathbf{u}(x,t) = \big(u_1(x_1,x_2,t), u_2(x_1,x_2,t)\big)$ stands for the velocity of the fluid particle located at x at time t, $\nabla\pi(x,t)$ stands for the pressure gradient, and $\mathbf{f} = \mathbf{f}(x,t)$ is an external forcing term. We endow (P) with initial and boundary conditions

$$\mathbf{u}(x,0) = \mathbf{u}_0(x), \qquad x\in\Omega, \tag{1.1}$$

$$\mathbf{u}(x,t)\cdot\mathbf{n}(x) = 0, \qquad x\in\partial\Omega,\ t\in(0,T); \tag{1.2}$$

we remark that the impermeability boundary condition (1.2) has to be properly reformulated when $\partial\Omega$ is not regular enough to admit a normal vector $\mathbf{n}$ almost everywhere.

This work is primarily motivated by the study of the well-posedness of the barotropic mode of the inviscid primitive equations of the atmosphere and the oceans [24]. As explained in [6, 7, 16], a certain vertical modal expansion of the primitive equations leads to an infinite system of coupled barotropic - baroclinic modes. In a first approximation, one can neglect the baroclinic modes and we obtain for the barotropic mode a system of equations

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very similar to the two-dimensional inviscid Euler equations in a rectangle. Henceforth this system is called the barotropic system; complementary developments regarding this very problem will appear elsewhere.

The study of the well-posedness of the barotropic system is thus very similar to the study of the well-posedness of the (inviscid) incompressible Euler equations in a rectangle Ω. When Ω is sufficiently regular, the well-posedness of the two-dimensional incompressible Euler equations has been studied by Kato [21] who obtains the existence of smooth solutions, up to $\mathscr{C}^\infty$ solutions with some additional work.

The notion of weak solution to the two-dimensional Euler equations has been introduced by Yudovich [33]. Yudovich, and later Bardos with a different (vanishing viscosity) approach [2] consider initial data with L^p vorticity, and show existence of weak solutions and, in the case of bounded initial vorticity, uniqueness. Among many references on the well-posedness of the incompressible Euler equations in a bounded smooth domain, let us quote the classical articles [2, 4, 34] (see also [29, 30]). Other weaker notions of solutions (initial datum in $\mathbf{L}^2(\Omega)$, with no assumption on the initial vorticity) have been considered by several authors, for instance [10, 25, 26].

In view of the lack of smoothness of Ω, we focus on the notion of weak solutions with L^p vorticity. Our main goal is to recover, in the case of a rectangular domain $\Omega = [0, L_1] \times [0, L_2]$, the known well-posedness theory for smooth domains, establishing existence of weak solutions with L^p vorticity and in addition uniqueness in the case $p = \infty$.

However, for the construction of a solution, we work in the more general case of a bounded convex domain Ω, as in the previous, related article [28], where, for $\frac{12}{7} < p \leq 2$, given a divergence-free initial datum $\mathbf{u}_0$ with $\operatorname{curl} \mathbf{u}_0 \in L^p(\Omega)$, an $L^\infty(0, T; \mathbf{W}^{1,p}(\Omega))$ weak solution is constructed as the limit of smooth solutions to (P) set on smooth convex subdomains Ω_n increasing to Ω. This principle has also been used in [15] for more general domains (complements of a finite number of compact connected sets with positive Sobolev capacity), albeit the notion of solution considered there is significantly weaker.

We follow the same principle, but make a more systematic study of the approximated problems (P_n). In particular we impose a uniform Lipschitz character to the domains Ω_n. This uniformity reflects on the uniform boundedness of the Leray projectors associated to each subdomain, which we exploit in our compactness arguments. As a result, our soluton is slightly more regular than the one constructed in [28], being continuous in time with values in $\mathbf{L}^2(\Omega)$ and belonging to $\mathbf{W}^{1,p}(\Omega)$ for each time $t \in [0, T]$, and satisfying (P) almost everywhere in $\Omega \times (0, T)$. The range of allowed exponents is $p \in [\frac{4}{3}, 2]$. The restriction $p \geq \frac{4}{3}$ can be lifted if one is interested in a weaker notion of solution than the one given in Subsection 4.1.2. The restriction $1 < p \leq 2$ corresponds to the sharp range of exponents for the regularity of the Biot-Savart law (see Theorem 2.2), giving the gradient of the velocity in terms of the vorticity, in a general convex domain, and seems unavoidable if one aims for a *reversible* existence result (in the sense that $\mathbf{u}(t)$ for $t > 0$ belongs to the same space where the data $\mathbf{u}(0)$ is required to be, so that we can solve the backward Euler equations with initial data $\mathbf{u}(t)$). The construction is summarized in Proposition 4.1 and Theorem 5.1.

For a rectangular Ω, or more generally, a convex polygonal-like domain (that is, $\partial\Omega$ is a connected finite union of $\mathscr{C}^2$ graphs) with internal angles $\alpha_i \leq \frac{\pi}{2}$, this result extends to $p \in (2, \infty)$ as well (Theorem 5.2). Indeed, for this more restricted class of domains, the range of exponents for the regularity of the Biot-Savart law is $(1,\infty)$, a consequence of Grisvard's results [18, 19]. See Remark 2.2 for a brief discussion of this point.

Theorem 5.2 also contains the main result of uniqueness of solutions with bounded initial vorticity in a rectangle. The proof follows Yudovich's energy method, and relies on the endpoint $L^\infty(\Omega) \to \mathbf{W}^{2,\mathrm{bmo}}(\Omega)$ regularity result for the solution to the Dirichlet problem on a rectangle, Proposition 3.1, which appears to be new. We do not dwell on this point, but unbounded initial vorticities with $\log\log$-type blowup of the L^p-norms as $p \to \infty$, like in [34], would also suffice for uniqueness.

Plan of the paper. In Section 2, we develop the necessary tools for the analysis of (P) on a bounded convex non-smooth domain Ω. Section 3 contains the endpoint-type regularity result for the Dirichlet problem on a rectangle, which will be instrumental in establishing uniqueness of solutions. In Section 4 we give a weak formulation of (P), and construct a weak solution to (P) on a bounded convex domain (Proposition 4.1). Section 5 contains the statements and the proofs of the main results, Theorems 5.1 and 5.2.

Notation. Given a domain $\Omega \subset \mathbb{R}^2$, and scalar functions $u, v : \Omega \to \mathbb{R}$, vector valued functions $\mathbf{u}, \mathbf{v} : \Omega \to \mathbb{R}^2$, we denote

$$(u,v)_\Omega := \int_\Omega u\,v, \qquad (\mathbf{u},\mathbf{v})_\Omega := \int_\Omega \mathbf{u}\cdot\mathbf{v}.$$

Throughout, for $p \in [1,\infty]$, we use the notations

$$p' = \frac{p}{p-1}, \qquad p^* = \begin{cases} \frac{2p}{2-p} & 1 \le p < 2 \\ \infty & p \ge 2 \end{cases}$$

respectively for the Hölder and Sobolev conjugate exponents of p.

Bump functions. Let $d \ge 1$, and $\phi^d : \mathbb{R}^d \to \mathbb{R}$ be a smooth nonnegative radial function supported in $\{|x| \le \frac{1}{2}\}$, equal to 1 in $\{|x| \le \frac{1}{4}\}$, and with $\int_{\mathbb{R}^d} \phi^d = 1$. We will make use of the functions

$$\phi^d_\varepsilon(x) = (\mathrm{Dil}^1_\varepsilon \phi^d)(x) := \varepsilon^{-d}\phi\left(\varepsilon^{-1}x\right), \qquad \varepsilon > 0,$$

and refer to them as L^1-bump functions (or L^1-bumps). Indeed, note that $\int_{\mathbb{R}^d} \phi^d_\varepsilon(x)\,\mathrm{d}x = 1$.

2. Elliptic regularity in a bounded convex domain

In this section, we set the foundation for our analysis of the Euler system (P). Throughout the section, Ω is an open, bounded, convex subset of $\mathbb{R}^2$ which contains the origin. In Section 3, we will specialize to the case of a rectangular domain and develop further elliptic regularity results.

We first recall the analytic properties of the boundary $\partial\Omega$ and construct an approximation of Ω by an increasing sequence of convex smooth subdomains with uniformly Lipschitz boundary. Then, we describe the normal trace operator on Ω, introduce the class of tangential vector fields, and establish the Helmholtz decomposition of $\mathbf{L}^p(\Omega)$, for $1 < p < \infty$. Finally, we discuss some regularity results for the Dirichlet problem in Ω, which we exploit to define the spaces in which the evolution of the Euler system (P) will take place.

2.1. **Regularity and approximation of bounded convex domains.** We begin with a proposition.

Proposition 2.1. *There exist positive constants* M_Ω, δ_Ω, *and a finite collection of open squares* $\{Q_i : i = 1, \ldots, N_\Omega\}$ *of diameter* δ *such that:*

- $\overline{\Omega} \subset \bigcup_{i=1}^{N_\Omega} Q_i, \quad Q_i = c(Q_i) + Q_0, \quad Q_0 = \{|y_1|, |y_2| < \delta_\Omega, \},$
- *whenever* $Q_i \cap \partial\Omega \neq \emptyset$, *there exists a function* $\beta_i : (-\delta_\Omega, \delta_\Omega) \to \mathbb{R}$, *in the coordinates with origin the center of* Q_i *and oriented along the sides of* Q_i, *with the properties*
  - β_i *is convex and Lipschitz with constant* M_Ω*;*
  - $Q_i \cap \partial\Omega = \{(y_1, \beta_i(y_1)) : y_1 \in (-\delta_\Omega, \delta_\Omega)\}$,
  - $Q_i \cap \Omega = \{(y_1, y_2) \in Q_i : y_2 > \beta_i(y_1)\}$,
- Ω *has the strong local Lipschitz property with constants* $M_\Omega, \delta_\Omega, N_\Omega$.

Proof. It is known (see [18, Corollary 1.2.2.3] that Ω bounded and convex implies that Ω has a Lipschitz boundary (in the sense of [18, Definition 1.2.2.1]), which is exactly what is described in the first two assertions. The fact that β_i is convex is a consequence of the fact that its epigraph $Q_i \cap \Omega$ is convex. Finally, the first two assertions imply the strong local Lipschitz property with constants $M_\Omega, \delta_\Omega, N_\Omega$ as described in [1, IV.4.2]. □

2.1.1. *Approximation by smooth convex subdomains.* We construct a sequence of smooth convex domains Ω_n increasing to Ω, that is

$$\Omega_n \text{ smooth convex}, \qquad \Omega_n \Subset \Omega_{n+1} \Subset \cdots \Subset \Omega, \qquad \Omega = \bigcup_n \Omega_n. \tag{2.1}$$

and with the property that

$$\text{the constants in the strong local Lipschitz property of } \Omega \text{ are uniform in } n. \tag{2.2}$$

We introduce the Minkowski functional of Ω

$$\mu_\Omega : \overline{\Omega} \to [0, \infty), \qquad \mu_\Omega(x) = \inf\{\lambda > 0 : \lambda^{-1}x \in \overline{\Omega}\}.$$

The function μ_Ω is a convex function on $\overline{\Omega}$ (see [20, pp. 57-59]). A convex function on a compact subset of any normed space is globally Lipschitz (see [23] for a simple proof): thus, call L_Ω the Lipschitz constant of μ_Ω. The function $\rho = \mu_\Omega - 1$ is convex and Lipschitz with the same constant L_Ω, and

$$\Omega = \{x \in \overline{\Omega} : \rho(x) < 0\}, \qquad \partial\Omega = \{x \in \overline{\Omega} : \rho(x) = 0\}$$

For $\varepsilon > 0$, let Ω^ε be the ε-neighborhood of Ω. It is easy to verify that the mollification (see Section 1 for notation)

$$\rho_\varepsilon : \Omega^\varepsilon \to [-1, 0], \qquad \rho_\varepsilon := \rho * \phi_\varepsilon^2$$

is smooth and convex, and moreover that the Lipschitz constants of $\{\rho_\varepsilon : \varepsilon > 0\}$ are uniformly bounded by L_Ω. Finally, we have that, for $\varepsilon \to 0, \rho_\varepsilon \to \rho$ in the uniform Lipschitz norm, that is

$$\sup_{x,y\in\Omega} \frac{|\rho_\varepsilon(x) - \rho(x) - \rho_\varepsilon(y) + \rho(y)|}{|x-y|} \to 0, \qquad \varepsilon \to 0.$$

Choose a subsequence n_k with

$$\sup_{x\in\Omega}\left|\rho_{\frac{1}{n_k}}(x)-\rho_{\frac{1}{n_{k+1}}}(x)\right|<(n_k)^{-3},$$

and define

$$\Omega_k:=\left\{x\in\Omega:\rho_{\frac{1}{n_k}}(x)<-\tfrac{1}{n_k}\right\};$$

it follows that Ω_k is a convex open set, $\Omega_k\Subset\Omega_{k+1}\Subset\Omega$, and that $\bigcup\Omega_k=\Omega$. Moreover the $\partial\Omega_k$ are smooth, and uniformly Lipschitz (with respect to k). Thus each Ω_k has the strong local Lipschitz property with Lipschitz constant uniformly bounded in k. The construction (2.2) is thus completed.

Several important consequences of (2.1)-(2.2) will be derived in the next subsections. Here, we mention that (2.2) guarantees that the implicit constants appearing in the Sobolev embeddings and trace theorems on Ω_n, which depend on the constants in the strong local Lipschitz property (see Theorem IV.4.1 and its proof in [1] for instance) are uniform in n (they do depend on Ω however, through $M_\Omega,\delta_\Omega,N_\Omega$ and L_Ω).

2.2. **Normal vector, normal traces, tangential vector fields.** Maintaining the notation of Proposition 2.1, we observe that for each i, $\beta_i'(y_1)$ is defined almost everywhere on $(-\delta_\Omega,\delta_\Omega)$ and $|\beta_i'(y_1)|\le M_\delta$ wherever defined. We can thus define the normal vector almost everywhere on $Q_i\cap\partial\Omega$ by

$$\mathbf{n}(x)=\mathbf{n}(y_1,\beta_i(y_1)):=\frac{(-\beta_i'(y_1),1)}{\sqrt{1+(\beta_i'(y_1))^2}},\qquad x=(y_1,\beta_i(y_1))\in Q_i\cap\partial\Omega.$$

This definition can be extended to all of Q_i by $\mathbf{n}(y_1,y_2)=\mathbf{n}(y_1,\beta_i(y_1))$, and using a partition of unity subordinated to the covering of $\overline{\Omega}$ by the Q_i, to a bounded vector field on all of $\overline{\Omega}$.

