Insensitizing controls with one vanishing component for the Navier-Stokes system

Nicolás Carreño*and Mamadou Gueye †

Abstract

In this paper we prove the existence of insensitizing controls, having one vanishing component, for the local L^2 norm of the solutions of the Navier-Stokes system. This problem can be recast as a null controllability problem for a nonlinear cascade system. We first prove a controllability result, with controls having one vanishing component, for a linear problem. Then, by means of an inverse mapping theorem, we deduce the controllability for the cascade system.

Resumé

Dans ce travail on prouve l'existence de contrôles insensibilisant, ayant une composante nulle, pour la norme L^2 locale de la solution du système de Navier -Stokes. Ce type de problème peut être reformuler comme un problème de contrôlabilité à zéro pour un système en cascade nonlinéaire. On prouve d'abord un résultat de controlabilité, avec un contrôle ayant une composante nulle, pour le problème linéarisé. La controlabilité du système en cascade s'en déduira par des arguments d'inversion locale.

Subject Classification: 34B15, 35Q30, 93C10, 93B05

Keywords: Navier-Stokes system, null controllability, Carleman inequalities, insensitizing controls

1 Introduction

Let Ω be a nonempty bounded connected open subset of $\mathbb{R}^N$ ($N = 2$ or 3) of class C^∞. Let $T > 0$ and let $\omega \subset \Omega$ be a (small) nonempty open subset which is the *control set*. We will use the notation $Q = \Omega \times (0,T)$ and $\Sigma = \partial\Omega \times (0,T)$. Let us also introduce another open set $\mathcal{O} \subset \Omega$ which is called the *observatory* or *observation set*.

Let us recall the definition of some usual spaces in the context of incompressible fluids:

$$V = \{y \in H_0^1(\Omega)^N : \nabla \cdot y = 0 \text{ in } \Omega\}$$

and

$$H = \{y \in L^2(\Omega)^N : \nabla \cdot y = 0 \text{ in } \Omega,\, y \cdot n = 0 \text{ on } \partial\Omega\}.$$

We introduce the following Navier-Stokes control system with incomplete data

$$\begin{cases} y_t - \Delta y + (y\cdot\nabla)y + \nabla p = f + v\mathbb{1}_\omega,\ \nabla\cdot y = 0 & \text{in } Q,\\ y = 0 & \text{on } \Sigma,\\ y(0) = y^0 + \tau\widehat{y}^0 & \text{in } \Omega, \end{cases} \tag{1.1}$$

*Université Pierre et Marie Curie, UMR 7598, Laboratoire Jacques-Louis Lions, F-75005, Paris, France; **ncarreno@ann.jussieu.fr**

†Université Pierre et Marie Curie, UMR 7598, Laboratoire Jacques-Louis Lions, F-75005, Paris, France; **gueye@ann.jussieu.fr**

where $v = (v_j)_{1\le j\le N}$ is the control function, $f \in L^2(Q)^N$ is a given externally applied force and the initial state $y(0)$ is partially unknown in the following sense:

- $y^0 \in H$ is known,
- $\widehat{y}^0 \in H$ is unknown with $\|\widehat{y}^0\|_{L^2(\Omega)^N} = 1$, and
- τ is a small unknown real number.

We observe the solution of system (1.1) via some functional $J_\tau(y)$, which is called the *sentinel*. Here, the sentinel is given by the square of the local L^2-norm of the state variable:

$$J_\tau(y) := \frac{1}{2} \iint\limits_{\mathcal{O}\times(0,T)} |y|^2 dx\, dt. \tag{1.2}$$

The insensitizing control problem is to find v such that the uncertainty in the initial data does not affect the measurement J_τ, at least at the first order, i.e.,

$$\left.\frac{\partial J_\tau(y)}{\partial \tau}\right|_{\tau=0} = 0 \quad \forall \widehat{y}^0 \in L^2(\Omega)^N \quad \text{such that} \quad \|\widehat{y}^0\|_{L^2(\Omega)^N} = 1. \tag{1.3}$$

If (1.3) holds, we say that v insensitizes the functional J_τ. This kind of problem was first considered by J.-L. Lions in [17]. This particular form of the sentinel J_τ allows us to reformulate the insensitizing problem as a controllability problem for a cascade system (for more details, see [2] or [16], for instance). In particular, condition (1.3) is equivalent to $z(0) = 0$ in Ω, where z, together with w, solves the following coupled system:

$$\begin{cases} w_t - \Delta w + (w\cdot\nabla)w + \nabla p^0 = f + v\mathbb{1}_\omega,\ \nabla\cdot w = 0 & \text{in } Q,\\ -z_t - \Delta z + (z\cdot\nabla^t)w - (w\cdot\nabla)z + \nabla q = w\mathbb{1}_{\mathcal{O}},\ \nabla\cdot z = 0 & \text{in } Q,\\ w = z = 0 & \text{on } \Sigma,\\ w(0) = y^0,\ z(T) = 0 & \text{in } \Omega. \end{cases} \tag{1.4}$$

Here, (w, p^0) is the solution of system (1.1) for $\tau = 0$, the equation of z corresponds to a formal adjoint of the equation satisfied by the derivative of y with respect to τ at $\tau = 0$ and we have denoted:

$$((z,\nabla^t)w)_i = \sum_{j=1}^N z_j \partial_i w_j \quad i = 1,\dots,N.$$

Most known results around this type of controllability problem concern parabolic system of the heat kind. In [2], the authors proved the existence of ε-insensitizing controls (i.e., such that $|\partial_\tau J_\tau(y)|_{\tau=0}| \le \varepsilon$) for solutions of a semilinear heat system with C^1 and globally Lipschitz nonlinearities and in [22], the author proved the existence of insensitizing controls for the same system. In [10], the author treated the case of a different type of sentinel, namely the gradient of the solution of a heat equation with potentials.

For the Stokes system, the first results were obtained in [11] when the sentinel is given by (1.2) or by the curl of the solution. In [12], the author proved the existence of insensitizing controls for the Navier-Stokes system. The main goal of this paper is to establish the existence of insensitizing controls for the Navier-Stokes system (1.1) having one vanishing component, that is, $v_i \equiv 0$ for any given $i \in \{1,\dots,N\}$.

In this subject, the first results were obtained in [8] for the local exact controllability to the trajectories of the Navier-Stokes and Boussinesq system when the closure of the control set ω intersects the boundary of Ω. Later, this geometric assumption was removed for the Stokes system in [5], for the local null controllability of the Navier-Stokes system in [4] and in [3] for the Boussinesq system. See also [6] for local null controllability of the 2-dimensional Navier-Stokes system in a torus with controls having one vanishing component.

We will suppose that $\omega \cap \mathcal{O} \neq \emptyset$, which is a condition that has always been imposed as long as insensitizing controls are concerned. However, in [23], it has been proved that this is not a necessary condition for ε-insensitizing controls for some linear parabolic equations (see also [19]).

In [22], the author proved for the linear heat equation that we cannot expect insensitivity to hold for all initial data, except when the control acts everywhere in Ω. Thus, we shall assume that $y^0 \equiv 0$ which is a classical hypothesis in insensitization problems.

