ON THE EXTENSION PROPERTY OF REIFENBERG-FLAT DOMAINS.

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Abstract. We provide a detailed proof of the fact that any domain which is sufficiently flat in the sense of Reifenberg is also Jones-flat, and hence it is an extension domain. We discuss various applications of this property, in particular we obtain L^∞ estimates for the eigenfunctions of the Laplace operator with Neumann boundary conditions. We also compare different ways of measuring the "distance" between two sufficiently close Reifenberg-flat domains. These results are pivotal to the quantitative stability analysis of the spectrum of the Neumann Laplacian performed in [26].

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1. Introduction

The main goal of the present paper is establishing extension and geometric properties for a class of domains whose boundaries satisfy a fairly weak regularity requirement introduced by Reifenberg [28]. In particular, we show that any domain that is sufficiently flat in the sense of Reifenberg enjoys the so-called extension property and we discuss applications that are relevant for the analysis of PDEs defined in these domains. We also compare different ways of measuring the "distance" between two sufficiently close Reifenberg-flat domains X and Y, in particular we discuss the relations between the Hausdorff distances $d_H(X, Y)$, $d_H(\mathbb{R}^N \setminus X, \mathbb{R}^N \setminus Y)$ and $d_H(\partial X, \partial Y)$ and the measure of the symmetric difference $|X \triangle Y|$.

Although we are confident our results can find different applications, our original motivation was the quantitative stability analysis of the spectrum of the Laplace operator with Neumann boundary conditions defined in Reifenberg-flat domains, see [26].

The notion of Reifenberg-flat sets was first introduced in 1960 by Reifenberg [28] when he was working on the Plateau problem, and has since then played an important role in the study of minimal surfaces. More recently, the works by David [11, 12] about the regularity for 2-dimensional minimal sets in $\mathbb{R}^N$ rely on the Reifenberg parametrization and the specific 3-dimensional results by David, De Pauw and Toro [13]. Also, Reifenberg-flat set are relevant in the study of the harmonic measure (see Kenig and Toro [19, 20, 21] and Toro [31, 32]) and of the regularity for free boundary problems, like the minimization of the Mumford-Shah functional (see [22, 23]). Elliptic and parabolic equations defined in Reifenberg-flat domains have been recently investigated by Byun, Wang and Zhou [2, 3, 4], by Lemenant, Milakis and Spinolo and by Milakis and Toro [24, 25, 26, 27]. Finally, we mention that Reifenberg-flat domains are in particular NTA domains in the sense of Jerison and Kenig [17].

We now provide the precise definition. We denote by d_H the classical Hausdorff distance between two sets X and Y,

$$d_H(X,Y) := \max\big\{ \sup_{x\in X} d(x,Y), \sup_{y\in Y} d(y,X)\big\}. \tag{1.1}$$

Definition 1. *Let ε, r_0 be two real numbers satisfying $0<\varepsilon<1/2$ and $r_0>0$. An (ε, r_0)-Reifenberg-flat domain $\Omega \subseteq \mathbb{R}^N$ is a nonempty open set satisfying the following two conditions:*

i) for every $x \in \partial\Omega$ and for every $r \le r_0$, there is a hyperplane $P(x,r)$ containing x which satisfies

$$\frac{1}{r} d_H\big(\partial\Omega \cap B(x,r), P(x,r)\cap B(x,r)\big) \le \varepsilon. \tag{1.2}$$

ii) For every $x\in\partial\Omega$, one of the connected component of

$$B(x,r_0) \cap \big\{x: \ dist(x, P(x,r_0)) \ge 2\varepsilon r_0\big\}$$

is contained in Ω and the other one is contained in $\mathbb{R}^N \setminus \Omega$.

Condition $i)$ states that the boundary of Ω is an (ε, r_0)-Reifenberg-flat set. A Reifenberg-flat set enjoys local separability properties (see e.g. Theorem 4.1. in [15]), however we observe that condition $ii)$ in the definition is not in general implied by condition $i)$, as the example of $\Omega = \mathbb{R}^N \setminus \partial B(0,1)$ shows (here $\partial B(0,1)$ denotes the boundary of the unit ball). However, a consequence of the analysis in David [10] is that $i)$ implies $ii)$ under some further topological assumption, for instance the implication holds if Ω and $\partial\Omega$ are both connected. Note furthermore that a straightforward consequence of the definition is that, if $\varepsilon_1 < \varepsilon_2$, then any (ε_1, r_0)-Reifenberg-flat domain is also an (ε_2, r_0)-Reifenberg-flat domain. Finally, note that we only impose the separability requirement $ii)$ at scale r_0 but it simply follows from the definition that it also holds at any scale $r \le r_0$ (see [19, Proposition 2.2] or Lemma 5 below).

In [28] Reifenberg proved the so-called topological disk theorem which states that, provided ε is small enough, any (ε, r_0)-Reifenberg-flat set in the unit N-ball is the bi-Hölderian image of an $(N-1)$-dimensional disk. Also, any Lipschitz domain with sufficiently small Lipschitz constant is Reifenberg-flat for a suitable choice of the regularity parameter ε (the choice depends on the Lipschitz constant). On the other hand, the "flat" Koch snowflake with sufficiently small angle is Reifenberg-flat (see Toro [31]) and hence it is an example of a Reifenberg-flat set which is *not* Lipschitz, and with Hausdorff dimension greater than $N-1$.

The main goal of this paper is providing a complete and detailed proof of the fact that Reifenberg-flat domains are extension domains. This fact is relevant for the study of elliptic problems and was already known and used in the literature (see e.g. the introduction of [2]). However, to the best of our knowledge, an explicit proof was so far missing. We recall that the so called *extension problem* can be formulated as follows: given an open set Ω, we denote by $W^{1,p}$ the classical Sobolev space and we wonder whether or not one can define a bounded linear operator (the so-called extension operator)

$$E : W^{1,p}(\Omega) \to W^{1,p}(\mathbb{R}^N)$$

such that $E(u) \equiv u$ on Ω. If $\partial\Omega$ is Lipschitz, Calderon [5] established the existence of an extension operator in the case when $1 < p < \infty$, while Stein [30] considered the cases $p = 1, \infty$. Jones [18] proved the existence of extension operators for a new class of domains, the so-called (ε, δ)-Jones flat domains (the precise definition is recalled in Section 2). In the present work we prove that sufficiently flat Reifenberg domains are indeed Jones flat domains. Our main result concerning the extension problem is as follows.

Theorem 2. *Any* $(1/600, r_0)$*-Reifenberg flat domain is a* $(1/450, r_0/7)$*-Jones flat domain.*

As direct consequence of Theorem 2 we get that one can define extension operators for $(1/600, r_0)$-Reifenberg flat domains (see Corollary 8 for a precise statement). Some relevant features of this result are the following: first, we provide an explicit and universal threshold on the coefficient ε for the extension property to hold (namely, $\varepsilon \leq 1/600$). Second, 1/600 is fairly big compared to the usual threshold needed to apply Reifenberg's topological disk theorem (for e.g. the threshold is 10^{-15} in [13], see also [16] for an interesting alternative proof).

As a consequence of the extension extension property, we obtain that the classical Rellich-Kondrachov Theorem applies to Reifenberg-flat domains (see Proposition 10), that the Neumann Laplacian has a discrete spectrum and that the eigenfunctions are bounded (see Proposition 11). Also, by combining Theorem 2 with the works by Chua [7, 8, 9] and Christ [6] we get that one can define extension operators for weighted Sobolev spaces and Sobolev spaces of fractional order (see Remark 9 in the present paper).

We conclude the paper by establishing results unrelated to the extension problem, namely we study the relation between different ways of measuring the "distance" between sets of $\mathbb{R}^N$. In particular, for two general open sets X and Y, neither the Hausdorff distance $d_H(X, Y)$ nor the Hausdorff distance between the complements $d_H(\mathbb{R}^N \setminus X, \mathbb{R}^N \setminus Y)$ is, in general, controlled by the Lebesgue measure of the symmetric difference $|X \triangle Y|$. However, we prove that they are indeed controlled provided that X, Y are Reifenberg flat and close enough, in a suitable sense. This result will be as well applied in [26] to the stability analysis of the spectrum of the Laplace operator with Neumann boundary conditions.

The paper is organized as follows: in Section 2 we prove that sufficiently flat Reifenberg domains are Jones-flat, in Section 3 we show that if these domains are also connected, then they enjoy the extension property. In Section 3 we also discuss some applications of the extension property. In Section 4 we investigate how to handle domains that are not connected and finally in Section 5 we investigate the relation between different ways of measuring the "distance" between Reifenberg-flat domains.