Thus, for $\mathbf{v}\in\mathscr{D}(\overline{\Omega};\mathbb{R}^2)$,

$$\gamma_{\mathbf{n}}\mathbf{v}(x):=\mathbf{v}(x)\cdot\mathbf{n}(x),\quad \gamma_{\mathbf{t}}\mathbf{v}(x):=\mathbf{v}^\perp(x)\cdot\mathbf{n}(x),\qquad x\in\partial\Omega$$

is defined almost everywhere on $\partial\Omega$. We are interested in the spaces

$$\mathbf{L}^p_{\mathrm{div}}(\Omega)=\{\mathbf{v}\in\mathbf{L}^p(\Omega):\operatorname{div}\mathbf{v}\in L^p(\Omega)\},\qquad 1\le q\le\infty.$$

The classical $W^{1,p}(\Omega)\to W^{1-\frac{1}{q},q}(\partial\Omega)$ trace theorem due to Gagliardo [13] yields the lemma below, arguing along the same lines of [31, Theorem I.1.2]. For a definition of Sobolev spaces (of fractional order) on Lipschitz submanifolds of $\mathbb{R}^d$, see Subsection 1.3.3 in [18]. Note that if $f\in\mathscr{D}(\overline{\Omega})$ (that is, f is restriction to $\overline{\Omega}$ of a function in $\mathscr{C}^\infty(\mathbb{R}^2)$), then the restriction of f to $\partial\Omega$ is a Lipschitz function on the Lipschitz submanifold $\partial\Omega$.

Lemma 2.1. *Let $\Omega\subset\mathbb{R}^2$ be a bounded convex domain and $p\in(1,\infty)$. The map $\gamma_{\mathbf{n}}:\mathbf{v}\in\mathscr{D}(\overline{\Omega};\mathbb{R}^2)\to\gamma_{\mathbf{n}}\mathbf{v}\in\mathrm{Lip}(\partial\Omega)$ extends as a linear bounded map*

$$\gamma_{\mathbf{n}}:\mathbf{L}^p_{\mathrm{div}}(\Omega)\to W^{-\frac{1}{p'},p'}(\partial\Omega)$$

and the generalized Stokes formula

$$(\mathbf{v},\nabla\varphi)_\Omega+(\operatorname{div}\mathbf{v},\varphi)_\Omega=\langle\gamma_{\mathbf{n}}\mathbf{v},\gamma_0\varphi\rangle \tag{2.3}$$

holds for every $\mathbf{v}\in\mathbf{L}^p_{\mathrm{div}}(\Omega)$ and $\varphi\in W^{1,p'}(\Omega)$.

Remark 2.1. If Ω_n is an approximating sequence of domains as in Subsection 2.1.1, the norms of the $\gamma_{\mathbf{n}} \in \mathscr{L}(\mathbf{L}^p_{\mathrm{div}}(\Omega)) \to W^{-1/p',p'}(\partial\Omega_n))$ are uniformly bounded in n. This uniformity descends from the uniformity of the constants in the Gagliardo trace theorem, [13].

2.2.1. $\mathbf{L}^p$*-tangential vector fields.* We say that $\mathbf{v} \in \mathbf{L}^1(\Omega)$ is a tangential divergence free vector field if

$$(\mathbf{v}, \nabla\varphi)_\Omega = 0 \qquad \forall \varphi \in \mathscr{D}(\overline{\Omega}). \tag{2.4}$$

As a consequence of Lemma 2.1, it follows that

$$\mathbf{L}^p_\tau(\Omega) := \{\mathbf{v} \in L^p(\Omega) : (2.4) \text{ holds}\} = \{\mathbf{v} \in \mathbf{L}^p_{\mathrm{div}}(\Omega) : \operatorname{div}\mathbf{v} = 0, \gamma_{\mathbf{n}}\mathbf{v} = 0\}. \tag{2.5}$$

2.3. **Helmholtz decomposition.** Let $\mathscr{V} := \{\mathbf{v} \in \mathscr{D}(\Omega; \mathbb{R}^2) : \operatorname{div}\mathbf{v} = 0\}$. It is well known (see for example Theorem I.1.4 in [31]) that for Ω Lipschitz

$$\mathbf{L}^2_\tau(\Omega) = \text{ the closure of } \mathscr{V} \text{ in } \mathbf{L}^2(\Omega), \qquad \mathbf{L}^2_\tau(\Omega)^\perp = \{\nabla\pi : \pi \in H^1(\Omega)\}. \tag{2.6}$$

Let us denote by $P_\Omega : \mathbf{L}^2(\Omega) \to \mathbf{L}^2_\tau(\Omega)$ the corresponding orthogonal projector. The following result, due to Geng and Shen [14] allows us to obtain (2.6) for all $1 < p < \infty$ and extend P_Ω boundedly to $\mathbf{L}^p(\Omega)$. Note that Theorem I.1.4 in [31] (i.e. the case $p = 2$ of Theorem 2.1 below) holds whenever Ω is Lipschitz; actually, the range $p \in \left(\frac{3}{2} - \varepsilon, 3 + \varepsilon\right)$ is known to be sharp for general Lipschitz domains [11].

Theorem 2.1. *Let $\Omega \subset \mathbb{R}^2$ be a bounded convex domain and $1 < p < \infty$. The following hold:*

$$\mathbf{L}^p_\tau(\Omega) = \text{ the closure of } \mathscr{V} \textit{ in } \mathbf{L}^p(\Omega); \tag{2.7}$$

P_Ω extends to a bounded linear operator

$$P_\Omega : \mathbf{L}^p(\Omega) \to \mathbf{L}^p_\tau(\Omega), \qquad P_\Omega|_{\mathbf{L}^p_\tau(\Omega)} = \mathrm{I}_{\mathbf{L}^p_\tau(\Omega)},$$

with operator norm only depending on M_Ω, δ_Ω, N_Ω, L_Ω. Moreover, for each $\mathbf{v} \in \mathbf{L}^p(\Omega)$, there exists $\pi \in W^{1,p}(\Omega)$, unique up to an additive constant such that

$$P^\perp_\Omega \mathbf{v} := (I - P_\Omega)\mathbf{v} = \nabla\pi. \tag{2.8}$$

Proof. The second and third statements appear almost verbatim in [14, Theorem 1.3]. Let us show how they imply (2.7); denote by $\mathbf{H}^p_\tau$ the space on the right hand side of (2.7). The backward inclusion is easy (see the proof of [31, Theorem I.1.4] for example). To get equality, we begin by characterizing the annihilator of $\mathbf{H}^p_\tau$ in $\mathbf{L}^p_\tau(\Omega)$ as

$$\left(\mathbf{H}^p_\tau\right)^\perp := \left\{F \in \mathscr{L}(\mathbf{L}^p_\tau(\Omega), \mathbb{R}) : F|_{\mathbf{H}^p_\tau} = 0\right\} = \nabla W^{1,p'}(\Omega), \quad \nabla W^{1,p'}(\Omega) := \{\nabla\pi : \pi \in W^{1,p'}(\Omega)\}. \tag{2.9}$$

Proof of (2.9). It is known (see [32] for a simple proof) that

$$F \in \mathscr{D}'(\Omega; \mathbb{R}^2), \qquad F(\mathbf{v}) = 0 \quad \forall \mathbf{v} \in \mathscr{V} \implies F = \nabla\pi,\ \pi \in \mathscr{D}'(\Omega).$$

Now from the Riesz representation theorem, for each $F \in \mathscr{L}(\mathbf{L}^p_\tau(\Omega), \mathbb{R})$ there exists a (possibly nonunique) $\mathbf{f} \in \mathbf{L}^{p'}(\Omega)$ such that $F(\mathbf{v}) = (\mathbf{f}, \mathbf{v})_\Omega$ for all $\mathbf{v} \in \mathbf{L}^p_\tau(\Omega)$. From the above characterization, it follows that $\mathbf{f} = \nabla\pi$, and thus $\pi \in W^{1,p'}(\Omega)$. □

With (2.9) in hand, we show that $\left(\mathbf{H}^p_\tau\right)^\perp = \{\mathbf{0}\}$, thus proving (2.7). Let $F \in \left(\mathbf{H}^p_\tau\right)^\perp$, and $\mathbf{f} \in \mathbf{L}^{p'}(\Omega)$ as above. By (2.8), it follows that $\mathbf{f} = (I - P_\Omega)\mathbf{f}$, i.e. $P_\Omega \mathbf{f} = 0$. Therefore

$$(\mathbf{f}, \mathbf{v})_\Omega = (\mathbf{f}, P_\Omega \mathbf{v})_\Omega = (P_\Omega \mathbf{f}, \mathbf{v})_\Omega = 0 \qquad \forall\, \mathbf{v} \in \mathbf{L}^p_\tau(\Omega),$$

and this proves the last claim. Here we used the same notation (with slight abuse) for both $P_\Omega : \mathbf{L}^p(\Omega) \to \mathbf{L}^p(\Omega)$ and $P_\Omega : \mathbf{L}^{p'}(\Omega) \to \mathbf{L}^{p'}(\Omega)$. □

2.4. **The Dirichlet problem and the Biot-Savart law.** Denote by

$$A_\Omega = -\Delta \text{ on } \Omega \text{ with Dirichlet boundary conditions}, \qquad \operatorname{dom}(A_\Omega) = H^2(\Omega) \cap H^1_0(\Omega).$$

We recall the following consequence of the Lax-Milgram lemma.

Proposition 2.2. *Let $f \in H^{-1}(\Omega)$ be given. Then there exists a unique $F \in H^1_0(\Omega)$ satisfying*

$$A_\Omega F = f, \qquad \|F\|_{H^1_0(\Omega)} \le \|f\|_{H^{-1}(\Omega)}. \tag{2.10}$$

Referring to (2.10), we use the notation $F = \mathrm{G}_\Omega f$. The classical theory of elliptic equations (see for example [17]) tells us that, when Ω is a bounded smooth domain, and $f \in L^p(\Omega)$, $\mathrm{G}_\Omega f$ has two derivatives in $L^p(\Omega)$ for any $1 < p < \infty$. This is no longer true in general if the domain Ω is merely bounded and convex. However, the above result still holds in the range $1 < p \le 2$: we state this precisely in the theorem below, due to Fromm [12].

Theorem 2.2. *Let $\Omega \subset \mathbb{R}^2$ be a bounded convex domain. Let $1 < p \le 2$ and $f \in L^p(\Omega)$ be given. Then, there exists a positive constant $C_{p,\Omega}$, depending only on p and on the Lipschitz character of Ω, such that*

$$\mathrm{G}_\Omega f \in W^{2,p}(\Omega) \cap W^{1,p}_0(\Omega), \qquad \|\mathrm{G}_\Omega f\|_{W^{2,p}(\Omega)} \le C_{p,\Omega} \|f\|_{L^p(\Omega)}. \tag{2.11}$$

Remark 2.2. We wish to make a remark on a more restricted class of domains, which however includes the physically interesting case of a rectangle. We say that $\Omega \subset \mathbb{R}^2$ is a *polygonal-like domain* if it is a bounded convex open set and $\partial\Omega$ is a piecewise $\mathscr{C}^2$ planar curve, with finitely many points $\{S_1, \ldots, S_N\}$ of discontinuity for the tangent vector $\mathbf{n}(x)$, $x \in \partial\Omega$; we call Γ_j the component of $\partial\Omega$ with end points $S_j, S_{j+1} (S_{N+1} = S_1)$. Then the tangent and normal vectors $\mathbf{t}_j, \mathbf{n}_j$ to Γ_j belong to $\mathscr{C}^1(\Gamma_j)$, for each $j = 1, \ldots, N$, and we write

$$\mathbf{n}_-(S_j) = \lim_{\substack{x \to S_j \\ x \in \Gamma_{j-1}}} \mathbf{n}_j(x), \qquad \mathbf{n}_+(S_j) = \lim_{\substack{x \to S_j \\ x \in \Gamma_j}} \mathbf{n}_j(x).$$

We assume for definiteness that $\mathbf{n}_-(S_j) \neq \mathbf{n}_+(S_j)$. Due to the convexity of Ω,

$$0 < \alpha_j := \operatorname{angle}(\mathbf{n}_-(S_j), \mathbf{n}_+(S_j)) < \pi, \qquad j = 1, \ldots, N, \tag{2.12}$$

and we set $\underline{\alpha} := \min\{\alpha_j\} > 0$, $\overline{\alpha} := \max\{\alpha_j\} < \pi$. For this class of domains, under the additional assumption that $\overline{\alpha} \le \frac{\pi}{2}$, given $1 < p < \infty$ and $f \in L^p(\Omega)$, the unique solution $\mathrm{G}_\Omega f \in H^1(\Omega)$ belongs to $W^{2,p}(\Omega)$ as well and satisfies the estimate

$$\|\mathrm{G}_\Omega f\|_{W^{2,p}(\Omega)} \le C_{p,\Omega} \|f\|_{L^p(\Omega)}. \tag{2.13}$$

If it is the case that $\alpha_j > \frac{\pi}{2}$ for some j, a singular part is present and analogues of (2.13) only hold in a range of exponents $1 < p < p_0$, with $p_0 > 2$ depending on $\overline{\alpha}$. These results are taken from Sections 4.1-4.4 of Grisvard's classical treatise [18]. See also [19].

Tracking the proof in [18], one finds that dependence on p of the constant $C_{p,\Omega}$ in (2.13) as $p \to \infty$ is $O(p^2)$, in contrast to the case of smooth domains, where $C_{p,\Omega} = O(p)$. As explained in Section 5, this dependence plays a key role in the uniqueness of solutions with bounded initial vorticity to the Euler system (P). A consequence of our analysis in Section 3 is that $C_{p,\Omega} = O(p)$ in the particular case of a rectangular domain.