The main result is stated in the following theorem:

Theorem 1.1. *Let $i \in \{1, \dots, N\}$ and $m \geq 10$ be a real number. Assume that $\omega \cap \mathcal{O} \neq \emptyset$ and $y^0 \equiv 0$. Then, there exist $\delta > 0$ and $C > 0$, depending on ω, Ω, $\mathcal{O}$ and T, such that for any $f \in L^2(Q)^N$ satisfying $\|e^{C/t^m} f\|_{L^2(Q)^N} < \delta$, there exists a control $v \in L^2(Q)^N$ with $v_i \equiv 0$ such that the corresponding solution (w, z) to (1.4) satisfies $z(0) = 0$ in Ω.*

To prove Theorem 1.1 we follow a standard approach introduced in [9] (see also [4], [7] and [13]). We first deduce a null controllability result for the linear system:

$$\begin{cases} w_t - \Delta w + \nabla p^0 = f^0 + v \mathbb{1}_\omega, \ \nabla \cdot w = 0 & \text{in } Q, \\ -z_t - \Delta z + \nabla q = f^1 + w \mathbb{1}_{\mathcal{O}}, \ \nabla \cdot z = 0 & \text{in } Q, \\ w = z = 0 & \text{on } \Sigma, \\ w(0) = 0, \ z(T) = 0 & \text{in } \Omega, \end{cases} \tag{1.5}$$

where f^0 and f^1 will be taken to decrease exponentially to zero at $t = 0$.

The main tool to prove this controllability result for system (1.5), and the second main result of this paper, is a suitable Carleman estimate for the solutions of its adjoint system, namely,

$$\begin{cases} -\varphi_t - \Delta \varphi + \nabla \pi = g^0 + \psi \mathbb{1}_{\mathcal{O}}, \ \nabla \cdot \varphi = 0, & \text{in } Q, \\ \psi_t - \Delta \psi + \nabla \kappa = g^1, \ \nabla \cdot \psi = 0, & \text{in } Q, \\ \varphi = 0, \ \psi = 0, & \text{on } \Sigma \\ \varphi(T) = 0, \ \psi(0) = \psi^0, & \text{in } \Omega, \end{cases} \tag{1.6}$$

where $g^0 \in L^2(Q)^N$, $g^1 \in L^2(0, T; V)$ and $\psi^0 \in H$. In fact, this Carleman inequality is of the form

$$\iint\limits_Q \widetilde{\rho}_1(t)(|\varphi|^2 + |\psi|^2) dx dt \leq C \left(\iint\limits_Q \widetilde{\rho}_2(t)(|g^0|^2 + |g^1|^2 + |\nabla g^1|^2) dx dt \right.$$
$$\left. + \sum_{j=1, j \neq i}^{N} \iint\limits_{\omega \times (0,T)} \widetilde{\rho}_3(t) |\varphi_j|^2 dx dt \right), \tag{1.7}$$

where $\widetilde{\rho}_k(t)$, $k \in \{1, 2, 3\}$, are positive weight functions and $C > 0$ only depends on Ω, ω, $\mathcal{O}$ and T.

The idea to prove (1.7) is to combine some observability inequalities for both φ and ψ and try to eliminate the local term in ψ. When proving a Carleman inequality for ψ, we will avoid having a local term of the kind

$$\iint\limits_{\widetilde{\omega} \times (0,T)} \widetilde{\rho}_4(x, t) |\psi|^2 dx \, dt.$$

The reason is that when we estimate this integral in terms of local terms in φ (using the equation of φ in (1.6)), a local term of the pressure π appears.

To overcome this issue, the author in [11] (see also [12]) proves a Carleman inequality for ψ with a local term in $\nabla \times \psi$. Even though this idea gets rid of the pressure term (again, using the equation satisfied by φ), it makes appear local terms in ψ_i and some of its derivatives, which for our purposes is not good.

This motivates finding a Carleman inequality for ψ with local terms in $\Delta\psi_j$, $j \neq i$. We base our strategy in the method introduced in [5] and the ideas in [4]. We decompose ψ in the form $\overline{\psi}+\check{\psi}$ in such a way that the pressure term associated to the more regular function, say $\check{\psi}$, is a harmonic function in Q. Then we apply the operator $(\nabla\nabla\Delta\cdot)$ to the equations satisfied by $\check{\psi}_j$, $j \neq i$, to have an equation in some derivatives of $\check{\psi}_j$ which do not depend neither on $\check{\psi}_i$ nor on the pressure. Note that by doing this, we lose all boundary conditions. At this point we would have

$$\sum_{j=1,j\neq i}^{N} \iint\limits_Q \widetilde{\rho}_6(x,t)|\Delta\check{\psi}_j|^2\,dx\,dt \\ \leq C\left(\iint\limits_Q \widetilde{\rho}_7(t)|g^1|^2 dx\,dt + \sum_{j=1,j\neq i}^{N} \iint\limits_{\widetilde{\omega}\times(0,T)} \widetilde{\rho}_6(x,t)|\Delta\check{\psi}_j|^2 dx\,dt + b.t.\right),$$

where "b.t." stands for boundary terms which have to be estimated. This is done using regularity estimates for the Stokes system. This will give additional integrals on $\check{\psi}_i$, g^1 and ∇g^1:

$$\sum_{j=1,j\neq i}^{N} \iint\limits_Q \widetilde{\rho}_6(x,t)|\Delta\check{\psi}_j|^2\,dx\,dt \leq C\left(\iint\limits_Q \widetilde{\rho}_7(t)(|g^1|^2+|\nabla g^1|^2)dx\,dt \right.\\ \left. + \sum_{j=1,j\neq i}^{N} \iint\limits_{\widetilde{\omega}\times(0,T)} \widetilde{\rho}_6(x,t)|\Delta\check{\psi}_j|^2 dx\,dt + \iint\limits_Q \widetilde{\rho}_8(t)|\check{\psi}_i|^2\,dx\,dt\right), \quad (1.8)$$

Now, using the divergence-free condition and the properties of the weight functions one can incorporate in the left-hand side of (1.8) a global term in $\check{\psi}$, which will be useful to absorb the last term in the right-hand side.

For φ we use a Carleman estimate proved in [4]:

$$\iint\limits_Q \widetilde{\rho}_9(t)|\varphi|^2 dx\,dt \leq C\left(\iint\limits_{\mathcal{O}\times(0,T)} \widetilde{\rho}_{10}(t)|\psi|^2 dx\,dt + \iint\limits_Q \widetilde{\rho}_{10}(t)|g^0|^2 dx\,dt \right.\\ \left. + \sum_{j=1,j\neq i}^{N} \iint\limits_{\omega\times(0,T)} \widetilde{\rho}_{11}(t)|\varphi_j|^2 dx\,dt\right).$$

Provided that $\widetilde{\rho}_{10} \leq \widetilde{\rho}_6$, we can absorb the first term in the right-hand side by the left-hand side of (1.8). At this point we arrive to:

$$\iint\limits_Q \widetilde{\rho}_1(t)(|\varphi|^2+|\psi|^2)dx\,dt \le C\left(\iint\limits_Q \widetilde{\rho}_2(t)(|g^0|^2+|g^1|^2+|\nabla g^1|^2)dx\,dt\right.$$
$$\left. + \sum_{j=1,j\neq i}^{N}\left[\iint\limits_{\widetilde{\omega}\times(0,T)} \widetilde{\rho}_6(t,x)|\Delta\psi_j|^2dx\,dt + \iint\limits_{\omega\times(0,T)} \widetilde{\rho}_{11}(t)|\varphi_j|^2dx\,dt\right]\right).$$

Finally, we estimate the local terms in $\Delta\psi_j$ in terms of local integrals of φ_j using that

$$\Delta\psi_j = -(\Delta\varphi_j)_t - \Delta(\Delta\varphi_j) + \partial_j\Delta\cdot g^0 - \Delta g_j^0 \text{ in } \widetilde{\omega}\times(0,T),$$

provided that $\widetilde{\omega}\subset\mathcal{O}$.

