KANIEL-SHINBROT ITERATION AND GLOBAL SOLUTIONS OF THE CAUCHY PROBLEM FOR THE BOLTZMANN EQUATION

CLAUDE BARDOS, IRENE GAMBA, AND C. DAVID LEVERMORE

ABSTRACT. Kaniel-Shinbrot iteration can be used to establish the existence of solutions to the Boltzmann equation over the spatial domain $\mathbb{R}^{\mathrm{D}}$ for some initial data that is pointwise bounded above and below by a class of global Maxwellians larger than previously considered. When these solutions are global in time, we obain results on their large-time asymptotics.

1. INTRODUCTION

We consider a kinetic density $F(v,x,t)$ over the velocity-position domain $\mathbb{R}^{\mathrm{D}} \times \mathbb{R}^{\mathrm{D}}$ that is governed by the Cauchy problem for the Boltzmann equation:

$$\partial_t F + v \cdot \nabla_x F = \mathcal{B}(F,F)\,, \qquad F\big|_{t=0} = F^{\mathrm{in}}\,. \tag{1.1}$$

We assume that the initial data $F^{\mathrm{in}}(v,x)$ is nonnegative and satisfies the bounds

$$0 < \iint_{\mathbb{R}^{\mathrm{D}}\times\mathbb{R}^{\mathrm{D}}} \big(1 + |v|^2 + |x|^2\big)\, F^{\mathrm{in}}\,\mathrm{d}v\,\mathrm{d}x < \infty\,. \tag{1.2}$$

The collision operator $\mathcal{B}(F,F)$ has the form

$$\mathcal{B}(F,F) = \iint_{\mathbb{S}^{\mathrm{D}-1}\times\mathbb{R}^{\mathrm{D}}} \big(F_*' F' - F_* F\big)\,\mathsf{b}\,\mathrm{d}\omega\,\mathrm{d}v_*\,, \tag{1.3}$$

where $\omega \in \mathbb{S}^{\mathrm{D}-1}$, $\mathsf{b}(\omega, v - v_*)$ is the collision kernel, while F_*, F', and F_*' denote $F(\,\cdot\,, x, t)$ evaluated at v_*, $v' = v - \omega\,\omega\cdot(v - v_*)$, and $v_*' = v_* + \omega\,\omega\cdot(v-v_*)$ respectively.

1.1. **Collision Kernels.** We will assume that the collision kernel has the separable form

$$\mathsf{b}(\omega, v - v_*) = \hat{\mathsf{b}}(\omega\cdot n)|v - v_*|^{\beta}\,, \qquad \text{where} \quad n = \frac{v - v_*}{|v - v_*|}\,, \tag{1.4}$$

for some $\beta \in (-\mathrm{D}, 2)$ while $\hat{\mathsf{b}}(\omega\cdot n)$ is positive almost everywhere and satisfies the weak small-deflection cutoff condition

$$\big\|\hat{\mathsf{b}}\big\|_{L^1(\mathrm{d}\omega)} = \int_{\mathbb{S}^{\mathrm{D}-1}} \hat{\mathsf{b}}(\omega\cdot n)\,\mathrm{d}\omega < \infty\,. \tag{1.5}$$

The conditions $\beta > -\mathrm{D}$ and (1.5) are required for $\mathsf{b}(\omega, v - v_*)$ given by (1.4) to be locally integrable with respect to $\mathrm{d}\omega\,\mathrm{d}v_*$. They allow us to decompose the collision operator (1.3) as

$$\mathcal{B}(F,F) = \mathcal{G}(F,F) - \mathcal{A}(F)\,F\,, \tag{1.6}$$

where the attenuation operator $\mathcal{A}(F)$ and the gain operator $\mathcal{G}(F,F)$ are defined by

$$\mathcal{A}(F) = \iint_{\mathbb{S}^{\mathrm{D}-1}\times\mathbb{R}^{\mathrm{D}}} F_*\,\mathsf{b}\,\mathrm{d}\omega\,\mathrm{d}v_*\,, \qquad \mathcal{G}(F,F) = \iint_{\mathbb{S}^{\mathrm{D}-1}\times\mathbb{R}^{\mathrm{D}}} F_*' F'\,\mathsf{b}\,\mathrm{d}\omega\,\mathrm{d}v_*\,. \tag{1.7}$$

Notes on our join work started at ICERM, draft December 4, 2011.

The form (1.4) arises from the classical scattering cross section calculation for identical hard spheres of mass m and diameter d_o, which yields

$$\mathsf{b}(\omega, v - v_*) = \frac{d_o^{\mathrm{D}-1}}{2m} \, |\omega \cdot (v - v_*)| \, . \tag{1.8}$$

This corresponds to the case $\beta = 1$ and $\hat{\mathsf{b}}(\omega \cdot n) = \frac{d_o^{\mathrm{D}-1}}{2m} |\omega \cdot n|$ in (1.4).

The form (1.4) also arises from the classical scattering cross-section calculation for an inverse-power interparticle potential after a small-deflection cutoff is imposed. For potentials proportional to r^{-k}, where r is the distance between the center of masses, one has

$$\beta = 1 - 2\,\frac{\mathrm{D}-1}{k} \qquad \text{for } k > 2\,\frac{\mathrm{D}-1}{\mathrm{D}+1} \, . \tag{1.9}$$

The cases $\beta \in (-\mathrm{D}, 0)$, $\beta = 0$, $\beta \in (0, 1]$, and $\beta \in (1, 2)$ are called respectively the "soft", "Maxwell", "hard", and "super-hard" cases. The super-hard cases do not arise from an inverse-power interparticle potential.

1.2. **Conservation Laws.** Any solution F of the Boltzmann equation (1.1) formally satisfies the local conservation law

$$\partial_t \int_{\mathbb{R}^\mathrm{D}} \xi \, F \, \mathrm{d}v + \nabla_x \cdot \int_{\mathbb{R}^\mathrm{D}} v \, \xi \, F \, \mathrm{d}v = 0 \, ,$$

when $\xi(v, x, t)$ is any quantity that satisfies

$$\xi(\,\cdot\,, x, t) \in \mathrm{Span}\{1, v_{_1}, v_{_2}, \cdots, v_{_\mathrm{D}}, |v|^2\} \, , \qquad \partial_t \xi + v \cdot \nabla_x \xi = 0 \, . \tag{1.10}$$

It has been known essentially since Boltzmann [4], who worked out the case $\mathrm{D} = 3$, that the only such quantities ξ are linear combinations of the $4 + 2\mathrm{D} + \frac{\mathrm{D}(\mathrm{D}-1)}{2}$ quantities

$$1\,, \qquad v\,, \qquad x - vt\,, \qquad \tfrac{1}{2}|v|^2\,, \qquad v \wedge x \qquad v \cdot (x - vt)\,, \qquad \tfrac{1}{2}|x - vt|^2\,, \tag{1.11}$$

where $v \wedge x = v\,x^T - x\,v^T$ is the skew tensor product. By integrating the coresponding local conservation laws over space and time, we formally obtain the global conservation laws

$$\iint_{\mathbb{R}^\mathrm{D} \times \mathbb{R}^\mathrm{D}} \begin{pmatrix} 1 \\ v \\ x - vt \\ \frac{1}{2}|v|^2 \\ v \wedge x \\ v \cdot (x - vt) \\ \frac{1}{2}|x - vt|^2 \end{pmatrix} F(v, x, t) \, \mathrm{d}v \, \mathrm{d}x = \iint_{\mathbb{R}^\mathrm{D} \times \mathbb{R}^\mathrm{D}} \begin{pmatrix} 1 \\ v \\ x \\ \frac{1}{2}|v|^2 \\ v \wedge x \\ v \cdot x \\ \frac{1}{2}|x|^2 \end{pmatrix} F^{\mathrm{in}}(v, x) \, \mathrm{d}v \, \mathrm{d}x \, , \tag{1.12}$$

where the right-hand sides above exist by the bounds (1.2). These conserved quantities are associated respectively with the conservation laws of mass, momentum, initial center of mass, energy, angular momentum, scalar momentum moment, and scalar inertial moment. The last two are not general physical laws, but are shared by the solutions of other kinetic equations.

1.3. **Global Maxwellians.** One consequence of Boltzmann's celebrated H-theorem [4] is that if $f(v)$ is any nonnegative integrable function that has appropriate behavior as $|v| \to \infty$ then $\mathcal{B}(f, f) = 0$ if and only if

$$f = \frac{\rho}{(2\pi\theta)^{\frac{\mathrm{D}}{2}}} \exp\!\left(- \frac{|v - u|^2}{2\theta} \right) \qquad \text{for some } (\rho, u, \theta) \in \mathbb{R}_+ \times \mathbb{R}^\mathrm{D} \times \mathbb{R}_+ \, . \tag{1.13}$$

Functions of the form (1.13) where ρ, u, and θ functions of (x,t) are called *local Maxwellians*. Any local Maxwellian that also satisfies the kinetic equation (1.1) is called a *global Maxwellian*. The family of all global Maxwellians over the spatial domain $\mathbb{R}^{\mathrm{D}}$ with positive mass, zero net momentum, and center of mass at the origin has the form

$$\mathcal{M} = \frac{m}{(2\pi)^{\mathrm{D}}} \sqrt{\det(Q)} \exp\big(- q(v,x,t)\big) \,, \qquad Q = (ac - b^2)I + B^2 \,, \tag{1.14a}$$

$$q(v,x,t) = \frac{1}{2} \begin{pmatrix} v \\ x - vt \end{pmatrix}^T \begin{pmatrix} cI & bI + B \\ bI - B & aI \end{pmatrix} \begin{pmatrix} v \\ x - vt \end{pmatrix} , \tag{1.14b}$$

with $m > 0$ and $(a, b, c, B) \in \Omega$ where Ω is the open cone in $\mathbb{R}\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}^{\mathrm{D}\wedge\mathrm{D}}$ defined by

$$\Omega = \big\{ (a,b,c,B) \in \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R}_+ \times \mathbb{R}^{\mathrm{D}\wedge\mathrm{D}} \,:\, (ac - b^2)I + B^2 > 0 \big\} \,. \tag{1.15}$$

Here $\mathbb{R}_+$ denotes the positive real numbers and $\mathbb{R}^{\mathrm{D}\wedge\mathrm{D}}$ denotes the skew-symmetric $\mathrm{D}\times\mathrm{D}$ real matrices. The form (1.14) can be derived from the fact that $\log(\mathcal{M})$ must satisfy

$$\log(\mathcal{M}) \in \mathrm{Span}\{1, v_1, v_2, \cdots, v_{\mathrm{D}}, |v|^2\} \,, \qquad \big(\partial_t + v\cdot\nabla_x\big)\log(\mathcal{M}) = 0 \,,$$

whereby $\log(\mathcal{M})$ must be a linear combination of the quantities (1.11). The form (1.14) then comes from the requirements that $\mathcal{M}$ have finite mass, zero net momentum, and center of mass at the origin. The larger family of global Maxwellians with positive mass is obtained from (1.14) by introducing translations in v and x; whereby it has a total of $4 + 2\mathrm{D} + \frac{\mathrm{D}(\mathrm{D}-1)}{2}$ parameters.

Remark. The positive definiteness condition (1.15) can be thought of as a bound on b and B in terms of ac by expressing it as

$$b^2 + \|B\|^2 < ac \,, \tag{1.16}$$

where $\|B\|$ is the ℓ^2-norm, which equals the spectral radius for the skew-symmetric matrix B.

We can bring $\mathcal{M}$ into the local Maxwellian form (1.13). By expanding (1.14b) we find that

$$q(v,x,t) = \tfrac{1}{2}\big(at^2 - 2bt + c\big)|v|^2 - \big(axt - bx - Bx\big)\cdot v + \tfrac{1}{2}a|x|^2 \,.$$

Because $a, c > 0$ and $ac > b^2$ by the remark above, we know that $at^2 - 2bt + c > 0$ for every t. We can therefore complete the square in v to obtain

$$q(v,x,t) = \frac{1}{2\theta(t)} \, |v - u(x,t)|^2 + \frac{\theta(t)}{2} \, x^T Q x \,,$$

where $\theta(t)$ and $u(x,t)$ are given by

$$\theta(t) = \frac{1}{at^2 - 2bt + c} \,, \qquad u(x,t) = \theta(t)\big(axt - bx - Bx\big) \,. \tag{1.17}$$

We thereby see that $\mathcal{M}$ has the local Maxwellian form

$$\mathcal{M} = \frac{\rho(x,t)}{\big(2\pi\theta(t)\big)^{\frac{\mathrm{D}}{2}}} \exp\!\left(- \frac{|v - u(x,t)|^2}{2\theta(t)} \right) , \tag{1.18}$$

where the temperature $\theta(t)$ and bulk velocity $u(x,t)$ are given by (1.17), while the mass density $\rho(x,t)$ is given by

$$\rho(x,t) = m \left(\frac{\theta(t)}{2\pi} \right)^{\frac{\mathrm{D}}{2}} \sqrt{\det(Q)} \exp\!\left(- \frac{\theta(t)}{2} \, x^T Q x \right) . \tag{1.19}$$

Because $Q > 0$ by (1.15), we see that $\rho(x,t)$ is integrable over $\mathbb{R}^{\mathrm{D}}$.

2. Properties of Global Maxwellians

2.1. **Moments of Global Maxwellians.** We can evaluate the zeroth, first, second, and any other moment of the global Maxwellian family $\mathcal{M}$ because they are Gaussian densities over the velocity-position space $\mathbb{R}^{\mathrm{D}} \times \mathbb{R}^{\mathrm{D}}$. We will use some basic facts about matrices in the form

$$\mathbf{A} = \begin{pmatrix} cI & bI + B \\ bI - B & aI \end{pmatrix}, \qquad \text{where } (a, b, c, B) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^{\mathrm{D} \wedge \mathrm{D}}. \tag{2.1}$$

Specifically, if $c \neq 0$ then the matrices $\mathbf{A}$ and $Q = (ac - b^2)I + B^2$ are related by the facts

$$\begin{aligned}
&\text{(i)} \quad \det(\mathbf{A}) = \det(Q)\,, \\
&\text{(ii)} \quad \mathbf{A} \text{ is invertible} \iff Q \text{ is invertible}\,, \\
&\qquad \text{in which case} \quad \mathbf{A}^{-1} = \begin{pmatrix} Q^{-1} & 0 \\ 0 & Q^{-1} \end{pmatrix} \begin{pmatrix} aI & -bI - B \\ -bI + B & cI \end{pmatrix}, \\
&\text{(iii)} \quad \mathbf{A} \geq 0 \iff Q \geq 0 \text{ and } c > 0\,, \\
&\text{(iv)} \quad \mathbf{A} > 0 \iff Q > 0 \text{ and } c > 0 \iff (a, b, c, B) \in \Omega\,.
\end{aligned} \tag{2.2}$$

These facts can all be seen from the block LDU factorization

$$\mathbf{A} = \begin{pmatrix} I & 0 \\ \frac{1}{c}(bI - B) & I \end{pmatrix} \begin{pmatrix} cI & 0 \\ 0 & \frac{1}{c}Q \end{pmatrix} \begin{pmatrix} I & \frac{1}{c}(bI + B) \\ 0 & I \end{pmatrix}.$$

This factorization holds for any $\mathbf{A}$ of the form (2.1) with $c \neq 0$ and $B \in \mathbb{R}^{\mathrm{D} \times \mathrm{D}}$. Fact (ii) follows from fact (i). The formula for $\mathbf{A}^{-1}$ can be checked by direct computation. Facts (iii) and (iv) require that $B \in \mathbb{R}^{\mathrm{D} \wedge \mathrm{D}}$, which insures that $\mathbf{A}$ and Q are symmetric. Fact (iv) shows that Ω is a cone. Because it is defined by strict inequalites, $\Omega \subset \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R}_+ \times \mathbb{R}^{\mathrm{D} \wedge \mathrm{D}}$ is open.