Let Ω_n be the approximating sequence to Ω constructed in (2.1). In this context, for a function $F : \Omega_n \to \mathbb{R}$, we denote by $\widetilde{F}$ its extension by zero to Ω. Note that

$$F \in W_0^{1,p}(\Omega_n) \implies \widetilde{F} \in W_0^{1,p}(\Omega) \quad \text{and} \quad \|\widetilde{F}\|_{W_0^{1,p}(\Omega)} = \|F\|_{W_0^{1,p}(\Omega_n)}. \tag{2.14}$$

We state and prove a so-called Γ-convergence type result. The restrictions $f_n, f \in L^1_{\mathrm{loc}}(\Omega)$ can be lifted, but we only need the particular case contained in the lemma below.

Lemma 2.2. *Let $f_n, f \in L^1_{\mathrm{loc}}(\Omega)$ and set $(f_n)_{\Omega_n} = f_n \mathbf{1}_{\Omega_n}$. Then*

$$f_n \to f \text{ in } H^{-1}(\Omega) \implies \widetilde{\mathrm{G}_{\Omega_n}(f_n)_{\Omega_n}} \to \mathrm{G}_\Omega f \text{ weakly in } H_0^1(\Omega). \tag{2.15}$$

Proof. By density it suffices to show that

$$\lim_{n\to\infty} \big(\nabla \widetilde{\mathrm{G}_{\Omega_n}(f_n)_{\Omega_n}} - \nabla \mathrm{G}_\Omega f, \nabla\varphi\big)_\Omega = 0 \qquad \forall \varphi \in \mathscr{D}(\Omega).$$

Fix $\varphi \in \mathscr{D}(\Omega)$ and choose N such that $\operatorname{supp}\varphi := D \Subset \Omega_N$. Then for $n \ge N$, $\operatorname{supp}\varphi \subset \Omega_n$ and

$$\begin{aligned}
\big(\nabla \widetilde{\mathrm{G}_{\Omega_n}(f_n)_{\Omega_n}}, \nabla\varphi\big)_\Omega &= \big(\nabla \widetilde{\mathrm{G}_{\Omega_n}(f_n)_{\Omega_n}}, \nabla\varphi\big)_{\Omega_n} = \big(\nabla \mathrm{G}_{\Omega_n}(f_n)_{\Omega_n}, \nabla\varphi\big)_{\Omega_n} \\
&= -(\Delta \mathrm{G}_{\Omega_n}(f_n)_{\Omega_n}, \varphi)_{\Omega_n} \\
&= -(f_n, \varphi)_{\Omega_n} = -(f_n, \varphi)_\Omega,
\end{aligned}$$

and similarly $\big(\nabla \mathrm{G}_\Omega f, \nabla\varphi\big)_\Omega = -(f, \varphi)_\Omega$. Thus

$$\big(\nabla \widetilde{\mathrm{G}_{\Omega_n}(f_n)_{\Omega_n}} - \nabla \mathrm{G}_\Omega f, \nabla\varphi\big)_\Omega = (f_n - f, \varphi)_\Omega = \langle f_n - f, \varphi\rangle \to 0,$$

and the lemma follows. □

We now make more explicit the connection between the Euler system and A_Ω. Set

$$\mathbf{K}_\Omega f := \nabla^\perp(\mathrm{G}_\Omega f), \qquad f \in H^{-1}(\Omega). \tag{2.16}$$

We have that

$$\mathbf{K}_\Omega \in \mathscr{L}(H^{-1}(\Omega), \mathbf{L}^2_\tau(\Omega)), \tag{2.17}$$

$$\mathbf{K}_\Omega \in \mathscr{L}(L^p(\Omega), \mathbf{W}^{1,p}_\tau(\Omega)), \qquad 1 < p \le 2. \tag{2.18}$$

Proof of (2.17)-(2.18). Due to (2.11) and (2.10), we are only left to verify that

$$\operatorname{div} \mathbf{K}_\Omega f = 0, \qquad \gamma_{\mathbf{n}} \mathbf{K}_\Omega f = 0, \qquad \forall f \in H^{-1}(\Omega). \tag{2.19}$$

Let $\varphi \in \mathscr{D}(\overline{\Omega})$. We then have, integrating by parts,

$$(\mathbf{K}_\Omega f, \nabla\varphi)_\Omega = (\nabla(\mathrm{G}_\Omega f), \nabla^\perp\varphi)_\Omega = -(\mathrm{G}_\Omega f, \operatorname{div}\nabla^\perp\varphi)_\Omega + \int_{\partial\Omega} (\mathrm{G}_\Omega f)\nabla^\perp\varphi \cdot \mathbf{n},$$

and both terms vanish in the right-hand side. We conclude by means of (2.5). □

The next lemma shows that curl is the left inverse of $\nabla^\perp \circ \mathrm{G}_\Omega$ on $L^p(\Omega)$, $1 < p \le 2$.

Lemma 2.3. *Let $1 < p \le 2$ and $f \in L^p(\Omega)$. Then*

$$f = \operatorname{curl} \mathbf{K}_\Omega f. \tag{2.20}$$

Proof. By density, it suffices to show that

$$(f,\varphi)_\Omega = \big(\operatorname{curl}\mathbf{K}_\Omega f,\varphi\big), \qquad \forall\varphi\in\mathscr{D}(\Omega). \tag{2.21}$$

Let now $\varphi\in\mathscr{D}(\Omega)$. We have

$$\begin{aligned}(f,\varphi)_\Omega &= (-\Delta \mathrm{G}_\Omega f,\varphi)_\Omega = (\mathrm{G}_\Omega f,-\Delta\varphi)_\Omega = (\nabla^\perp \mathrm{G}_\Omega f,\nabla^\perp\varphi)_\Omega \\ &= (\mathbf{K}_\Omega f,\nabla^\perp\varphi)_\Omega = (\operatorname{curl}\mathbf{K}_\Omega f,\varphi)_\Omega - \int_{\partial\Omega}\varphi(\mathbf{K}_\Omega f)^\perp\cdot\mathbf{n},\end{aligned}$$

and the last term on the right hand side is zero. We integrated by parts in the last equality, which is legitimate since $\mathbf{K}_\Omega f\in W^{1,p}(\Omega)$. □

2.5. **The spaces $\mathbf{V}^{1,p}$.** The evolution of our solution to the Euler system (P) will take place in the space of $\mathbf{L}^p$ tangential vector-fields with L^p vorticity. That is, we define, for $1<p<\infty$,

$$\mathbf{V}^{1,p}(\Omega) := \big\{\mathbf{v}\in\mathbf{L}^p_\tau(\Omega) : \operatorname{curl}\mathbf{v}\in L^p(\Omega)\big\}, \qquad \|\mathbf{v}\|_{\mathbf{V}^{1,p}(\Omega)} := \big(\|\mathbf{v}\|^p_{\mathbf{L}^q(\Omega)} + \|\operatorname{curl}\mathbf{v}\|^p_{L^p(\Omega)}\big)^{\frac{1}{p}}. \tag{2.22}$$

As a consequence of Lemma 2.3 and Theorem 2.2, when $1<p\le 2$, we have the continuous embedding $\mathbf{V}^{1,p}(\Omega)\hookrightarrow\mathbf{W}^{1,p}(\Omega)$:

$$\|\mathbf{v}\|_{\mathbf{W}^{1,p}(\Omega)} \le C_{p,\Omega}\|\mathbf{v}\|_{\mathbf{V}^{1,p}(\Omega)} \qquad \forall\mathbf{v}\in\mathbf{V}^{1,p}(\Omega),\ 1<p\le 2. \tag{2.23}$$

The embedding discussed above allows for an improvement of the boundary regularity of functions in $\mathbf{V}^{1,p}(\Omega)$, by further applying Gagliardo's trace theorem to the components of $\mathbf{v}$. More precisely,

$$\mathbf{v}|_{\partial\Omega}\in\mathbf{W}^{1-\frac{1}{p},p}(\partial\Omega), \qquad \forall 1<p\le 2. \tag{2.24}$$

In view of (2.24), we can therefore make sense of $\gamma_{\mathbf{n}}\mathbf{v}$ as $\mathbf{v}\cdot\mathbf{n}|_{\partial\Omega}$ whenever $\mathbf{v}\in\mathbf{V}^{1,p}(\Omega)$.

3. Elliptic regularity in a rectangle

We assume throughout this section that $\Omega = [0,L_1]\times[0,L_2]$. Hereafter, we develop an appropriate substitute of Theorem 2.2 in the range $2<p\le\infty$, with spaces of functions with bounded mean oscillation replacing L^∞.

3.1. **Local bmo spaces.** We denote by $\mathrm{bmo}(\mathbb{R}^2)$ the strict subspace of $\mathrm{BMO}(\mathbb{R}^2)$ normed by

$$\|f\|_{\mathrm{bmo}(\mathbb{R}^2)} = \sup_{|Q|<1}\frac{1}{|Q|}\int_Q |f(x)-f_Q|\,\mathrm{d}x + \sup_{|Q|\ge 1}\frac{1}{|Q|}\int_Q |f(x)|\,\mathrm{d}x \tag{3.1}$$

where the suprema above are taken over squares $Q\subset\mathbb{R}^2$ and f_Q denotes the average of f on the cube Q. See for example [5] for more details.

Let $D\subset\mathbb{R}^2$ be a domain. For a function $f:\overline{D}\to\mathbb{R}$, let $\overline{f}$ be its extension to zero outside $\overline{D}$, i.e. $\overline{f} := f\mathbf{1}_{\overline{D}}$. We define the Banach spaces

$$\mathrm{bmo}_z(D) = \{f:\overline{D}\to\mathbb{R} : \overline{f}\in\mathrm{bmo}(\mathbb{R}^2)\}, \qquad \|f\|_{\mathrm{bmo}_z(D)} := \|\overline{f}\|_{\mathrm{bmo}(\mathbb{R}^2)},$$

i.e. the space of functions on $\overline{D}$ whose trivial extension is in $\mathrm{bmo}(\mathbb{R}^2)$. The continuous embedding $L^\infty(D)\hookrightarrow\mathrm{bmo}_z(D)$ is immediate to verify; this, together with John-Nirenberg's inequality

$$\|f\|_{L^p(D)} \le C_D p\|f\|_{\mathrm{bmo}_z(D)}, \tag{3.2}$$

where the constant C_D is only dependent on diam(D), hints at the relevance of $\mathrm{bmo}_z(D)$ as a substitute for $L^\infty(D)$. We use the notation

$$W^{2,\mathrm{bmo}_z}(D)=\{f\in \mathrm{bmo}_z(D): D^2 f\in \mathrm{bmo}_z(D)^{2\times 2}\},\quad \|f\|_{W^{2,\mathrm{bmo}_z}(D)}:=\|f\|_{\mathrm{bmo}_z(D)}+\|D^2 f\|_{\mathrm{bmo}_z(D)^{2\times 2}}.$$

The next theorem, which we quote from [5], tells us that the solution to the Dirichlet problem $\mathrm{G}_D f$ is in $W^{2,\mathrm{bmo}_z}(D)$ whenever $f\in \mathrm{bmo}_z(D)$.

Theorem 3.1. *Let D be either a bounded domain of class $\mathscr{C}^2$ or the halfspace $\mathbb{R}^2_+$. Then there exists a constant $C_D>0$, depending only on D, such that*

$$\mathrm{G}_D: \mathrm{bmo}_z(D)\to W^{2,\mathrm{bmo}_z}(D),\qquad \|\mathrm{G}_D f\|_{W^{2,\mathrm{bmo}_z}(D)}\le C_D\|f\|_{\mathrm{bmo}_z(D)}. \tag{3.3}$$

3.2. **Theorem 3.1 in a rectangle.** We prove the following proposition, which is an extension of Theorem 3.1 to the (nonsmooth, convex) domain $\Omega=[0,L_1]\times[0,L_2]$.

Proposition 3.1. *Let $\Omega=[0,L_1]\times[0,L_2]$. There exists $C=C(L_1,L_2)$ such that*

$$\|\mathrm{G}_\Omega f\|_{W^{2,\mathrm{bmo}_z}(\Omega)}\le C\|f\|_{\mathrm{bmo}_z(\Omega)}. \tag{3.4}$$

Proof. In this proof, the almost inequality sign is hiding a positive constant, possibly varying from line to line and depending on Ω only. We also refer to Remark 2.2 for notation.