1.1. **Notations.** We denote by $C(a_1, \dots, a_h)$ a constant only depending on the variables $a_1, \dots, a_h$. Its precise value can vary from line to line. Also, we use the following notations:
$\mathcal{H}^N$: the N-dimensional Hausdorff measure.
ω_N : the Lebesgue measure of the unit ball in $\mathbb{R}^N$.
$|A|$: the Lebesgue measure of the Borel set $A \subseteq \mathbb{R}^N$.
A^c: the complement of the set A, $A^c := \mathbb{R}^N \setminus A$.
$\bar{A}$: the closure of the set A.
$W^{1,p}(\Omega)$: the Sobolev space of L^p functions whose derivatives are in L^p.
$\langle x, y \rangle$: the standard scalar product between the vectors $x, y \in \mathbb{R}^N$.
$|x|$: the norm of the vector $x \in \mathbb{R}^N$.
$d(x, y)$: the distance from the point x to the point y, $d(x, y) = |x - y|$.
$d(x, A)$: the distance from the point x to the set A.
$d_H(A, B)$: the Hausdorff distance from the set A to the set B.
$[x, y]$: the segment joining the points $x, y \in \mathbb{R}^N$.
$B(x, r)$: the open ball of radius r centered at x.
$\overline{B}(x, r)$: the closed ball of radius r centered at x.

2. Reifenberg-flat and Jones domains

In this section we show that any sufficiently flat Reifenberg domain is Jones-flat, in the sense of [18]. The extension property follows then as a corollary of the analysis in [18].

First, we provide the precise definition of Jones-flatness.

Definition 3. *An open and bounded set Ω is a (δ, R_0)-Jones-flat domain if for any $x, y \in \Omega$ such that $d(x, y) \leq R_0$ there is a rectifiable curve γ which connects x and y and satisfies*

$$\mathcal{H}^1(\gamma) \leq \delta^{-1} d(x, y) \tag{2.1}$$

and

$$d(z, \Omega^c) \geq \delta \, \frac{d(z, x) d(z, y)}{d(x, y)}, \quad \text{for all } z \in \gamma. \tag{2.2}$$

To investigate the relation between Jones flatness and Reifenberg flatness we need two preliminary lemmas.

Lemma 4. *Let $\Omega \subseteq \mathbb{R}^N$ be an $(\varepsilon, r_0)-$Reifenberg flat domain. Given $x \in \partial\Omega$ and $r \leq r_0$, we term ν_r the unit normal vector to the hyperplane $P(x, r)$ provided by the definition of Reifenberg-flatness. Given $M \geq 1$, for every $r \leq r_0/M$ we have*

$$|\langle \nu_r, \nu_{Mr} \rangle| \geq 1 - (M+1)\varepsilon. \tag{2.3}$$

Proof. We assume with no loss of generality that x is the origin. For simplicity, in the proof we denote by B_r the ball $B(0, r)$ and by P_r the hyperplane $P(0, r)$. From the definition of Reifenberg flatness we infer that

$$\begin{aligned} d_H(P_{Mr} \cap B_r, P_r \cap B_r) &\leq d_H(P_{Mr} \cap B_r, \partial\Omega \cap B_r) + d_H(\partial\Omega \cap B_r, P_r \cap B_r) \\ &\leq Mr\varepsilon + r\varepsilon \leq (M+1)r\varepsilon. \end{aligned}$$

Since P_{Mr} and P_r are linear spaces we deduce that

$$d_H(P_{Mr} \cap B_1, P_r \cap B_1) \leq (M+1)\varepsilon. \tag{2.4}$$

We term π_r and π_{Mr} the orthogonal projections onto P_r and P_{Mr}, respectively, and we fix an arbitrary point $y \in P_r \cap B_1$. Inequality (2.4) states that there is $z \in \bar{P}_{Mr} \cap \bar{B}_1$ satisfying

$$d(z, y) \leq (M+1)\varepsilon.$$

In particular, since $1 = |\nu_{Mr}| = \inf_{z \in P_{Mr}} d(\nu_{Mr}, z)$, we get

$$d(\nu_{Mr}, y) \geq d(\nu_{Mr}, z) - d(z, y) \geq 1 - (M+1)\varepsilon.$$

By taking the infimum for $y \in P_r \cap B_1$ we obtain

$$|\nu_{Mr} - \pi_r(\nu_{Mr})| \geq 1 - (M+1)\varepsilon,$$

and the proof is concluded by recalling that $|\langle \nu_{Mr}, \nu_r \rangle| = d(\nu_{Mr}, \pi_r(\nu_{Mr}))$. □

The following lemma discuss an observation due to Kenig and Toro [19, Proposition 2.2]. Note that the difference between Lemma 5 and part *ii*) in the definition of Reifenberg flatness is that in *ii*) we only require the separation property at scale r_0.

Lemma 5. *Let $\Omega \subseteq \mathbb{R}^N$ be an $(\varepsilon, r_0)-$Reifenberg flat domain. For every $x \in \partial\Omega$ and $r \in]0, r_0]$, one of the connected components of*

$$B(x,r) \cap \big\{x : \ dist(x, P(x,r)) \geq 2\varepsilon r\big\}$$

is contained in Ω and the other one is contained in $\mathbb{R}^N \backslash \Omega$. Here $P(x,r)$ is the same hyperplane as in part i) of the definition of Reifenberg-flatness.

Proof. We fix $\rho \in]0, r_0]$ and we assume that the separation property holds at scale ρ, namely that one of the connected components of

$$B(x,\rho) \cap \big\{x : \ dist(x, P(x,\rho)) \geq 2\varepsilon \rho\big\}$$

is contained in Ω and the other one is contained in $\mathbb{R}^N \setminus \Omega$. We now show that the same separation property holds at scale r for every $r \in]\rho/M, \rho]$ provided that $M \leq (1-\varepsilon)/3\varepsilon$. By iteration this implies that the separation property holds at any scale $r \in]0, r_0]$.

Let us fix $r \in]\rho/M, \rho]$ and denote by $B^+(x,r)$ one of the connected components of

$$B(x,r) \cap \big\{x : \ dist(x, P(x,r)) \geq 2\varepsilon r\big\}$$

and by $B^-(x,r)$ the other one. Also, we term Y^+ and Y^- the points of intersection of the line passing through x and perpendicular to $P(x,r)$ with the boundary of the ball $B(x,r)$.

By recalling (2.3) and the inequality $r \geq \rho/M$, we get that the distance of $Y^\pm$ from the hyperpane $P(x,\rho)$ satisfies the following inequality:

$$d(Y^\pm, P(x,\rho)) \geq r|\langle \nu_\rho, \nu_{\rho/M} \rangle| \geq r\big[1 - (M+1)\varepsilon\big] \geq \rho\frac{1-(M+1)\varepsilon}{M}.$$

Since by assumption $M \leq (1-\varepsilon)/3\varepsilon$, this implies that $d(Y^\pm, P(x,\rho)) \geq 2\varepsilon\rho$ and hence that one among Y^+ and Y^- belongs to $B^+(x,\rho)$ and the other one to $B^-(x,\rho)$. Since by assumption the separation property holds at scale ρ, this implies that one of them belongs to Ω and the other one to Ω^c.

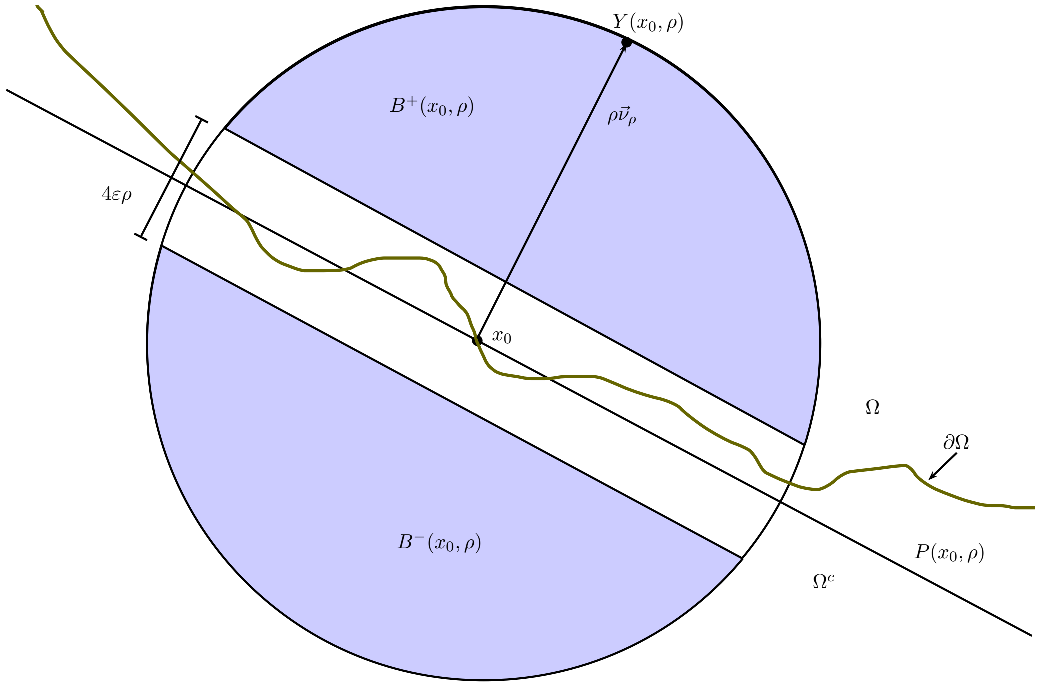

FIGURE 1. notations for the proof of Theorem 2

To conclude, note that part $i)$ in the definition of Reifenberg flatness implies that