The paper is organized as follows. In section 2, we present the technical results needed to prove inequality (1.7). In section 3, we prove a new Carleman inequality for the solutions of (1.6). In section 4, we deal with the null controllability of the linear system (1.5). Finally, in section 5 we prove Theorem 1.1.

2 Technical results

In this section we present all the technical results we need to prove inequality (1.7). It is based on suitable global Carleman estimates. In order to establish these inequalities, we are going to introduce some weight functions. Let ω_0 be a nonempty open subset of $\mathbb{R}^N$ such that $\omega_0 \Subset \omega\cap\mathcal{O}$ and $\eta\in C^2(\overline{\Omega})$ such that

$$|\nabla\eta|>0 \text{ in } \overline{\Omega\setminus\omega_0},\ \eta>0 \text{ in } \Omega \text{ and } \eta\equiv 0 \text{ on } \partial\Omega. \tag{2.1}$$

The existence of such a function η is given in [9]. Let also $\ell\in C^\infty([0,T])$ be a positive function in $(0,T)$ satisfying

$$\begin{aligned}&\ell(t)=t \quad \forall\, t\in[0,T/4],\ \ell(t)=T-t \quad \forall\, t\in[3T/4,T],\\ &\ell(t)\le \ell(T/2),\ \forall\, t\in[0,T].\end{aligned} \tag{2.2}$$

Then, for all $\lambda\ge 1$ and $m\ge 10$ we consider the following weight functions:

$$\begin{aligned}&\alpha(x,t)=\frac{e^{2\lambda\|\eta\|_\infty}-e^{\lambda\eta(x)}}{\ell(t)^m},\ \xi(x,t)=\frac{e^{\lambda\eta(x)}}{\ell(t)^m},\\ &\alpha^*(t)=\max_{x\in\overline{\Omega}}\alpha(x,t),\ \xi^*(t)=\min_{x\in\overline{\Omega}}\xi(x,t),\\ &\widehat{\alpha}(t)=\min_{x\in\overline{\Omega}}\alpha(x,t),\ \widehat{\xi}(t)=\max_{x\in\overline{\Omega}}\xi(x,t).\end{aligned} \tag{2.3}$$

The first result is a Carleman inequality for the Stokes system with right-hand side in $L^2(Q)^N$ proved in [4, Proposition 2.1] .

Lemma 2.1. *There exists a constant $\widehat{\lambda}_0>0$ such that for any $\lambda\ge\widehat{\lambda}_0$ there exists $C>0$ depending only on λ, Ω, ω, η and ℓ such that for any $i\in\{1,\dots,N\}$, any $g\in L^2(Q)^N$ and any $u^0\in H$, the solution of*

$$\begin{cases} u_t-\Delta u+\nabla p=g,\ \nabla\cdot u=0, & \textit{in } Q,\\ u=0, & \textit{on } \Sigma\\ u(0)=u^0, & \textit{in } \Omega,\end{cases}$$

satisfies

$$s^3 \sum_{j=1, j\neq i}^{N} \iint\limits_Q e^{-8/3s\alpha - 4s\alpha^*} \xi^3 |\Delta u_j|^2 dx\, dt + s^4 \iint\limits_Q e^{-20/3s\alpha^*} (\xi^*)^4 |u|^2 dx\, dt$$
$$\leq C \left(\iint\limits_Q e^{-4s\alpha^*} |g|^2 dx\, dt + s^7 \sum_{j=1, j\neq i}^{N} \iint\limits_{\omega\times(0,T)} e^{-8/3s\widehat{\alpha} - 4s\alpha^*} \widehat{\xi}^7 |u_j|^2 dx\, dt \right) \quad (2.4)$$

for every $s \geq C$.

Remark 2.2. *In [4], the weight functions* α *and* ξ *are given for* $m = 8$*, but the proof also holds for any* $m \geq 8$*. Additionally, the first term in the left-hand side of* (2.4) *does not appear explicitly in Proposition 2.1 of [4]. However, it is easily seen from its proof that this term can be added.*

The following result is a Carleman inequality for parabolic equations with non-homogeneous boundary conditions proved in [14, Theorem 2.1] :

Lemma 2.3. *Let* $f_0, f_1, \dots, f_N \in L^2(Q)$*. There exists a constant* $\widehat{\lambda}_1 > 0$ *such that for any* $\lambda \geq \widehat{\lambda}_1$ *there exists* $C > 0$ *depending only on* λ, Ω, ω_0, η *and* ℓ *such that for every* $u \in L^2(0,T; H^1(\Omega)) \cap H^1(0,T; H^{-1}(\Omega))$ *satisfying*

$$u_t - \Delta u = f_0 + \sum_{j=1}^{N} \partial_j f_j \ \text{ in } Q,$$

we have

$$s^{-1} \iint\limits_Q e^{-3s\alpha} \xi^{-1} |\nabla u|^2 dx\, dt + s \iint\limits_Q e^{-3s\alpha} \xi |u|^2 dx\, dt \leq C \left(s \iint\limits_{\omega_0\times(0,T)} e^{-3s\alpha} \xi |u|^2 dx\, dt \right.$$
$$+ s^{-1/2} \left\| e^{-3/2s\alpha^*} (\xi^*)^{-1/4} u \right\|^2_{H^{\frac14,\frac12}(\Sigma)} + s^{-1/2} \left\| e^{-3/2s\alpha^*} (\xi^*)^{-1/4+1/m} u \right\|^2_{L^2(\Sigma)}$$
$$\left. + s^{-2} \iint\limits_Q e^{-3s\alpha} \xi^{-2} |f_0|^2 dx\, dt + \sum_{j=1}^{N} \iint\limits_Q e^{-3s\alpha} |f_j|^2 dx\, dt \right), \quad (2.5)$$

for every $s \geq C$.

Recall that

$$\|u\|_{H^{\frac14,\frac12}(\Sigma)} = \left(\|u\|^2_{H^{1/4}(0,T;L^2(\partial\Omega))} + \|u\|^2_{L^2(0,T;H^{1/2}(\partial\Omega))} \right)^{1/2}.$$

Remark 2.4. *The usual notation for this space is actually* $H^{\frac12,\frac14}(\Sigma)$ *(see, for instance, [18]). However, we follow the notation used in [14].*

The next technical result corresponds to Lemma 3 in [5].

Lemma 2.5. *Let $r \in \mathbb{R}$. There exists $C > 0$ depending only on Ω, ω_0, η and ℓ such that, for every $T > 0$ and every $u \in L^2(0,T;H^1(\Omega))$,*

$$
\begin{aligned}
& s^2 \iint\limits_Q e^{-3s\alpha}\xi^{r+2}|u|^2 dx\, dt \\
& \le C \left(\iint\limits_Q e^{-3s\alpha}\xi^{r}|\nabla u|^2 dx\, dt + s^2 \iint\limits_{\omega_0\times(0,T)} e^{-3s\alpha}\xi^{r+2}|u|^2 dx\, dt \right), \qquad (2.6)
\end{aligned}
$$

for every $s \ge C$.