Denote by $F:=\mathrm{G}_\Omega f$. We preliminarily observe that, by (2.13) applied for $p=3$, and John-Nirenberg's inequality (3.2)

$$\|F\|_{W^{2,3}(\Omega)}\lesssim \|f\|_{L^3(\Omega)}\lesssim \|f\|_{\mathrm{bmo}_z(\Omega)}.$$

By the Sobolev embedding $W^{2,3}(\Omega)\subset \mathscr{C}^{1,\frac13}(\overline{\Omega})$, we have in particular that

$$\nabla F\in \mathscr{C}^{0,\frac13}(\overline{\Omega})^2,\qquad \|F\|_{L^\infty(\Omega)}+\|\nabla F\|_{\mathbf{L}^\infty(\Omega)}\lesssim \|f\|_{\mathrm{bmo}_z(\Omega)}. \tag{3.5}$$

The core of the argument begins now. Let $\mathscr{C}:=\big\{\Omega_0\cup\{B_i: i=1,\dots,4\}\big\}$ be an open cover of $\overline{\Omega}$ as follows: each B_i is an open ball centered at the corner S_i, of radius $\rho>0$ chosen small enough so that $B_i\cap B_j=\emptyset$ when $1\le i<j\le 4$; $\Omega_0\subset\Omega$ is an open set with smooth boundary and $\mathrm{dist}(\overline{\Omega_0},\{S_1,\dots,S_4\})>\frac{\rho}{2}$. For $i=1,\dots,4$, set $\Omega_i=B_i\cap\Omega$. Let $\{\mu_0,\mu_1,\dots,\mu_4\}$ be a partition of unity subordinated to the cover $\mathscr{C}$, and write $F_i=F\mu_i$. Then F_i solves the Dirichlet problem

$$\begin{cases}\Delta F_i=f_i & \text{on } \Omega_i,\\ F_i=0 & \text{on } \partial\Omega_i,\end{cases}\qquad f_i=\mu_i f+\nabla F\cdot\nabla\mu_i+F\Delta\mu_i,\qquad i=0,\dots,4. \tag{3.6}$$

We gather from (3.5) that

$$\|f_i\|_{\mathrm{bmo}_z(\Omega)}\lesssim \|\mu_i\|_{C^2(\Omega)}\big(\|f\|_{\mathrm{bmo}_z(\Omega)}+\|\nabla F\|_{\mathbf{L}^\infty(\Omega)}+\|F\|_{L^\infty(\Omega)}\big)\lesssim \|f\|_{\mathrm{bmo}_z(\Omega)}. \tag{3.7}$$

Since Ω_0 is a smooth domain, we apply Theorem 3.1 with $D=\Omega_0$, and get

$$\|F_0\|_{W^{2,\mathrm{bmo}_z}(\Omega)}=\|F_0\|_{W^{2,\mathrm{bmo}_z}(\Omega_0)}\lesssim \|f_0\|_{\mathrm{bmo}_z(\Omega_0)}\lesssim \|f\|_{\mathrm{bmo}_z(\Omega)}. \tag{3.8}$$

We now deal with the cases $i\ge 1$ and show that

$$\|F_i\|_{W^{2,\mathrm{bmo}_z}(\Omega)}\lesssim \|f_i\|_{\mathrm{bmo}_z(\Omega)},\qquad i=1,\dots,4 \tag{3.9}$$

which will be enough to obtain (3.4). The four corners are treated in the exact same way, so we fix $i=1$, and $S_1=(0,0)$. Note that F_1 solves the Dirichlet problem

$$\begin{cases} \Delta F_1 = f_1 & \text{on } \Gamma : (0,\infty)\times(0,\infty),\\ F_1(x_1,0)=0 & x_1\geq 0,\\ F_1(0,x_2)=0 & x_2 \geq 0.\end{cases}$$

Consider the even reflections of F_1 and f_1, defined by:

$$w(x_1,x_2) := F_1(|x_1|,x_2), \quad (x_1,x_2)\in\overline{\mathbb{R}^2_+}, \qquad h(x_1,x_2):= f_1(|x_1|,x_2), \quad (x_1,x_2)\in\mathbb{R}^2_+.$$

One sees that w is the solution to the Dirichlet problem on $\mathbb{R}^2_+$ with data h and zero boundary condition, i.e. $w = \mathrm{G}_{\mathbb{R}^2_+} h$. We note that $\|h\|_{\mathrm{bmo}_z(\mathbb{R}^2_+)} \leq 2\|f_1\|_{\mathrm{bmo}_z(\Gamma)}$; indeed, this is the same as $\|\overline{h}\|_{\mathrm{bmo}(\mathbb{R}^2)} \leq 2\|\overline{f_1}\|_{\mathrm{bmo}(\mathbb{R}^2)}$; recall that the bar denotes extension by zero to $\mathbb{R}^2$. This is clear, since $\overline{h}(x_1,x_2) = \overline{f_1}(x_1,x_2)+\overline{f_1}(-x_1,x_2)$. Therefore, we can apply Theorem 3.1 in the case of $D=\mathbb{R}^2_+$, and deduce that

$$\|w\|_{W^{2,\mathrm{bmo}}(\mathbb{R}^2_+)} \lesssim \|h\|_{\mathrm{bmo}_z(\mathbb{R}^2_+)} \lesssim \|f_1\|_{\mathrm{bmo}_z(\Gamma)}. \tag{3.10}$$

Since w restricted to Γ coincides with F_1, (3.9) follows from the above display, and the proof of Proposition 3.1 is complete. □

Remark 3.1. The estimate (3.4) also holds for a slightly more general class of domains, that is, polygonal-like domains, as described in Remark 2.2, with angles of the type $\alpha_j = \frac{\pi}{m(j)}$ for some integer $m(j)\geq 2$. The outline of the proof is the same: after localization and a suitable change of coordinates which straightens up the boundary, one studies the resulting (general) elliptic problem in the corner $\mathbb{R}\times(0,\alpha_j)$, and applies a similar reflection argument to reach the halfspace. The proof is finished by applying the analogue of Theorem 3.1 for a general elliptic operator of class $\mathscr{C}^2$.

3.3. **The space $\mathbf{V}^{1,\infty}(\Omega)$.** In view of the elliptic regularity result (3.4), for $\Omega=[0,L_1]\times[0,L_2]$ it makes sense to extend the scale of spaces $\mathbf{V}^{1,p}(\Omega)$ to the endpoint $p=\infty$ by setting

$$\mathbf{V}^{1,\infty}(\Omega) := \left\{\mathbf{v}\in\mathbf{L}^2_\tau(\Omega)\cap\mathbf{L}^\infty(\Omega) : \operatorname{curl}\mathbf{v}\in L^\infty(\Omega)\right\}, \qquad \|\mathbf{v}\|_{\mathbf{V}^{1,\infty}(\Omega)} := \|\mathbf{v}\|_{\mathbf{L}^\infty(\Omega)} + \|\operatorname{curl}\mathbf{v}\|_{L^\infty(\Omega)}. \tag{3.11}$$

As a consequence of Proposition 3.1, we have the continuous embedding

$$\mathbf{V}^{1,\infty}(\Omega)\hookrightarrow\mathbf{W}^{1,\mathrm{bmo}_z}(\Omega) = \{\mathbf{v}\in\mathrm{bmo}_z(\Omega)^2 : \nabla\mathbf{v}\in\mathrm{bmo}_z(\Omega)^{2\times 2}\} \tag{3.12}$$

and in turn, John-Nirenberg inequality gives in particular that

$$\|\nabla\mathbf{v}\|_{L^p(\Omega)^{2\times2}} \leq Cp\|\mathbf{v}\|_{\mathbf{V}^{1,\infty}(\Omega)} \qquad \forall p\in[1,\infty). \tag{3.13}$$

A further remark is that standard interpolation theory [27] yields the embedding $\mathbf{V}^{1,p}(\Omega)\hookrightarrow\mathbf{W}^{1,p}(\Omega)$ for all $1<p<\infty$ as well when Ω is a rectangle. This can also be inferred, as a particular case, from our Remark 2.2.

4. The two-dimensional Euler equations in a bounded convex domain

In this section, we formulate problem (P) in a suitable weak sense and construct a weak solution.

4.1. **Weak formulation of** (P) **in bounded convex domains.** Our first task is to provide a weak formulation of (P) when Ω is a bounded convex domain, which we continue to assume throughout this section. For convenience, we say that a triple of exponents (r_1, r_2, r_3) is a Hölder triplet if

$$1 \le r_1, r_2, r_3 \le \infty, \quad \tfrac{1}{r_1} + \tfrac{1}{r_2} + \tfrac{1}{r_3} = 1.$$

4.1.1. *The trilinear form.* We define the trilinear form

$$b_\Omega(\mathbf{u}, \mathbf{v}, \mathbf{w}) = \int_\Omega (\mathbf{u} \cdot \nabla)\mathbf{u} \cdot \mathbf{w} = ((\mathbf{u} \cdot \nabla)\mathbf{u}, \mathbf{w})_\Omega, \qquad \mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathscr{D}(\overline{\Omega}; \mathbb{R}^2).$$

Lemma 4.1. *Let* (r_1, r_2, r_3) *be a Hölder triplet. Then the form* b_Ω *extends to a trilinear continuous form*

$$b_\Omega : \mathbf{L}^{r_1}_\tau(\Omega) \times \mathbf{W}^{1,r_2}(\Omega) \times \mathbf{L}^{r_3}(\Omega);$$

moreover, we have

$$b_\Omega(\mathbf{u}, \mathbf{v}, \mathbf{w}) + b_\Omega(\mathbf{u}, \mathbf{v}, \mathbf{w}) = 0, \qquad \forall \mathbf{u} \in \mathbf{L}^{r_1}_\tau(\Omega), \quad \mathbf{v}, \mathbf{w} \in \mathbf{W}^{1,r_2}(\Omega) \cap \mathbf{L}^{r_3}(\Omega). \tag{4.1}$$

Proof. Note that, as a a mere consequence of Hölder's inequality

$$|b_\Omega(\mathbf{u}, \mathbf{v}, \mathbf{w})| \le \|\mathbf{u}\|_{\mathbf{L}^{r_1}(\Omega)} \|\nabla \mathbf{v}\|_{\mathbf{L}^{r_2}(\Omega)} \|\mathbf{w}\|_{\mathbf{L}^{r_3}(\Omega)}, \qquad \forall\, \mathbf{u} \in \mathscr{V}, \mathbf{v}, \mathbf{w} \in \mathscr{D}(\overline{\Omega}; \mathbb{R}^2). \tag{4.2}$$

Thus the first claim follows by density of $\mathscr{V}$ in $\mathbf{L}^{r_1}(\Omega)$ and of $\mathscr{D}(\overline{\Omega}; \mathbb{R}^2)$ in $\mathbf{W}^{1,r_2}(\Omega) \cap \mathbf{L}^{r_3}(\Omega)$. To show (4.1), we employ Stokes' formula for $\mathbf{u} \in \mathscr{V}$ and $\mathbf{v} \in \mathscr{D}(\overline{\Omega}; \mathbb{R}^2)$ to find

$$2b_\Omega(\mathbf{u}, \mathbf{v}, \mathbf{v}) = 2\sum_{j=1}^{2} (\mathbf{u}, \nabla(v_j)^2)_\Omega = -\sum_{j=1}^{2} (\operatorname{div} \mathbf{u}, (v_j)^2)_\Omega + \sum_{j=1}^{2} \int_{\partial\Omega} (v_j)^2 \mathbf{u} \cdot \mathbf{n} = 0,$$

and (4.1) then follows by trilinearity and density. □

Remark 4.1. This remark, together with Remark 4.3 below, motivates why we are mostly interested in the range $p \in [\frac{4}{3}, \infty)$. Define

$$s(p) = \begin{cases} \frac{2p}{4-p}, & \frac{4}{3} \le p < 2, \\ \text{any } s \in [1,2) & p = 2 \\ p & p > 2 \end{cases}, \qquad z(p) = (s(p))'. \tag{4.3}$$

Assume $\mathbf{u} \in \mathbf{V}^{1,p}(\Omega)$. If $p \neq 2$, taking $r_1 = p^*, r_2 = p, r_3 = z(p)$ in (4.2), and exploiting the Sobolev embedding $\mathbf{V}^{1,p}(\Omega) \subset \mathbf{L}^{p^*}(\Omega)$ yields

$$\left\| \mathbf{v} \mapsto b_\Omega(\mathbf{u}, \mathbf{u}, \mathbf{v}) \right\|_{\mathscr{L}(\mathbf{L}^{z(p)}(\Omega), \mathbb{R})} \le C_\Omega \|\mathbf{u}\|^2_{\mathbf{V}^{1,p}(\Omega)}. \tag{4.4}$$

By the Riesz representation theorem, moreover, it follows that $(\mathbf{u} \cdot \nabla)\mathbf{u} \in \mathbf{L}^{s(p)}(\Omega)$ whenever $p \in \left(\frac{4}{3}, \infty\right)$. A similar argument gives the same result for any choice of $s(p) \in [1,2)$ when $p = 2$.

Let $T > 0$ be fixed. We will also make use of the functional

$$B_\Omega(\mathbf{u}, \mathbf{v}, \mathbf{w}) = \int_0^T b_\Omega(\mathbf{u}(t), \mathbf{v}(t), \mathbf{w}(t))\, \mathrm{d}t \tag{4.5}$$

initially defined on $\mathbf{u},\mathbf{v},\mathbf{w}\in\mathscr{C}^\infty([0,T]\times\overline{\Omega})$. In consequence of Lemma 4.1 and Hölder's inequality, whenever $\{s_i\}_{i=1}^3,\{r_i\}_{i=1}^3$ are Hölder triplets, the estimates

$$|B_\Omega(\mathbf{u},\mathbf{v},\mathbf{w})|\le\|\mathbf{u}\|_{L^{s_1}(0,T;\mathbf{L}^{r_1}_\tau(\Omega))}\|\mathbf{v}\|_{L^{s_2}(0,T;\mathbf{W}^{r_2}(\Omega))}\|\mathbf{w}\|_{L^{s_3}(0,T;\mathbf{L}^{r_3}(\Omega))} \tag{4.6}$$

hold for all $\mathbf{u},\mathbf{v},\mathbf{w}\in\mathscr{C}^\infty([0,T]\times\overline{\Omega})$ and thus B_Ω extends to a trilinear continuous form on the triple of spaces appearing in the right-hand side of (4.6). Analogously,

$$B_\Omega(\mathbf{u},\mathbf{v},\mathbf{w})=-B_\Omega(\mathbf{u},\mathbf{w},\mathbf{v})\ \forall\mathbf{u}\in L^{s_1}(0,T;\mathbf{L}^{r_1}_\tau(\Omega)),\,\mathbf{v},\mathbf{w}\in L^{s_2}(0,T;\mathbf{W}^{1,r_2}(\Omega))\cap L^{s_3}(0,T;\mathbf{L}^{r_3}(\Omega)). \tag{4.7}$$

4.1.2. *Weak solutions.* Let $p\in\left[\frac{4}{3},\infty\right)$, $\mathbf{u}_0\in\mathbf{V}^{1,p}(\Omega)$, $\mathbf{f}\in L^p(0,T;\mathbf{W}^{1,p}(\Omega))$ be given. A function

$$\mathbf{u}:\Omega\times(0,T)\to\mathbb{R}^2$$

is a *weak solution* to problem (P) if

$$\mathbf{u}\in L^\infty\big(0,T;\mathbf{V}^{1,p}(\Omega)\big); \tag{4.8}$$

$$\mathbf{u}\in\mathscr{C}\big([0,T];\mathbf{L}^p_\tau(\Omega)\big)\quad\text{and}\quad\mathbf{u}(0)=\mathbf{u}_0, \tag{4.9}$$

$$-\int_0^T\big(\mathbf{u}(t),\mathbf{v}\psi'(t)\big)_\Omega\,\mathrm{d}t+B_\Omega(\mathbf{u},\mathbf{u},\mathbf{v}\otimes\psi)=\int_0^T\big(\mathbf{f}\,(t),\mathbf{v}\psi(t)\big)_\Omega, \tag{4.10}$$

for all $\mathbf{v}\in\mathscr{V}$, and $\psi\in\mathscr{D}(0,T)$.