$$B^{\pm}(x,r) \cap \partial\Omega = \emptyset$$

and hence both $B^+(x,r)$ and $B^-(x,r)$ are entirely contained in either Ω or Ω^c. By recalling that one among Y^+ and Y^- belongs to Ω and the other one to Ω^c, we conclude the proof of the lemma. □

We are now ready to establish the main result of this section, namely Theorem 2.

Proof of Theorem 2. We assume $\varepsilon \leq 1/600$, we fix an (ε, r_0)-Reifenberg flat domain $\Omega \subseteq \mathbb{R}^N$ and we proceed according to the following steps.

⋄ STEP 1. We first introduce some notations (see Figure 1 for a representation).

For any $x_0 \in \partial\Omega$ and $\rho \leq r_0$, we denote as usual by $P(x_0,\rho)$ the hyperplane provided by the definition of Reifenberg flatness, and by $\vec{\nu}_\rho$ its normal. By Lemma 5, we can choose the orientation of $\vec{\nu}_\rho$ in such a way that

$$B^+(x_0,\rho) := \{z + t\vec{\nu}_\rho : \; z \in P(x_0,\rho), \, t \geq 2\varepsilon\rho\} \cap B(x_0,\rho) \subseteq \Omega$$

and

$$B^-(x_0,\rho) := \{z - t\vec{\nu}_\rho : \ z \in P(x_0,\rho),\ t \geq 2\varepsilon\rho\} \cap B(x_0,\rho) \subseteq \Omega^c.$$

Also, we define the hyperplanes $P^+(x_0,\rho)$ and $P^-(x_0,\rho)$ by setting

$$P^+(x_0,\rho) := \{z + 2\varepsilon\rho\vec{\nu}_\rho : \ z \in P(x_0,\rho)\}$$

and

$$P^-(x_0,\rho) := \{z - 2\varepsilon\rho\vec{\nu}_\rho : \ z \in P(x_0,\rho)\}$$

and we denote by $Y(x_0,\rho)$ the point

$$Y(x_0,\rho) := x_0 + \rho\vec{\nu}_\rho.$$

Finally, for any $x \in \Omega$, we denote by $x_0 \in \partial\Omega$ the point such that $d(x,\Omega^c) = d(x,x_0)$ (if there is more than one such x_0, we arbitrarily fix one).

⋄ STEP 2. We provide a preliminary construction: more precisely, given

- $x \in \Omega$ such that $d(x,\Omega^c) \leq 2r_0/7$ and
- r satisfying $d(x,\Omega^c)/2 \leq r \leq r_0/7$,

the curve $\gamma_{x,r}$ is defined as follows.

(I) If $d(x,\Omega^c)/2 \leq r \leq 2d(x,\Omega^c)$, then $\gamma_{x,r}$ is simply the segment $[x, Y(x_0,r)]$.

(II) If $2d(x,\Omega^c) < r \leq r_0/7$, we denote by $k_0 \geq 1$ the biggest natural number k satisfying $2^{-k}r \geq d(x,\Omega^c)$ and we set

$$\gamma_{x,r} := [x, Y(x_0, 2^{-k_0}r)] \cup \bigcup_{k=0}^{k_0-1} [Y(x_0,2^{-k}r), Y(x_0,2^{-(k+1)}r)].$$

⋄ STEP 3. We prove that in both cases (I) and (II) we have

$$\mathcal{H}^1(\gamma_{x,r}) \leq 4r. \tag{2.5}$$

To handle case (I) we just observe that, since by assumption $d(x,\Omega^c) = d(x,x_0) \leq 2r$, then, by recalling $d(x_0, Y(x_0,r)) = r$, property (2.5) follows.

To handle case (II), we first observe that, since $d(x,x_0) \leq 2^{-k_0}r$, then both x and $Y(x_0,2^{-k_0}r)$ belong to the closure of $B(x_0,2^{-k_0}r)$. Also, by construction both $Y(x_0,2^{-k}r)$ and $Y(x_0,2^{-(k+1)}r)$ belong to the closure of $B(x_0,2^{-k}r)$ and by combining these observations we conclude that

$$\begin{aligned}\mathcal{H}^1(\gamma_{x,r}) &\leq d(x, Y(x_0,2^{-k_0}r)) + \sum_{k=0}^{k_0-1} d(Y(x_0,2^{-k}r), Y(x_0,2^{-(k+1)}r)) \\ &\leq 2\cdot 2^{-k_0}r + \sum_{k=0}^{k_0-1} 2\cdot 2^{-k}r \leq 2r\sum_{k\in\mathbb{N}} 2^{-k} = 4r. \qquad (2.6)\end{aligned}$$

$\diamond$ STEP 4. We prove that for every $z \in \gamma_{x,r}$

$$d(z, \Omega^c) \geq \frac{29}{240}\, d(z, x). \tag{2.7}$$

We start by handling case (I): we work in the ball $B(x_0, 4r)$ and we recall the definition of $B^+(x_0, 4r)$ and of $B^-(x_0, 4r)$, given at STEP 1. Since by assumption $\varepsilon \leq 1/32$, we have

$$16\varepsilon r \leq \frac{r}{2} \leq d(x, \Omega^c) \leq d(x, B^-(x_0, 4r))$$

and hence $x \in B^+(x_0, 4r)$. Let β denotes the angle between ν_r and ν_{4r}, then by Lemma 4 applied with $M = 4$ we get that provided $\varepsilon \leq 1/9$, then $4\varepsilon r \leq r \cos\beta$, so that $Y(x_0, r) \in B^+(x_0, 4r)$. By recalling that $x \in B^+(x_0, 4r)$, we conclude that $[x, Y(x_0, r)] \subseteq B^+(x_0, 4r)$.

We are now ready to establish (2.7), so we fix $z \in [x, Y(x_0, r)]$. To provide a bound from above on $d(z, x)$, we simply observe that, since both x and $Y(x_0, r)$ belong to the closure of $B(x_0, 2r)$, then so does z and hence

$$d(z, x) \leq 4r. \tag{2.8}$$

Next, we provide a bound from below on $d(z, \Omega^c)$: since $z \in B^+(x_0, 4r) \subseteq \Omega$, then

$$d(z, \Omega^c) \geq d(z, \partial B^+(x_0, 4r)) = \min\Big\{ d(z, P^+(x_0, 4r)), d(z, \partial B(x_0, 4r)) \Big\}. \tag{2.9}$$

First, we recall that $z \in B(x_0, 2r)$ and we provide a bound on the distance from z to the spherical part of $\partial B^+(x_0, 4r)$:

$$d(z, \partial B(x_0, 4r)) = 4r - d(z, x_0) \geq 4r - 2r = 2r.$$

Next, we observe that

$$d(z, P^+(x_0, 4r)) = d(z, P(x_0, 4r)) - 8\varepsilon r$$

and, since $z \in [x, Y(x_0, r)]$, then

$$d(z, P(x_0, 4r)) \geq \min\Big\{ d(x, P(x_0, 4r), d(Y(x_0, r), P(x_0, 4r)) \Big\}.$$

Note that $d(Y(x_0, r), P(x_0, 4r)) = r\cos\beta$ and, using Lemma 4, we conclude that

$$d(Y(x_0, r), P(x_0, 4r)) \geq r/2$$

because $\varepsilon \leq 1/10$. Also, since $B^-(x_0, 4r) \subseteq \Omega^c$, then

$$r/2 \leq d(x, \Omega^c) \leq d(x, B^-(x_0, 4r)) = d(x, P(x_0, 4r)) + 2\varepsilon r$$

By recalling (2.9) and the inequality $\varepsilon \le 1/600$ and by combining all the previous observations we conclude that

(2.10)
$$d(z,\Omega^c) \ge d(z, \partial B^+(x_0,4r)) \ge \min\{d(z, P^+(x_0,4r), 2r\} = \min\{d(z, P(x_0,4r) - 8\varepsilon r, 2r\} \ge$$
$$\ge \min\Big\{ \min\big\{ d\big(x, P(x_0,4r)\big), d\big(Y(x_0,r), P(x_0,4r)\big)\big\} - 8\varepsilon r, 2r \Big\} \ge$$
$$\ge \min\Big\{ \min\big\{ r/2 - 2\varepsilon r, r/2 \big\} - 8\varepsilon r, 2r \Big\} = \frac{r}{2} - 10\varepsilon r \ge \frac{29}{60} r.$$

Finally, by comparing (2.10) and (2.8) we obtain (2.7).