By evaluating the Gaussian integral and using facts (i) and (iv) of (2.2) we see that

$$\iint_{\mathbb{R}^{\mathrm{D}} \times \mathbb{R}^{\mathrm{D}}} \mathcal{M} \, \mathrm{d}v \, \mathrm{d}x = \int_{\mathbb{R}^{\mathrm{D}}} \rho(x, t) \, \mathrm{d}x = m\,.$$

Hence, m is the total mass of $\mathcal{M}$. Because $\mathcal{M}$ is an even function of $\begin{pmatrix} v & x - vt \end{pmatrix}^T$, we see that

$$\iint_{\mathbb{R}^{\mathrm{D}} \times \mathbb{R}^{\mathrm{D}}} \begin{pmatrix} v \\ x - vt \end{pmatrix} \mathcal{M} \, \mathrm{d}v \, \mathrm{d}x = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

This reflects the facts that $\mathcal{M}$ has no net momentum and that its center of mass is at the origin. In fact, every odd-order moment will vanish because $\mathcal{M}$ has even symmetry.

The second moments of $\mathcal{M}$ may be computed by evaluating the Gaussian integrals and by using the formula for $\mathbf{A}^{-1}$ given in fact (ii) of (2.2) as

$$\begin{aligned}
&\iint_{\mathbb{R}^{\mathrm{D}} \times \mathbb{R}^{\mathrm{D}}} \begin{pmatrix} v\,v^T & v\,(x - vt)^T \\ (x - vt)\,v^T & (x - vt)\,(x - vt)^T \end{pmatrix} \mathcal{M} \, \mathrm{d}v \, \mathrm{d}x \\
&\quad = \iint_{\mathbb{R}^{\mathrm{D}} \times \mathbb{R}^{\mathrm{D}}} \begin{pmatrix} v \\ x - vt \end{pmatrix} \begin{pmatrix} v \\ x - vt \end{pmatrix}^T \mathcal{M} \, \mathrm{d}v \, \mathrm{d}x \\
&\quad = m\mathbf{A}^{-1} = m \begin{pmatrix} aQ^{-1} & -bQ^{-1} - Q^{-1}B \\ -bQ^{-1} + Q^{-1}B & cQ^{-1} \end{pmatrix}.
\end{aligned} \tag{2.3}$$

In particular, the values of the quadratic conserved quantities from (1.12) are given by

$$\begin{aligned}
&\iint_{\mathbb{R}^{\mathrm{D}}\times\mathbb{R}^{\mathrm{D}}} |v|^2 \mathcal{M}\,\mathrm{d}v\,\mathrm{d}x = m\,\mathrm{tr}\big(Q^{-1}\big)a\,,\\
&\iint_{\mathbb{R}^{\mathrm{D}}\times\mathbb{R}^{\mathrm{D}}} v\cdot(x-vt)\,\mathcal{M}\,\mathrm{d}v\,\mathrm{d}x = -m\,\mathrm{tr}\big(Q^{-1}\big)b\,,\\
&\iint_{\mathbb{R}^{\mathrm{D}}\times\mathbb{R}^{\mathrm{D}}} |x-vt|^2 \mathcal{M}\,\mathrm{d}v\,\mathrm{d}x = m\,\mathrm{tr}\big(Q^{-1}\big)c\,,\\
&\iint_{\mathbb{R}^{\mathrm{D}}\times\mathbb{R}^{\mathrm{D}}} v\wedge x\,\mathcal{M}\,\mathrm{d}v\,\mathrm{d}x = -2mQ^{-1}B\,.
\end{aligned} \tag{2.4}$$

Every even-order moment of $\mathcal{M}$ may also be computed, but we will not do so here.

2.2. **Conserved Quantities and Global Maxwellians.** Every Cauchy problem (1.1) with initial data $F^{\mathrm{in}}(v,x)$ that satisfies the bounds (1.2) can be associated with a unique global Maxwellian determined by the values of the conserved quantities computed from F^{in}. By choosing an appropriate rescaling and Galilean frame, we may assume without loss of generality that

$$\iint_{\mathbb{R}^{\mathrm{D}}\times\mathbb{R}^{\mathrm{D}}} F^{\mathrm{in}}\,\mathrm{d}v\,\mathrm{d}x = 1\,,\qquad \iint_{\mathbb{R}^{\mathrm{D}}\times\mathbb{R}^{\mathrm{D}}} v\,F^{\mathrm{in}}\,\mathrm{d}v\,\mathrm{d}x = \iint_{\mathbb{R}^{\mathrm{D}}\times\mathbb{R}^{\mathrm{D}}} x\,F^{\mathrm{in}}\,\mathrm{d}v\,\mathrm{d}x = 0\,. \tag{2.5}$$

The following theorem was proved in [8].

Theorem 2.1. *Let $F^{\mathrm{in}}(v,x)$ be a nonnegative function that satisfies the bounds (1.2) and the normalizations (2.5). Let a_*, b_*, c_*, and B_* be given by*

$$\begin{aligned}
&\iint_{\mathbb{R}^{\mathrm{D}}\times\mathbb{R}^{\mathrm{D}}} |v|^2 F^{\mathrm{in}}\,\mathrm{d}v\,\mathrm{d}x = a_*\,, &&\iint_{\mathbb{R}^{\mathrm{D}}\times\mathbb{R}^{\mathrm{D}}} v\cdot x\,F^{\mathrm{in}}\,\mathrm{d}v\,\mathrm{d}x = b_*\,,\\
&\iint_{\mathbb{R}^{\mathrm{D}}\times\mathbb{R}^{\mathrm{D}}} |x|^2 F^{\mathrm{in}}\,\mathrm{d}v\,\mathrm{d}x = c_*\,, &&\iint_{\mathbb{R}^{\mathrm{D}}\times\mathbb{R}^{\mathrm{D}}} v\wedge x\,F^{\mathrm{in}}\,\mathrm{d}v\,\mathrm{d}x = B_*\,.
\end{aligned} \tag{2.6}$$

Then $(a_*,b_*,c_*,B_*)\in\Omega_*$, where Ω_* is the open cone in $\mathbb{R}\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}^{\mathrm{D}\wedge\mathrm{D}}$ defined by

$$\Omega_* = \left\{(a_*,b_*,c_*,B_*)\in\mathbb{R}_+\times\mathbb{R}\times\mathbb{R}_+\times\mathbb{R}^{\mathrm{D}\wedge\mathrm{D}}\;:\;\tfrac{1}{2}\mathrm{tr}(|B_*|)<\sqrt{a_*c_*-b_*^2}\right\}, \tag{2.7}$$

with $|B_*|=\sqrt{B_*^TB_*}=\sqrt{-B_*^2}$.

Conversely, if $(a_*,b_*,c_*,B_*)\in\Omega_*$ then there exists a unique (up to a time shift) global Maxwellian $\mathcal{M}$ given by (1.14) with $m=1$ and $(a,b,c,B)\in\Omega$ such that the quadratic converved quantities associated with $\mathcal{M}$ through (2.4) have values (a_*,b_*,c_*,B_*) — i.e. such that

$$a_*=\mathrm{tr}\big(Q^{-1}\big)a\,,\quad b_*=-\mathrm{tr}\big(Q^{-1}\big)b\,,\quad c_*=\mathrm{tr}\big(Q^{-1}\big)c\,,\quad B_*=-2Q^{-1}B\,, \tag{2.8}$$

where $Q=(ac-b^2)I+B^2$ and the set Ω was defined by (1.15).

Remark. The first part of this theorem states that Ω_* contains the set of values that can be realized by the conserved quantities given by (2.6). The second part asserts that every point in Ω_* can be so realized. Therefore Ω_* characterizes all such values.

Remark. Because B_* is skew-symmetric, its nonzero eigenvalues will come in conjugate pairs. When B_* has at most two nonzero eigenvalues then $\|B_*\|=\frac{1}{2}\mathrm{tr}(|B_*|)$, where $\|B_*\|$ is the ℓ^2 matrix norm, which equals the spectral radius of B_*. If $\mathrm{D}=2$ or $\mathrm{D}=3$ then B_* can have at most two nonzero eigenvalues and we see by (1.16) that $\Omega_*=\Omega$. If $\mathrm{D}>3$ then Ω_* will be a proper subset of Ω because $\|B_*\|<\frac{1}{2}\mathrm{tr}(|B_*|)$ when B_* has more than two nonzero eigenvalues.

2.3. **Ordering Global Maxwellians.** The following lemma characterizes when one global Maxwellian with finite mass bounds another that has the same net momentum and center of mass. By choosing an appropriate Galilean frame, we can assume without loss of generality that the net momentum is zero and the center of mass is the origin. In that case the global Maxwellians will have the form (1.14).

Lemma 2.1. *Let $\mathcal{M}_1$ and $\mathcal{M}_2$ be global Maxwellians of the form (1.14) with parameters given by $(m_1, a_1, b_1, c_1, B_1) \in \mathbb{R}_+ \times \Omega$ and $(m_2, a_2, b_2, c_2, B_2) \in \mathbb{R}_+ \times \Omega$ respectively. Then $\mathcal{M}_1 \leq \mathcal{M}_2$ for every (v, x, t) if and only if*

$$\begin{pmatrix} c_2 I & b_2 I + B_2 \\ b_2 I - B_2 & a_2 I \end{pmatrix} \leq \begin{pmatrix} c_1 I & b_1 I + B_1 \\ b_1 I - B_1 & a_1 I \end{pmatrix} . \tag{2.9}$$

$$m_1 \sqrt{\det\big((a_1 c_1 - b_1^2) I + B_1^2\big)} \leq m_2 \sqrt{\det\big((a_2 c_2 - b_2^2) I + B_2^2\big)} \,, \tag{2.10}$$

Similarly, $\mathcal{M}_1 < \mathcal{M}_2$ for every (v, x, t) if and only if (2.9) holds and

$$m_1 \sqrt{\det\big((a_1 c_1 - b_1^2) I + B_1^2\big)} < m_2 \sqrt{\det\big((a_2 c_2 - b_2^2) I + B_2^2\big)} \,, \tag{2.11}$$

Proof. It follows from (1.14) that

$$\begin{aligned} \log\left(\frac{\mathcal{M}_2}{\mathcal{M}_1}\right) &= \log\left(\frac{m_2 \sqrt{\det\big((a_2 c_2 - b_2^2) I + B_2^2\big)}}{m_1 \sqrt{\det\big((a_1 c_1 - b_1^2) I + B_1^2\big)}}\right) \\ &\quad + \frac{1}{2} \begin{pmatrix} v \\ x - vt \end{pmatrix}^T \begin{pmatrix} (c_1 - c_2) I & (b_1 - b_2) I + (B_1 - B_2) \\ (b_1 - b_2) I - (B_1 - B_2) & (a_1 - a_2) I \end{pmatrix} \begin{pmatrix} v \\ x - vt \end{pmatrix} . \end{aligned}$$

The assertions of the lemma can be read off from this. □

Remark. The matrix inequality (2.9) is equivalent to $a_2 \leq a_1$, $c_2 \leq c_1$, and

$$\big((a_1 - a_2)(c_1 - c_2) - (b_1 - b_2)^2\big) I + (B_1 - B_2)^2 \geq 0 \,. \tag{2.12}$$

This positive definiteness condition can be thought of as a bound on $b_1 - b_2$ and $B_1 - B_2$ in terms of $(a_1 - a_2)(c_1 - c_2)$ by expressing it as

$$(b_1 - b_2)^2 + \|B_1 - B_2\|^2 < (a_1 - a_2)(c_1 - c_2) \,,$$

where $\|B_1 - B_2\|$ is the matrix ℓ^2-norm, which equals the spectral radius for $B_1 - B_2$.

Remark. The matrix inequality (2.9) implies that

$$\sqrt{\det\big((a_2 c_2 - b_2^2) I + B_2^2\big)} \leq \sqrt{\det\big((a_1 c_1 - b_1^2) I + B_1^2\big)} \,,$$

with equality if and only if

$$\begin{pmatrix} c_2 I & b_2 I + B_2 \\ b_2 I - B_2 & a_2 I \end{pmatrix} = \begin{pmatrix} c_1 I & b_1 I + B_1 \\ b_1 I - B_1 & a_1 I \end{pmatrix} .$$

Therefore if $\mathcal{M}_1 \leq \mathcal{M}_2$ then $m_1 < m_2$ unless $\mathcal{M}_1 = \mathcal{M}_2$.

Remark. In this section we assumed that $\mathcal{M}_1$ and $\mathcal{M}_2$ have the same net momentum and the same center of mass. It is clear that if $\mathcal{M}_1 \leq \mathcal{M}_2$ then $\mathcal{M}_1$ and $\mathcal{M}_2$ must have the same net momentum, but they need not have the same same center of mass. Allowing for different centers of mass does not change condition (2.9), but does modify condition (2.10).

2.4. **Bounds by Global Maxwellians.** Let $\mathcal{M}_1$ and $\mathcal{M}_2$ be global Maxwellians of the form (1.14) with parameters given by $(m_1, a_1, b_1, c_1, B_1) \in \mathbb{R}_+ \times \Omega$ and $(m_2, a_2, b_2, c_2, B_2) \in \mathbb{R}_+ \times \Omega$ respectively such that $\mathcal{M}_1 \leq \mathcal{M}_2$ for every (v, x, t). Let $\mathcal{M}_1(t)$ and $\mathcal{M}_2(t)$ denote their values at time t. Let $F^{\mathrm{in}}(v, x)$ satisfy the pointwise bounds

$$\mathcal{M}_1(s) < F^{\mathrm{in}} < \mathcal{M}_2(s) \quad \text{over } \mathbb{R}^{\mathrm{D}} \times \mathbb{R}^{\mathrm{D}} \text{ for some } s \in \mathbb{R}\,. \tag{2.13}$$

and the normalizations

$$\iint_{\mathbb{R}^{\mathrm{D}}\times\mathbb{R}^{\mathrm{D}}} F^{\mathrm{in}}\,\mathrm{d}v\,\mathrm{d}x = 1\,, \qquad \iint_{\mathbb{R}^{\mathrm{D}}\times\mathbb{R}^{\mathrm{D}}} v\,F^{\mathrm{in}}\,\mathrm{d}v\,\mathrm{d}x = \iint_{\mathbb{R}^{\mathrm{D}}\times\mathbb{R}^{\mathrm{D}}} x\,F^{\mathrm{in}}\,\mathrm{d}v\,\mathrm{d}x = 0\,. \tag{2.14}$$

We would like to show that under certain conditions there exists a unique solution $F(v, x, t)$ of the kinetic equation posed over the spatial domain $\mathbb{R}^{\mathrm{D}}$ with initial data $F^{\mathrm{in}}(v, x)$.