Remark 4.2. Note that $\mathscr{V}$ is certainly not dense in $\mathbf{V}^{1,p}(\Omega)$. In fact, one can see that the closure of $\mathscr{V}$ in $\mathbf{V}^{1,p}(\Omega)$ is $\mathbf{V}^{1,p}(\Omega)\cap\mathbf{W}^{1,p}_0(\Omega)$. In a smooth domain Ω, one can obtain $\mathbf{V}^{1,p}(\Omega)$ as the closure of

$$\mathscr{W}:=\{\mathbf{v}\in\mathscr{C}^\infty(\overline{\Omega}):\operatorname{div}\mathbf{v}=0\text{ in }\Omega,\,\mathbf{v}\cdot\mathbf{n}|_{\partial\Omega}=0\}=\mathscr{C}^\infty(\overline{\Omega})\cap\mathbf{V}^{1,p}(\Omega).$$

This is not known to be true (and may very well be false) in a general convex domain. Ultimately, we are interested in testing the equation (4.10) with functions of $\mathbf{V}^{1,p}(\Omega)$, thus the choice of $\mathscr{W}$ as a test function space over $\mathscr{V}$ does not give any advantage. We explain how to circumvent this difficulty in Remark 4.3.

Remark 4.3. Assume that $p\ge\frac{4}{3}$ is such that the embedding $\mathbf{V}^{1,p}(\Omega)\hookrightarrow\mathbf{W}^{1,p}(\Omega)$ holds, and $\mathbf{u}$ is a weak solution of (P). Note that each of the three functionals

$$\mathbf{v}\mapsto\int_0^T\big(\mathbf{u},\mathbf{v}\psi'\big)_\Omega,\quad\mathbf{v}\mapsto\int_0^T\big(\mathbf{f}\,,\mathbf{v}\psi\big)_\Omega,\quad\mathbf{v}\mapsto B_\Omega(\mathbf{u},\mathbf{u},\mathbf{v}\otimes\psi),$$

are linear and continuous on $\mathbf{L}^{z(p)}_\tau(\Omega)$: the only nontrivial verification is that, thanks to (4.4) and the embedding

$$\begin{aligned}|B_\Omega(\mathbf{u},\mathbf{u},\mathbf{v}\otimes\psi)|&\le C_\Omega\|\mathbf{u}\|_{L^\infty(0,T;\mathbf{L}^p_\tau(\Omega))}\|\mathbf{u}\|_{L^p(0,T;\mathbf{W}^{1,p}(\Omega))}\|\mathbf{v}\otimes\psi\|_{L^{p'}(0,T;\mathbf{L}^{z(p)}_\tau(\Omega))}\\&\le C\|\mathbf{u}\|^2_{L^p(0,T;\mathbf{V}^{1,p}(\Omega))}\|\psi\|_{L^{p'}(0,T)}\|\mathbf{v}\|_{\mathbf{L}^{z(p)}_\tau(\Omega)}.\end{aligned}$$

This shows that (4.10) makes sense for $\mathbf{u}$ as in (4.8). By density of $\mathscr{V}$ in $\mathbf{L}^{z(p)}_\tau(\Omega)$, the equality (4.10) actually holds for all $\mathbf{v}\in\mathbf{L}^{z(p)}_\tau(\Omega)$. Further observe that $p\ge\frac{3}{2}$ ensures the continuity of the embedding $\mathbf{W}^{1,p}(\Omega)\hookrightarrow\mathbf{L}^{z(p)}(\Omega)$. Hence, when $p\ge\frac{3}{2}$, it is legitimate to take $\mathbf{v}\in\mathbf{V}^{1,p}(\Omega)$ as a test function in (4.10).

We summarize the construction of a weak solution to the Euler system (P) in the proposition below.

Proposition 4.1. *Let $\Omega \subset \mathbb{R}^2$ be a bounded convex domain and $p \in \left[\frac{4}{3}, 2\right]$. Then Problem* (P) *has a weak solution $\mathbf{u} = \mathbf{u}(x,t) \in L^\infty\big(0,T;\mathbf{V}^{1,p}(\Omega)\big)$ in the sense of* (4.8)-(4.10)*, which possesses the further regularity*

$$\mathbf{u} \in \mathscr{C}\big([0,T];\mathbf{L}^2_\tau(\Omega)\big), \qquad \mathbf{u}(t) \in \mathbf{V}^{1,p}(\Omega)\ \forall t \in [0,T], \tag{4.11}$$

$$\operatorname{curl}\mathbf{u} \in \mathscr{C}\big([0,T];\mathrm{w}-L^p(\Omega)\big), \tag{4.12}$$

$$\partial_t \mathbf{u} \in L^p\big(0,T;\mathbf{L}^{s(p)}(\Omega)). \tag{4.13}$$

Remark 4.4. Proposition 4.1 is a slight improvement of [28, Proposition 7.1]. The proof follows the same general scheme of construction of a weak solution to (P) on Ω as a limit of smooth solutions on a sequence of approximating domains. However, in [28], no continuity-in-time results analogous to (4.11)-(4.12) are given.

4.2. **Proof of Proposition 4.1.** We assume throughout the proof that Ω is a bounded convex domain, $\frac{4}{3} \le p \le 2$, and

$$\mathbf{u}_0 \in \mathbf{V}^{1,p}(\Omega), \qquad \mathbf{f} \in L^p(0,T;\mathbf{W}^{1,p}(\Omega)), \qquad \Big(\|\mathbf{u}_0\|^p_{\mathbf{V}^{1,p}(\Omega)} + \|\mathbf{f}\|^p_{L^p(0,T;\mathbf{W}^{1,p}(\Omega))}\Big)^{\frac{1}{p}} \le R.$$

The symbol $\mathscr{Q}$ will stand for a positive increasing function of its argument, depending only on p and Ω, which can be explicitly computed and is allowed to vary from line to line. Also, all the constants implied by the almost inequality signs ($\lesssim$) are allowed to depend on p and Ω without explicit mention. The proof will proceed through several steps.

4.2.1. *Approximation of the domain and of the data.* We approximate Ω by a sequence of smooth convex domains Ω_n as in (2.1). As discussed in Section 2, the implicit constants in the Sobolev embeddings for Ω_n, as well as the $\mathbf{L}^p(\Omega_n)$ norm of the Leray projector P_{Ω_n} are uniform in n and depend only on (the Lipschitz character of) Ω. Our use of almost-inequalities and of the function $\mathscr{Q}(\cdot)$ will reflect this uniformity.

We turn to the approximation of the data in (P). Let $\omega_0 = \operatorname{curl}\mathbf{u}_0$. We set

$$\omega_0^{(n)}(x) = \Big(\omega_0 \mathbf{1}_{\Omega_{n-1}}\Big) * \phi^2_{\varepsilon_n}(x), \qquad x \in \Omega, \tag{4.14}$$

$$\mathbf{f}^{(n)}(x,t) = \Big(\big(\mathbf{f}\,\mathbf{1}_{\Omega_{n-1}x}\mathbf{1}_{(\varepsilon_n, T-\varepsilon_n)t}\Big) * \big(\phi^2_{\varepsilon_n} \otimes \phi^1_{\varepsilon_n}\big)(x,t), \quad x \in \Omega,\ t \in [0,T], \tag{4.15}$$

with $\varepsilon_n > 0$, $\varepsilon_n \to 0$ chosen small enough to have

$$\operatorname{supp}\omega_0^{(n)} \Subset \Omega_n, \qquad \operatorname{supp}\mathbf{f}^{(n)} \Subset \Omega_n \times (0,T). \tag{4.16}$$

Note that

$$\omega_0^{(n)} \in \mathscr{D}(\Omega_n), \qquad \|\omega_0^{(n)}\|_{L^p(\Omega)} \le \|\omega_0\|_{L^p(\Omega)}, \qquad \widetilde{\omega_0^{(n)}} \to \omega_0 \text{ in } L^p(\Omega), \tag{4.17}$$

$$\mathbf{f}^{(n)} \in \mathscr{D}(\Omega_n \times (0,T)), \quad \|\mathbf{f}^{(n)}\|_{L^p(0,T;\mathbf{W}^{1,p}(\Omega_n))} \le \|\mathbf{f}\|_{L^p(0,T;\mathbf{W}^{1,p}(\Omega))}, \tag{4.18}$$

$$\|\mathbf{f}^{(n)} - \mathbf{f}\|_{L^p(0,T;\mathbf{W}^{1,p}(\Omega_n))} \to 0, \qquad n \to \infty. \tag{4.19}$$

Lemma 4.2. *Define*

$$\mathbf{u}_0^{(n)} : \Omega_n \to \mathbb{R}^2, \qquad \mathbf{u}_0^{(n)} = \mathbf{K}_{\Omega_n}\omega_0^{(n)}.$$

We then have

$$\mathbf{u}_0^{(n)} \in \mathscr{C}^\infty(\overline{\Omega}_n, \mathbb{R}^2), \tag{4.20}$$

$$\nabla\cdot\mathbf{u}_0^{(n)} = 0 \ \text{ in } \ \Omega_n, \quad \mathbf{u}_0^{(n)}\cdot\mathbf{n}|_{\partial\Omega_n} = 0; \tag{4.21}$$

$$\|\mathbf{u}_0^{(n)}\|_{\mathbf{W}^{1,p}(\Omega_n)} \le \mathscr{Q}(R). \tag{4.22}$$

Proof. Immediate from (4.17) and Theorem 2.2. □

4.2.2. *The approximating problems and* a priori *estimates.* We consider the following evolution problem:

$$(\mathrm{P}_n)\qquad \begin{cases} \partial_t\mathbf{u}(x,t)+(\mathbf{u}\cdot\nabla)\mathbf{u}(x,t)+\nabla\pi(x,t)=\mathbf{f}^{(n)}(x,t), & x\in\Omega_n, t\in(0,T)\\ \nabla\cdot\mathbf{u}(x,t)=0 & x\in\Omega_n, t\in(0,T)\\ \mathbf{u}\cdot\mathbf{n}(x,t)=0, & x\in\partial\Omega_n, t\in(0,T)\\ \mathbf{u}(x,0)=\mathbf{u}_0^{(n)}(x) & x\in\Omega_n \end{cases}$$

Due to the smoothness (4.20) of the data $\mathbf{u}_0^{(n)}, \mathbf{f}^{(n)}$ and of the domain Ω_n, the classical result of Kato [21] (see also [2, 3]) yields the existence of a unique classical solution of (P_n):

$$(\mathbf{u}^{(n)},\pi^{(n)}) \in \mathscr{C}^\infty(\overline{\Omega_n}\times[0,T])^2\times\mathscr{C}^\infty(\overline{\Omega_n}\times[0,T]).$$

Taking curl of the equation in (P_n), we obtain the following equation for $\omega^{(n)}$:

$$\begin{cases} \partial_t\omega^{(n)}(x,t)+(\mathbf{u}^{(n)}\cdot\nabla)\omega^{(n)}(x,t)=g^{(n)}(x,t), & x\in\Omega_n, t\in(0,T),\\ \omega^{(n)}(x,0)=\omega_0^{(n)}(x) & x\in\Omega_n, \end{cases} \tag{4.23}$$

where $g^{(n)} = \operatorname{curl}\mathbf{f}^{(n)}$. We now derive *a priori* estimates on the smooth solutions $\mathbf{u}^{(n)}$ via the fundamental estimate on the vorticity $\omega^{(n)}$ in the lemma below.

Lemma 4.3. *The following estimate holds:*

$$\|\omega^{(n)}\|_{L^\infty(0,T;L^p(\Omega_n))} \le e^T R.$$

Proof. Multiplying (4.23)by $p\omega^{(n)}|\omega^{(n)}|^{p-2}$, and integrating on Ω_n, we obtain

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\|\omega^{(n)}(t)\|^p_{L^p(\Omega_n)} &= p(g^{(n)}(t),\omega^{(n)}|\omega^{(n)}|^{p-2})_{\Omega_n} \le \|g^{(n)}(t)\|^p_{L^p(\Omega_n)}+(p-1)\|\omega^{(n)}(t)\|^p_{L^p(\Omega_n)}\\ &\le \|g^{(n)}(t)\|^p_{L^p(\Omega_n)}+(p-1)\|\omega^{(n)}(t)\|^p_{L^p(\Omega_n)},\end{aligned}$$

so that the desired conclusion follows from the Gronwall lemma and (4.17)-(4.19). We used that

$$p\big((\mathbf{u}^{(n)}\cdot\nabla)\omega^{(n)},\omega^{(n)}|\omega^{(n)}|^{p-2}\big)_{\Omega_n} = \big(\mathbf{u}^{(n)},\nabla|\omega^{(n)}|^p\big)_{\Omega_n} = 0$$

which stems from $\mathbf{u}^{(n)}$ being divergence free and with zero normal component on $\partial\Omega_n$. □

Via an application of (2.18), Lemma 4.3 entails the following a priori estimate on the solutions $\mathbf{u}^{(n)}$:

$$\|\mathbf{u}^{(n)}\|_{L^\infty(0,T;\mathbf{V}^{1,p}(\Omega_n))} \le \mathscr{Q}(R). \tag{4.24}$$

We then derive an equicontinuity result for the vorticity.

Lemma 4.4. *For each $\psi \in L^{p'}(\Omega)$, the sequence of functions*

$$t \in [0,T] \mapsto \big(\widetilde{\omega^{(n)}}(t), \psi\big)_\Omega,$$

is equicontinuous on $[0,T]$.