◇ STEP 5. We now establish (2.7) in case (II).

If $z \in [x, Y(x_0, 2^{-k_0} r]$, then we can repeat the argument we used in STEP 4 by replacing r with $2^{-k_0} r$, which satisfies

$$d(x,\Omega^c) \le 2^{-k_0} r \le 2d(x,\Omega^c).$$

Hence, we are left to consider the case when $z \in [Y(x_0, 2^{-k} r), Y(x_0, 2^{-(k+1)} r)]$ for some natural number $k \le k_0 - 1$. We set $\rho := 2^{-k} r$ and we work in the ball $B(x_0, 2\rho)$. We denote by α the angle between $\nu_{2\rho}$ and ν_ρ, and by β the angle between $\nu_{2\rho}$ and $\nu_{\rho/2}$. Due to Lemma 4 applied with $M = 2$ and $M = 4$, we know that, if $\varepsilon \le 1/13$, then

$$\rho \cos\alpha \ge 4\varepsilon\rho \qquad \frac{1}{2}\rho\cos\beta \ge 4\varepsilon\rho,$$

so that both $Y(x_0,\rho)$ and $Y(x_0,\rho/2)$ belong to $B^+(x_0,2\rho)$. Hence, given

$$z \in [Y(x_0,\rho), Y(x_0,\rho/2)] \subseteq B^+(x_0,2\rho) \subseteq \Omega,$$

we have $d(z,\Omega^c) \ge d(z, \partial B^+(x_0,2\rho))$. The distance from z to the spherical part of $\partial B^+(x_0,2\rho)$ is bounded from below by ρ, while the distance from z to $P^+(x_0,2\rho)$ is bounded from below by $\frac{1}{2}\rho - 4\varepsilon\rho \ge \frac{1}{4}\rho$ provided that $\varepsilon \le 1/16$. Hence, $d(z,\Omega^c) \ge \rho/4$. To provide an upper bound on $d(z,x)$ we observe that, since $d(x,x_0) = d(x,\Omega^c) \le 2^{-k} r$, then both z and x belong to the closure of $B(x_0,\rho)$. Hence, $d(x,z) \le 2\rho$ and (2.7) holds.

◇ STEP 6. We are finally ready to show that Ω is a Jones-flat domain. Given $x, y \in \Omega$ satisfying $d(x,y) \le r_0/7$, there are two possible cases:

(1) if either $d(x,\Omega^c) \ge 2d(x,y)$ or $d(y,\Omega^c) \ge 2d(x,y)$, then we set $\gamma := [x,y]$. To see that γ satisfies (2.2), let us assume that $d(x,\Omega^c) \ge 2d(x,y)$ (the other case is completely analogous), then $y \in B(x, d(x,y)) \subseteq \Omega$ and $[x,y] \subseteq \Omega$. Also, since

$$\sup_{z\in[x,y]} \frac{d(z,x)d(z,y)}{d(x,y)} = \frac{1}{4} d(x,y), \tag{2.11}$$

then for any $z \in \gamma$,

$$d(z, \Omega^c) \geq d(x, \Omega^c) - d(z, x) \geq d(x, y) \geq 4d(z, x)d(z, y)/d(x, y).$$

Hence, γ satisfies (2.2) provided that $\delta = 4$.

(2) we are left to consider the case when both $d(x, \Omega^c) < 2d(x, y)$ and $d(y, \Omega^c) < 2d(x, y)$. Denote by $x_0 \in \partial\Omega$ a point such that $d(x, \Omega^c) = d(x, x_0)$ and $y_0 \in \partial\Omega$ a point such that $d(y, y_0) = d(y, \Omega^c)$ and set $r := d(x, y) \leq r_0/7$. We define

$$\gamma := \gamma_{x,r} \cup \gamma_{y,r} \cup [Y(x_0, r), Y(y_0, r)]. \tag{2.12}$$

STEP 7 is devoted to showing that γ satisfies (2.1) and (2.2).

◇ STEP 7. First, we establish (2.1): we observe that

$$d\Big(Y(x_0, r), Y(y_0, r)\Big) \leq d\Big(Y(x_0, r), x_0\Big) + d(x_0, x) + d(x, y) + d(y, y_0) + d\Big(y_0, Y(y_0, r)\Big) \leq 7r$$

and hence by using (2.5)

$$\mathcal{H}^1(\gamma) \leq \mathcal{H}^1(\gamma_{x,r}) + d\Big(Y(x_0, r), Y(y_0, r)\Big) + \mathcal{H}^1(\gamma_{y,r}) \leq 15r$$

which proves (2.1).

Next, we establish (2.2): we denote by d_γ the geodesic distance on the curve γ and we observe that

$$\frac{d(z, y)}{15d(x, y)} \leq \frac{d_\gamma(z, y)}{d_\gamma(x, y)} \leq 1. \tag{2.13}$$

Hence, if $z \in \gamma_{x,r}$, then by using (2.7) we obtain

$$d(z, \Omega^c) \geq \frac{29}{240} d(z, x) \geq \frac{29}{240 \cdot 15} \left(\frac{d(z, x)d(z, y)}{d(x, y)} \right)$$

and we next observe $29/240 \cdot 15 \geq 5/60 \cdot 15 = 1/180$. Since the same argument works in the case when $z \in \gamma_{y,r}$, then we are left to esablish (2.2) in the case when z lies on the segment $[Y(x_0, r), Y(y_0, r)]$.

We first observe that

$$d(x_0, Y(y_0, r)) \leq d(x_0, x) + d(x, y) + d(y, y_0) + d(y_0, Y(y_0, r)) \leq 6r \tag{2.14}$$

and hence $[Y(x_0, r), Y(y_0, r)] \subseteq B(x_0, 7r)$. Next, we note that $7r \leq r_0$ and we use (2.10) to get

$$\frac{29}{60} r \leq d(Y(x_0, r), \Omega^c) \leq d(Y(x_0, r), P^-(x_0, 7r)), \tag{2.15}$$

hence since ε is so small that $28\varepsilon r \leq 29r/60$, then we have $d(Y(x_0, r), P^-(x_0, 7r)) \geq 28\varepsilon r$, which means that $Y(x_0, r) \in B^+(x_0, 7r)$. By repeating the same argument we get $Y(x_0, r) \in B^+(x_0, 7r)$ and hence $[Y(x_0, r), Y(y_0, r)] \subseteq B^+(x_0, 7r)$.

We fix $z \in [Y(x_0, r), Y(y_0, r)]$ and we observe that

$$\begin{aligned} d(z, x) &\leq d(z, Y(x_0, r)) + d(Y(x_0, r), x_0) + d(x_0, x) \\ &\leq d(Y(y_0, r), Y(x_0, r)) + d(Y(x_0, r), x_0) + d(x_0, x) \leq 7r + r + 2r = 10r. \end{aligned} \tag{2.16}$$

Also,

$$d(z, \Omega^c) \geq d(z, \partial B^+(x_0, 7r)) \geq \min\Big\{ d(z, \partial B(x_0, 7r)); d(z, P^+(x_0, 7r)) \Big\} \tag{2.17}$$

and by using (2.14) we get

$$d(z, \partial B(x_0, 7r)) \geq r.$$

Also, we have

$$d(z, P^+(x_0, 7r)) \geq \min\Big\{ d(Y(x_0, r), P^+(x_0, 7r)), d(Y(y_0, r), P^+(x_0, 7r)) \Big\}$$

and by recalling (2.15) we get that

$$d(Y(x_0, r), P^+(x_0, 7r)) = d(Y(x_0, r), P^-(x_0, 7r)) - 28\varepsilon r \geq \frac{29}{60} r - 28\varepsilon r \geq \frac{r}{3}.$$

Since $Y(y_0, r)$ satisfies the same estimate, then by recalling (2.13), (2.16) and (2.17) we get

$$d(z, \Omega^c) \geq \frac{r}{3} \geq \frac{1}{3 \cdot 10} d(z, x) \geq \frac{1}{3 \cdot 10 \cdot 15} \frac{d(z, x) d(z, y)}{d(x, y)},$$

which concludes the proof because $3 \cdot 10 \cdot 15 = 450$. □

Remark 6. There are Jones-flat domains that are not Reifenberg-flat, for instance a Lipschitz domain with sufficiently big constant (for example a heavily non convex polygonal domain). Actually, Jones [18, Theorem 3] proved that, for a simply connected domain in dimension 2, being Jones-flat is equivalent to being an extension domain, which is also known to be equivalent to the fact that the boundary is a quasicircle (see the introduction of [18]).