In addition, we hope to show that there exists a global Maxwellian $\mathcal{M}$ of the form (1.14) with $m = 1$ and $(a, b, c, B) \in \Omega$ such that

$$F(v, x, t) \sim \mathcal{M}(v, x, t) \quad \text{as } t \to \infty \text{ in some topology}\,.$$

If we let $(a_*, b_*, c_*, B_*) \in \Omega_*$ be given by

$$\begin{aligned} &\iint_{\mathbb{R}^{\mathrm{D}}\times\mathbb{R}^{\mathrm{D}}} |v|^2 F^{\mathrm{in}}\,\mathrm{d}v\,\mathrm{d}x = a_*\,, && \iint_{\mathbb{R}^{\mathrm{D}}\times\mathbb{R}^{\mathrm{D}}} v\cdot x\,F^{\mathrm{in}}\,\mathrm{d}v\,\mathrm{d}x = b_*\,, \\ &\iint_{\mathbb{R}^{\mathrm{D}}\times\mathbb{R}^{\mathrm{D}}} |x|^2 F^{\mathrm{in}}\,\mathrm{d}v\,\mathrm{d}x = c_*\,, && \iint_{\mathbb{R}^{\mathrm{D}}\times\mathbb{R}^{\mathrm{D}}} v\wedge x\,F^{\mathrm{in}}\,\mathrm{d}v\,\mathrm{d}x = B_*\,, \end{aligned} \tag{2.15}$$

then by Theorem 2.1 there exists $(a, b, c, B) \in \Omega$ such that

$$a_* = \mathrm{tr}\big(Q^{-1}\big)a\,, \qquad b_* = -\mathrm{tr}\big(Q^{-1}\big)b\,, \qquad c_* = \mathrm{tr}\big(Q^{-1}\big)c\,, \qquad B_* = -2Q^{-1}B\,, \tag{2.16}$$

where $Q = (ac - b^2)I + B^2$.

The global Maxwellian bounds (2.13) implies that (a_*, b_*, c_*, B_*) can be bounded by the corresponding quantities associated with $m_1^{\mathrm{in}}\mathcal{M}_1^{\mathrm{in}}$ and $m_2^{\mathrm{in}}\mathcal{M}_2^{\mathrm{in}}$, which are

$$\begin{aligned} a_{1*} &= m_1\,\mathrm{tr}\big(Q_1^{-1}\big)a_1\,, & a_{2*} &= m_2\,\mathrm{tr}\big(Q_2^{-1}\big)a_2\,, \\ b_{1*} &= -m_1\,\mathrm{tr}\big(Q_1^{-1}\big)b_1\,, & b_{2*} &= -m_2\,\mathrm{tr}\big(Q_2^{-1}\big)b_2\,, \\ c_{1*} &= m_1\,\mathrm{tr}\big(Q_1^{-1}\big)c_1\,, & c_{2*} &= m_2\,\mathrm{tr}\big(Q_2^{-1}\big)c_2\,, \\ B_{1*} &= -2m_1\,Q_1^{-1}B_1\,, & B_{2*} &= -2m_2\,Q_2^{-1}B_2\,, \end{aligned} \tag{2.17}$$

where $Q_1 = (a_1c_1 - b_1^2)I + B_1^2$ and $Q_2 = (a_2c_2 - b_2^2)I + B_2^2$. Theorem 2.1 implies that

$$\begin{aligned} &\big(a_* - a_{1*}\,,\, b_* - b_{1*}\,,\, c_* - c_{1*}\,,\, B_* - B_{1*}\big) \in \Omega_*\,, \\ &\big(a_{2*} - a_*\,,\, b_{2*} - b_*\,,\, c_{2*} - c_*\,,\, B_{2*} - B_*\big) \in \Omega_*\,, \\ &\big(a_{2*} - a_{1*}\,,\, b_{2*} - b_{1*}\,,\, c_{2*} - c_{1*}\,,\, B_{2*} - B_{1*}\big) \in \Omega_*\,, \end{aligned} \tag{2.18}$$

which yields the bounds

$$\begin{aligned} a_{1*} < a_* < a_{2*}\,, \qquad & c_{1*} < c_* < c_{2*}\,, \\ \tfrac{1}{2}\,\mathrm{tr}(|B_* - B_{1*}|) &< \sqrt{(a_* - a_{1*})(c_* - c_{1*}) - (b_* - b_{1*})^2}\,, \\ \tfrac{1}{2}\,\mathrm{tr}(|B_{2*} - B_*|) &< \sqrt{(a_{2*} - a_*)(c_{2*} - c_*) - (b_{2*} - b_*)^2}\,, \\ \tfrac{1}{2}\,\mathrm{tr}(|B_{2*} - B_{1*}|) &< \sqrt{(a_{2*} - a_{1*})(c_{2*} - c_{1*}) - (b_{2*} - b_{1*})^2}\,. \end{aligned} \tag{2.19}$$

3. Kaniel-Shinbrot Iteration

3.1. **Kaniel-Shinbrot Theorem.** Kaniel-Shinbrot iteration can be used to prove the existence of solutions F to the Boltzmann initial-value problem posed over the spatial domain $\mathbb{R}^{\mathrm{D}}$ with initial data $F^{\mathrm{in}}(v,x)$ that satisfies certain pointwise bounds. More specifically, it is used to construct two sequences of approximate solutions, $\{F_j^L\}_{j\in\mathbb{N}}$ and $\{F_j^U\}_{j\in\mathbb{N}}$, the first of which converges monotonically to F from below, while the second converges monotonically to F from above. In other words, for some $T\in(0,\infty]$ these opposing monotone sequences should satisfy the order relationships

$$F_j^L \le F_{j+1}^L \le F_{j+1}^U \le F_j^U \qquad \text{over } \mathbb{R}^{\mathrm{D}}\times\mathbb{R}^{\mathrm{D}}\times[0,T) \text{ for every } j\in\mathbb{N}\,. \tag{3.1}$$

The expectation is that these sequences will converge to $F(v,x,t)$ over $\mathbb{R}^{\mathrm{D}}\times\mathbb{R}^{\mathrm{D}}\times[0,T)$. When this construction can be carried out for $T=\infty$ then the solution F will be global in time.

Kaniel-Shinbrot iteration constructs sequences as follows. Given F_{j-1}^L and F_{j-1}^U for any $j\in\mathbb{Z}_+$ we define the Kaniel-Shinbrot iterates F_j^L and F_j^U to be the solution of the linear system

$$\begin{aligned} \partial_t F_j^U + v\cdot\nabla_x F_j^U + \mathcal{A}\big(F_{j-1}^L\big)\,F_j^U &= \mathcal{G}\big(F_{j-1}^U, F_{j-1}^U\big)\,, \\ \partial_t F_j^L + v\cdot\nabla_x F_j^L + \mathcal{A}\big(F_{j-1}^U\big)\,F_j^L &= \mathcal{G}\big(F_{j-1}^L, F_{j-1}^L\big)\,, \end{aligned} \tag{3.2a}$$

$$F_j^U\big|_{t=0} = F_j^L\big|_{t=0} = F^{\mathrm{in}}\,. \tag{3.2b}$$

The existence of F_j^L and F_j^U can be inferred from the mild formulation of system (3.2).

The form of the Kaniel-Shinbrot iteration (3.2) is motivated by the fact that $\mathcal{A}$ and $\mathcal{G}$ have the monotonicity properties

$$F\le G \quad\Longrightarrow\quad \mathcal{A}(F)\le\mathcal{A}(G) \quad\text{and}\quad \mathcal{G}(F,F)\le\mathcal{G}(G,G)\,. \tag{3.3}$$

The following lemma shows that this form insures that Kaniel-Shinbrot iterates preserve the order relationships that appear in (3.1).

Lemma 3.1. (Order Preservation Lemma) *If for some $T\in(0,\infty]$ the Kaniel-Shinbrot iterates F_{j-1}^L, F_{j-1}^U, F_j^L, and F_j^U satisfy*

$$F_{j-1}^L \le F_j^L \le F_j^U \le F_{j-1}^U \qquad \textit{over } \mathbb{R}^{\mathrm{D}}\times\mathbb{R}^{\mathrm{D}}\times[0,T)\,, \tag{3.4}$$

then the Kaniel-Shinbrot iterates F_{j+1}^L and F_{j+1}^U satisfy

$$F_j^L \le F_{j+1}^L \le F_{j+1}^U \le F_j^U \qquad \textit{over } \mathbb{R}^{\mathrm{D}}\times\mathbb{R}^{\mathrm{D}}\times[0,T)\,. \tag{3.5}$$

Proof. By (3.2a) of the Kaniel-Shinbrot construction we see that

$$\begin{aligned} \big(\partial_t + v\cdot\nabla_x + \mathcal{A}\big(F_j^U\big)\big)\,\big(F_{j+1}^L - F_j^L\big) &= \mathcal{G}\big(F_j^L,F_j^L\big) - \mathcal{G}\big(F_{j+1}^L,F_{j+1}^L\big) + \mathcal{A}\big(F_{j-1}^U - F_j^U\big)\,F_j^L\,, \\ \big(\partial_t + v\cdot\nabla_x + \mathcal{A}\big(F_j^L\big)\big)\,\big(F_j^U - F_{j+1}^U\big) &= \mathcal{G}\big(F_{j-1}^U,F_{j-1}^U\big) - \mathcal{G}\big(F_j^U,F_j^U\big) + \mathcal{A}\big(F_j^L - F_{j-1}^L\big)\,F_j^U\,. \end{aligned}$$

By the monotonicity (3.3) of $\mathcal{A}$ and $\mathcal{G}$ and hypothesis (3.4) the right-hand sides above are nonnegative over $\mathbb{R}^{\mathrm{D}}\times\mathbb{R}^{\mathrm{D}}\times[0,T)$, while (3.2b) of the Kaniel-Shinbrot construction implies that

$$\big(F_{j+1}^L - F_j^L\big)\big|_{t=0} = \big(F_j^U - F_{j+1}^U\big)\big|_{t=0} = 0\,, \qquad \text{over } \mathbb{R}^{\mathrm{D}}\times\mathbb{R}^{\mathrm{D}}\,.$$

The maximum principle then implies that

$$\big(F_{j+1}^L - F_j^L\big)\ge 0\,, \qquad \big(F_j^U - F_{j+1}^U\big)\ge 0\,, \qquad \text{over } \mathbb{R}^{\mathrm{D}}\times\mathbb{R}^{\mathrm{D}}\times[0,T)\,. \tag{3.6}$$

This yields two of the three inequalities asserted by (3.5).

Similarly, by (3.2a) of the Kaniel-Shinbrot construction we have

$$\big(\partial_t + v\cdot\nabla_x + \mathcal{A}(F_j^U)\big)\,\big(F_{j+1}^U - F_{j+1}^L\big) = \mathcal{G}\big(F_j^U, F_j^U\big) - \mathcal{G}\big(F_j^L, F_j^L\big) + \mathcal{A}\big(F_j^U - F_j^L\big)\,F_{j+1}^U\,.$$

By the monotonicity (3.3) of $\mathcal{A}$ and $\mathcal{G}$ and hypothesis (3.4) the right-hand side above is non-negative over $\mathbb{R}^{\mathrm{D}}\times\mathbb{R}^{\mathrm{D}}\times[0,T)$, while by (3.2b) of the Kaniel-Shinbrot construction we see that

$$\big(F_{j+1}^U - F_{j+1}^L\big)\big|_{t=0} = 0\,, \qquad \text{over } \mathbb{R}^{\mathrm{D}}\times\mathbb{R}^{\mathrm{D}}\,.$$

The maximum principle then implies that

$$\big(F_{j+1}^U - F_{j+1}^L\big) \geq 0\,, \qquad \text{over } \mathbb{R}^{\mathrm{D}}\times\mathbb{R}^{\mathrm{D}}\times[0,T)\,.$$

When this inequality is combimed with the two in (3.6), we see that assertion (3.5) holds. □

Induction, Lemma 3.1, the Lebesgue Monotone Convergence Theorem, and a stability bound can be used [7, 6] to prove the following.

Theorem 3.1. (Kaniel-Shinbrot) *If for some $T\in(0,\infty]$ the Kaniel-Shinbrot iterates F_0^L, F_0^U, F_1^L, and F_1^U satisfy the so-called beginning condition*

$$F_0^L \leq F_1^L \leq F_1^U \leq F_0^U \qquad \text{over } \mathbb{R}^{\mathrm{D}}\times\mathbb{R}^{\mathrm{D}}\times[0,T)\,, \tag{3.7}$$

then the Kaniel-Shinbrot iteration yields opposing monotone sequences $\{F_j^L\}_{j\in\mathbb{N}}$ and $\{F_j^U\}_{j\in\mathbb{N}}$ over $\mathbb{R}^{\mathrm{D}}\times\mathbb{R}^{\mathrm{D}}\times[0,T)$ — i.e. sequences that satisfy the order relationship (3.1). These sequences converge to a unique mild solution of the initial-value problem (1.1) for the Boltzmann equation.

Remark. The hard part of applying the Kaniel-Shinbrot Theorem is being sure that the first two iterates satisfy the beginning condition (3.7).

3.2. **Beginning with Local Maxwellians.** When the initial Kaniel-Shinbrot iterates are local Maxwellians then there is a simple criterion that insures the beginning condition (3.7) is satisfied for some $T\in(0,\infty]$.