The next result concerns the regularity of the solutions to the Stokes system which can be found in [15] (see also [20]):

Lemma 2.6. *For every $T > 0$ and every $f \in L^2(Q)^N$, there exists a unique solution*

$$u \in L^2(0,T;H^2(\Omega)^N) \cap H^1(0,T;H)$$

to the Stokes system

$$
\begin{cases}
u_t - \Delta u + \nabla p = f & \text{in } Q, \\
\nabla \cdot u = 0 & \text{in } Q, \\
u = 0 & \text{on } \Sigma, \\
u(0) = 0 & \text{in } \Omega,
\end{cases} \qquad (2.7)
$$

for some $p \in L^2(0,T;H^1(\Omega))$, and there exists a constant $C > 0$ depending only on Ω such that

$$\|u\|^2_{L^2(0,T;H^2(\Omega)^N)} + \|u\|^2_{H^1(0,T;L^2(\Omega)^N)} \le C\, \|f\|^2_{L^2(Q)^N}. \qquad (2.8)$$

The following regularity result can be found in [21] (see also [15]).

Lemma 2.7. *For every $T > 0$ and every*

$$f \in L^2(0,T;V) \cup \left(L^2(0,T;H^1(\Omega)^N) \cap H^1(0,T;V')\right),$$

the unique solution to the Stokes system (2.7) satisfies

$$u \in L^2(0,T;H^3(\Omega)^N) \cap H^1(0,T;V)$$

and there exists a constant $C > 0$ depending only on Ω such that (depending on where f is taken)

$$\|u\|^2_{L^2(0,T;H^3(\Omega)^N)} + \|u\|^2_{H^1(0,T;V)} \le C\, \|f\|^2_{L^2(0,T;V)} \qquad (2.9)$$

or

$$\|u\|^2_{L^2(0,T;H^3(\Omega)^N)} + \|u\|^2_{H^1(0,T;V)} \le C \left(\|f\|^2_{L^2(0,T;H^1(\Omega)^N)} + \|f_t\|^2_{L^2(0,T;V')} \right). \qquad (2.10)$$

Furthermore, let us assume that

$$f \in \left(L^2(0,T;H^3(\Omega)^N) \cap H^1(0,T;V)\right) \cup \left(L^2(0,T;H^3(\Omega)^N) \cap H^1(0,T;H^1(\Omega)^N) \cap H^2(0,T;V')\right)$$

and satisfies the following compatibility condition:

$$\nabla p_f = f(0) \text{ on } \partial\Omega,$$

where p_f is any solution of the Neumann boundary-value problem

$$\begin{cases} \Delta p_f = \nabla \cdot f(0) & \text{in } \Omega, \\ \dfrac{\partial p_f}{\partial n} = f(0) \cdot n & \text{on } \partial\Omega. \end{cases}$$

Then, $u \in L^2(0,T;H^5(\Omega)^N) \cap H^1(0,T;H^3(\Omega)^N) \cap H^2(0,T;V)$ and there exists a constant $C > 0$ depending only on Ω such that

$$\begin{aligned} &\|u\|^2_{L^2(0,T;H^5(\Omega)^N)} + \|u\|^2_{H^1(0,T;H^3(\Omega)^N)} + \|u\|^2_{H^2(0,T;V)} \\ &\qquad \leq C\left(\|f\|^2_{L^2(0,T;H^3(\Omega)^N)} + \|f_t\|^2_{L^2(0,T;V)}\right) \end{aligned} \tag{2.11}$$

or

$$\begin{aligned} &\|u\|^2_{L^2(0,T;H^5(\Omega)^N)} + \|u\|^2_{H^1(0,T;H^3(\Omega)^N)} + \|u\|^2_{H^2(0,T;V)} \\ &\qquad \leq C\left(\|f\|^2_{L^2(0,T;H^3(\Omega)^N)} + \|f_t\|^2_{L^2(0,T;H^1(\Omega)^N)} + \|f_{tt}\|^2_{L^2(0,T;V')}\right). \end{aligned} \tag{2.12}$$

3 Carleman estimates

In this section we prove a new Carleman estimate for the Stokes coupled system

$$\begin{cases} -\varphi_t - \Delta\varphi + \nabla\pi = g^0 + \psi \mathbb{1}_{\mathcal{O}},\ \nabla\cdot\varphi = 0, & \text{in } Q, \\ \psi_t - \Delta\psi + \nabla\kappa = g^1,\ \nabla\cdot\psi = 0, & \text{in } Q, \\ \varphi = 0,\ \psi = 0, & \text{on } \Sigma \\ \varphi(T) = 0,\ \psi(0) = \psi^0, & \text{in } \Omega, \end{cases} \tag{3.1}$$

where $g^0 \in L^2(Q)^N$, $g^1 \in L^2(0,T;V)$ and $\psi^0 \in H$. It is given by the following proposition:

Proposition 3.1. *Assume that $\omega \cap \mathcal{O} \neq \emptyset$. Then, there exists a constant λ_0, such that for any $\lambda \geq \lambda_0$ there exists a constant $C > 0$ depending only on λ, Ω, ω and ℓ such that for any $i \in \{1, \dots, N\}$, any $g^0 \in L^2(Q)^N$, any $g^1 \in L^2(0,T;V)$ and any $\psi^0 \in H$, the solution (φ, ψ) of (3.1) satisfies*

$$\begin{aligned} &s^4 \iint\limits_Q e^{-7s\alpha^*}(\xi^*)^4|\varphi|^2 dx\,dt + s^5 \iint\limits_Q e^{-4s\alpha^*}(\xi^*)^5|\psi|^2 dx\,dt \\ &\qquad \leq C\left(s^9 \iint\limits_Q e^{-3s\alpha - s\alpha^*}\xi^9|g^0|^2 dx\,dt + \iint\limits_Q e^{-s\alpha^*}(|g^1|^2 + |\nabla g^1|^2) dx\,dt \right. \\ &\qquad\qquad \left. + s^{13}\sum_{j=1, j\neq i}^{N} \iint\limits_{\omega\times(0,T)} e^{-3s\alpha - s\alpha^*}\xi^{13}|\varphi_j|^2 dx\,dt\right), \end{aligned} \tag{3.2}$$

for every $s \geq C$.

The proof of Proposition 3.1 is divided in three parts. In the first part, we prove a general Carleman inequality for ψ with local terms only in $\Delta\psi_j$, $j \neq i$ (see Proposition 3.2 below). In the second part, we deduce a Carleman estimate for the equation of ψ in (3.1). In the third and final part, we combine it with the Carleman estimate in Lemma 2.1 for φ and do the final estimates to obtain (3.2).

3.1 New Carleman estimate for Stokes systems with $\Delta(\cdot)_j, (j \neq i)$ as local terms

Before proving the Carleman estimate for ψ, let us prove a more general inequality which has its own interest. The new Carleman inequality for ψ will be deduced from it.

We consider the Stokes system

$$\begin{cases} \phi_t - \Delta\phi + \nabla h = f + g, \ \nabla \cdot \phi = 0, & \text{in } Q, \\ \phi = 0, & \text{on } \Sigma \\ \phi(0) = \phi^0, & \text{in } \Omega, \end{cases} \tag{3.3}$$

where $\phi^0 \in H$ and

$$(f, g) \in \left(L^2(0,T; H^3(\Omega)^N) \cap H^1(0,T;V)\right) \times \left(L^2(0,T;H^3(\Omega)^N) \cap H^1(0,T;H^1(\Omega)^N) \cap H^2(0,T;V')\right).$$

We prove the following estimate for the solutions of system (3.3).