3. Extension properties of Reifenberg-flat domains and applications

In this section we combine the analysis in [18] with Theorem 2 to prove that domains that are sufficiently flat in the sense of Reifenberg satisfy the extension property. We also discuss some direct consequences. Note that in this section we always assume that Ω is connected, as Jones did in [18]. In Section 4 we prove that the connectedness assumption can be actually removed in the case of Reifenberg flat domains. Note also that, before providing the precise extension result, we have to introduce a preliminary lemma comparing different notions of "radius" of a given domain Ω.

3.1. **"Inner radius", "outer radius" and "diameter" of a given domain.** We term outer radius of a nonempty set $\Omega \subseteq \mathbb{R}^N$ the quantity

$$Rad(\Omega) := \inf_{x\in\Omega} \sup_{y\in\Omega} d(x,y), \tag{3.1}$$

and we term inner radius the quantity

$$rad(\Omega) := \sup_{x\in\Omega} \sup\{r > 0 : \; B(x,r) \subset \Omega\}. \tag{3.2}$$

The inner radius is the radius of the biggest ball that could fit inside Ω, whereas the outer radius, as seen below, is the radius of the smallest ball, centered in $\overline{\Omega}$, that contains Ω.

Also, we recall that $Diam(\Omega)$ denotes the diameter of Ω, namely

$$Diam(\Omega) := \sup_{x,y\in\Omega} d(x,y).$$

For the convenience of the reader, we collect some consequences of the definition in the following lemma.

Lemma 7. *Let Ω be a nonempty subset of $\mathbb{R}^N$, then the following properties hold:*

(i) *We have the formula*

$$Rad(\Omega) = \inf_{x\in\Omega} \inf\{r > 0 : \; \Omega \subset B(x,r)\}. \tag{3.3}$$

Also, if $Rad(\Omega) < +\infty$, then there is a point $x \in \overline{\Omega}$ such that $\Omega \subseteq B(x, Rad(\Omega))$.

(ii) $rad(\Omega) \le Rad(\Omega) \le \mathrm{Diam}(\Omega)$.

(iii) *If Ω is an (ε, r_0)-Reifenberg-flat domain for some $r_0 > 0$ and some ε satisfying $0 < \varepsilon < 1/2$, then $r_0/4 \le rad(\Omega) \le Rad(\Omega) \le \mathrm{Diam}(\Omega)$.*

Proof. To establish property (i), we first observe that, if Ω is not bounded, then $Rad(\Omega) = +\infty$ and formula (3.3) is trivially satisfied. Also, the assumption $Rad(\Omega) < +\infty$ implies that the closure $\overline{\Omega}$ is compact. Hence, if $Rad(\Omega) < +\infty$, then

$$Rad(\Omega) = \min_{x\in\overline{\Omega}} \sup_{y\in\Omega} d(x,y) \tag{3.4}$$

and if we term $x_0 \in \overline{\Omega}$ any point that realizes the minimum in (3.4) we have $\Omega \subset \overline{B}(x_0, Rad(\Omega))$. This establishes the inequality

$$Rad(\Omega) \ge \inf_{x\in\Omega} \inf\{r > 0 : \; \Omega \subset B(x,r)\}.$$

To establish the reverse inequality we observe that if $x \in \Omega$ is any arbitrary point and $r > 0$ is such that $\Omega \subset B(x, r)$, then $\sup_{y \in \Omega} d(x, y) \le r$. By taking the infimum in x and r we conclude. This ends the proof of property (i).

To establish (ii), we focus on the case when $Rad(\Omega) < +\infty$, because otherwise Ω is unbounded and (ii) trivially holds. Hence, by relying on (i) we infer that $\Omega \subseteq B := B(x_0, Rad(\Omega))$ for some point $x_0 \in \Omega$. Given $x \in \Omega$ and $r > 0$ satisfying $B(x, r) \subset \Omega$, we have $B(x, r) \subset B(x_0, Rad(\Omega))$. Hence, $d(x, x_0) + r \le Rad(\Omega)$ and hence $r \le Rad(\Omega)$. By taking the supremum in r and x we get finally $rad(\Omega) \le Rad(\Omega)$. The inequality $Rad(\Omega) \le \mathrm{Diam}(\Omega)$ directly follows from the two definitions.

Given (ii), establishing property (iii) amounts to show that

$$rad(\Omega) \ge r_0/4. \tag{3.5}$$

We can assume with no loss of generality that $\partial\Omega \neq \emptyset$, otherwise $\Omega = \mathbb{R}^N$ and (3.5) trivially holds in this case (we recall that the case $\Omega = \emptyset$ is ruled out by the definition of Reifenberg-flat domain).

Hence, we fix $y \in \partial\Omega$, denote by $P(y, r_0)$ the hyperplane in the definition and let $\vec{\nu}$ be its normal vector. We choose the orientation of $\vec{\nu}$ in such a way that

$$\{z + t\nu : \ z \in P(y, r_0), \ t \ge 2\varepsilon r\} \cap B(y, r_0) \subseteq \Omega. \tag{3.6}$$

Since $d_H(P(y, r_0) \cap B(y, r_0), \partial\Omega \cap B(y, r_0)) \le \varepsilon r$, then from (3.6) we infer that actually

$$\{z + t\nu : \ z \in P(y, r_0), \ t \ge \varepsilon r\} \cap B(y, r_0) \subseteq \Omega.$$

By recalling $\varepsilon < 1/2$, we infer that there is $x \in \Omega$ such that $B(x, r_0/4) \subset \Omega$ and this establishes (3.5). □

3.2. **Extension properties and applications.** The following extension property of Reifenberg flat domains is established by combining Theorem 2 above with Jones'analysis (Theorem 1 in [18]).

Corollary 8. *Let $\Omega \subseteq \mathbb{R}^N$ be a connected, (ε, r_0)-Reifenberg-flat domain. If $\varepsilon \le 1/600$, then, for every $p \in [1, +\infty]$, there is an extension operator $E : W^{1,p}(\Omega) \to W^{1,p}(\mathbb{R}^N)$ satisfying*

$$\|E(u)\|_{W^{1,p}(\mathbb{R}^N)} \le C \|u\|_{W^{1,p}(\Omega)},$$

where the constant C only depends on N, p, and r_0.

Proof. The corollary is a direct application of [18, Theorem 1].

The only nontrivial point we have to address is that, in general, the norm of the extension operator E depends on $Rad(\Omega)$, see for examples the statements of Jones' Theorem provided in the paper by Chua [7] and in the very recent preprint by Brewster, D. Mitrea, I. Mitrea and M. Mitrea [1]. Note that in Jones' original statement the dependence on the radius was not mentioned because the radius was fixed (see the remark at the top of page 76 in [18]).

However, by applying for example the remarks in [1, pages 9 and 10] to Reifenberg-flat domains, we get that the norm of E is bounded by $C(N, p, r_0, M)$ if $1/Rad(\Omega) \leq M$. By recalling that $r_0/4 \leq Rad(\Omega)$, we finally infer that the bound on the norm of the extension operator only depends on N, p and r_0 and this concludes the proof. □

Remark 9. To simplify the exposition, we chose to only state the extension property for classical Sobolev Spaces. However, the extension property also applies to other classes of spaces. For instance, Chua [7, 8, 9], extended Jones' Theorem to weighted Sobolev spaces. These spaces are defined by replacing the Lebesgue measure by a weighted measure ωdx, where ω is a function satisfying suitable growth conditions and Poincaré inequalities. Also, Christ [6] established the extension property for Sobolev spaces of fractional order.

The results of both Christ [6] and Chua [7, 8, 9] apply to Jones-flat domains, hence by relying on Theorem 2 we infer that they apply to $(1/600, r_0)$-Reifenberg-flat domains as well.

As a consequence of Corollary 8 we get that the classical Rellich-Kondrachov Theorem holds in Reifenberg-flat domains.