Proposition 3.1. (Local Maxwellian Beginning Lemma) *Suppose M^L and M^U are local Maxwellians that satisfy*

$$M^L\big|_{t=0} \leq M^U\big|_{t=0} \qquad \text{over } \mathbb{R}^{\mathrm{D}}\times\mathbb{R}^{\mathrm{D}}\,, \tag{3.8a}$$

and for some $T\in(0,\infty]$ satisfy

$$\begin{aligned} \partial_t M^U + v\cdot\nabla_x M^U &\geq \mathcal{A}(M^U - M^L)M^U \qquad &&\text{over } \mathbb{R}^{\mathrm{D}}\times\mathbb{R}^{\mathrm{D}}\times[0,T)\,,\\ -\,\partial_t M^L - v\cdot\nabla_x M^L &\geq \mathcal{A}(M^U - M^L)M^L \qquad &&\text{over } \mathbb{R}^{\mathrm{D}}\times\mathbb{R}^{\mathrm{D}}\times[0,T)\,, \end{aligned} \tag{3.8b}$$

Then for every initial data F^{in} such that

$$M^L\big|_{t=0} \leq F^{\mathrm{in}} \leq M^U\big|_{t=0} \qquad \text{over } \mathbb{R}^{\mathrm{D}}\times\mathbb{R}^{\mathrm{D}}\,,$$

the Kaniel-Shinbrot iterates obtained by setting $F_0^L = M^L$ and $F_0^U = M^U$ satisfy the beginning condition (3.7) over $[0,T)$. Moreover, the Kaniel-Shinbrot Theorem (3.1) yields the unique mild solution $F(v,x,t)$ of the initial-value problem (1.1) for the Boltzmann equation that satisfies the bounds

$$M^L(v,x,t) \leq F(v,x,t) \leq M^U(v,x,t) \qquad \text{over } \mathbb{R}^{\mathrm{D}}\times\mathbb{R}^{\mathrm{D}}\times[0,T)\,. \tag{3.9}$$

Proof. Because $F_0^L = M^L$ and $F_0^U = M^U$, the Kaniel-Shinbrot construction (3.2) yields

$$\begin{aligned} \partial_t F_1^U + v\cdot\nabla_x F_1^U + \mathcal{A}(M^L) F_1^U &= \mathcal{G}(M^U, M^U)\,, \\ \partial_t F_1^L + v\cdot\nabla_x F_1^L + \mathcal{A}(M^U) F_1^L &= \mathcal{G}(M^L, M^L)\,, \\ F_1^U\big|_{t=0} = F_1^L\big|_{t=0} &= F^{\mathrm{in}}\,. \end{aligned}$$

The above right-hand sides satisfy $\mathcal{G}(M^L, M^L) = \mathcal{A}(M^L)M^L$ and $\mathcal{G}(M^U, M^U) = \mathcal{A}(M^U)M^U$ because M^L and M^U are local Maxwellians. Therefore

$$\begin{aligned} \big(\partial_t + v\cdot\nabla_x + \mathcal{A}(M^L)\big)\big(M^U - F_1^U\big) &= \partial_t M^U + v\cdot\nabla_x M^U - \mathcal{A}(M^U - M^L)M^U\,, \\ \big(\partial_t + v\cdot\nabla_x + \mathcal{A}(M^U)\big)\big(F_1^L - M^L\big) &= -\,\partial_t M^L - v\cdot\nabla_x M^L - \mathcal{A}(M^U - M^L)M^L\,. \end{aligned}$$

By hypothesis (3.8b) the right-hand sides above are nonnegative over $\mathbb{R}^{\mathrm{D}}\times\mathbb{R}^{\mathrm{D}}\times[0,T)$, while by hypothesis (3.8a)

$$\big(M^U - F_1^U\big)\big|_{t=0} \geq 0\,, \qquad \big(F_1^L - M^L\big)\big|_{t=0} \geq 0\,, \qquad \text{over } \mathbb{R}^{\mathrm{D}}\times\mathbb{R}^{\mathrm{D}}\,.$$

The maximum principle then implies that

$$\big(M^U - F_1^U\big) \geq 0\,, \qquad \big(F_1^L - M^L\big) \geq 0\,, \qquad \text{over } \mathbb{R}^{\mathrm{D}}\times\mathbb{R}^{\mathrm{D}}\times[0,T)\,. \tag{3.10}$$

This yields two of the three inequalities that are required by the beginning condition (3.7).

Similarly, we have

$$\big(\partial_t + v\cdot\nabla_x + \mathcal{A}(M^U)\big)\big(F_1^U - F_1^L\big) = \mathcal{G}(M^U, M^U) - \mathcal{G}(M^L, M^L) + \mathcal{A}(M^U - M^L)F_1^U\,.$$

The right-hand side above is nonnegative by the monotonicity (3.3) of $\mathcal{A}$ and $\mathcal{G}$, while we see by (3.2b) of the Kaniel-Shinbrot construction that

$$\big(F_1^U - F_1^L\big)\big|_{t=0} = 0 \qquad \text{over } \mathbb{R}^{\mathrm{D}}\times\mathbb{R}^{\mathrm{D}}\,.$$

The maximum principle then implies that

$$\big(F_1^U - F_1^L\big) \geq 0 \qquad \text{over } \mathbb{R}^{\mathrm{D}}\times\mathbb{R}^{\mathrm{D}}\times[0,T)\,.$$

When this inequality is combined with the two in (3.10), we see that the beginning condition (3.7) is satisfied. The Kaniel-Shinbrot Theorem (3.1) thereby yields a unique mild solution of the initial-value problem (1.1) for the Boltzmann equation that satisfies (3.9). □

Proposition 3.1 requires us to find local Maxwellians M^L and M^U that meet criterion (3.8) for some $T \in (0,\infty]$. When $M^L > 0$ this criterion may be recast as

$$\begin{aligned} \big(\partial_t + v\cdot\nabla_x\big)\log\big(M^U\big) &\geq \mathcal{A}\big(M^U - M^L\big) \qquad \text{over } \mathbb{R}^{\mathrm{D}}\times\mathbb{R}^{\mathrm{D}}\times[0,T)\,, \\ -\big(\partial_t + v\cdot\nabla_x\big)\log\big(M^L\big) &\geq \mathcal{A}\big(M^U - M^L\big) \qquad \text{over } \mathbb{R}^{\mathrm{D}}\times\mathbb{R}^{\mathrm{D}}\times[0,T)\,, \end{aligned} \tag{3.11a}$$

$$M^L\big|_{t=0} \leq M^U\big|_{t=0} \qquad \text{over } \mathbb{R}^{\mathrm{D}}\times\mathbb{R}^{\mathrm{D}}\,. \tag{3.11b}$$

When $M^L = 0$ criterion (3.8) reduces to simply

$$\big(\partial_t + v\cdot\nabla_x\big)\log(M^U) \geq \mathcal{A}\big(M^U\big) \qquad \text{over } \mathbb{R}^{\mathrm{D}}\times\mathbb{R}^{\mathrm{D}}\times[0,T)\,. \tag{3.12}$$

We will construct such local Maxwellians from the family of global Maxwellians given by (1.14). When we can do this with $T = \infty$, the Kaniel-Shinbrot theorem will yield global solutions.

3.3. **Constructing Local Maxwellians from Global Ones.** Let $\mathcal{M}_1$ and $\mathcal{M}_2$ be global Maxwellians in the form (1.14) that are given by parameters $(m_1, a_1, b_1, c_1, B_1) \in \mathbb{R}_+ \times \Omega$ and $(m_2, a_2, b_2, c_2, B_2) \in \mathbb{R}_+ \times \Omega$ respectively, and that satisfy $\mathcal{M}_1 \leq \mathcal{M}_2$ for every (v, x, t). Let $\mathcal{M}_1(t)$ and $\mathcal{M}_2(t)$ denote their values at time t. Set

$$
\begin{aligned}
Q_1 &= (a_1c_1 - b_1^2)I + B_1^2\,, \qquad Q_2 = (a_2c_2 - b_2^2)I + B_2^2\,, \\
q_1(v,x,t) &= \frac{1}{2}\begin{pmatrix} v \\ x - vt \end{pmatrix}^T \begin{pmatrix} c_1I & b_1I + B_1 \\ b_1I - B_1 & a_1I \end{pmatrix} \begin{pmatrix} v \\ x - vt \end{pmatrix} . \\
q_2(v,x,t) &= \frac{1}{2}\begin{pmatrix} v \\ x - vt \end{pmatrix}^T \begin{pmatrix} c_2I & b_2I + B_2 \\ b_2I - B_2 & a_2I \end{pmatrix} \begin{pmatrix} v \\ x - vt \end{pmatrix} ,
\end{aligned}
\tag{3.13}
$$

so that

$$
\mathcal{M}_1 = \frac{m_1}{(2\pi)^{\mathrm{D}}} \sqrt{\det(Q_1)}\, \exp\big(-q_1(v,x,t)\big)\,, \qquad \mathcal{M}_2 = \frac{m_2}{(2\pi)^{\mathrm{D}}} \sqrt{\det(Q_2)}\, \exp\big(-q_2(v,x,t)\big)\,.
$$

Because $\mathcal{M}_1 \leq \mathcal{M}_2$, we know that $q_1(v,x,t) \geq q_2(v,x,t)$.

We will construct local Maxwellians M^U and M^L in the form

$$
M^U(v,x,t) = \gamma(t)\, \frac{m_2}{(2\pi)^{\mathrm{D}}} \sqrt{\det(Q_2)}\, \exp\left(-\frac{q_2(v,x,s+t)}{\eta(t)}\right)\,, \tag{3.14a}
$$

$$
M^L(v,x,t) = \frac{1}{\gamma(t)}\, \frac{m_1}{(2\pi)^{\mathrm{D}}} \sqrt{\det(Q_1)}\, \exp\left(-\left(2 - \frac{1}{\eta(t)}\right) q_1(v,x,s+t)\right)\,, \tag{3.14b}
$$

where the functions $\gamma(t)$ and $\eta(t)$ satisfy

$$
\gamma(0) = \eta(0) = 1\,, \qquad \gamma'(t) > 0\,, \qquad \eta'(t) \geq 0\,.
$$

Clearly, inequality (3.11b) is satisfied because

$$
M^L\big|_{t=0} = \mathcal{M}_1(s) \leq \mathcal{M}_2(s) = M^U\big|_{t=0}\,.
$$

By direct calculation we find that

$$
\begin{aligned}
\big(\partial_t + v\cdot\nabla_x\big)\log\big(M^U\big) &= \frac{\gamma'(t)}{\gamma(t)} + \frac{\eta'(t)}{\eta(t)}\,\frac{q_2(v,x,s+t)}{\eta(t)}\,, \\
-\big(\partial_t + v\cdot\nabla_x\big)\log\big(M^L\big) &= \frac{\gamma'(t)}{\gamma(t)} + \frac{\eta'(t)}{\eta(t)}\,\frac{q_1(v,x,s+t)}{\eta(t)}\,.
\end{aligned}
$$

Because $q_1(v,x,s+t) \geq q_2(v,x,s+t)$, we see that

$$
-\big(\partial_t + v\cdot\nabla_x\big)\log\big(M^L\big) \geq \big(\partial_t + v\cdot\nabla_x\big)\log\big(M^U\big)\,,
$$

therefore criterion (3.11) will be satisfied if

$$
\frac{\gamma'(t)}{\gamma(t)} + \frac{\eta'(t)}{\eta(t)}\,\frac{q_2(v,x,s+t)}{\eta(t)} \geq \mathcal{A}\big(M^U - M^L\big)\,. \tag{3.15}
$$

Similarly, if $M^L = 0$ while M^U is given by (3.14a) then criterion (3.12) will be satisfied if

$$
\frac{\gamma'(t)}{\gamma(t)} + \frac{\eta'(t)}{\eta(t)}\,\frac{q_2(v,x,s+t)}{\eta(t)} \geq \mathcal{A}\big(M^U\big)\,. \tag{3.16}
$$

This extends the basic inequality derived in [7] to a more general class of local Maxwellians. We can treat criterion (3.16) as criterion (3.15) with $M^L = 0$, The right-hand sides of (3.15) and (3.16) contain expressions of the form $\mathcal{A}(M)$ where M is some local Maxwellian. If $\rho(x,t)$, $u(x,t)$, and $\theta(x,t)$ are the mass density, bulk velocity, and temperature associated with M. Then

$$
\begin{aligned}
\mathcal{A}(M) &= \iint_{\mathbb{S}^{\mathrm{D}-1}\times\mathbb{R}^{\mathrm{D}}} M_* \, \mathsf{b}(\omega, v-v_*)\, \mathrm{d}\omega\, \mathrm{d}v_* \\
&= \big\|\hat{\mathsf{b}}\big\|_{L^1(\mathrm{d}\omega)} \int_{\mathbb{R}^{\mathrm{D}}} |v-v_*|^\beta \frac{\rho}{(2\pi\theta)^{\frac{\mathrm{D}}{2}}} \exp\!\left(-\frac{|v_*-u|^2}{2\theta}\right) \mathrm{d}v_* \\
&= \rho\, \theta^{\frac{\beta}{2}}\, \mathsf{a}\!\left(\frac{v-u}{\sqrt{\theta}}\right),
\end{aligned}
\tag{3.17}
$$

where the attenuation coefficient $\mathsf{a}(w)$ is defined by

$$
\mathsf{a}(w) = \big\|\hat{\mathsf{b}}\big\|_{L^1(\mathrm{d}\omega)} \frac{1}{(2\pi)^{\frac{\mathrm{D}}{2}}} \int_{\mathbb{R}^{\mathrm{D}}} |w-w_*|^\beta \exp\!\big(-\tfrac{1}{2}|w_*|^2\big)\, \mathrm{d}w_* \,. \tag{3.18}
$$

By rotation invariance $\mathsf{a}(w)$ is a function of only $|w|$. It is a bounded function when $\beta \in (-\mathrm{D}, 0]$, and is an unbounded function when $\beta \in (0,2]$. This fundamental difference in the behavior of $\mathsf{a}(w)$ requires a different analysis for each of the two cases. These will be presented in the subsequent two subsections.

Remark. For any local Maxwellian M with mass density $\rho(x,t)$, bulk velocity $u(x,t)$, and temperature θ one has

$$
\begin{aligned}
\big(\partial_t + v\cdot\nabla_x\big)\log(M) &= \frac{\partial_t\rho + u\cdot\nabla_x\rho + \rho\nabla_x\cdot u}{\rho} + (v-u)\cdot\frac{\rho(\partial_t u + u\cdot\nabla_x u) + \theta\nabla_x\rho}{\rho\theta} \\
&\quad + \left(\frac{|v-u|^2}{2\theta} - \frac{\mathrm{D}}{2}\right)\frac{\partial_t\theta + u\cdot\nabla_x\theta + \frac{2}{\mathrm{D}}\theta\nabla_x\cdot u}{\theta} \\
&\quad + \left(\frac{(v-u)\times(v-u)}{\theta} - \frac{1}{\mathrm{D}}\frac{|v-u|^2}{\theta} I\right) : \nabla_x u \\
&\quad + \left(\frac{|v-u|^2}{2\theta} - \frac{\mathrm{D}}{2}\right)\frac{(v-u)\cdot\nabla_x\theta}{\theta} \,.
\end{aligned}
\tag{3.19}
$$

For this to be positive everywhere, we must impose the constraints

$$
\nabla_x\theta = 0\,, \qquad \nabla_x u + (\nabla_x u)^T - \tfrac{1}{\mathrm{D}}\nabla_x\cdot u I = 0\,.
$$

For this to be a function of $|v-u|$, we must impose the constraint

$$
\rho(\partial_t u + u\cdot\nabla_x u) + \theta\nabla_x\rho = 0\,.
$$

The form of (3.19) thereby simplifies to

$$
\big(\partial_t + v\cdot\nabla_x\big)\log(M) = \frac{\partial_t\rho + u\cdot\nabla_x\rho + \rho\nabla_x\cdot u}{\rho} + \left(\frac{|v-u|^2}{2\theta} - \frac{\mathrm{D}}{2}\right)\frac{\partial_t\theta + \frac{2}{\mathrm{D}}\theta\nabla_x\cdot u}{\theta}\,.
$$

This quadratic function of $|v-u|$ is similar to those obtained in (3.15) and (3.16). Moreover, the constraints we imposed above imply that the spatial dependence of ρ, u, and θ has the same form as for a global Maxwellian. This means the forms that we have assumed in (3.14) are close to being general.