Proposition 3.2. *Let $\widehat{\omega} \subset \Omega$ be a nonempty open set such that $\omega_0 \Subset \widehat{\omega}$. Then, there exists a constant $\widehat{\lambda}_2$, such that for any $\lambda \geq \widehat{\lambda}_2$ there exists a constant $C(\lambda) > 0$ such that for any $i \in \{1, \ldots, N\}$, any $\phi^0 \in H$ and any*

$$(f, g) \in \left(L^2(0,T; H^3(\Omega)^N \cap V) \times L^2(0,T; H^3(\Omega)^N)\right) \cap \left(H^1(0,T;V) \times H^2(0,T;V')\right),$$

the solution of (3.3) *satisfies*

$$\begin{aligned}
&\sum_{j=1, j\neq i}^{N} \left[\iint_Q e^{-3s\alpha} (s^5\xi^5 |\Delta\phi_j|^2 + s^3\xi^3 |\nabla\Delta\phi_j|^2 + s\xi|\nabla\nabla\Delta\phi_j|^2 + s^{-1}\xi^{-1}|\nabla\nabla\nabla\Delta\phi_j|^2) dx\, dt \right] \\
&\quad + s^5 \iint_Q e^{-3s\alpha^*} (\xi^*)^5 |\phi|^2 dx\, dt \leq C \Bigg(s^{5/2} \iint_Q e^{-3s\alpha^*} (\xi^*)^{3-2/m} (|f|^2 + |g|^2)\, dx\, dt \\
&\quad + s^{3/2} \int_0^T \left\| \left(e^{-3/2s\alpha^*} (\xi^*)^{1-3/(2m)} g \right)_t \right\|_{V'}^2 dt + s^{-1/2} \int_0^T e^{-3s\alpha^*} (\xi^*)^{-5/m} \|(f,g)\|^2_{H^3(\Omega)^{2N}} dt \\
&\quad + s^{-1/2} \int_0^T \left\| \left(e^{-3/2s\alpha^*} (\xi^*)^{-5/(2m)} (f,g) \right)_t \right\|^2_{H^1(\Omega)^{2N}} dt + s^{-1/2} \int_0^T \left\| \left(e^{-3/2s\alpha^*} (\xi^*)^{-5/(2m)} g \right)_{tt} \right\|^2_{V'} dt \\
&\quad + \sum_{j=1, j\neq i}^{N} \iint_Q e^{-3s\alpha} |\nabla\Delta(f_j + g_j)|^2 dx\, dt + \iint_Q e^{-3s\alpha} |\nabla\nabla(\nabla \cdot g)|^2 dx\, dt \\
&\quad + s^5 \sum_{j=1, j\neq i}^{N} \iint_{\widehat{\omega}\times(0,T)} e^{-3s\alpha} \xi^5 |\Delta\phi_j|^2 dx\, dt \Bigg),
\end{aligned} \tag{3.4}$$

for every $s \geq C$.

Remark 3.3. *For our purpose, we will take $g = 0$. See the proof of Proposition 3.5 below for more details.*

Remark 3.4. *For the sake of simplicity, from now on we consider $N = 2$ and $i = 2$ for the proofs of Proposition 3.1, Proposition 3.2 and Proposition 3.5. The arguments are easily adapted to the general case.*

Proof. First, following the method introduced in [5], we apply the divergence operator to the equation (3.22) to obtain

$$\Delta h = \nabla \cdot g \text{ in } Q.$$

Then, applying the operator $(\nabla\nabla\Delta\cdot)$ to the equation satisfied by ϕ_1 we have

$$(\nabla\nabla\Delta\phi_1)_t - \Delta(\nabla\nabla\Delta\phi_1) = \nabla\nabla\Delta(f_1 + g_1) - \partial_1\nabla\nabla(\nabla \cdot g).$$

Notice that the right-hand side of this equation belongs to $L^2(0,T;H^{-1}(\Omega)^4)$, thus we can apply Lemma 2.3 to this equation to obtain

$$\begin{aligned} s^{-1}\iint\limits_Q e^{-3s\alpha}\xi^{-1}|\nabla\nabla\nabla\Delta\phi_1|^2 dx\,dt + s\iint\limits_Q e^{-3s\alpha}\xi|\nabla\nabla\Delta\phi_1|^2 dx\,dt \\ \leq C\Bigg(s^{-1/2}\left\| e^{-3/2s\alpha^*}(\xi^*)^{-1/4+1/m}\nabla\nabla\Delta\phi_1\right\|^2_{L^2(\Sigma)^4} + s^{-1/2}\left\| e^{-3/2s\alpha^*}(\xi^*)^{-1/4}\nabla\nabla\Delta\phi_1\right\|^2_{H^{\frac14,\frac12}(\Sigma)^4} \\ + \iint\limits_Q e^{-3s\alpha}|\nabla\Delta(f_1+g_1)|^2 dx\,dt + \iint\limits_Q e^{-3s\alpha}|\nabla\nabla(\nabla\cdot g)|^2 dx\,dt \\ + s\iint\limits_{\omega_0\times(0,T)} e^{-3s\alpha}\xi|\nabla\nabla\Delta\phi_1|^2 dx\,dt \Bigg), \end{aligned} \tag{3.5}$$

for every $s \geq C$.
We divide the rest of the proof in several steps:

- In step 1, we add some global terms in the left-hand side of (3.5), but by doing so we add some local terms in the right-hand side. These terms will be estimated in step 3.
- In step 2, we estimate the boundary terms which appear in the right-hand side of (3.5).
- In step 3, we estimate the undesirable local terms.

In the following, C denotes a constant depending only on λ, Ω, ω_0, $\mathcal{O}$ and ℓ.

Step 1. We apply Lemma 2.5 with $r = 1$ and $u := \nabla\Delta\phi_1$ to obtain

$$\begin{aligned} s^3\iint\limits_Q e^{-3s\alpha}\xi^3|\nabla\Delta\phi_1|^2 dx\,dt \\ \leq C\left(s\iint\limits_Q e^{-3s\alpha}\xi|\nabla\nabla\Delta\phi_1|^2 dx\,dt + s^3\iint\limits_{\omega_0\times(0,T)} e^{-3s\alpha}\xi^3|\nabla\Delta\phi_1|^2 dx\,dt\right), \end{aligned} \tag{3.6}$$

for every $s \geq C$, and another time with $r = 3$ and $u := \Delta\phi_1$. We have

$$\begin{aligned} s^5\iint\limits_Q e^{-3s\alpha}\xi^5|\Delta\phi_1|^2 dx\,dt \\ \leq C\left(s^3\iint\limits_Q e^{-3s\alpha}\xi^3|\nabla\Delta\phi_1|^2 dx\,dt + s^5\iint\limits_{\omega_0\times(0,T)} e^{-3s\alpha}\xi^5|\Delta\phi_1|^2 dx\,dt\right), \end{aligned} \tag{3.7}$$

for every $s \geq C$.