Proposition 10. *Let $\Omega \subseteq \mathbb{R}^N$ be a bounded, connected (ε, r_0)-Reifenberg-flat domain and assume $0 < \varepsilon \leq 1/600$.*

If $1 \leq p < N$, set $p^* := \dfrac{Np}{N-p}$. Then the Sobolev space $W^{1,p}(\Omega)$ is continuously embedded in the space $L^{p^*}(\Omega)$ and is compactly embedded in $L^q(\Omega)$ for every $1 \leq q < p^*$.

If $p \geq N$, then the Sobolev space $W^{1,N}(\Omega)$ is continuously embedded in the space $L^\infty(\Omega)$ and is compactly embedded $L^q(\Omega)$ for every $q \in [1, +\infty[$.

Also, the norm of the above embedding operators only depends on N, r_0, q, p and $Rad(\Omega)$.

Proof. We first use the extension operator provided by Corollary 8 and then we apply the classical Embedding Theorem in a ball of radius $Rad(\Omega)$ containing Ω (see property (i) in the statement of Lemma 7). □

As an example of application of Proposition 10, we establish a uniform bound on the L^∞ norm of Neumann eigenfunctions defined in Reifenberg-flat domains. We use this bound in the companion paper [26]. Here is the precise statement. We recall that we term "Neumann eigenfunction" an eigenfunction for the Laplace operator subject to homogeneous Neumann conditions on the boundary of the domain.

Proposition 11. *Let $\Omega \subseteq \mathbb{R}^N$ be a bounded, connected, (ε, r_0)-Reifenberg-flat domain and let u be a Neumann eigenfunction associated to the eigenvalue μ. If $\varepsilon \leq 1/600$, then u is bounded and*

$$\|u\|_{L^\infty(\Omega)} \leq C(1+\sqrt{\mu})^{\gamma(N)} \|u\|_{L^2(\Omega)}, \tag{3.7}$$

where $\gamma(N) = \max\left\{\frac{N}{2}, \frac{2}{N-1}\right\}$ and $C = C(N, r_0, Rad(\Omega))$.

Proof. By using classical techniques coming from the regularity theory for elliptic operators, Ross [29, Proposition 3.1] established (3.7) in the case of Lipschitz domains. However, in [29] the only reason why one needs the regularity assumption on the domain Ω is to use the Sobolev inequality

$$\|u\|_{L^{2^*}(\Omega)} \leq C(\|u\|_{L^2(\Omega)} + \|\nabla u\|_{L^2(\Omega)}), \qquad C = C(N, r_0, Rad(\Omega)) \tag{3.8}$$

as the starting point for a bootstrap argument. Since Proposition 10 states that (3.8) holds if Ω is a bounded Reifenberg-flat domain, then the proof in [29] can be extended to the case of Reifenberg-flat domains. □

Remark 12. An inequality similar to (3.7) holds for Dirichlet eigenfunctions. We emphasize that the boundedness of Dirichlet eigenfunctions, unlike the boundedness of Neumann eigenfunctions, does not require any regularity assumption on the domain Ω, see for instance [14, Lemma 3.1.] for a precise statement.

4. Connected components of Reifenberg-flat domains

In the previous section we have always assumed that the domain Ω is connected. We now show that the results we have established can be extended to general (i.e., not necessarily connected) Reifenberg-flat domains. Although extension of the result of Jones [18] to non-connected domains were already widely known in the literature, we decided to provide here a self-contained proof. In this way, we obtain results on the structure of Reifenberg-flat domains that may be of independent interest.

We first show that any sufficiently flat Reifenberg-flat domain is finitely connected and we establish a quantitative bound on the Hausdorff distance between two connected components.

Proposition 13. *Let* $\Omega \subseteq \mathbb{R}^N$ *be a bounded,* (ε, r_0)*-Reifenberg flat domain and we assume* $\varepsilon \le 20^{-N}$*. Then* Ω *has a finite number of nonempty, open and disjoint connected components* $U_1, \dots, U_n$*, where*

$$n \le \frac{20^N}{\omega_N} \frac{|\Omega|}{r_0^N}. \tag{4.1}$$

Moreover, if $i \neq j$*, then for every* $z \in \partial U_i$ *we have*

$$d(z, U_j) > r_0/70. \tag{4.2}$$

Proof. We proceed according to the following steps.
◇ Step 1 We recall that any nonempty open set $\Omega \subseteq \mathbb{R}^N$ can be decomposed as

$$\Omega := \bigcup_{i \in I} U_i, \tag{4.3}$$

where the connected components U_i satisfy

- for every $i \in I$, U_i is a nonempty, open, arcwise connected set which is also closed in Ω. Hence, in particular, $\partial U_i \subseteq \partial\Omega$.
- $U_i \cap U_j = \emptyset$ if $i \neq j$.

Indeed, for any $x \in \Omega$ we can define

$$U_x := \big\{ y \in \Omega : \text{ there is a continuous curve } \gamma : [0,1] \to \Omega \text{ such that } \gamma(0) = x \text{ and } \gamma(1) = y \big\}$$

and observe that any U_x is a nonempty, open, arcwise connected set which is also closed in Ω. Also, given two points $x, y \in \mathbb{R}^N$, we have either $U_x = U_y$ or $U_x \cap U_y = \emptyset$.
◇ Step 2 Let Ω as in the statement of the proposition, and let the family $\{U_i\}_{i \in I}$ be as in (4.3). We fix $i \in I$ and we prove that $|U_i| \ge C(r_0, N)$. This straightforwardly implies that $\sharp I \le C(|\Omega|, r_0, N)$.

Since U_i is bounded, then $\partial U_i \neq \emptyset$: hence, we can fix a point $\tilde{x} \in \partial U_i$, and a sequence $\{x_n\}_{n \in \mathbb{N}}$ such that $x_n \in U_i$ and $x_n \to \tilde{x}$ as $n \to +\infty$. We recall that $\partial U_i \subseteq \partial\Omega$ and we infer that, for any $n \in \mathbb{N}$, the following chain of inequalities holds:

$$d(x_n, \partial U_i) = d(x_n, U_i^c) \le d(x_n, \Omega^c) = d(x_n, \partial\Omega) \le d(x_n, \partial U_i),$$

which implies $d(x_n, \Omega^c) = d(x_n, \partial U_i)$. We fix n sufficiently large such that $d(x_n, \tilde{x}) \le r_0/7$, so that

$$d(x_n, \Omega^c) = d(x_n, \partial U_i) \le r_0/7.$$

We term $\Gamma := \gamma_{x_n, r_0/7}$ the polygonal curve constructed as in Step 2 of the proof of Theorem 2 and we observe that, if $\varepsilon \le 1/32$, then (2.7) holds and $\Gamma \subseteq \Omega$ and hence, by definition of U_i, $\Gamma \subseteq U_i$. We use the same notation as in STEP 1 of the proof of Theorem 2 and we recall that Γ connects x_n to some point $Y(x_0, r_0/7)$, defined with some $x_0 \in \partial\Omega$. Hence, in particular, $Y(x_0, r_0/7) \in U_i$ and this implies that $B^+(x_0, r_0/7) \subseteq U_i$ because $B^+(x_0, r_0/7)$ is connected. This finally yields

$$|U_i| \ge |B^+(x_0, r_0/7)| \ge \omega_N \Big(\frac{r_0}{14}(1-2\varepsilon)\Big)^N \ge \omega_N \Big(\frac{9r_0}{140}\Big)^N \ge \omega_N \Big(\frac{r_0}{20}\Big)^N,$$

because $\varepsilon \le 1/20$. We deduce that

$$\sharp I \le \frac{20^N}{\omega_N} \frac{|\Omega|}{r_0^N}.$$

◇ STEP 3 We establish the separation property (4.2).