3.4. **Soft and Maxwell Cases.** We now complete the construction of Section 3.3 for cases when the collision kernel has the separable form (1.4) with $\beta \in (-\mathrm{D}, 0]$. In other words, for cases when the kernel arises from either soft or Maxwell potentials. Because (3.18) shows the right-hand sides of (3.15) and (3.16) are bounded functions of v when $\beta \in (-\mathrm{D}, 0]$, we take $\eta(t) = 1$ in the forms of M^U and M^L given by (3.14). Criterion (3.15) then reduces to

$$\frac{\gamma'(t)}{\gamma(t)} \geq \mathcal{A}\big(M^U - M^L\big) \,, \qquad \gamma(0) = 1 \,. \tag{3.20}$$

Below we show how this criterion can be met in certain cases.

3.4.1. *Near Vacuum Case.* The easiest case to treat is when the lower bound is the vacuum.

Proposition 3.2. *Let $\mathcal{M}(t)$ be the global Maxwellian in the form (1.14) given by $m > 0$ and $(a, b, c, B) \in \Omega$. Let $F^{\mathrm{in}}(v, x)$ be any initial data such that*

$$0 \leq F^{\mathrm{in}} \leq \mathcal{M}(s) \qquad \text{for some } s \in \mathbb{R} \,. \tag{3.21}$$

Let $\beta \in (-\mathrm{D}, 0]$. Let

$$N_s(t) = \|\mathsf{a}\|_{L^\infty(\mathrm{d}w)} \sqrt{\det\big(\tfrac{1}{2\pi}\, Q\big)} \int_0^t \theta(s + t')^{\frac{\mathrm{D}+\beta}{2}} \, \mathrm{d}t' \,, \tag{3.22}$$

where $Q = (ac - b^2)I + B^2$ and $\theta(t)$ is given by (1.17).

Then there exists a mild solution $F(v, x, t)$ to the initial-value problem (1.1) for the Boltzmann equation over $[0, T)$ that satisfies the bounds

$$0 \leq F(t) \leq \gamma(t)\, \mathcal{M}(s + t) \qquad \text{over } \mathbb{R}^{\mathrm{D}} \times \mathbb{R}^{\mathrm{D}} \times [0, T) \,, \tag{3.23}$$

where

$$\gamma(t) = \frac{1}{1 - mN_s(t)} \quad \text{over } [0, T) \,, \qquad T = \sup\{t > 0 \,:\, mN_s(t) < 1\} \,. \tag{3.24}$$

In particular, this solution is global when $\beta \in (1 - \mathrm{D}, 0]$ and

$$m\, N_s^\infty \leq 1 \,, \qquad \text{where } N_s^\infty = \lim_{t \to \infty} N_s(t) \,. \tag{3.25}$$

and is globally bounded by a global Maxwellian when strict inequality holds.

Remark. The local result above is in the spirit of Kaniel and Shinbrot [7], while the global result is in the spirit of Illner and Shinbrot [6]. The bounds obtained here are different because of our use of the global Maxwelian family (1.14).

Proof. Set $M^L = 0$ and $M^U = \gamma(t)\mathcal{M}(s + t)$. Then criterion (3.20) reduces to

$$\frac{\gamma'}{\gamma} \geq \gamma \mathcal{A}(\mathcal{M}(s + t)) \,, \qquad \gamma(0) = 1 \,.$$

Because by (3.18) and (1.19)

$$\begin{aligned} \mathcal{A}(\mathcal{M}(s + t)) &= m\theta(s + t)^{\frac{\mathrm{D}+\beta}{2}} \sqrt{\det\big(\tfrac{1}{2\pi} Q\big)} \, \exp\!\left(-\frac{\theta}{2}\, x^T Q x\right) \mathsf{a}\!\left(\frac{v - u}{\sqrt{\theta}}\right) \\ &\leq m\theta(s + t)^{\frac{\mathrm{D}+\beta}{2}} \sqrt{\det\big(\tfrac{1}{2\pi} Q\big)} \, \|\mathsf{a}\|_{L^\infty(\mathrm{d}w)} = mN_s'(t) \,, \end{aligned}$$

this criterion will be satisfied if

$$\gamma' = \gamma^2 m N_s'(t) \,, \qquad \gamma(0) = 1 \,.$$

But (3.24) gives the solution $\gamma(t)$ of this initial-value problem along with its interval of existence $[0,T)$, whereby criterion (3.20) is satisfied over this interval. The bounds (3.23) thereby follow from (3.9) of Proposition 3.1.

Because $\theta(t)$ given by (1.17) decays like t^{-2} as $t\to\infty$, the integrand in definition (3.22) of $N_s(t)$ decays like $t^{-(\mathrm{D}+\beta)}$ as $t\to\infty$. It is therefore clear that $N_s^\infty=\lim_{t\to\infty}N_s(t)<\infty$ if and only if $\beta\in(1-\mathrm{D},0]$. The bound (3.25) on m then follows from (3.24). □

3.4.2. *Near Global Maxwellian Case.* The next easiest case to treat is when the lower and upper bounds are proportional to the same global Maxwellian.

Proposition 3.3. *Let $\mathcal{M}(t)$ be the global Maxwellian in the form (1.14) given by $m=1$ and some $(a,b,c,B)\in\Omega$. Let $m_2>m_1>0$. Let $F^{\mathrm{in}}(v,x)$ be any initial data such that*

$$m_1\mathcal{M}(s)\le F^{\mathrm{in}}\le m_2\mathcal{M}(s)\qquad \textit{for some } s\in\mathbb{R}\,. \tag{3.26}$$

Let $\gamma_{\min}=\sqrt{m_1/m_2}$. Let $\beta\in(-\mathrm{D},0]$. Let $N_s(t)$ be given by (3.22).

Then there exists a mild solution $F(v,x,t)$ to the initial-value problem (1.1) for the Boltzmann equation over $[0,T)$ that satisfies the bounds

$$\frac{m_1}{\gamma(t)}\,\mathcal{M}(s+t)\le F(t)\le\gamma(t)\,m_2\,\mathcal{M}(s+t)\qquad \textit{over } \mathbb{R}^{\mathrm{D}}\times\mathbb{R}^{\mathrm{D}}\times[0,T)\,, \tag{3.27}$$

where

$$\begin{aligned}\gamma(t)&=\gamma_{\min}\,\frac{1+\gamma_{\min}+(1-\gamma_{\min})e^{2\gamma_{\min}m_2N_s(t)}}{1+\gamma_{\min}-(1-\gamma_{\min})e^{2\gamma_{\min}m_2N_s(t)}}\qquad \textit{over } [0,T)\,,\\ T&=\sup\{t>0\,:\,(1-\gamma_{\min})e^{2\gamma_{\min}m_2N_s(t)}<1+\gamma_{\min}\}\,.\end{aligned} \tag{3.28}$$

In particular, this solution is global when $\beta\in(1-\mathrm{D},0]$ and

$$m_2\,N_s^\infty\le\frac{1}{2\gamma_{\min}}\,\log\!\left(\frac{1+\gamma_{\min}}{1-\gamma_{\min}}\right),\qquad \textit{where } N_s^\infty=\lim_{t\to\infty}N_s(t)\,, \tag{3.29}$$

and is globally bounded by a global Maxwellian when strict inequality holds.

Remark. Condition (3.29) yields global solutions with larger mass by picking $\gamma_{\min}$ closer to 1.

Remark. This result is in the spirit of Toscani [11]. The bounds obtained here are different because of our use of the global Maxwelian family (1.14). In particular, we can treat initial data with significant rotation that are excluded by the hypotheses in [11].

Proof. Set $M^L=\frac{m_1}{\gamma(t)}\mathcal{M}(s+t)$ and $M^U=\gamma(t)m_2\mathcal{M}(s+t)$. Criterion (3.20) then reduces to

$$\frac{\gamma'}{\gamma}\ge\left(\gamma-\frac{\gamma_{\min}^2}{\gamma}\right)m_2\mathcal{A}(\mathcal{M}(s+t))\,,\qquad \gamma(0)=1\,.$$

Because $\mathcal{A}(\mathcal{M}(s+t))\le N_s'(t)$, this criterion will be satisfied if

$$\gamma'=\left(\gamma^2-\gamma_{\min}^2\right)m_2\,N_s'(t)\,,\qquad \gamma(0)=1\,.$$

But (3.28) gives the solution $\gamma(t)$ of this initial-value problem along with its interval of existence $[0,T)$, whereby criterion (3.20) is satisfied over this interval. The bounds (3.27) thereby follow from (3.9) of Proposition 3.1.

By arguing as in the proof of Proposition 3.2, we see that $N_s^\infty=\lim_{t\to\infty}N_s(t)<\infty$ if and only if $\beta\in(1-\mathrm{D},0]$. The bound (3.29) on m_2 then follows from (3.28). □

3.4.3. *Between Global Maxwellians Case.* We now treat the case is when the lower and upper bounds are proportional to possibly different global Maxwellians. The question addressed below is whether or not the only bounds on global solutions that one can obtain are the ones already recovered by the arguments in the foregoing subsections.

Let $\beta \in (-\mathrm{D}, 0]$. Let $\mathcal{M}_1$ and $\mathcal{M}_2$ be the global Maxwellians in the form (1.14) that are given by some $(m_1, a_1, b_1, c_1, B_1)$ and $(m_2, a_2, b_2, c_2, B_2)$ in $\mathbb{R}_+ \times \Omega$ respectively. We assume that $\mathcal{M}_1 \leq \mathcal{M}_2$. This implies that $m_1\sqrt{\det(Q_1)} \leq m_2\sqrt{\det(Q_2)}$, where $Q_1 = (a_1c_1 - b_1^2)I + B_1^2 > 0$ and $Q_2 = (a_2c_2 - b_2^2)I + B_2^2 > 0$. It also implies that $q_1(v, x, t) \geq q_2(v, x, t)$, which means that $a_1 \geq a_2$, $c_1 \geq c_2$, and

$$(b_1 - b_2)^2 + \|B_1 - B_2\|^2 \leq (a_1 - a_2)(c_1 - c_2)\,.$$

Because by (1.17) we have

$$\theta_1(t) = \frac{1}{a_1t^2 - 2b_1t + c_1} > 0\,, \qquad \theta_1(t) = \frac{1}{a_2t^2 - 2b_2t + c_2} > 0\,,$$

while by (2.12) we have $a_1 \geq a_2$ and $(a_1 - a_2)(c_1 - c_2) \geq (b_1 - b_2)^2$, we see that

$$\frac{1}{\theta_1(t)} - \frac{1}{\theta_2(t)} = (a_1 - a_2)t^2 - 2(b_1 - b_2)t + (c_1 - c_2) \geq 0\,.$$

Hence, $\theta_1(t) \leq \theta_2(t)$. It is also clear that $\rho_1(x, t) \leq \rho_2(x, t)$.

Now set $M^L = \frac{1}{\gamma(t)}\mathcal{M}_1(s + t)$ and $M^U = \gamma(t)\mathcal{M}_2(s + t)$. Then because

$$\begin{aligned}\big(\partial_t + v\cdot\nabla_x\big)\log(M^U) &= -\big(\partial_t + v\cdot\nabla_x\big)\log(M^L) = \frac{\gamma'}{\gamma}\,,\\ \mathcal{A}\big(M^U - M^L\big) &= \gamma\,\mathcal{A}(\mathcal{M}_2(s + t)) - \frac{1}{\gamma}\,\mathcal{A}(\mathcal{M}_1(s + t))\,,\end{aligned}$$

criterion (3.20) becomes

$$\frac{\gamma'}{\gamma} \geq \gamma\,\mathcal{A}\big(\mathcal{M}_2(s + t)\big) - \frac{1}{\gamma}\,\mathcal{A}\big(\mathcal{M}_1(s + t)\big)\,, \qquad \gamma(0) = 1\,.$$

The above differential inequality has the form

$$\gamma' \geq \left(\gamma^2 - \frac{\mathcal{A}\big(\mathcal{M}_1(s + t)\big)}{\mathcal{A}\big(\mathcal{M}_2(s + t)\big)}\right)\mathcal{A}\big(\mathcal{M}_2(s + t)\big)\,.$$

By using (3.18) followed by (1.17) and (1.19), we find that the ratio above is

$$\frac{\mathcal{A}(\mathcal{M}_1)}{\mathcal{A}(\mathcal{M}_2)} = \frac{\rho_1\,\theta_1^{\frac{\beta}{2}}\,\mathsf{a}\left(\dfrac{v - u_1}{\sqrt{\theta_1}}\right)}{\rho_2\,\theta_2^{\frac{\beta}{2}}\,\mathsf{a}\left(\dfrac{v - u_2}{\sqrt{\theta_2}}\right)} = \frac{m_1\,\sqrt{\det(Q_1)}\,\theta_1^{\frac{\mathrm{D}+\beta}{2}}\,\exp\!\left(-\dfrac{\theta_1}{2}x^TQ_1x\right)\mathsf{a}\left(\dfrac{v - u_1}{\sqrt{\theta_1}}\right)}{m_2\,\sqrt{\det(Q_2)}\,\theta_2^{\frac{\mathrm{D}+\beta}{2}}\,\exp\!\left(-\dfrac{\theta_2}{2}x^TQ_2x\right)\mathsf{a}\left(\dfrac{v - u_2}{\sqrt{\theta_2}}\right)}\,.$$

We would like to know when this ratio is bounded below by a positive number.

The ratio is bounded below by a positive number when $(a_1, b_1, c_1, B_1) = (a_2, b_2, c_2, B_2)$, in which case

$$\frac{\mathcal{A}\big(\mathcal{M}_1(s + t)\big)}{\mathcal{A}\big(\mathcal{M}_2(s + t)\big)} = \frac{m_1}{m_2}\,.$$

This is just the "Toscani case" treated in the last subsection.