At this point, combining (3.5), (3.6) and (3.7), we get

$$\begin{aligned}
&\iint\limits_Q e^{-3s\alpha}(s^{-1}\xi^{-1}|\nabla\nabla\nabla\Delta\phi_1|^2 + s\xi|\nabla\nabla\Delta\phi_1|^2 + s^3\xi^3|\nabla\Delta\phi_1|^2 + s^5\xi^5|\Delta\phi_1|^2)dx\,dt \\
&\leq C\Bigg(s^{-1/2}\left\|e^{-3/2s\alpha^*}(\xi^*)^{-1/4+1/m}\nabla\nabla\Delta\phi_1\right\|^2_{L^2(\Sigma)} + s^{-1/2}\left\|e^{-3/2s\alpha^*}(\xi^*)^{-1/4}\nabla\nabla\Delta\phi_1\right\|^2_{H^{\frac{1}{4},\frac{1}{2}}(\Sigma)} \\
&\qquad + \iint\limits_Q e^{-3s\alpha}|\nabla\Delta(f_1+g_1)|^2 dx\,dt + \iint\limits_Q e^{-3s\alpha}|\nabla\nabla\nabla\cdot g|^2 dx\,dt \\
&\qquad + \iint\limits_{\omega_0\times(0,T)} e^{-3s\alpha}(s\xi|\nabla\nabla\Delta\phi_1|^2 + s^3\xi^3|\nabla\Delta\phi_1|^2 + s^5\xi^5|\Delta\phi_1|^2)dx\,dt\Bigg), \qquad (3.8)
\end{aligned}$$

for every $s \geq C$.

Estimate of ϕ_2. Now we would like to introduce in the left-hand side a term in $\phi = (\phi_1, \phi_2)$. From the divergence free condition, we get

$$\begin{aligned}
s^5\iint\limits_Q e^{-3s\alpha^*}(\xi^*)^5|\partial_2\phi_2|^2 dx\,dt &= s^5\iint\limits_Q e^{-3s\alpha^*}(\xi^*)^5|\partial_1\phi_1|^2 dx\,dt \\
&\leq s^5\iint\limits_Q e^{-3s\alpha^*}(\xi^*)^5|\nabla\phi_1|^2 dx\,dt.
\end{aligned}$$

Since $\phi_2|_{\partial\Omega} = 0$ and Ω is bounded, we have

$$\int_\Omega |\phi_2|^2 dx \leq C(\Omega)\int_\Omega |\partial_2\phi_2|^2 dx.$$

Finally, notice that α^* and ξ^* do not depend on the space variable x, so that

$$s^5\iint\limits_Q e^{-3s\alpha^*}(\xi^*)^5|\phi_2|^2 dx\,dt \leq C(\Omega)s^5\iint\limits_Q e^{-3s\alpha^*}(\xi^*)^5|\partial_2\phi_2|^2 dx\,dt,$$

and therefore

$$s^5\iint\limits_Q e^{-3s\alpha^*}(\xi^*)^5|\phi_2|^2 dx\,dt \leq Cs^5\iint\limits_Q e^{-3s\alpha^*}(\xi^*)^5|\nabla\phi_1|^2 dx\,dt.$$

Now, since $\|\Delta\cdot\|_{L^2(\Omega)}$ is an equivalent norm to $\|\cdot\|_{H^2(\Omega)}$ in the space of functions with null trace, and using the definition of α^* and ξ^* (see (2.3)), we obtain from this last inequality

$$s^5\iint\limits_Q e^{-3s\alpha^*}(\xi^*)^5|\phi_2|^2 dx\,dt \leq Cs^5\iint\limits_Q e^{-3s\alpha}\xi^5|\Delta\phi_1|^2 dx\,dt. \qquad (3.9)$$

Combining (3.8) and (3.9) we have for the moment

$$
\begin{aligned}
I_1(s,\phi) :=& \iint\limits_Q e^{-3s\alpha}\left(s^{-1}\xi^{-1}|\nabla\nabla\nabla\Delta\phi_1|^2 + s\xi|\nabla\nabla\Delta\phi_1|^2 + s^3\xi^3|\nabla\Delta\phi_1|^2\right)dx\,dt\\
&+ s^5\iint\limits_Q e^{-3s\alpha}\xi^5|\Delta\phi_1|^2 dx\,dt + s^5\iint\limits_Q e^{-3s\alpha^*}(\xi^*)^5|\phi|^2 dx\,dt\\
\leq C\Bigg(& s^{-1/2}\left\|e^{-3/2s\alpha^*}(\xi^*)^{-1/4+1/m}\nabla\nabla\Delta\phi_1\right\|^2_{L^2(\Sigma)^4} + s^{-1/2}\left\|e^{-3/2s\alpha^*}(\xi^*)^{-1/4}\nabla\nabla\Delta\phi_1\right\|^2_{H^{\frac14,\frac12}(\Sigma)^4}\\
&+ \iint\limits_Q e^{-3s\alpha}|\nabla\Delta(f_1+g_1)|^2 dx\,dt + \iint\limits_Q e^{-3s\alpha}|\nabla\nabla\nabla\cdot g|^2 dx\,dt\\
&+ \iint\limits_{\omega_0\times(0,T)} e^{-3s\alpha}\left(s\xi|\nabla\nabla\Delta\phi_1|^2 + s^3\xi^3|\nabla\Delta\phi_1|^2 + s^5\xi^5|\Delta\phi_1|^2\right)dx\,dt\Bigg),
\end{aligned}
\tag{3.10}
$$

for every $s \geq C$.

Step 2. In this step we deal with the boundary terms in (3.10). We begin with the first one. Notice that the minimum of the weight functions $e^{-3/2s\alpha}$ and ξ is reached at the boundary $\partial\Omega$, where $\alpha = \alpha^*$ and $\xi = \xi^*$ do not depend on x. Since $m \geq 10$, $(\xi^*)^{-1/4+1/m}$ is bounded in $(0,T)$, thus we obtain

$$
\begin{aligned}
&s^{-1/2}\left\|e^{-3/2s\alpha^*}(\xi^*)^{-1/4+1/m}\nabla\nabla\Delta\phi_1\right\|^2_{L^2(\Sigma)^4} \leq Cs^{-1/2}\left\|e^{-3/2s\alpha^*}\nabla\nabla\Delta\phi_1\right\|^2_{L^2(\Sigma)^4}\\
&\leq Cs^{-1/2}\Bigg(\left\|s^{1/2}e^{-3/2s\alpha^*}(\xi^*)^{1/2}\nabla\nabla\Delta\phi_1\right\|_{L^2(Q)^4}\left\|s^{-1/2}e^{-3/2s\alpha^*}(\xi^*)^{-1/2}\nabla\nabla\nabla\Delta\phi_1\right\|_{L^2(Q)^8}\\
&\qquad + \left\|e^{-3/2s\alpha^*}\nabla\nabla\Delta\phi_1\right\|^2_{L^2(Q)^4}\Bigg)\\
&\leq Cs^{-1/2}\iint\limits_Q e^{-3s\alpha}(s\xi|\nabla\nabla\Delta\phi_1|^2 + s^{-1}\xi^{-1}|\nabla\nabla\nabla\Delta\phi_1|^2)dx\,dt.
\end{aligned}
\tag{3.11}
$$

Therefore, this boundary term can be absorbed by the left-hand side of (3.10) for $s \geq C$.