We set $r_1 := r_0/70$ and we argue by contradiction, assuming that there are $z \in \partial U_i$, $y \in \partial U_j$ such that

$$d(z, U_j) = d(z, \partial U_j) = d(z, y) \le r_1.$$

Let $\{z_n\}_{n\in\mathbb{N}}$ and $\{y_n\}_{n\in\mathbb{N}}$ be sequences in U_i and U_j converging to z and y, respectively. We fix n sufficiently large such that

$$d(z_n, \partial U_i) \le d(z_n, z) \le r_1 \le r_0/14$$

and we term $\bar z$ be a point in ∂U_i satisfying $d(z_n, \bar z) = d(z_n, \partial U_i)$ (if there is more than one such $\bar z$, we arbitrarily fix one). By arguing as in STEP 2, we infer that $B^+(\bar z, r_0/14) \subseteq U_i$. Next, we do the same for U_j, namely we fix m sufficiently large that

$$d(y_m, \partial U_j) \le d(y_m, y) \le r_1 \le r_0/7,$$

we let $\bar y$ be a point in ∂U_j satisfying $d(y_m, \bar y) = d(y_m, \partial U_j)$ and, by arguing as in STEP 2, we get that $B^+(\bar y, r_0/7) \subseteq U_j$. Also, we note that

$$d(\bar z, \bar y) \le d(\bar z, z_n) + d(z_n, z) + d(z, y) + d(y, y_m) + d(y_m, \bar y) \le 5r_1.$$

Since $r_1 = r_0/70$, then $B^+(\bar z, r_0/14) \subseteq B(\bar z, r_0/14) \subseteq B(\bar y, r_0/7)$. We observe that

$$B^+(\bar z, r_0/14) \cap B^-(\bar y, r_0/7) = \emptyset \tag{4.4}$$

since by construction $B^+(\bar z, r_0/14) \subseteq \Omega$ and $B^-(\bar y, r_0/7) \subseteq \Omega^c$. Also, by recalling that

$$B^+(\bar z, r_0/14) \subseteq U_i, \qquad B^+(\bar y, r_0/7) \subseteq U_j \quad \text{and} \quad U_i \cap U_j = \emptyset,$$

we have that

$$B^+(\bar z, r_0/14) \cap B^+(\bar y, r_0/7) = \emptyset \tag{4.5}$$

By combining (4.4) and (4.5) we get

$$B^+(\bar z, r_0/14) \subseteq B(\bar y, r_0/7) \setminus \big(B^+(\bar y, r_0/7) \cup B^-(\bar y, r_0/7)\big). \tag{4.6}$$

We now use the inequality

$$\omega_N \geq \omega_{N-1} \frac{1}{2^{N-1}}, \tag{4.7}$$

which will be proven later. By relying on (4.7) and by recalling that $\varepsilon \leq 20^{-N} \leq 1/20$ we obtain

$$|B^+(\bar z, r_0/14)| \geq \omega_N \left(\frac{r_0}{28}(1-2\varepsilon)\right)^N \geq 2\omega_{N-1}\left(\frac{9r_0}{560}\right)^N$$

and

$$\left| B(\bar y, r_0/7) \setminus \big(B^+(\bar y, r_0/7) \cup B^-(\bar y, r_0/7)\big) \right| \leq 4\varepsilon \omega_{N-1}\left(\frac{r_0}{7}\right)^N \leq 2\omega_{N-1}\left(\frac{2r_0}{140}\right)^N,$$

which contradicts (4.6) since $2/140 < 9/560$.

To finish the proof we are thus left to establish (4.7). To do this, we use the relation

$$\omega_N = \omega_{N-1} \int_{-1}^{1} \big(\sqrt{1-x^2}\big)^{N-1} dx.$$

This implies that, for any $\lambda \in (0,1)$, we have

$$\begin{aligned} \omega_N &\geq \omega_{N-1} 2 \int_0^{\lambda} \big(\sqrt{1-x^2}\big)^{N-1} dx \\ &\geq \omega_{N-1} 2\lambda \big(\sqrt{1-\lambda^2}\big)^{N-1} \end{aligned}$$

By choosing $\lambda = \sqrt{3}/2$ we obtain the inequality

$$\omega_N \geq \omega_{N-1} \frac{\sqrt{3}}{2^{N-1}} \geq \omega_{N-1} \frac{1}{2^{N-1}},$$

and this concludes the proof. □

By relying on Proposition 13 we can now remove the connectedness assumption in the statement of Proposition 8.

Corollary 14. *Let* $N \ge 2$ *and* $\Omega \subseteq \mathbb{R}^N$ *be a bounded,* (ε, r_0)*-Reifenberg flat domain with* $\varepsilon \le \min(20^{-N}, 1/600)$*. Then for every* $p \in [1, +\infty]$ *there is an extension operator*

$$E : W^{1,p}(\Omega) \to W^{1,p}(\mathbb{R}^N) \tag{4.8}$$

whose norm is bounded by a constant which only depends on N*,* p*, and* r_0*.*

Proof. We employ the same notation as in the statement of Proposition 13 and we fix a connected component U_i. By recalling that $\partial U_i \subseteq \partial\Omega$ and the separation property (4.2), we infer that U_i is itself a $(\varepsilon, r_0/140)$-Reifenberg flat domain. Since by definition U_i is connected, we can apply Proposition 8 which says that, for every $p \in [1, +\infty]$, there is an extension operator

$$E_i : W^{1,p}(U_i) \to W^{1,p}(\mathbb{R}^N)$$

whose norm is bounded by a constant which only depends on N, p and r_0.

In order to "glue together" the extension operators $E_1, \dots, E_n$ we proceed as follows. Given $i = 1, \dots, n$, we set $\delta := r_0/280$ and we introduce the notation

$$U_i^\delta := \big\{ x \in \mathbb{R}^N : \ d(x, U_i) < \delta \big\}.$$

Note that the separation property (4.2) implies that $U_i^{2\delta} \cap U_j^{2\delta} = \emptyset$ if $i \neq j$.

We now construct suitable cut-off functions φ_i, $i = 1, \dots, n$. Let $\ell : [0, +\infty[\to [0, 1]$ be the auxiliary function defined by setting

$$\ell(t) := \begin{cases} 1 & \text{if } t \le \delta \\ 1 + \dfrac{\delta - t}{\delta} & \text{if } \delta \le t \le 2\delta \\ 0 & \text{if } t \ge 2\delta \end{cases}$$

We set $\varphi_i(x) := \ell\big(d(x, U_i)\big)$ and we recall that the function $x \mapsto d(x, U_i)$ is 1-Lipschitz and that $\delta = r_0/280$. Hence, the function φ_i satisfies the following properties:

$$0 \le \varphi_i(x) \le 1, \ \ |\nabla \varphi_i(x)| \le C(r_0) \ \ \forall x \in \mathbb{R}^N, \quad \varphi_i \equiv 1 \text{ on } U_i, \quad \varphi_i \equiv 0 \text{ on } \mathbb{R}^N \setminus U_i^{2\delta}. \tag{4.9}$$

We then define $E : W^{1,p}(\Omega) \to W^{1,p}(\mathbb{R}^N)$ by setting

$$E(u) := \sum_{i=1}^n E_i(u)(x) \varphi_i(x).$$

We recall that the sets $U_1, \dots, U_n$ are all pairwise disjoint, we focus on the case $p < +\infty$ and we get

$$\begin{aligned}
\|E(u)\|_{L^p(\mathbb{R}^N)} &= \left(\int_{\mathbb{R}^N} \left| \sum_{i=1}^n E_i(u)(x)\varphi_i(x) dx \right|^p \right)^{1/p} \leq \sum_{i=1}^n \left(\int_{U_i^{2\delta}} |E_i(u)(x)\varphi_i(x)|^p dx \right)^{1/p} \\
&\leq \sum_{i=1}^n \|E_i(u)\|_{L^p(\mathbb{R}^N)} \leq \sum_{i=1}^n C(N,p,r_0)\|u\|_{W^{1,p}(U_i)} \\
&\leq C(N,p,r_0)\|u\|_{W^{1,p}(\Omega)}.
\end{aligned}$$

Also, by using the bound on $|\nabla\varphi_i|$ provided by (4.9), we get

$$\begin{aligned}
\|\nabla E(u)\|_{L^p(\mathbb{R}^N)} &= \left(\int_{\mathbb{R}^N} \left| \sum_{i=1}^n \big(\nabla E_i(u)(x)\varphi_i(x) + E_i(u)(x)\nabla\varphi_i(x)\big) dx \right|^p \right)^{1/p} \\
&\leq \sum_{i=1}^n \left(\int_{U_i^{2\delta}} |\nabla E_i(u)(x)\varphi_i(x)|^p dx \right)^{1/p} + \sum_{i=1}^n \left(\int_{U_i^{2\delta}} |E_i(u)(x)\nabla\varphi_i(x)|^p dx \right)^{1/p} \\
&\leq \sum_{i=1}^n \|\nabla E_i(u)\|_{L^p(\mathbb{R}^N)} + C(r_0)\sum_{i=1}^n \|E_i(u)\|_{L^p(\mathbb{R}^N)} \\
&\leq C(N,p,r_0)\|u\|_{W^{1,p}(\Omega)}.
\end{aligned}$$

The proof in the case $p = \infty$ is a direct consequence of the bounds on the norm of E_i and on the uniform norms of φ_i and $\nabla\varphi_i$. This concludes the proof of the corollary. □

5. On the Hausdorff distance between Reifenberg-flat domains

We end this paper by comparing different ways of measuring the "distance" between Reifenberg-flat domains.