Proof. Let $\varepsilon > 0$ and $\psi \in L^{p'}(\Omega)$ be given; we have to show that

$$\exists \delta = \delta(R,\varepsilon,\psi), N = N(\varepsilon,\psi) : |t_2 - t_1| < \delta \implies \sup_{n \geq N} \Big| \big(\widetilde{\omega^{(n)}}(t_2), \psi\big)_\Omega - \widetilde{\omega^{(n)}}(t_1), \psi\big)_\Omega \Big| < \varepsilon. \tag{4.25}$$

We first assume $\psi = \varphi \in \mathscr{D}(\Omega)$. Take $N = N(\varphi)$ large enough so that $\operatorname{supp} \varphi \Subset \Omega_N$. Then, we multiply by φ the equation in (4.23) and integrate over Ω_n: observing that

$$\int_{\Omega_n} (\mathbf{u}^{(n)} \cdot \nabla \omega^{(n)} \varphi) + \int_{\Omega_n} (\mathbf{u}^{(n)} \cdot \nabla \varphi) \omega^{(n)} = -\int_{\Omega_n} (\operatorname{div} \mathbf{u}^{(n)} \nabla \varphi) \omega^{(n)} - \int_{\partial \Omega_n} (\varphi \omega^{(n)}) \mathbf{u}^{(n)} \cdot \mathbf{n} = 0,$$

since $\mathbf{u}^{(n)}$ is divergence free and has zero normal component on $\partial\Omega_n$, and integrating on (t_1, t_2), one finds

$$\begin{aligned}
&\Big| \big(\widetilde{\omega^{(n)}}(t_2), \varphi\big)_\Omega - \widetilde{\omega^{(n)}}(t_1), \varphi\big)_\Omega \Big| = \Big| \big(\omega^{(n)}(t_2), \psi\big)_{\Omega_n} - \omega^{(n)}(t_1), \psi\big)_{\Omega_n} \Big| \\
&\leq \Big| \int_{t_1}^{t_2} \big(\mathbf{u}^{(n)} \cdot \nabla \varphi, \omega^{(n)}\big)_{\Omega_n} \Big| + \Big| \int_{t_1}^{t_2} \big(g^{(n)}, \varphi\big)_{\Omega_n} \Big| \\
&\leq |t_1 - t_2| \|\mathbf{u}^{(n)}\|_{L^\infty(0,T;\mathbf{L}^{p'}(\Omega_n))} \|\nabla \varphi\|_{L^\infty(\Omega_n)} \|\omega^{(n)}\|_{L^\infty(0,T;L^p(\Omega_n))} + |t_1 - t_2|^{\frac{1}{p'}} \|g^{(n)}\|_{L^p(0,T;L^p(\Omega_n))} \|\varphi\|_{L^{p'}(\Omega_n)} \\
&\leq T^{\frac{1}{p}} |t_1 - t_2|^{p'} \Big(\|\mathbf{u}^{(n)}\|_{L^\infty(0,T;\mathbf{V}^{1,p}(\Omega_n))} \|\nabla \varphi\|_{L^\infty(\Omega_n)} \|\omega^{(n)}\|_{L^\infty(0,T;L^p(\Omega_n))} + \|g^{(n)}\|_{L^p(0,T;L^p(\Omega_n))} \|\varphi\|_{L^{p'}(\Omega_n)} \Big) \\
&\leq T^{\frac{1}{p}} |t_1 - t_2|^{p'} \mathscr{Q}(R) \big(\|\nabla \varphi\|_{L^\infty(\Omega_n)} + \|\varphi\|_{L^{p'}(\Omega_n)} \big).
\end{aligned}$$

We used the Sobolev embedding $\mathbf{V}^{1,p}(\Omega_n) \hookrightarrow \mathbf{L}^{p'}(\Omega_n)$ and (4.24), as well as Lemma 4.3. Now, for $\psi \in L^{p'}(\Omega)$, take $\varphi \in \mathscr{D}(\Omega)$ with $\|\varphi - \psi\|_{L^{p'}(\Omega)} < \varepsilon / 4 \|\omega^{(n)}\|_{L^\infty(0,T;L^p(\Omega))}$. Set

$$Q(\varepsilon, \psi, R) := T^{\frac{1}{p}} \big(\|\nabla \varphi\|_{L^\infty(\Omega)} + \|\varphi\|_{L^{p'}(\Omega)} \big).$$

Then for $n \geq N(\varphi) = N(\varepsilon, \psi)$,

$$\begin{aligned}
\Big| \big(\widetilde{\omega^{(n)}}(t_2) - \widetilde{\omega^{(n)}}(t_1), \psi\big)_\Omega \Big| &\leq \Big| \big(\widetilde{\omega^{(n)}}(t_2) - \widetilde{\omega^{(n)}}(t_1), \varphi\big)_\Omega \Big| + \Big| \big(\widetilde{\omega^{(n)}}(t_1) - \widetilde{\omega^{(n)}}(t_2), \varphi - \psi\big)_\Omega \Big| \\
&\leq |t_1 - t_2|^{p'} Q(\varepsilon, \psi, R) + 2 \|\omega^{(n)}\|_{L^\infty(0,T;L^p(\Omega_n))} \|\varphi - \psi\|_{L^{p'}(\Omega)} < \varepsilon,
\end{aligned}$$

if $|t_1 - t_2| \leq \delta := \big(\frac{\varepsilon}{2} Q(\varepsilon, \psi, R)\big)^{\frac{1}{p'}}$. Thus (4.25) is proven. □

Denote $\Psi^{(n)} := \mathrm{G}_{\Omega_n} \omega^{(n)}$. In view of Lemma 4.3, the elliptic regularity (2.11) gives

$$\Psi^n(t) \in W^{2,p}(\Omega_n) \cap W_0^{1,p}(\Omega_n) \qquad \text{a.e. } t \in (0,T), \tag{4.26}$$

$$\|\Psi^{(n)}\|_{L^\infty(0,T;W^{2,p}(\Omega_n))} \lesssim \|\omega^{(n)}\|_{L^\infty(0,T;L^p(\Omega_n))} \leq \mathscr{Q}(R). \tag{4.27}$$

Note that the embedding $W^{2,p}(\Omega_n) \cap W_0^{1,p}(\Omega_n) \hookrightarrow W_0^{1,r}(\Omega_n)$ is continuous (with uniformity in n, see Section 2) whenever $1 \leq r < p^*$. Thus, we use the extension property (2.14) to derive from (4.26)-(4.26) that

$$\|\widetilde{\Psi^{(n)}}\|_{L^\infty(0,T;W_0^{1,r}(\Omega))} = \|\Psi^{(n)}\|_{L^\infty(0,T;W_0^{1,r}(\Omega_n))} \leq \mathscr{Q}(R), \qquad 1 \leq r < p^*. \tag{4.28}$$

We will actually use (4.28) with $r = p$ and $r = 2$. Lastly, we derive a uniform estimate on the time-derivative of $\mathbf{u}^{(n)}$.

Lemma 4.5. *Let $s(p)$ be as in* (4.3)*. We have the estimate*

$$\|\partial_t \widetilde{\Psi^{(n)}}\|_{L^p(0,T;W_0^{1,s(p)}(\Omega))} = \|\partial_t \widetilde{\mathbf{u}^{(n)}}\|_{L^p(0,T;\mathbf{L}^{s(p)}(\Omega))} \leq \mathscr{Q}(R). \tag{4.29}$$

Proof. We set

$$\mathbf{w}^{(n)} = \nabla\Psi^{(n)} = (\mathbf{u}^{(n)})^{\perp}.$$

Observe that, since $\mathbf{w}^{(n)}$ is a gradient, $P^{\perp}_{\Omega_n}\partial_t \mathbf{w}^{(n)} = \partial_t P^{\perp}_{\Omega_n}\mathbf{w}^{(n)} = \partial_t \mathbf{w}^{(n)}$. At this point, note that, for any $\varphi \in \mathscr{D}(\Omega_n)$,

$$\begin{aligned}
\big(\partial_t \mathbf{w}^{(n)}, \nabla\varphi\big)_{\Omega_n} &= \frac{\mathrm{d}}{\mathrm{d}t}\big(\mathbf{w}^{(n)}, \nabla\varphi\big)_{\Omega_n} = -\frac{\mathrm{d}}{\mathrm{d}t}\big(\Delta\Psi^{(n)}, \varphi\big)_{\Omega_n} = -\big(\partial_t\omega^{(n)}, \varphi\big)_{\Omega_n} \\
&= \big(\mathbf{u}^{(n)}\cdot\nabla\omega^{(n)}, \varphi\big)_{\Omega_n} - \big(g^{(n)}, \varphi\big)_{\Omega_n} \\
&= \big(-\omega^{(n)}\mathbf{u}^{(n)} + \nabla\mathbf{f}^{(n)}, \nabla\varphi\big)_{\Omega_n}.
\end{aligned}$$

We integrated by parts the first term in the next to last line to arrive at the last line. Thus, we obtain the equation

$$\partial_t \mathbf{w}^{(n)} = -P^{\perp}_{\Omega_n}\big[-\omega^{(n)}\mathbf{u}^{(n)}\big] + \nabla\mathbf{f}^{(n)}, \qquad \text{in } \Omega_n \times (0,T). \tag{4.30}$$

Observe that, by Sobolev embeddings and (4.27),

$$\|\mathbf{u}^{(n)}\|_{L^\infty(0,T;\mathbf{L}^r(\Omega_n))} \lesssim \|\Psi^{(n)}\|_{L^\infty(0,T;W^{2,p}(\Omega_n))} \lesssim \mathscr{Q}(R), \qquad r = \begin{cases} p^*, & \frac{4}{3} \leq p < 2, \\ s(p)^* & p = 2. \end{cases} \tag{4.31}$$

Theorem 2.1 ensures that $P^{\perp}_{\Omega_n}$ is a linear bounded operator on $\mathbf{L}^s(\Omega)$, with bound independent on n. This, together with Hölder's inequality and Sobolev embeddings gives

$$\begin{aligned}
\Big\|P^{\perp}_{\Omega_n}\big[\omega^{(n)}\mathbf{u}^{(n)}\big]\Big\|_{L^\infty(0,T;\mathbf{L}^{s(p)}(\Omega_n))} &\lesssim \Big\|\omega^{(n)}\mathbf{u}^{(n)}\Big\|_{L^\infty(0,T;\mathbf{L}^{s(p)}(\Omega_n))} \\
&\leq \|\omega^{(n)}\|_{L^\infty(0,T;L^p(\Omega_n))}\|\mathbf{u}^{(n)}\|_{L^\infty(0,T;\mathbf{L}^r(\Omega_n))} \\
&\leq \mathscr{Q}(R).
\end{aligned} \tag{4.32}$$

To obtain the last inequality we used Lemma 4.3 and (4.31). Since we also have

$$\|\nabla\mathbf{f}^{(n)}\|_{L^p(0,T;\mathbf{L}^s(\Omega_n))} \lesssim \|\nabla\mathbf{f}^{(n)}\|_{L^p(0,T;\mathbf{L}^p(\Omega_n))} \lesssim \|\mathbf{f}\|_{L^p(0,T;\mathbf{L}^p(\Omega)} \leq \mathscr{Q}(R),$$

we obtain that

$$\|\partial_t \mathbf{u}^{(n)}\|_{L^p((0,T),\mathbf{L}^{s(p)}(\Omega_n))} = \|\partial_t \mathbf{w}^{(n)}\|_{L^p((0,T),\mathbf{L}^{s(p)}(\Omega_n))} \leq \mathscr{Q}(R),$$

by comparison of (4.30) with (4.32) and with the last display. The lemma follows since $\widetilde{\partial_t \mathbf{u}^{(n)}} = \partial_t \widetilde{\mathbf{u}^{(n)}}$ in $\Omega \times [0,T]$ and the extension by zero to Ω preserves the L^p norms. □

4.2.3. *Conclusion of the proof.* We will proceed in steps.

STEP 1. COMPACTNESS. In light of Lemmata 4.3 and 4.4, an application of Ascoli-Arzelà's Theorem yields that the sequence $\{\widetilde{\omega^{(n)}}\}$ is precompact in $\mathscr{C}([0,T];\mathrm{w}-L^p(\Omega))$, so that up to a subsequence

$$\widetilde{\omega^{(n)}} \to \omega \ \text{ in } \ \mathscr{C}([0,T];\mathrm{w}-L^p(\Omega)). \tag{4.33}$$

We derive some consequences from (4.33).

Lemma 4.6. *We have that*

$$\omega \in \mathscr{C}([0,T];\mathrm{w}-L^p(\Omega)) \cap L^\infty(0,T;L^p(\Omega)) \cap \mathscr{C}([0,T];H^{-1}(\Omega)), \tag{4.34}$$

$$\omega(0) = \omega_0 \in L^p(\Omega). \tag{4.35}$$

$$\omega^{(n)}(t) \to \omega(t) \quad \text{in } \ H^{-1}(\Omega) \qquad \forall t \in [0,T]. \tag{4.36}$$

Proof. The first two inclusions in (4.34) have just been shown. Actually the first inclusion implies the stronger property

$$\omega(t) \in L^p(\Omega)) \qquad \forall t \in [0,T]. \tag{4.37}$$

The third follows from the first and the continuity of the embedding $\mathrm{w}-L^p(\Omega) \hookrightarrow H^{-1}(\Omega))$ [1]. Then, (4.33) yields in particular that $\omega(0)$ is the $\mathrm{w}-L^p(\Omega)$-limit of $\{\widetilde{\omega^{(n)}(0)}\}$; but, in view of (4.17), this implies $\omega(0) = \omega_0 \in L^p(\Omega)$, so that (4.35) follows. Finally, (4.36) follows from (4.33) and the continuity of the embedding $\mathrm{w}-L^p(\Omega) \hookrightarrow H^{-1}(\Omega)$. □

Define

$$\Psi : \Omega \times [0,T] \to \mathbb{R}, \qquad \Psi := \mathrm{G}_\Omega \omega.$$

Then, an application of (2.2) and (4.34) yields

$$\Psi(t) \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega) \quad \forall\, t \in [0,T], \qquad \|\Psi\|_{L^\infty(0,T;W^{2,p}(\Omega))} \lesssim \|\omega\|_{L^\infty(0,T;L^p(\Omega))} \le \mathscr{Q}(R). \tag{4.38}$$

Lemma 4.7 (Convergence of the stream functions)**.** *We have that*

$$\widetilde{\Psi^{(n)}} \to \Psi := \mathrm{G}_\Omega \omega \qquad \text{in } \ L^2(0,T;\mathrm{w}-H_0^1(\Omega)). \tag{4.39}$$

Proof. We need to show that

$$\big(\nabla \widetilde{\Psi^{(n)}}(\cdot), \nabla\varphi\big)_\Omega \to \big(\nabla\Psi(\cdot), \nabla\varphi\big)_\Omega \quad \text{in } L^2(0,T) \qquad \forall \varphi \in H_0^1(\Omega), \|\varphi\|_{H_0^1(\Omega)} = 1.$$

Using Lemma 2.2, it follows from (4.36) that $\widetilde{\Psi^{(n)}}(t) \to \Psi(t)$ in $\mathrm{w}-H_0^1(\Omega)$ for all t, that is

$$\big(\nabla \widetilde{\Psi^{(n)}}(t), \nabla\varphi\big)_\Omega \to \big(\nabla\Psi(t), \nabla\varphi\big)_\Omega \qquad \forall t \in [0,T].$$

Then, the case $r = 2$ of (4.28) yields

$$\Big|\big(\nabla \widetilde{\Psi^{(n)}}(t), \nabla\varphi\big)_\Omega\Big| \le \|\varphi\|_{H_0^1(\Omega)} \|\widetilde{\Psi^n}\|_{L^\infty(0,T;H_0^1(\Omega))} \lesssim \mathscr{Q}(R), \qquad \forall t \in [0,T],$$

and a similar bound holds for Ψ, in view of (4.38). We conclude with the dominated convergence theorem. □

[1] The continuity of the embedding $\mathrm{w}-L^p(\Omega) \hookrightarrow H^{-1}(\Omega)$ for $1 < p < \infty$ is a consequence of the compactness of $L^p(\Omega) \hookrightarrow H^{-1}(\Omega)$. This in turn follows from the adjoint compact embedding $H_0^1(\Omega) \hookrightarrow L^{p'}(\Omega)$.