We turn to the second boundary term: $s^{-1/2}\left\|e^{-3/2s\alpha^*}(\xi^*)^{-1/4}\nabla\nabla\Delta\phi_1\right\|^2_{H^{\frac14,\frac12}(\Sigma)^4}$. To this end, let us define

$$(\Phi^1, h_1) := s^{3/2}e^{-3/2s\alpha^*}(\xi^*)^{3/2-1/m}(\phi, h) =: \zeta_1(t)(\phi, h)$$

Then, from (3.3), (Φ^1, h_1) is the solution of the Stokes system:

$$
\begin{cases}
\Phi^1_t - \Delta\Phi^1 + \nabla h_1 = \zeta_1(f+g) + \zeta_1'\phi,\ \nabla\cdot\Phi^1 = 0, & \text{in } Q,\\
\Phi^1 = 0, & \text{on } \Sigma\\
\Phi^1(0) = 0, & \text{in } \Omega.
\end{cases}
$$

Using the regularity estimate (2.8) for this system, we have

$$\left\|\Phi^1\right\|^2_{L^2(0,T;H^2(\Omega)^2)\cap H^1(0,T;L^2(\Omega)^2)} \leq C\left(\|\zeta_1(f+g)\|^2_{L^2(Q)^2} + \|\zeta_1'\,\phi\|^2_{L^2(Q)^2}\right).$$

From (2.3), we see that

$$|\zeta_1'| \leq Cs^{5/2}e^{-3/2s\alpha^*}(\xi^*)^{5/2},$$

for every $s \geq C$. Thus, we obtain

$$\left\|\Phi^1\right\|^2_{L^2(0,T;H^2(\Omega)^2)\cap H^1(0,T;L^2(\Omega)^2)} \leq C\left(\|\zeta_1(f+g)\|^2_{L^2(Q)^2} + \left\|s^{5/2}e^{-3/2s\alpha^*}(\xi^*)^{5/2}\,\phi\right\|^2_{L^2(Q)^2}\right). \tag{3.12}$$

Now, notice that, from an interpolation argument between the espaces $L^2(Q)^2$ and $L^2(0,T;H^2(\Omega)^2)$, we obtain

$$\begin{aligned}\left\|s^2e^{-3/2s\alpha^*}(\xi^*)^{2-1/(2m)}\,\phi\right\|^2_{L^2(0,T;V)} &\leq \left\|\Phi^1\right\|_{L^2(0,T;H^2(\Omega)^2)}\left\|s^{5/2}e^{-3/2s\alpha^*}(\xi^*)^{5/2}\,\phi\right\|_{L^2(Q)^2}\\ &\leq C\left(\|\zeta_1(f+g)\|^2_{L^2(Q)^2} + \left\|s^{5/2}e^{-3/2s\alpha^*}(\xi^*)^{5/2}\,\phi\right\|^2_{L^2(Q)^2}\right).\end{aligned} \tag{3.13}$$

Next, we introduce:

$$(\Phi^2, h_2) := se^{-3/2s\alpha^*}(\xi^*)^{1-3/(2m)}(\phi, h) =: \zeta_2(t)(\phi, h). \tag{3.14}$$

Then, (Φ^2, h_2) is the solution of the Stokes system:

$$\begin{cases} \Phi^2_t - \Delta\Phi^2 + \nabla h_2 = \zeta_2(f+g) + \zeta_2'\,\phi,\ \nabla\cdot\Phi^2 = 0, & \text{in } Q,\\ \Phi^2 = 0, & \text{on } \Sigma\\ \Phi^2(0) = 0, & \text{in } \Omega.\end{cases}$$

Using the regularity results (2.9) and (2.10), we find:

$$\begin{aligned}\left\|\Phi^2\right\|^2_{L^2(0,T;H^3(\Omega)^2)\cap H^1(0,T;V)} \leq C\Big(&\|\zeta_2 f\|^2_{L^2(0,T;V)} + \|\zeta_2 g\|^2_{L^2(0,T;H^1(\Omega)^2)\cap H^1(0,T;V')}\\ &+ \|\zeta_2'\,\phi\|^2_{L^2(0,T;V)}\Big).\end{aligned}$$

Using the estimate

$$|\zeta_2'| \leq Cs^2e^{-3/2s\alpha^*}(\xi^*)^{2-1/(2m)},$$

for every $s \geq C$, and (3.13) we get

$$\begin{aligned}\left\|\Phi^2\right\|^2_{L^2(0,T;H^3(\Omega)^2)\cap H^1(0,T;V)} \leq C\Big(&\|\zeta_1(f+g)\|^2_{L^2(Q)^2} + \|\zeta_2 f\|^2_{L^2(0,T;V)}\\ &+ \|\zeta_2 g\|^2_{L^2(0,T;H^1(\Omega)^2)\cap H^1(0,T;V')} + \left\|s^{5/2}e^{-3/2s\alpha^*}(\xi^*)^{5/2}\,\phi\right\|^2_{L^2(Q)^2}\Big).\end{aligned} \tag{3.15}$$

Finally, let

$$(\Phi^3, h_3) := e^{-3/2s\alpha^*}(\xi^*)^{-5/(2m)}(\phi, h) =: \zeta_3(t)(\phi, h). \tag{3.16}$$

Then, (Φ^3, h_3) is the solution of the Stokes system:

$$\begin{cases} \Phi^3_t - \Delta\Phi^3 + \nabla h_3 = \zeta_3\,(f+g) + \zeta_3'\,\phi,\ \nabla\cdot\Phi^3 = 0, & \text{in } Q,\\ \Phi^3 = 0, & \text{on } \Sigma\\ \Phi^3(0) = 0, & \text{in } \Omega.\end{cases}$$

Using the regularity results (2.11) and (2.12) (note that the compatibility condition is trivially satisfied) and estimates for the weight functions, we have

$$\begin{aligned}\left\|\Phi^3\right\|^2_{L^2(0,T;H^5(\Omega)^2)\cap H^1(0,T;H^3(\Omega)^2)\cap H^2(0,T;V)} \leq C\Big(&\|\zeta_3\,f\|^2_{L^2(0,T;H^3(\Omega)^2)\cap H^1(0,T;H^1(\Omega)^2)}\\ &+ \|\zeta_3\,g\|^2_{L^2(0,T;H^3(\Omega)^2)\cap H^1(0,T;H^1(\Omega)^2)\cap H^2(0,T;V')}\\ &+ \left\|\Phi^2\right\|^2_{L^2(0,T;H^3(\Omega)^2)\cap H^1(0,T;V)} + \left\|s^2e^{-3/2s\alpha^*}(\xi^*)^{2-1/(2m)}\,\phi\right\|^2_{L^2(0,T;V)}\Big),\end{aligned}$$

and combining this with (3.15) and (3.13), we get

$$\|\Phi^3\|^2_{L^2(0,T;H^5(\Omega)^2)\cap H^1(0,T;H^3(\Omega)^2)\cap H^2(0,T;V)} \leq C\Big(\|\zeta_1(f+g)\|^2_{L^2(Q)^2} + \|\zeta_2 f\|^2_{L^2(0,T;V)} + \|\zeta_2 g\|^2_{L^2(0,T;H^1(\Omega)^2)\cap H^1(0,T;V')} + \|\zeta_3 f\|^2_{L^2(0,T;H^3(\Omega)^2)\cap H^1(0,T;H^1(\Omega)^2)} + \|\zeta_3 g\|^2_{L^2(0,T;H^3(\Omega)^2)\cap H^1(0,T;H^1(\Omega)^2)\cap H^2(0,T;V')} + \|s^{5/2}e^{-3/2s\alpha^*}(\xi^*)^{5/2}\phi\|^2_{L^2(Q)^2}\Big). \quad (3.17)$$