In general, it is clear that

$$\frac{m_1 \sqrt{\det(Q_1)}\, \theta_1^{\frac{D+\beta}{2}}}{m_2 \sqrt{\det(Q_2)}\, \theta_2^{\frac{D+\beta}{2}}} \qquad \text{is bounded above and below by positive numbers}\,.$$

Because $q_1(v,x,t) \geq q_2(v,x,t)$, we see that

$$\frac{|v-u_1|^2}{\theta_1} + \theta_1\, x^T Q_1 x = 2q_1(v,x,t) \geq 2q_2(v,x,t) = \frac{|v-u_2|^2}{\theta_2} + \theta_2\, x^T Q_2 x\,.$$

By setting $v = u_1$ into this inequality we obtain

$$\theta_1\, x^T Q_1 x \geq \frac{|u_1-u_2|^2}{\theta_2} + \theta_2\, x^T Q_2 x\,. \tag{3.30}$$

Hence, $\theta_1 Q_1 - \theta_2 Q_2$ is nonnegative definite. Because $\mathsf{a}(w) \sim |w|^\beta$ as $|w| \to \infty$, we see that

$$\frac{\exp\left(-\frac{\theta_1}{2} x^T Q_1 x\right) \mathsf{a}\left(\frac{v-u_1}{\sqrt{\theta_1}}\right)}{\exp\left(-\frac{\theta_2}{2} x^T Q_2 x\right) \mathsf{a}\left(\frac{v-u_2}{\sqrt{\theta_2}}\right)} \qquad \text{is not bounded away from zero unless } \theta_1 Q_1 = \theta_2 Q_2\,.$$

Now suppose that $\theta_1 Q_1 = \theta_2 Q_2$. Because Q_1 and Q_2 are positive definite and constant while $\theta_1 \leq \theta_2$, this can only be the case if $\theta_1(t) = \kappa\theta_2(t)$ for some $\kappa \in (0,1]$. However, this implies $Q_2 = \kappa Q_1$ and that $a_2 = \kappa a_1$, $b_2 = \kappa b_1$, $c_2 = \kappa c_1$. Moreover, from (3.30) we see $u_1 = u_2$ when $\theta_1 Q_1 = \theta_2 Q_2$. The fact $u_1 = u_2$ then implies that $B_2 = \kappa B_1$. We thereby see that $Q_2 = \kappa^2 Q_1$ and $Q_2 = \kappa Q_1$. Therefore $\kappa = 1$, and we are in the Toscani case. In other words,

$$\frac{\mathcal{A}\big(\mathcal{M}_1(s+t)\big)}{\mathcal{A}\big(\mathcal{M}_2(s+t)\big)} \qquad \text{is not bounded away from zero except in the Toscani case}\,.$$

The Toscani case seems to be the only one that satifies the hypotheses in the theorem of Alonso and Gamba. For all other cases we seem to be led to the type of bounds in Illner and Shinbrot. Have I made a mistake in this analysis?

3.5. **Hard and Super-Hard Cases.** We now complete the construction of Section 3.3 for cases when the collision kernel has the separable form (1.4) with $\beta \in (0,2]$. Such kernels arise from hard potentials when $\beta \in (0,1)$, and from hard spheres when $\beta = 1$. When $\beta \in (1,2]$ the kernel is said to be super-hard. The super-hard case has only theoretical interest.

Recall from (3.14) and (3.15) that we are seeking to construct M^U and M^L in the form

$$M^U(v,x,t) = \gamma(t)\, \frac{m_2}{(2\pi)^D} \sqrt{\det(Q_2)}\, \exp\left(-\frac{q_2(v,x,s+t)}{\eta(t)}\right)\,, \tag{3.31a}$$

$$M^L(v,x,t) = \frac{1}{\gamma(t)}\, \frac{m_1}{(2\pi)^D} \sqrt{\det(Q_1)}\, \exp\left(-\left(2 - \frac{1}{\eta(t)}\right) q_1(v,x,s+t)\right)\,, \tag{3.31b}$$

where $q_1(v,x,t)$ and $q_2(v,x,t)$ are given by (3.13), while the functions $\gamma(t)$ and $\eta(t)$ satisfy

$$\frac{\gamma'(t)}{\gamma(t)} + \frac{\eta'(t)}{\eta(t)}\, \frac{q_2(v,x,s+t)}{\eta(t)} \geq \mathcal{A}\big(M^U - M^L\big)\,, \tag{3.32a}$$

$$\gamma(0) = \eta(0) = 1\,, \qquad \gamma'(t) > 0\,, \qquad \eta'(t) \geq 0\,. \tag{3.32b}$$

Because (3.18) shows the right-hand side of (3.32) is an unbounded function of v when $\beta \in (0, 2]$, we can no longer set $\eta(t) = 1$ as we did for the soft and Maxwell cases. Below we show how criterion (3.32) can be met in the case when the lower bound is the vacuum.

Proposition 3.4. *Let $\mathcal{M}$ be the global Maxwellian in the form (1.14) given by $m > 0$ and $(a, b, c, B) \in \Omega$, so that*

$$\mathcal{M}(v,x,t) = \frac{m}{(2\pi)^{\mathrm{D}}} \sqrt{\det(Q)} \exp\big(- q(v,x,t)\big) , \qquad Q = (ac - b^2)I + B^2 , \tag{3.33a}$$

$$q(v,x,t) = \frac{1}{2} \begin{pmatrix} v \\ x - vt \end{pmatrix}^T \begin{pmatrix} cI & bI + B \\ bI - B & aI \end{pmatrix} \begin{pmatrix} v \\ x - vt \end{pmatrix} . \tag{3.33b}$$

Let $F^{\mathrm{in}}(v,x)$ be any initial data such that

$$0 \le F^{\mathrm{in}}(v,x) \le \mathcal{M}(v,x,s) \qquad \text{over } \mathbb{R}^{\mathrm{D}} \times \mathbb{R}^{\mathrm{D}} \text{ for some } s \in \mathbb{R} . \tag{3.34}$$

Let $\beta \in (0, 2]$ and

$$N_s(t) = m \sqrt{\det\big(\tfrac{1}{2\pi} Q\big)} \, \big\|\hat{\mathsf{b}}\big\|_{L^1(\mathrm{d}\omega)} \int_0^t \left(\frac{1}{a(s+t')^2 - 2b(s+t') + c} \right)^{\frac{\mathrm{D}+\beta}{2}} \mathrm{d}t' . \tag{3.35}$$

Then there exists a mild solution $F(v,x,t)$ to the initial-value problem (1.1) for the Boltzmann equation over $[0,T)$ that satisfies the bounds

$$0 \le F(t) \le \gamma(t) \frac{m}{(2\pi)^{\mathrm{D}}} \sqrt{\det(Q)} \exp\left(- \frac{q(v,x,s+t)}{\eta(t)} \right) \qquad \text{over } \mathbb{R}^{\mathrm{D}} \times \mathbb{R}^{\mathrm{D}} \times [0,T) , \tag{3.36}$$

where

$$\gamma(t) = \left(1 - \frac{N_s(t)}{\tau} \right)^{-\mu} , \qquad \eta(t) = \left(1 - \frac{N_s(t)}{\tau} \right)^{-\nu} , \tag{3.37a}$$

$$\mu = 1 - \frac{\beta}{2} \frac{\mathrm{D}+\beta}{2\mathrm{D}+\beta} , \qquad \nu = \frac{\beta}{2\mathrm{D}+\beta} , \qquad \tau = (2\mathrm{D}+\beta)^{-\frac{\beta}{2}} , \tag{3.37b}$$

$$T = \sup\{ t > 0 \, : \, N_s(t) < \tau \} . \tag{3.37c}$$

In particular, this solution is global when

$$N_s^\infty \le \tau , \qquad \text{where } N_s^\infty = \lim_{t \to \infty} N_s(t) , \tag{3.38}$$

and is globally bounded by a global Maxwellian when strict inequality holds.

Remark. The local result above is in the spirit of Kaniel and Shinbrot [7], while the global result is in the spirit of Illner and Shinbrot [6]. The bounds obtained here are different because of our use of the global Maxwelian family (1.14) and because we employ a sharper bound on the attenuation coefficient. If we set $\beta = 0$ into the results here they are identical to those of Proposition 3.2 for Maxwell case.

Proof. Set $M^L(v,x,t) = 0$ and

$$M^U(v,x,t) = \gamma(t) \frac{m}{(2\pi)^{\mathrm{D}}} \sqrt{\det(Q)} \exp\left(- \frac{q(v,x,s+t)}{\eta(t)} \right) . \tag{3.39}$$

Criterion (3.32) then reduces to

$$\frac{\gamma'(t)}{\gamma(t)} + \frac{\eta'(t)}{\eta(t)}\,\frac{q(v,x,s+t)}{\eta(t)} \geq \mathcal{A}\big(M^U\big)\,, \tag{3.40a}$$

$$\gamma(0)=\eta(0)=1\,,\qquad \gamma'(t)>0\,,\qquad \eta'(t)\geq 0\,. \tag{3.40b}$$

The fluid dynamical variables associated with the local Maxwellian M^U are

$$\theta(t) = \frac{\eta(t)}{a(s+t)^2-2b(s+t)+c}\,,\qquad u(x,t)=\frac{ax(s+t)-bx-Bx}{a(s+t)^2-2b(s+t)+c}\,, \tag{3.41a}$$

$$\rho(x,t) = m\,\gamma(t)\,\theta(t)^{\frac{\mathrm{D}}{2}}\sqrt{\det\big(\tfrac{1}{2\pi}\,Q\big)}\,\exp\!\left(-\frac{\theta(t)}{2\,\eta(t)^2}\,x^TQx\right)\,. \tag{3.41b}$$

We may express $q(v,x,s+t)$ in terms of the fluid variables as

$$\frac{q(v,x,s+t)}{\eta(t)} = \frac{|v-u(x,t)|^2}{2\,\theta(t)} + \frac{\theta(t)}{2\,\eta(t)^2}\,x^TQx\,.$$

We may use (3.18) to express $\mathcal{A}(M^U)$ in terms of the fluid variables as

$$\begin{aligned}\mathcal{A}(M^U) &= \rho(x,t)\,\theta(t)^{\frac{\beta}{2}}\,\mathsf{a}\!\left(\frac{v-u(x,t)}{\sqrt{\theta(t)}}\right)\\ &= m\,\theta(t)^{\frac{\mathrm{D}+\beta}{2}}\sqrt{\det\big(\tfrac{1}{2\pi}\,Q\big)}\,\exp\!\left(-\frac{\theta(t)}{2\,\eta(t)^2}\,x^TQx\right)\mathsf{a}\!\left(\frac{v-u(x,t)}{\sqrt{\theta(t)}}\right)\,.\end{aligned}$$

When $\beta\in[0,2]$ we may bound $\mathsf{a}(w)$ above by

$$\begin{aligned}\mathsf{a}(w) &= \big\|\hat{\mathsf{b}}\big\|_{L^1(\mathrm{d}\omega)}\int_{\mathbb{R}^\mathrm{D}}|w-w_*|^\beta\,\frac{1}{(2\pi)^{\frac{\mathrm{D}}{2}}}\,\exp\!\big(-\tfrac12|w_*|^2\big)\,\mathrm{d}w_*\\ &\leq \big\|\hat{\mathsf{b}}\big\|_{L^1(\mathrm{d}\omega)}\left(\int_{\mathbb{R}^\mathrm{D}}|w-w_*|^2\,\frac{1}{(2\pi)^{\frac{\mathrm{D}}{2}}}\,\exp\!\big(-\tfrac12|w_*|^2\big)\,\mathrm{d}w_*\right)^{\frac{\beta}{2}}\\ &= \big\|\hat{\mathsf{b}}\big\|_{L^1(\mathrm{d}\omega)}\big(|w|^2+\mathrm{D}\big)^{\frac{\beta}{2}}\,.\end{aligned} \tag{3.42}$$

Therefore the differential inequality (3.40a) of criterion (3.40) will be satisfied if

$$\frac{\gamma'(t)}{\gamma(t)} + \frac{\eta'(t)}{\eta(t)}\,\frac{|w|^2}{2} \geq \gamma(t)\,\eta(t)^{\frac{\mathrm{D}+\beta}{2}}\big(|w|^2+\mathrm{D}\big)^{\frac{\beta}{2}}\,N_s'(t)\qquad\text{for every } w\in\mathbb{R}^\mathrm{D}\,, \tag{3.43}$$

where $N_s(t)$ is given by (3.35).

It is easily checked by using (3.43) that criterion (3.40) has solutions of the form (3.37a), where $\tau>0$, $\mu\in(0,1]$ and $\nu = 2\frac{1-\mu}{\mathrm{D}+\beta}$, provided that τ and μ satisfy the inequality

$$\mu + \frac{1-\mu}{\mathrm{D}+\beta}\,y \geq \tau\,(y+\mathrm{D})^{\frac{\beta}{2}}\qquad\text{for every } y\geq 0\,. \tag{3.44}$$

We seek all values of $\tau>0$ and $\mu\in(0,1]$ that satisfy this inequality for any given $\beta\in[0,2]$. Moreover, we seek to maximize τ among such values. It will be shown in Lemma 3.2 that these optimal values are those given by (3.37b). Therefore (3.37) gives a solution of the differential inequality (3.43) along with its interval of existence $[0,T)$, whereby criterion (3.40) is satisfied over this interval. The bounds (3.36) thereby follow from (3.9) of Proposition 3.1.

Because the integrand in definition (3.35) of $N_s(t)$ decays like $t^{-(\mathrm{D}+\beta)}$ as $t \to \infty$, it is clear that $N_s^\infty = \lim_{t\to\infty} N_s(t) < \infty$ for every $\beta \in (0,2]$. The bound (3.38) on N_s^∞ therefore follows from (3.37). $\square$

Remark. The upper bound on the attenuation coefficient given by (3.42) is exact for $\beta = 0$ and $\beta = 2$. It is also sharp for all $\beta \in (0,2)$ in the sense that

$$\lim_{|w|\to\infty} \mathsf{a}(w)\big(|w|^2 + \mathrm{D}\big)^{-\frac{\beta}{2}} = \big\|\hat{\mathsf{b}}\big\|_{L^1(\mathrm{d}\omega)}\,.$$

Its simple form enables us to obtain the explicit formulas given in (3.37b).

The following is the kind of lemma whose proof often is omitted in articles because it is both elementary and complicated. This omission is unfortunate, as it will sometimes cost the interested reader more time than need be to check the claimed result.

Lemma 3.2. *Let $\beta \in [0,2]$ and $\mathrm{D} > 0$. Let*

$$\mu_{\max}(\beta) = 1 - \frac{\beta}{2}\,\frac{\mathrm{D}+\beta}{2\mathrm{D}+\beta}\,, \qquad \tau_{\max}(\beta) = (2\mathrm{D}+\beta)^{-\frac{\beta}{2}}\,. \tag{3.45}$$

Then $\mu = \mu_{\max}(\beta)$ and $\tau = \tau_{\max}(\beta)$ satisfies the inequality

$$\mu + \frac{1-\mu}{\mathrm{D}+\beta}\,y \geq \tau\,(y+\mathrm{D})^{\frac{\beta}{2}} \qquad \text{for every } y \geq 0\,. \tag{3.46}$$

Moreover, if $\mu \in (0,1]$ and $\tau > 0$ satisfy (3.46) and $\mu \neq \mu_{\max}(\beta)$ then $\tau < \tau_{\max}(\beta)$.