5.1. **Comparison between different Hausdorff distances.** This subsections aims at comparing the Hausdorff distances $d_H(X,Y)$, $d_H(X^c,Y^c)$ and $d_H(\partial X, \partial Y)$, where X and Y are subsets of $\mathbb{R}^N$.

First, we exhibit two examples showing that, in general, neither $d_H(X,Y)$ controls $d_H(X^c,Y^c)$ nor $d_H(X^c,Y^c)$ controls $d_H(X,Y)$. We term $B := B(1,\vec{0})$ the unit ball and we consider the two perturbations A and C as represented in Figure 2.

Next, we exhibit an example showing that, in general, $d_H(\partial X, \partial Y)$ controls neither $d_H(X,Y)$ nor $d_H(X^c,Y^c)$. Let $X := B(R,\vec{0})$ and $Y := B(R+\varepsilon,\vec{0}) \setminus B(R,\vec{0})$, then

$$\varepsilon = d_H(\partial X, \partial Y) << d_H(X,Y) = d_H(X^c,Y^c) = R.$$

$\varepsilon \simeq d_H(A^c, B^c) << d_H(A, B) \simeq 1$ $\qquad$ $\varepsilon \simeq d_H(C, B) << d_H(C^c, B^c) \simeq 1$

Figure 2

Also, note that the examples represented in Figure 2 show that, in general, neither $d_H(X, Y)$ nor $d_H(X^c, Y^c)$ controls $d_H(\partial X, \partial Y)$. Indeed, $d_H(\partial A, \partial B) \simeq 1$ and $d_H(\partial C, \partial B) \simeq 1$.

However, if X and Y are two sufficiently close Reifenberg-flat domains, then we have the following result.

Lemma 15. *Let X and Y be two (ε, r_0)-Reifenberg-flat domains satisfying $d_H(\partial X, \partial Y) \leq 2r_0$. Then*

$$d_H(\partial X, \partial Y) \leq \frac{4}{1-2\varepsilon} \min\big\{d_H(X, Y), d_H(X^c, Y^c)\big\}. \tag{5.1}$$

Proof. Just to fix the ideas, assume that $d_H(\partial X, \partial Y) = \sup_{x \in \partial X} d(x, \partial Y)$. Since by assumption $d_H(\partial X, \partial Y) < +\infty$, then for every $h > 0$ there is $x_h \in \partial X$ such that

$$d_H(\partial X, \partial Y) - h \leq d_h := d(x_h, \partial Y) \leq d_H(\partial X, \partial Y).$$

Note that $\partial Y \cap B(x_h, d_h/2) = \emptyset$ and hence either (i) $B(x_h, d_h/2) \subseteq Y$ or (ii) $B(x_h, d_h/2) \subseteq Y^c$.

First, consider case (i): let $P(x_h, d_h/2)$ be the hyperplane prescribed by the definition of Reifenberg flatness, then by Lemma 5 we can choose the orientation of the normal vector ν in such a way that

$$B^-(x_h, d_h/2) := \{z + t\nu : \ z \in P(x_h, d_h/2),\ t \geq \varepsilon d_h\} \cap B(x_h, d_h/2) \subseteq X^c$$

and

$$B^+(x_h, d_h/2) := \{z - t\nu : \ z \in P(x_h, d_h/2),\ t \geq \varepsilon d_h\} \cap B(x_h, d_h/2) \subseteq X.$$

Fix the point

$$\bar{z} := x_h + \frac{(1+2\varepsilon)d_h}{4}\nu,$$

then we have

$$B\left(\bar{z}, \frac{(1-2\varepsilon)d_h}{4}\right) \subseteq B^-(x_h, d_h/2) \subseteq X^c \cap Y$$

and hence

$$d_H(X^c, Y^c) \geq \sup_{z \in X^c} d(z, Y^c) \geq d(\bar{z}, Y^c) \geq \frac{(1-2\varepsilon)d_h}{4}$$

and

$$d_H(X,Y) \geq \sup_{z\in Y} d(z,X) \geq d(\bar{z},X) \geq \frac{(1-2\varepsilon)d_h}{4}.$$

Since case (ii) can be tackled in an entirely similar way, by the arbitrariness of h we deduce that

$$d_H(\partial X,\partial Y) \leq \frac{4}{1-2\varepsilon} d_H(X,Y). \tag{5.2}$$

The proof of (5.1) is concluded by making the following observations:

- if X is an (ε, r_0)-Reifenberg flat domain, then X^c is also an (ε, r_0)-Reifenberg flat domain.
- $\partial X = \partial X^c$ and $\partial Y = \partial Y^c$.

Hence, by replacing in (5.2) X with X^c and Y with Y^c we obtain (5.1). □

5.2. **Comparison between the Hausdorff distance and the measure of the symmetric difference.** This subsection aims at comparing the Hausdorff distances $d_H(X,Y)$ and $d_H(X^c,Y^c)$ with the Lebesgue measure of the symmetric difference, $|X\triangle Y|$. As usual, X and Y are subsets of $\mathbb{R}^N$. The results we state are applied in [26] to the stability analysis of the spectrum of the Laplace operator with Neumann boundary conditions.

First, we observe that the examples illustrated in Figure 2 show that, in general, $|X\triangle Y|$ controls neither $d_H(X,Y)$ nor $d_H(X^c,Y^c)$. Indeed, $|A\triangle B| \simeq \varepsilon$ and $|C\triangle B| \simeq \varepsilon$. However, if X and Y are two sufficiently close Reifenberg-flat domains, then the following result hold.

Lemma 16. *Let X and Y be two (ε, r_0)-Reifenberg-flat domains in $\mathbb{R}^N$.*

Then the following implications hold:

(1) *if $d_H(X,Y) \leq 4r_0$, then*

$$d_H(X,Y) \leq \frac{8}{(1-2\varepsilon)} \left(\frac{|X\triangle Y|}{\omega_N}\right)^{1/N}. \tag{5.3}$$

(2) *If $d_H(X^c,Y^c) \leq 4r_0$, then*

$$d_H(X^c,Y^c) \leq \frac{8}{(1-2\varepsilon)} \left(\frac{|X\triangle Y|}{\omega_N}\right)^{1/N}. \tag{5.4}$$

In both the previous expressions, ω_N denotes the measure of the unit ball in $\mathbb{R}^N$.

Proof. The argument relies on ideas similar to those used in the proof of Lemma 15.

We first establish (5.3). Just to fix the ideas, assume that $d_H(X,Y) = \sup_{x\in X} d(x,Y)$ and note that by assumption $d_H(X,Y) < +\infty$. Hence, for every $h > 0$ there is $x_h \in X$ such that

$$d_H(X,Y) - h \leq d_h := d(x_h,Y) \leq d_H(X,Y)$$

Note that, by the very definition of $d(x_h, Y)$, we have $B(x_h, d_h) \subseteq Y^c$. We now separately consider two cases: if $B(x_h, d_h/2) \subseteq X$, then

$$B(x_h, d_h/2) \subseteq X \cap Y^c \subseteq |X \triangle Y|$$

and hence

$$\omega_N \left(\frac{d_h}{2} \right)^N \leq |X \triangle Y|,$$

and by the arbitrariness of h this implies (5.3).

Hence, we are left to consider the case when there is $x_0 \in B(x_h, d_h/2) \cap \partial X$. We make the following observations: first,

$$B(x_0, d_h/4) \subseteq B(x_h, d_h) \subseteq Y^c. \tag{5.5}$$

Second, since $d_h/4 \leq d_H(X, Y)/4 \leq r_0$, then we can apply the definition of Reifenberg-flatness in the ball $B(x_0, d_h/4)$. Let $P(x_0, d_h/4)$ be the hyperplane provided by property (i) in the definition, and let ν_0 denote the normal vector. By relying on Lemma 5 we infer that we can choose the orientation of ν_0 in such a way that

$$B\left(x_0 + \frac{(1+2\varepsilon)d_h}{8} \nu_0, \frac{(1-2\varepsilon)d_h}{8} \right) \subseteq X \cap B(x_0, d_h/4) \tag{5.6}$$

By combining (5.5) and (5.6) we infer that

$$\omega_N \left(\frac{(1-2\varepsilon)d_h}{8} \right)^N \leq |X \cap Y^c| \leq |X \triangle Y|$$

and by the arbitrariness of h this completes the proof of (5.3).

Estimate (5.4) follows from (5.3) by relying on the following two observations:

- $X \triangle Y = (X^c \cap Y) \cup (X \cap Y^c) = X^c \triangle Y^c$.
- if X is an (ε, r_0)-Reifenberg flat domain, then X^c is also an (ε, r_0)-Reifenberg flat domain.

Hence, by replacing in (5.3) X with X^c and Y with Y^c we get (5.4). □

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