We read from (4.28) with $r=2$ and (4.29) that

$$\{\widetilde{\Psi^{(n)}}\} \text{ is bounded in } L^\infty(0,T;H^1_0(\Omega)), \qquad \{\widetilde{\partial_t \Psi^{(n)}}\} \text{ is bounded in } L^p(0,T;L^{s(p)}(\Omega)). \tag{4.40}$$

Thus, via Aubin-Lions compactness lemma, the convergence (4.39) improves to the following two results:

$$\widetilde{\Psi^{(n)}} \to \Psi \qquad \text{in } L^2(0,T;L^2(\Omega)), \tag{4.41}$$

$$\partial_t \Psi \in L^p(0,T;W^{1,s(p)}(\Omega)). \tag{4.42}$$

STEP 2. CONSTRUCTION OF THE SOLUTION TO (P). We now introduce

$$\mathbf{u}=\mathbf{u}(x,t):\Omega\to\mathbb{R}^2, \qquad \mathbf{u}(t):=\mathbf{K}_\Omega\omega(t),\ t\in[0,T].$$

As a consequence of (2.18), Lemma 4.6, and (4.42) we have

$$\mathbf{u}(t)\in\mathbf{W}^{1,p}_\tau(\Omega) \qquad \forall t\in[0,T], \tag{4.43}$$

$$\|\mathbf{u}\|_{\mathbf{L}^\infty(0,T;\mathbf{W}^{1,p}_\tau(\Omega))} \lesssim \|\omega\|_{L^\infty(0,T;L^p(\Omega))} \le \mathcal{Q}(R), \tag{4.44}$$

$$\mathbf{u}\in\mathscr{C}([0,T];\mathbf{L}^2(\Omega)), \tag{4.45}$$

$$\partial_t\mathbf{u}\in L^p(0,T;\mathbf{L}^{s(p)}(\Omega)), \tag{4.46}$$

$$\mathbf{u}(0)=\mathbf{L}^2-\lim_{t=0}\mathbf{u}(t)=\mathbf{K}_\Omega\omega_0=\mathbf{u}_0 \tag{4.47}$$

Finally, we preliminarily deduce from (4.39) that

$$\widetilde{\mathbf{u}^{(n)}}\to\mathbf{u} \qquad \text{in } L^2(0,T;\mathrm{w}-\mathbf{L}^2(\Omega)). \tag{4.48}$$

Due to the convergences (4.41) and (4.33)

$$\begin{aligned}
\|\widetilde{\mathbf{u}^{(n)}}\|^2_{L^2(0,T;\mathbf{L}^2(\Omega))} &= \int_0^T \big(\nabla^\perp\Psi^{(n)}(t),\nabla^\perp\Psi^{(n)}(t)\big)\Omega_n\,\mathrm{d}t \\
&= -\int_0^T \big(\Psi^{(n)}(t),\Delta\Psi^{(n)}(t)\big)_{\Omega_n}\mathrm{d}t = \int_0^T \big(\Psi^{(n)}(t),\omega^{(n)}(t)\big)_\Omega\,\mathrm{d}t \\
&\to \int_0^T \big(\Psi(t),\omega(t)\big)_{\Omega_n}\mathrm{d}t = \|\mathbf{u}\|^2_{L^2(0,T;\mathbf{L}^2(\Omega))}.
\end{aligned}$$

Therefore (4.48) upgrades to

$$\widetilde{\mathbf{u}^{(n)}}\to\mathbf{u} \qquad \text{in } L^2(0,T;\mathbf{L}^2(\Omega)). \tag{4.49}$$

STEP 3. PASSAGE TO THE LIMIT. Thanks to (4.44)-(4.47) we see that (4.8)-(4.9) and (4.13) hold true. To show that $\mathbf{u}$ is a weak solution to (P) we are left with proving that the distributional equality (4.10) holds for any $\mathbf{v}\in\mathscr{V}$, $\psi\in\mathscr{D}(0,T)$.

Multiplying and integrating by parts in (P_n) leads to the equation

$$-\int_0^T \big(\mathbf{u}^{(n)}(t),\mathbf{v}\psi'(t)\big)_{\Omega_n}\mathrm{d}t + B_{\Omega_n}(\mathbf{u}^{(n)},\mathbf{u}^{(n)},\mathbf{v}\otimes\psi) = \int_0^T \big(\mathbf{f}(t),\mathbf{v}\psi(t)\big)_{\Omega_n}\mathrm{d}t. \tag{4.50}$$

Due to (4.49) and (4.19) respectively, it is immediate to see that the first term on the left hand side, and the right hand side of (4.50) converge to the homologous terms in (4.10).

We now treat the nonlinear term. We have that, using (4.7), and subsequently (4.6) with $s_1 = r_1 = s_3 = r_3 = 2$, $s_2 = r_2 = \infty$,

$$\begin{aligned}&\left|B_{\Omega_n}(\mathbf{u}^{(n)},\mathbf{u}^{(n)},\mathbf{v}\psi) - B_\Omega((\mathbf{u},\mathbf{u},\mathbf{v}\psi)\right| = \left|B_\Omega(\mathbf{u}^{(n)},\mathbf{v}\psi,\mathbf{u}^{(n)}) - B_\Omega(\mathbf{u},\mathbf{v}\psi,\mathbf{u})\right|\\ &\le \left|B_\Omega(\mathbf{u}-\widetilde{\mathbf{u}^{(n)}},\mathbf{v}\psi,\mathbf{u}^{(n)})\right| + \left|B_\Omega(\mathbf{u},\mathbf{v}\psi,\mathbf{u}-\widetilde{\mathbf{u}^{(n)}})\,\mathrm{d}t\right|\\ &\lesssim \|\mathbf{u}-\widetilde{\mathbf{u}^{(n)}}\|_{L^2(0,T;\mathbf{L}^2(\Omega))}\big(\|\widetilde{\mathbf{u}^{(n)}}\|_{L^2(0,T;\mathbf{L}^2(\Omega))} + \|\mathbf{u}\|_{L^2(0,T;\mathbf{L}^2(\Omega))}\big)\|\psi\|_{L^\infty(0,T)}\|\nabla\mathbf{v}\|_{L^\infty(\Omega)},\end{aligned}$$

so that (4.49) allows us to conclude that (4.10) holds. Note that $\{\widetilde{u^{(n)}}\}$ is a bounded sequence in $L^2(0,T;\mathbf{L}^2(\Omega))$. Also note that for n sufficiently large, $\operatorname{supp}\mathbf{v}\subset\Omega_n$, thus we could replace B_{Ω_n} by B_Ω in the second step. The proof of Proposition 2.2 is therefore complete.

5. Main Results

Our first main result is an improvement of Proposition 4.1: we exploit the extra regularity (4.13) of the weak solution to Problem (P) given in Proposition 4.1 to recover the pressure π and show that the pair $(\mathbf{u},\pi)$ thus obtained satisfies (P) almost everywhere in $\Omega\times(0,T)$.

Theorem 5.1. *Let $\Omega\subset\mathbb{R}^2$ be a bounded convex domain and $p\in\left(\frac{4}{3},2\right]$. Given*

$$\mathbf{u}_0\in\mathbf{V}^{1,p}(\Omega),\qquad \mathbf{f}\in L^p(0,T;\mathbf{W}^{1,p}(\Omega)),$$

there exists a pair $(\mathbf{u}:\Omega\times[0,T]\to\mathbb{R}^2, \pi:\Omega\times[0,T]\to\mathbb{R})$, with

$$\begin{aligned}&\mathbf{u}\in L^\infty\big(0,T;\mathbf{V}^{1,p}(\Omega)\big),\\ &\mathbf{u}\in\mathscr{C}\big([0,T];\mathbf{L}^2_\tau(\Omega)\big),\quad \mathbf{u}(t)\in\mathbf{V}^{1,p}(\Omega)\ \forall t\in[0,T],\\ &\operatorname{curl}\mathbf{u}\in\mathscr{C}\big([0,T];\mathrm{w}-L^p(\Omega)\big),\\ &\partial_t\mathbf{u}\in L^p\big(0,T;\mathbf{L}^{s(p)}(\Omega)),\\ &\pi\in L^p\big(0,T;W^{1,s(p)}(\Omega)\big),\\ &\mathbf{u}(0)=\mathbf{u}_0,\end{aligned}$$

satisfying the equation

$$\partial_t\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla\pi = \mathbf{f}\qquad \text{in } L^p\big(0,T;\mathbf{L}^{s(p)}(\Omega)\big), \tag{5.1}$$

and the two equations of Problem (P) *hold almost everywhere in $\Omega\times(0,T)$.*

Remark 5.1. When $p\ge\frac{3}{2}$, the embedding $L^\infty(0,T;\mathbf{V}^{1,p}(\Omega))\hookrightarrow L^{p'}(0,T;\mathbf{L}^{z(p)}_\tau(\Omega))$ is continuous (see also Remark 4.3). Therefore, we can multiply (5.1) by $\mathbf{u}$ and integrate on $\Omega\times(0,T)$. The convective term $b_\Omega(\mathbf{u},\mathbf{u},\mathbf{u})$ vanishes identically, so that we get that the solution $\mathbf{u}$ of Theorem 5.1 satisfies the energy equality

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\mathbf{u}(t)\|^2_{\mathbf{L}^2(\Omega)} - \big(\mathbf{f}(t),\mathbf{u}(t)\big)_\Omega = 0,\qquad t\in(0,T). \tag{5.2}$$

Remark 5.2. In relation with (5.2), we recall that the issue of conservation of energy for the solutions of the Euler equations is an important issue related to turbulence, although not in the focus of this article. The relation of the conservation of energy for the Euler equations in relation with the Onsager conjecture [22] is well explained in e.g. the article by Schnirelman, [26] and its review in MathSciNet. In this direction Uriel Frisch just draw our attention to the most recent result of De Lellis and Szï£¡kelyhidi Jr., [9] who proved the existence of solutions of the Euler equations which are Holder with exponent $< 1/10$ and which do not satisfy the conservation of energy; see also the earlier results [8].

Proof of Theorem 5.1. Assume $p \in (\frac{4}{3}, 2]$ and let $\mathbf{u}$ be the solution to Problem (P) given by Proposition 4.1. Note that, in view of Remark 4.1 the functional $\mathbf{v} \mapsto B_\Omega(\mathbf{u},\mathbf{u},\mathbf{v})$ extends boundedly to $\mathbf{L}^{p'}(0,T;\mathbf{L}^{z(p)}_\tau(\Omega))$, the function $(\mathbf{u}\cdot\nabla)\mathbf{u} \in L^\infty(0,T;\mathbf{L}^{s(p)}(\Omega))$, and we have the equality

$$B_\Omega(\mathbf{u},\mathbf{u},\mathbf{v}) = \int_0^T \big((\mathbf{u}(t)\cdot\nabla)\mathbf{u}(t),\mathbf{v}(t)\big)_\Omega \,\mathrm{d}t, \qquad \forall \mathbf{v} \in L^{p'}(0,T;\mathbf{L}^{z(p)}_\tau(\Omega)), \tag{5.3}$$

by density of $\mathscr{V}\otimes\mathscr{D}(0,T)$ in $L^{p'}(0,T;\mathbf{L}^{z(p)}_\tau(\Omega))$. Due to (4.13), we also have $\partial_t\mathbf{u} \in L^p\big(0,T;\mathbf{L}^{s(p)}(\Omega)\big)$, and the equality

$$\int_0^T (\partial_t \mathbf{u},\mathbf{v}\psi)_\Omega = -\int_0^T (\mathbf{u},\mathbf{v}\psi')_\Omega = -B_\Omega(\mathbf{u},\mathbf{u},\mathbf{v}\otimes\psi) + \int_0^T (\mathbf{f},\mathbf{v}\psi), \qquad \forall \mathbf{v}\otimes\psi \in \mathscr{V}\otimes\mathscr{D}(0,T),$$

carries over by density to

$$\int_0^T \Big(\partial_t\mathbf{u}(t) + (\mathbf{u}(t)\cdot\nabla)\mathbf{u}(t) - \mathbf{f}(t),\mathbf{v}(t)\Big)_\Omega \,\mathrm{d}t = 0, \qquad \forall \mathbf{v} \in \mathbf{L}^{p'}(0,T;\mathbf{L}^{z(p)}_\tau(\Omega)). \tag{5.4}$$

Now we define

$$\mathbf{w} : \Omega\times(0,T) \to \mathbb{R}^2, \qquad \mathbf{w} := \partial_t\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} - \mathbf{f}\,; \tag{5.5}$$

we have $\mathbf{w} \in L^p(0,T;\mathbf{L}^{s(p)}(\Omega))$, and we read from (5.4) that $\mathbf{w}$ belongs to the annihilator $X_{p'}$ of $\mathbf{L}^{p'}(0,T;\mathbf{L}^{z(p)}_\tau(\Omega))$ in $\mathbf{L}^{p'}(0,T;\mathbf{L}^{z(p)}(\Omega))$. Using (2.9), we see that $X_{p'}$ is the $L^p(0,T;\mathbf{L}^{s(p)}(\Omega))$-closure of $(\mathbf{H}^{z(p)}_\tau)^\perp\otimes\mathscr{D}(0,T)$, that is, $X_{p'} = L^p(0,T;\nabla W^{1,s(p)}(\Omega))$. Therefore, we proved that

$$\exists \pi \in L^p(0,T;W^{1,s(p)}(\Omega)) : \mathbf{w} = -\nabla\pi. \tag{5.6}$$

Summarizing (4.13), (5.3), (5.5), (5.6), Theorem 5.1 is proven. □

We now come to the second main theorem, which deals with (P) on the rectangle $\Omega = [0,L_1]\times[0,L_2]$ in the range of exponents $2 < p \le \infty$ not covered by Theorem 5.1. In particular, this theorem contains the uniqueness of weak solutions to (P) on a rectangle with bounded initial vorticity.