Proof. For every $\tau > 0$ and $\mu \in (0,1]$ define the function $h(y)$ over $[0,\infty)$ by

$$h(y) = \mu + \frac{1-\mu}{\mathrm{D}+\beta}\,y - \tau\,(y+\mathrm{D})^{\frac{\beta}{2}}\,. \tag{3.47}$$

Inequality (3.46) is satisfied if and only if $h(y) \geq 0$ over $[0,\infty)$.

When $\beta = 0$ we see from (3.47) that

$$h(y) = \mu - \tau + \frac{1-\mu}{\mathrm{D}}\,y\,.$$

Clearly h has a minimum over $[0,\infty)$ if and only if $\mu \leq 1$, in which case its minimum value is $h_{\min} = h(0) = \mu - \tau$. This value will be nonnegative if and only if $\tau \leq \mu$. Therefore when $\beta = 0$ the maximum τ is realized only when $\tau = \mu = 1$. Hence, we see from (3.45) that the claimed result holds for $\beta = 0$.

When $\beta = 2$ we see from (3.47) that

$$h(y) = (\mu - \tau\,\mathrm{D}) + \left(\frac{1-\mu}{\mathrm{D}+2} - \tau\right) y\,.$$

Clearly h has a minimum over $[0,\infty)$ if and only if $\tau \leq \frac{1-\mu}{\mathrm{D}+2}$, in which case its minimum value is $h_{\min} = h(0) = \mu - \tau\,\mathrm{D}$. This value will be nonnegative if and only if $\tau \leq \frac{\mu}{\mathrm{D}}$. Hence, $h(y)$ is nonnegative if and only if

$$\tau \leq \min\left\{\tfrac{\mu}{\mathrm{D}}\,,\, \tfrac{1-\mu}{\mathrm{D}+2}\right\}$$

Therefore when $\beta = 2$ the maximum τ is realized only when

$$\mu = \frac{\mathrm{D}}{2\mathrm{D}+2}\,, \qquad \tau = \frac{1}{2\mathrm{D}+2}\,.$$

Hence, we see from (3.45) that the claimed result holds for $\beta = 2$.

When $\beta \in (0,2)$ we see from (3.47) that if $\mu = 1$ then $h(y) \to -\infty$ as $y \to \infty$. We therefore require the $\mu \in (0,1)$, so that $h(y) \to \infty$ as $y \to \infty$. Moreover, we see from (3.47) that

$$h'(y) = \frac{1-\mu}{\mathrm{D}+\beta} - \tau\,\frac{\beta}{2}\,(y+\mathrm{D})^{\frac{\beta-2}{2}}\,, \qquad h''(y) = \tau\,\frac{\beta}{2}\frac{2-\beta}{2}(y+\mathrm{D})^{\frac{\beta-4}{2}} > 0\,.$$

Hence, h is strictly convex over $[0,\infty)$, whereby it has a unique minimum value over $[0,\infty)$. This minimum value is given by

$$h_{\min} = \begin{cases} h(0) = \mu - \tau\,\mathrm{D}^{\frac{\beta}{2}} & \text{if} \quad \tau \leq \dfrac{2}{\beta}\,\dfrac{1-\mu}{\mathrm{D}+\beta}\,\mathrm{D}^{\frac{2-\beta}{2}}\,, \\ h(y_{\min}) & \text{if} \quad \tau > \dfrac{2}{\beta}\,\dfrac{1-\mu}{\mathrm{D}+\beta}\,\mathrm{D}^{\frac{2-\beta}{2}}\,, \end{cases} \tag{3.48}$$

where the minimizer $y_{\min} > 0$ for the second case is found by setting $h'(y_{\min}) = 0$, which yields

$$y_{\min} + \mathrm{D} = \left(\tau\,\frac{\beta}{2}\,\frac{\mathrm{D}+\beta}{1-\mu}\right)^{\frac{2}{2-\beta}}\,.$$

For the first case in (3.48) the value of $h_{\min}$ is nonnegative if and only if $\tau \leq \mu/\mathrm{D}^{\frac{\beta}{2}}$. Hence, $h(y)$ is nonnegative in this case if and only if

$$\tau \leq \min\left\{\frac{\mu}{\mathrm{D}^{\frac{\beta}{2}}}\,,\,\frac{2}{\beta}\,\frac{1-\mu}{\mathrm{D}+\beta}\,\mathrm{D}^{\frac{2-\beta}{2}}\right\}\,.$$

Therefore when $\beta \in (0,2)$ the maximum τ arising from this case is realized when $\mu = \mu_0(\beta)$ and $\tau = \tau_0(\beta)$, where we define

$$\mu_0(\beta) = \frac{2\mathrm{D}}{2\mathrm{D}+\beta\,\mathrm{D}+\beta^2}\,, \qquad \tau_0(\beta) = \frac{2\mathrm{D}^{\frac{2-\beta}{2}}}{2\mathrm{D}+\beta\,\mathrm{D}+\beta^2}\,. \tag{3.49}$$

For the second case in (3.48) the value of $h_{\min}$ is

$$\begin{aligned} h_{\min} = h(y_{\min}) &= \mu - \frac{1-\mu}{\mathrm{D}+\beta}\,\mathrm{D} + \frac{1-\mu}{\mathrm{D}+\beta}\,\left(\tau\,\frac{\beta}{2}\,\frac{\mathrm{D}+\beta}{1-\mu}\right)^{\frac{2}{2-\beta}} - \tau\,\left(\tau\,\frac{\beta}{2}\,\frac{\mathrm{D}+\beta}{1-\mu}\right)^{\frac{\beta}{2-\beta}} \\ &= \frac{(2\mathrm{D}+\beta)\mu - \mathrm{D}}{\mathrm{D}+\beta} - \frac{1-\mu}{\mathrm{D}+\beta}\,\frac{2-\beta}{\beta}\,\left(\tau\,\frac{\beta}{2}\,\frac{\mathrm{D}+\beta}{1-\mu}\right)^{\frac{2}{2-\beta}}\,. \end{aligned}$$

This value is nonnegative if and only if

$$\mu > \frac{\mathrm{D}}{2\mathrm{D}+\beta}\,, \qquad \tau \leq \frac{2}{\beta}\,\frac{1-\mu}{\mathrm{D}+\beta}\,\left(\frac{\beta}{2-\beta}\,\frac{(2\mathrm{D}+\beta)\mu-\mathrm{D}}{1-\mu}\right)^{\frac{2-\beta}{2}}\,.$$

Hence, $h(y)$ is nonnegative in this case if and only if

$$\mu > \frac{\mathrm{D}}{2\mathrm{D}+\beta}\,, \qquad \frac{2}{\beta}\,\frac{1-\mu}{\mathrm{D}+\beta}\,\mathrm{D}^{\frac{2-\beta}{2}} < \tau \leq \frac{2}{\beta}\,\frac{1-\mu}{\mathrm{D}+\beta}\,\left(\frac{\beta}{2-\beta}\,\frac{(2\mathrm{D}+\beta)\mu-\mathrm{D}}{1-\mu}\right)^{\frac{2-\beta}{2}}\,. \tag{3.50}$$

In particular, these inequalities imply that μ must satisfy the constraints

$$\mu > \frac{\mathrm{D}}{2\mathrm{D}+\beta}\,, \qquad 1 < \frac{\beta}{2-\beta}\,\frac{(2\mathrm{D}+\beta)\mu-\mathrm{D}}{(1-\mu)\mathrm{D}}\,.$$

Recalling definition (3.49) of $\mu_0(\beta)$, we find that the second of the above constaints reduces to

$$\mu > \frac{2\mathrm{D}}{2\mathrm{D}+\beta\,\mathrm{D}+\beta^2} = \mu_0(\beta)\,, \tag{3.51}$$

which implies the first constraint.

Now let $\bar{\tau}(\mu,\beta)$ denote the upper bound on τ given in (3.50) — namely, let

$$\bar{\tau}(\mu,\beta) = \frac{2}{\beta}\,\frac{1-\mu}{\mathrm{D}+\beta}\left(\frac{\beta}{2-\beta}\,\frac{(2\mathrm{D}+\beta)\mu-\mathrm{D}}{1-\mu}\right)^{\frac{2-\beta}{2}}\,.$$

One can easily check from definition (3.49) of $\mu_0(\beta)$ and $\tau_0(\beta)$ that

$$\bar{\tau}\big(\mu_0(\beta),\beta\big) = \frac{2}{\beta}\,\frac{1-\mu_0(\beta)}{\mathrm{D}+\beta}\,\mathrm{D}^{\frac{2-\beta}{2}} = \tau_0(\beta)\,.$$

Moreover, upon maximizing $\bar{\tau}(\mu,\beta)$ over $\mu\in[\mu_0(\beta),1]$, we find that when $\beta\in(0,2)$ the maximum $\bar{\tau}(\mu,\beta)$ is realized by the unique maximizer $\mu=\mu_{\max}(\beta)$, at which

$$\bar{\tau}\big(\mu_{\max}(\beta),\beta\big) = \tau_{\max}(\beta)\,,$$

where $\mu_{\max}(\beta)$ and $\tau_{\max}(\beta)$ are given by (3.45). In particular, $\mu_{\max}(\beta)$ satisfies constraint (3.51) and $\tau_{\max}(\beta) > \tau_0(\beta)$. The claimed result therefore holds when $\beta\in(0,2)$. □

3.6. **Attenuation Coefficient.** Here we record expressions for the attenuation coefficient, which was defined by

$$\mathsf{a}(w) = \big\|\hat{\mathsf{b}}\big\|_{L^1(\mathrm{d}\omega)}\iint_{\mathbb{S}^{\mathrm{D}-1}\times\mathbb{R}^{\mathrm{D}}}|w-w_*|^{\beta}\frac{1}{(2\pi)^{\frac{\mathrm{D}}{2}}}\exp\big(-\tfrac{1}{2}|w_*|^2\big)\,\mathrm{d}\omega\,\mathrm{d}w_*\,. \tag{3.52}$$

By expressing the above integral in polar coordinates we see that $\mathsf{a}(w)$ has the form

$$\mathsf{a}(w) = \big\|\hat{\mathsf{b}}\big\|_{L^1(\mathrm{d}\omega)}\,\frac{|\mathbb{S}^{\mathrm{D}-1}|}{(2\pi)^{\frac{\mathrm{D}}{2}}}\,\alpha(|w|)\,, \tag{3.53a}$$

$$\alpha(r) = \int_0^\infty \frac{1}{2}\int_{-1}^{1}\big(r^2+r_*^2-2\,r\,r_*\,\xi\big)^{\frac{\beta}{2}}\,\mathrm{d}\xi\,\exp\big(-\tfrac{1}{2}r_*^2\big)\,r_*^{\mathrm{D}-1}\,\mathrm{d}r_*\,. \tag{3.53b}$$

For every $0<a<b$ and $\beta\neq-2$ we have

$$\frac{1}{2}\int_{-1}^{1}(b-a\,\xi)^{\frac{\beta}{2}}\,\mathrm{d}\xi = \frac{1}{2a}\int_{-a}^{a}(b-\zeta)^{\frac{\beta}{2}}\,\mathrm{d}\zeta = -\frac{(b-\zeta)^{\frac{\beta+2}{2}}}{(\beta+2)a}\bigg|_{-a}^{a} = \frac{(b+a)^{\frac{\beta+2}{2}}-(b-a)^{\frac{\beta+2}{2}}}{(\beta+2)a}\,,$$

while

$$\frac{1}{2}\int_{-1}^{1}(b-a\,\xi)^{-1}\,\mathrm{d}\xi = \frac{1}{2a}\int_{-a}^{a}\frac{1}{b-\zeta}\,\mathrm{d}\zeta = -\frac{\log(b-\zeta)}{2a}\bigg|_{-a}^{a} = \frac{1}{2a}\,\log\!\left(\frac{b+a}{b-a}\right)\,.$$

By setting $a=2\,r\,r_*$ and $b=r^2+r_*^2$ we obtain

$$\frac{1}{2}\int_{-1}^{1}\big(r^2+r_*^2-2\,r\,r_*\,\xi\big)^{\frac{\beta}{2}}\,\mathrm{d}\xi = \begin{cases}\dfrac{(r+r_*)^{\beta+2}-|r-r_*|^{\beta+2}}{(\beta+2)\,2\,r\,r_*} & \text{for } \beta\neq-2\,,\\[2ex] \dfrac{1}{2\,r\,r_*}\,\log\!\left(\dfrac{r+r_*}{|r-r_*|}\right) & \text{for } \beta=-2\,.\end{cases}$$

Hence,

$$\alpha(r) = \begin{cases}\displaystyle\int_0^\infty \frac{(r+r_*)^{\beta+2}-|r-r_*|^{\beta+2}}{(\beta+2)\,2\,r\,r_*}\,\exp\big(-\tfrac{1}{2}r_*^2\big)\,r_*^{\mathrm{D}-1}\,\mathrm{d}r_* & \text{for } \beta\neq-2\,,\\[2ex] \displaystyle\int_0^\infty \frac{1}{2\,r\,r_*}\,\log\!\left(\frac{r+r_*}{|r-r_*|}\right)\exp\big(-\tfrac{1}{2}r_*^2\big)\,r_*^{\mathrm{D}-1}\,\mathrm{d}r_* & \text{for } \beta=-2\,.\end{cases}$$

4. Asymptotic Behavior

In this section we explore the large-time asymptotic behavior displayed by certain solutions of the Cauchy problem (1.1).

4.1. Stability of the Collision Operator.

We begin with a rough L^1 stability estimate on the collision operator $\mathcal{B}(F,F)$ that holds for any F that is bounded above by a global Maxwellian.

Lemma 4.1. *Let the collision kernel* b *have the separated form (1.4) for some* $\beta \in (-\mathrm{D}, 2]$. *Let* $\mathcal{M}$ *be the global Maxwellian given by (1.14) for some* $(m,a,b,c,B) \in \mathbb{R}_+ \times \Omega$. *Let* F *be any measurable function that satisfies the pointwise bounds*

$$0 \leq F(v,x,t) \leq \mathcal{M}(v,x,s+t) \qquad \text{over } \mathbb{R}^{\mathrm{D}} \times \mathbb{R}^{\mathrm{D}} \times [0,\infty) \text{ for some } s \in \mathbb{R}\,. \tag{4.1}$$

Then for every $[t_1,t_2] \subset [0,\infty)$ *one has the* L^1*-bound*

$$\int_{t_1}^{t_2} \iiiint |F'_*F' - F_*F| \, \mathsf{b} \, \mathrm{d}\omega \, \mathrm{d}v_* \, \mathrm{d}v \, \mathrm{d}x \, \mathrm{d}t \leq C_1 \int_{t_1}^{t_2} \theta(s+t)^{\frac{\beta+\mathrm{D}}{2}} \, \mathrm{d}t\,, \tag{4.2}$$

where C_1 *is a constant and* $\theta(t)$ *is given by (1.17). Here all the four-fold integrals are understood to be taken over the domain* $\mathbb{S}^{\mathrm{D}-1} \times \mathbb{R}^{\mathrm{D}} \times \mathbb{R}^{\mathrm{D}} \times \mathbb{R}^{\mathrm{D}}$.

Proof. Let $[t_1,t_2] \subset [0,\infty)$. Set $M(v,x,t) = \mathcal{M}(v,x,s+t)$. It follows from bound (4.1), the triangle inequality, and the properties of Maxwellians that

$$|F'_*F' - F_*F| \leq F'_*F' + F_*F \leq M'_*M' + M_*M = 2M_*M\,. \tag{4.3}$$

Therefore

$$\int_{t_1}^{t_2} \iiiint |F'_*F' - F_*F| \, \mathsf{b} \, \mathrm{d}\omega \, \mathrm{d}v_* \, \mathrm{d}v \, \mathrm{d}x \, \mathrm{d}t \leq \int_{t_1}^{t_2} \iiiint 2M_*M\,, \mathsf{b} \, \mathrm{d}\omega \, \mathrm{d}v_* \, \mathrm{d}v \, \mathrm{d}x \, \mathrm{d}t\,. \tag{4.4}$$

Let ρ, u, and θ be the mass density, bulk velocity, and temperature for $M = \mathcal{M}(v,x,s+t)$ as given by (1.19) and (1.17). Then from the definition (1.7) of the attenuation operator $\mathcal{A}$, and the evaluation of $\mathcal{A}(M)$ given by (3.17), we see that

$$\begin{aligned}
\iiiint M_*\, M \, \mathsf{b} \, \mathrm{d}\omega \, \mathrm{d}v_* \, \mathrm{d}v \, \mathrm{d}x &= \iint_{\mathbb{R}^{\mathrm{D}}\times\mathbb{R}^{\mathrm{D}}} \mathcal{A}(M)\, M \, \mathrm{d}v \, \mathrm{d}x \\
&= \iint_{\mathbb{R}^{\mathrm{D}}\times\mathbb{R}^{\mathrm{D}}} \rho(x,s+t)\, \theta(s+t)^{\frac{\beta}{2}} \, \mathsf{a}\left(\frac{v-u(x,s+t)}{\sqrt{\theta(s+t)}}\right) M \, \mathrm{d}v \, \mathrm{d}x \\
&= C_{\mathsf{a}}\, \theta(s+t)^{\frac{\beta}{2}} \int_{\mathbb{R}^{\mathrm{D}}} \rho(x,s+t)^2 \, \mathrm{d}x\,,
\end{aligned}$$

where

$$C_{\mathsf{a}} = \int_{\mathbb{R}^{\mathrm{D}}} \mathsf{a}(w)\, \frac{1}{(2\pi)^{\frac{\mathrm{D}}{2}}} \exp\bigl(-\tfrac{1}{2}|w|^2\bigr) \, \mathrm{d}w\,.$$

Finally, we see from (1.19) that

$$\begin{aligned}
\int_{\mathbb{R}^{\mathrm{D}}} \rho(x,s+t)^2 \, \mathrm{d}x &= \int_{\mathbb{R}^{\mathrm{D}}} m^2 \left(\frac{\theta(s+t)}{2\pi}\right)^{\mathrm{D}} \det(Q)\, \exp\bigl(-\theta(s+t)\, x^T Q x\bigr) \, \mathrm{d}x \\
&= m^2 \sqrt{\det(Q)} \left(\frac{\theta(s+t)}{4\pi}\right)^{\frac{\mathrm{D}}{2}}\,.
\end{aligned}$$

By combining the last two equalities under a time integral, we find that

$$\int_{t_1}^{t_2} \iiiint M_* \, M \, \mathsf{b} \, \mathrm{d}\omega \, \mathrm{d}v_* \, \mathrm{d}v \, \mathrm{d}x \, \mathrm{d}t = C_{\mathsf{a}} \, m^2 \sqrt{\det\left(\tfrac{1}{4\pi} Q\right)} \int_{t_1}^{t_2} \theta(s+t)^{\frac{\beta+\mathrm{D}}{2}} \, \mathrm{d}t \, .$$

When this is placed into the right-hand side of inequality (4.4), we obtain the bound (4.2) with $C_1 = 2C_{\mathsf{a}} \, m^2 \sqrt{\det\left(\tfrac{1}{4\pi} Q\right)}$. □

An immediate consequence of the foregoing lemma is the following.

Proposition 4.1. *Let the collision kernel have the separated form (1.4) for some $\beta \in (1-\mathrm{D}, 2]$. Let $F(v,x,t)$ satisfy the hypotheses of Lemma 4.1. Then*

$$\int_0^\infty \left\| \mathcal{B}(F,F) \right\|_{L^1(\mathrm{d}v \, \mathrm{d}x)} \mathrm{d}t < \infty \, . \tag{4.5}$$

Proof. The definition (1.3) of $\mathcal{B}$, the triangle inequality, and bound (4.2) of Lemma 4.1 show that for every $t_2 \in \mathbb{R}_+$ we have the bound

$$\begin{aligned}
\int_0^{t_2} \left\| \mathcal{B}(F,F) \right\|_{L^1(\mathrm{d}v \, \mathrm{d}x)} \mathrm{d}t &= \int_0^{t_2} \iint_{\mathbb{R}^{\mathrm{D}} \times \mathbb{R}^{\mathrm{D}}} \left| \iint_{\mathbb{S}^{\mathrm{D}-1} \times \mathbb{R}^{\mathrm{D}}} \left(F_*' F' - F_* F \right) \mathsf{b} \, \mathrm{d}\omega \, \mathrm{d}v_* \right| \mathrm{d}v_* \, \mathrm{d}x \, \mathrm{d}t \\
&\leq \int_0^{t_2} \iiiint \left| F_*' F' - F_* F \right| \mathsf{b} \, \mathrm{d}\omega \, \mathrm{d}v_* \, \mathrm{d}v \, \mathrm{d}x \, \mathrm{d}t \\
&\leq C_1 \int_0^{t_2} \theta(s+t)^{\frac{\beta+\mathrm{D}}{2}} \, \mathrm{d}t \, ,
\end{aligned}$$

where C_1 is a constant and $\theta(t)$ is given by (1.17). We see from (1.17) that $\theta(t)$ decays like t^{-2} as $t \to \infty$. Therefore the integrand on the right-hand side of the above bound decays like $t^{-(\mathrm{D}+\beta)}$ as $t \to \infty$. Hence, the bound remains finite as $t_2 \to \infty$ if and only if $\beta \in (1-\mathrm{D}, 2]$. □

4.2. **Scattering for Boltzmann Solutions.** The advection operator $\mathsf{A} = -v \cdot \nabla_x$ generates the group $e^{t\mathsf{A}}$ that acts on every function F^{in} that is defined almost everywhere by the formula

$$e^{t\mathsf{A}} F^{\mathrm{in}}(v,x) = F^{\mathrm{in}}(v, x - vt) \, .$$

When F^{in} is locally integrable then $F = e^{t\mathsf{A}} F^{\mathrm{in}}$ is the unique distribution solution of the initial-value problem

$$\partial_t F + v \cdot \nabla_x F = 0 \, , \qquad F\Big|_{t=0} = F^{\mathrm{in}} \, . \tag{4.6}$$

The main result of this section states that certain global solutions of the Cauchy problem for the Boltzmann equation will behave like solutions of (4.6) as $t \to \infty$. This kind of large-time asymptotic result is often called a scattering result.

Theorem 4.1. *Let the collision kernel b have the separated form (1.4) for some $\beta \in (1-\mathrm{D}, 2]$. Let $F(v,x,t)$ be a global mild solution of the Cauchy problem (1.1) for the Boltzmann equation that also satisfies the hypotheses of Lemma 4.1. There exists a unique $F^\infty(v,x)$ such that*

$$\lim_{t \to \infty} \left\| F(t) - e^{t\mathsf{A}} F^\infty \right\|_{L^1(\mathrm{d}v \, \mathrm{d}x)} = 0 \, , \tag{4.7}$$

and F^∞ satisfies the bounds

$$0 \leq F^\infty(v,x) \leq \mathcal{M}(v,x,s) \qquad \textit{almost everywhere over } \mathbb{R}^{\mathrm{D}} \times \mathbb{R}^{\mathrm{D}} \, . \tag{4.8}$$

Proof. The fact that F is a global mild solution of the Cauchy problem for the Boltzmann equation means that for every $t \in [0, \infty)$

$$F(t) = e^{t\mathsf{A}} F^{\mathrm{in}} + \int_0^t e^{(t-t')\mathsf{A}} \mathcal{B}(F(t'), F(t'))\, \mathrm{d}t' . \tag{4.9}$$

Because $\beta \in (1 - \mathrm{D}, 2]$ while F satisfies the hypotheses of Lemma 4.1, Proposition 4.1 implies that

$$\int_0^\infty \|e^{-t'\mathsf{A}} \mathcal{B}(F(t'), F(t'))\|_{L^1(\mathrm{d}v\, \mathrm{d}x)}\, \mathrm{d}t' = \int_0^\infty \|\mathcal{B}(F(t'), F(t'))\|_{L^1(\mathrm{d}v\, \mathrm{d}x)}\, \mathrm{d}t' < \infty . \tag{4.10}$$

This implies (by the Cauchy criterion for families) that

$$\lim_{t\to\infty} \int_0^t e^{-t'\mathsf{A}} \mathcal{B}(F(t'), F(t'))\, \mathrm{d}t' \qquad \text{exists in } L^1(\mathrm{d}v\, \mathrm{d}x) .$$

We then define $F^\infty \in L^1(\mathrm{d}v\, \mathrm{d}x)$ by

$$F^\infty = F^{\mathrm{in}} + \int_0^\infty e^{-t'\mathsf{A}} \mathcal{B}(F(t'), F(t'))\, \mathrm{d}t' . \tag{4.11}$$

Upon solving this relationship for F^{in} and placing the result into (4.9), we find that

$$F(t) = e^{t\mathsf{A}} F^\infty - \int_t^\infty e^{(t-t')\mathsf{A}} \mathcal{B}(F(t'), F(t'))\, \mathrm{d}t' . \tag{4.12}$$

Because $e^{(t-t')\mathsf{A}}$ is an isometry in $L^1(\mathrm{d}v\, \mathrm{d}x)$ we see that

$$\big\|F(t) - e^{t\mathsf{A}} F^\infty\big\|_{L^1(\mathrm{d}v\, \mathrm{d}x)} \leq \int_t^\infty \big\|\mathcal{B}(F(t'), F(t'))\big\|_{L^1(\mathrm{d}v\, \mathrm{d}x)}\, \mathrm{d}t' .$$

But the right-hand side above vanishes as $t \to \infty$ by (4.10), whereby limit (4.7) is established.

Because by hypothesis $0 \leq F(t) \leq \mathcal{M}(s + t)$, it follows that

$$0 \leq e^{-t\mathsf{A}} F(t) \leq e^{-t\mathsf{A}} \mathcal{M}(s + t) = \mathcal{M}(s) \qquad \text{over } \mathbb{R}^{\mathrm{D}} \times \mathbb{R}^{\mathrm{D}} .$$

By passing to the limit as $t \to \infty$ in these inequalities, we obtain

$$0 \leq F^\infty = \lim_{t\to\infty} e^{-t\mathsf{A}} F(t) \leq \mathcal{M}(s) \qquad \text{almost everywhere over } \mathbb{R}^{\mathrm{D}} \times \mathbb{R}^{\mathrm{D}} .$$

This establishes bound (4.8), and thereby completes our proof. □

Remark. At this point we have been unable to show that F^∞ is a global Maxwellian. If it were, and we could show that it has the same values for its formally conserved quantities as F^{in}, then F^∞ would be uniquely determined up to a time shift by the values of those quantities. We have spent some effort trying to determine when F^∞ might be a global Maxwellian. We have not spent much effort trying to show that the class of solutions we are studying indeed conserve the formally conserved quantities. One approach to this last question is to show that our solutions inherit regularity through their approximation by Kaniel-Shinbrot iterates.

References

[1] R.J. Alonso amd I. Gamba, *Distributional and Classical Solutions to the Cauchy Boltzmann Problem for Solt Potentials with Integrable Angular Cross Section*, Jour. Stat. Phys. **137** (2009), 1147–1165.

[2] R.J. Alonso amd I. Gamba, *A Revision on Classical Solutions to the Cauchy Boltzmann Problem for Solt Potentials*, Jour. Stat. Phys. (submitted 2011).

[3] N. Bellomo and G. Toscani, *On the Cauchy Problem for the Nonlinear Boltzmann Equation: Global Existence, Uniqueness, and Asymptotic Stability*, Jour. Math. Phys. **26** (1985), 334–338.

[4] L. Boltzmann *Lectures on Gas Dynamics*, Dover, New York, 1995.

[5] S.-Y. Ha, M. Yamazaki, and S.-B. Yun, *Uniform L^p-Stability Estimate for the Spatially Inhomogeneous Boltzmann Equation near Vacuum*, Jour. Hyperbolic Diff. Equ. **5** (2008), 713–739.

[6] R. Illner and M. Shinbrot, *The Boltzmann Equation: Global Existence for a Rare Gas in an Infinite Vacuum*, Commun. Math. Phys. **95** (1984), 217–226.

[7] S. Kaniel and M. Shinbrot, *The Boltzmann Equation I: Uniqueness and Local Existence*, Commun. Math. Phys. **58** (1978), 65–84.

[8] C.D. Levermore, *Global Maxwellians over All Space and Their Relation to Conserved Quantites of Classical Kinetic Equations*, preprint, 2011.

[9] J. Polewczak, *Classical Solution of the Nonlinear Boltzmann Equation in All $\mathbb{R}^3$: Asymptotic Behavior of Solutions*, Jour. Stat. Phys. **50** (1988), 611–632.

[10] G. Toscani, *On the Non-linear Boltzmann Equation in Unbounded Domains*, Arch. Rational Mech & Anal. **95** (1986), 37–49.

[11] G. Toscani, *Global Solution of the Initial Value Problem for the Boltzmann Equation near a Local Maxwellian*, Arch. Rational Mech & Anal. **102** (1988), 231–241.

(CB) Laboratoire Jacques-Louis Lions, Université Denis Diderot (retired), Paris, France
E-mail address: claude.bardos@gmail.com

(IG) Department of Mathematics, University of Texas, Austin, TX, USA
E-mail address: gamba@math.utexas.edu

(CDL) Department of Mathematics & Institute for Physical Science and Technology (IPST), University of Maryland, College Park, MD 20742-4015, USA
E-mail address: lvrmr@math.umd.edu