Theorem 5.2. *Let* $\Omega = [0,L_1]\times[0,L_2]$ *and* $2 < p \le \infty$*. Given*

$$\mathbf{u}_0 \in \mathbf{V}^{1,p}(\Omega), \qquad \mathbf{f} \in L^p\big(0,T;\mathbf{W}^{1,p}(\Omega)\big),$$

there exists a pair $(\mathbf{u}:\Omega\times[0,T]\to\mathbb{R}^2, \pi:\Omega\times[0,T]\to\mathbb{R})$*, with*

$$\mathbf{u}\in L^\infty\big(0,T;\mathbf{V}^{1,p}(\Omega)\big),\quad \mathbf{u}(t)\in\mathbf{V}^{1,p}(\Omega)\quad \forall t\in[0,T], \tag{5.7}$$

$$\partial_t\mathbf{u}\in\begin{cases} L^p\big(0,T;\mathbf{L}^p(\Omega)\big) & 2<p<\infty,\\ L^\infty\big(0,T;\mathbf{L}^q(\Omega)\big)\ \forall q<p & p=\infty,\end{cases} \tag{5.8}$$

$$\pi\in\begin{cases} L^p\big(0,T;W^{1,p}(\Omega)\big) & 2<p<\infty,\\ L^\infty\big(0,T;W^{1,q}(\Omega)\big)\ \forall q<p & p=\infty,\end{cases} \tag{5.9}$$

$$\mathbf{u}(0)=\mathbf{u}_0, \tag{5.10}$$

satisfying the equation

$$\partial_t\mathbf{u}+(\mathbf{u}\cdot\nabla)\mathbf{u}+\nabla\pi=\mathbf{f}\qquad\text{in}\quad\begin{cases} L^p\big(0,T;\mathbf{L}^p(\Omega)\big) & 2<p<\infty,\\ L^\infty\big(0,T;\mathbf{L}^q(\Omega)\big)\quad\forall q<p & p=\infty,\end{cases}$$

and the two equations of Problem (P) *hold almost everywhere in* $\Omega\times(0,T)$*. Moreover,* $\mathbf{u}$ *satisfies the energy equality* (5.2)*. Finally, when* $p=\infty$*, the solution* $(\mathbf{u},\pi)$ *is unique (up to an additive constant in* π*).*

Remark 5.3. The proof of the existence part in Theorem 5.2 only relies on the same techniques used for Theorem 5.1 and on the elliptic estimate (2.13), which holds for polygonal-like domains with $\overline{\alpha}\le\frac{\pi}{2}$ as described in Remark 2.2. Thus Theorem 5.2 in the range $2<p<\infty$ also holds for this wider class of domains. The endpoint case $p=\infty$ also holds for the subclass of polygonal-like domains described in Remark 3.1.

Proof of the existence in Theorem 5.2. Let for the moment $2<p<\infty$ and $\mathbf{u}_0,\mathbf{f}$ be given as in the theorem. The proof is carried out by an approximation procedure totally analogous to the one described in Subsections 4.2.1 and 4.2.2: with the same notation, the sequence $\{\widetilde{\omega}^{(n)}\}$ is equicontinuous in $\mathscr{C}([0,T];\mathrm{w}-L^p(\Omega))$ and obeys the bound

$$\|\widetilde{\omega}^{(n)}\|_{L^\infty(0,T;L^p(\Omega))}\le \mathrm{e}^T\big(\|\mathbf{u}_0\|_{\mathbf{V}^{1,p}(\Omega)}+\|\mathbf{f}\|_{L^p((0,T);\mathbf{W}^{1,p}(\Omega))}\big)^{\frac{1}{p}}:=\mathrm{e}^T R_p. \tag{5.11}$$

The approximate solutions $\mathbf{u}^{(n)}$ are given by $\mathbf{u}^{(n)}=\mathbf{K}_{\Omega_n}\omega^{(n)}$. Thus we have that, uniformly in n, $\mathbf{u}^{(n)}\in L^\infty(0,T;\mathbf{V}^{1,p}(\Omega_n))$. In particular, the (uniform in n, see Section 2) embedding $\mathbf{V}^{1,p}(\Omega_n)\hookrightarrow\mathbf{W}^{1,p}(\Omega_n)\hookrightarrow\mathbf{L}^\infty(\Omega_n)$, the second half of which is the Sobolev embedding, then entails

$$\|\mathbf{u}^{(n)}\|_{L^\infty(0,T;\mathbf{L}^\infty(\Omega_n))}\le\mathscr{Q}(R_p). \tag{5.12}$$

This allows to modify (4.32) appropriately and obtain the uniform estimate on the time-derivatives

$$\|\partial_t\widetilde{\mathbf{u}^{(n)}}\|_{L^p(0,T;\mathbf{L}^p(\Omega))}\le\mathscr{Q}(R_p). \tag{5.13}$$

Letting ω be a limit point of $\{\widetilde{\omega}^{(n)}\}$ in $\mathscr{C}([0,T];\mathrm{w}-L^p(\Omega))$ and defining the candidate solution $\mathbf{u}:=\mathbf{K}_\Omega\omega$, an application of (2.13) also yields

$$\mathbf{u}\in L^\infty(0,T;\mathbf{V}^{1,p}(\Omega))\hookrightarrow L^\infty(0,T;\mathbf{W}^{1,p}(\Omega)).$$

The same exact arguments of Subsection 4.2.3 yield that $\mathbf{u}$ is a weak solution to (P), satisfying in addition (5.7)-(5.8). Repeating the proof of Theorem 5.1 finally gives the remaining statements in the case $2<p<\infty$.

The case $p = \infty$ can be dealt with as follows. One constructs an approximating sequence of problems as in Subsection 4.2.1, with data

$$\widetilde{\omega_0^{(n)}} \to \omega_0 \quad \text{in } L^\infty(\Omega), \qquad \|\mathbf{f}^{(n)} - \mathbf{f}\|_{L^\infty(0,T;\mathbf{W}^{1,\infty}(\Omega_n))} \to 0, \quad n \to \infty.$$

Since the R_p in (5.11) are bounded uniformly in $2 < p < \infty$ by

$$R := R(\mathbf{u}_0, \mathbf{f}) := \left(\|\mathbf{u}_0\|_{\mathbf{V}^{1,\infty}(\Omega)} + \|\mathbf{f}\|_{L^\infty(0,T;\mathbf{W}^{1,\infty}(\Omega))}\right)$$

one can pass to the lim sup as $p \to \infty$, and obtain that

$$\|\widetilde{\omega^{(n)}}\|_{L^\infty(0,T;L^\infty(\Omega))} \le \mathrm{e}^T R. \tag{5.14}$$

Arguing similarly to what we did in Lemma 4.4 will give that the sequence $\{\widetilde{\omega^{(n)}}\}$ is precompact in $\mathscr{C}([0,T]; \mathrm{w}^\star - L^\infty(\Omega))$.

One then chooses once again a limit point ω of $\{\widetilde{\omega^{(n)}}\}$ in $\mathscr{C}([0,T]; \mathrm{w}^\star - L^\infty(\Omega))$ and defines the candidate solution $\mathbf{u} := \mathbf{K}_\Omega \omega$. An application of Proposition 3.1 then entails $\mathbf{u} \in L^\infty(0,T;\mathbf{V}^{1,\infty}(\Omega)) \hookrightarrow L^\infty(0,T;\mathbf{W}^{1,\mathrm{bmo}_z}(\Omega))$; in particular,

$$\|\mathbf{u}\|_{L^\infty(0,T;\mathbf{W}^{1,\mathrm{bmo}_z}(\Omega))} \le C R(\mathbf{u}_0, \mathbf{f}) \mathrm{e}^T. \tag{5.15}$$

Here, one is not able to obtain a uniform $L^\infty(0,T;\mathbf{L}^\infty(\Omega))$ estimate on $\partial_t \widetilde{\mathbf{u}^{(n)}}$. We recall that $\partial_t \widetilde{\mathbf{u}^{(n)}}^\perp$ is given by equation (4.30). Despite $\omega^{(n)}\mathbf{u}^{(n)}$ being indeed uniformly bounded in $L^\infty(0,T;\mathbf{L}^\infty(\Omega_n))$, we do not have the (uniform) $\mathbf{L}^\infty(\Omega_n)$ boundedness of the Leray projector, nor do we have the substitute $P_{\Omega_n} \in \mathscr{L}(\mathbf{bmo}_z(\Omega_n))$ uniformly. However, the estimate (5.13) holds for all $p = q < \infty$. This is sufficient to carry on the arguments of Subsection 4.2.3 and conclude the proof of existence in the case $p = \infty$. □

Proof of the uniqueness in Theorem 5.2. The proof is analogous to Yudovich's proof in the case of a smooth domain. With K we denote a positive constant, possibly varying from line to line and depending only on $(\mathbf{u}_0, \mathbf{f})$. Assume that there exist two solutions $(\mathbf{u}^1, \pi^1), (\mathbf{u}^2, \pi^2)$ corresponding to the same data $(\mathbf{u}_0, \mathbf{f})$. Preliminarily observe that, from (3.2), Proposition 3.1, and (5.15),

$$\begin{aligned} \|\nabla \mathbf{u}^j\|_{\mathbf{L}^p(\Omega)} &\le C_\Omega p \|\nabla \mathbf{u}^j\|_{\mathrm{bmo}_z(\Omega)^{2\times 2}} \le C_\Omega p \|\operatorname{curl} \mathbf{u}^j\|_{\mathrm{bmo}_z(\Omega)} \\ &\le C_\Omega p \|\operatorname{curl} \mathbf{u}^j\|_{L^\infty(\Omega)} \le C_\Omega p R(\mathbf{u}_0, \mathbf{f}) p := K p \end{aligned} \tag{5.16}$$

for all $1 \le p < \infty$. The difference $\mathbf{u} = \mathbf{u}^2 - \mathbf{u}^1$, $\pi = \pi^2 - \pi^1$ then satisfies the equation

$$\partial_t \mathbf{u} + (\mathbf{u} \cdot \nabla)\mathbf{u}^1 + (\mathbf{u}^2 \cdot \nabla)\mathbf{u} + \nabla \pi = 0 \tag{5.17}$$

almost everywhere in $\Omega \times (0,T)$. Denote by $Y(t) := \|\mathbf{u}(t)\|^2_{\mathbf{L}^2(\Omega)}$. We take the scalar product of (5.17) with $\mathbf{u}$ and integrate on Ω. After (legitimately) integrating by parts, applying Hölder's inequality with an arbitrary $p > 2$ to be chosen later, and using interpolation one obtains the differential inequality

$$\frac{\mathrm{d}}{\mathrm{d}t} Y = -\big((\mathbf{u} \cdot \nabla)\mathbf{u}^1, \mathbf{u}\big)_\Omega \le K \|\nabla \mathbf{u}^1\|_{L^p(\Omega)} \|\mathbf{u}\|_{\mathbf{L}^{2p'}(\Omega)} \le K \|\nabla \mathbf{u}^1\|_{L^p(\Omega)} \|\mathbf{u}\|^{\frac{1}{p'}}_{\mathbf{L}^2(\Omega)} \|\mathbf{u}\|^{\frac{1}{p}}_{\mathbf{L}^\infty(\Omega)} \le K Y^{\frac{1}{p'}};$$

in the last step, we used (5.15) to bound $\|\mathbf{u}\|_{\mathbf{L}^\infty(\Omega)}$, and (5.16). The bound $Y(t) \le K$ for all $t \in [0,T]$ makes possible to choose a constant M large enough, depending only on $(\mathbf{u}_0, \mathbf{f})$ and

Ω, such that $p = p(t) = \log \frac{M}{Y(t)} > 2$ for all t. The above differential inequality then turns into

$$\frac{\mathrm{d}}{\mathrm{d}t} Y(t) \leq K Y(t) \log \frac{M}{Y(t)}.$$

Let $\varepsilon > 0$ be given; integrating on (ε, t), we obtain that

$$Y(t) \leq M\big(Y(\varepsilon)/M\big)^{\mathrm{e}^{-K(t-\varepsilon)}}, \qquad \forall t > \varepsilon. \tag{5.18}$$

Due to the continuity $\mathbf{u}^i \in \mathscr{C}\big([0,T]; \mathbf{L}^2_\tau(\Omega)\big)$, the functional Y is continuous on $[0,T]$ and $Y(0) = 0$. Passing to the limit as $\varepsilon \to 0$ in (5.18) we obtain that $Y(t) \equiv 0$ on $[0,T]$. Thus $\mathbf{u}^1 = \mathbf{u}^2$, which forces $\nabla\pi^1 = \nabla\pi^2$ too. This completes the proof. □

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