For $m \geq 10$, we have in particular that

$$e^{-3/2s\alpha^*}(\xi^*)^{-1/4}\nabla\nabla\Delta\,\phi_1 \in L^2(0,T;H^1(\Omega)^4) \cap H^1(0,T;H^{-1}(\Omega)^4)$$

and, using a trace inequality (see, for instance, [18]) we deduce

$$\left\|e^{-3/2s\alpha^*}(\xi^*)^{-1/4}\nabla\nabla\Delta\,\phi_1\right\|^2_{H^{\frac14,\frac12}(\Sigma)^4} \leq C\left(\left\|e^{-3/2s\alpha^*}(\xi^*)^{-5/(2m)}\nabla\nabla\Delta\,\phi_1\right\|^2_{L^2(0,T;H^1(\Omega)^4)} + \left\|e^{-3/2s\alpha^*}(\xi^*)^{-5/(2m)}\nabla\nabla\Delta\,\phi_1\right\|^2_{H^1(0,T;H^{-1}(\Omega)^4)}\right).$$

From (3.17), we find

$$\begin{aligned}
&s^{-1/2}\left\|e^{-3/2s\alpha^*}(\xi^*)^{-1/4}\nabla\nabla\Delta\,\phi_1\right\|^2_{H^{\frac14,\frac12}(\Sigma)^4}\\
&\leq Cs^{-1/2}\left(\left\|e^{-3/2s\alpha^*}(\xi^*)^{-5/(2m)}\,\phi\right\|^2_{L^2(0,T;H^5(\Omega)^2)} + \left\|e^{-3/2s\alpha^*}(\xi^*)^{-5/(2m)}\,\phi\right\|^2_{H^1(0,T;H^3(\Omega)^2)}\right)\\
&\leq Cs^{-1/2}\Big(s^5\|e^{-3/2s\alpha^*}(\xi^*)^{5/2}\phi\|_{L^2(Q)^2} + \|\zeta_1\,(f+g)\|^2_{L^2(Q)^2} + \|\zeta_2\,f\|^2_{L^2(0,T;V)}\\
&\quad + \|\zeta_2\,g\|^2_{L^2(0,T;H^1(\Omega)^2)\cap H^1(0,T;V')} + \|\zeta_3\,f\|^2_{L^2(0,T;H^3(\Omega)^2)\cap H^1(0,T;H^1(\Omega)^2)}\\
&\quad + \|\zeta_3\,g\|^2_{L^2(0,T;H^3(\Omega)^2)\cap H^1(0,T;H^1(\Omega)^2)\cap H^2(0,T;V')}\Big).
\end{aligned}$$

This inequality, combined with (3.10) and (3.11) gives

$$\begin{aligned}
I_1(s,\phi) \leq C\Bigg(&\iint\limits_{\omega_0\times(0,T)} e^{-3s\alpha}(s\xi|\nabla\nabla\Delta\phi_1|^2 + s^3\xi^3|\nabla\Delta\phi_1|^2)dx\,dt\\
&+ s^5\iint\limits_{\omega_0\times(0,T)} e^{-3s\alpha}\xi^5|\Delta\phi_1|^2dx\,dt + \iint\limits_Q e^{-3s\alpha}|\nabla\Delta(f_1+g_1)|^2dx\,dt\\
&+ \iint\limits_Q e^{-3s\alpha}|\nabla\nabla\nabla\cdot g|^2dx\,dt + \left\|s^{-1/4}\zeta_1(f+g)\right\|^2_{L^2(Q)^2}\\
&+ \left\|s^{-1/4}\zeta_2(f,g)\right\|^2_{L^2(0,T;H^1(\Omega)^4)} + \left\|s^{-1/4}\zeta_2\,g\right\|^2_{H^1(0,T;V')}\\
&+ \left\|s^{-1/4}\zeta_3\,(f,g)\right\|^2_{L^2(0,T;H^3(\Omega)^4)\cap H^1(0,T;H^1(\Omega)^4)} + \left\|s^{-1/4}\zeta_3\,g\right\|^2_{H^2(0,T;V')}\Bigg), \quad (3.18)
\end{aligned}$$

for every $s \geq C$.

Step 3. The last part of the proof consists of estimating the local terms in the right-hand side of (3.18) by local terms of $\Delta\phi_1$ and $I_1(s,\phi)$ multiplied by small constants.

Let us begin with the first term in the right-hand side of (3.18). Let ω_1 be a nonempty open set such that $\omega_0 \Subset \omega_1 \Subset \widehat{\omega}$ and $\theta_1 \in C_0^2(\omega_1)$ be a nonnegative function with $\theta_1 \equiv 1$ in ω_0.

Integration by parts gives

$$s \iint\limits_{\omega_0\times(0,T)} e^{-3s\alpha}\xi|\nabla\nabla\Delta\phi_1|^2 dx\,dt \le s \iint\limits_{\omega_1\times(0,T)} \theta_1 e^{-3s\alpha}\xi|\nabla\nabla\Delta\phi_1|^2 dx\,dt$$
$$= -s \iint\limits_{\omega_1\times(0,T)} \theta_1 e^{-3s\alpha}\xi\nabla\Delta^2\phi_1\nabla\Delta\phi_1 dx\,dt + \frac{s}{2} \iint\limits_{\omega_1\times(0,T)} \Delta(\theta_1 e^{-3s\alpha}\xi)|\nabla\Delta\phi_1|^2 dx\,dt.$$

Using the estimate

$$|\Delta(\theta_1 e^{-3s\alpha}\xi)| \le Cs^2 e^{-3s\alpha}\xi^3,$$

for every $s \ge C$, and Young's inequality, we get

$$\begin{aligned} & s \iint\limits_{\omega_0\times(0,T)} e^{-3s\alpha}\xi|\nabla\nabla\Delta\phi_1|^2 dx\,dt \\ & \le \varepsilon s^{-1} \iint\limits_{\omega_1\times(0,T)} e^{-3s\alpha}\xi^{-1}|\nabla\Delta^2\phi_1|^2 dx\,dt + C(\varepsilon)s^3 \iint\limits_{\omega_1\times(0,T)} e^{-3s\alpha}\xi^3|\nabla\Delta\phi_1|^2 dx\,dt, \end{aligned} \tag{3.19}$$

for every $s \ge C$ and any $\varepsilon > 0$.

In an analogous way, we estimate the local term in $\nabla\Delta\phi_1$. Indeed, let $\theta_2 \in C_0^2(\widehat{\omega})$ be a nonnegative function with $\theta_2 \equiv 1$ in ω_1. Integration by parts gives

$$s^3 \iint\limits_{\omega_1\times(0,T)} e^{-3s\alpha}\xi^3|\nabla\Delta\phi_1|^2 dx\,dt \le s^3 \iint\limits_{\widehat{\omega}\times(0,T)} \theta_2 e^{-3s\alpha}\xi^3|\nabla\Delta\phi_1|^2 dx\,dt$$
$$= -s^3 \iint\limits_{\widehat{\omega}\times(0,T)} \theta_2 e^{-3s\alpha}\xi^3\Delta^2\phi_1\Delta\phi_1 dx\,dt + \frac{s^3}{2} \iint\limits_{\widehat{\omega}\times(0,T)} \Delta(\theta_2 e^{-3s\alpha}\xi^3)|\Delta\phi_1|^2 dx\,dt.$$

Using the estimate

$$|\Delta(\theta_2 e^{-3s\alpha}\xi^3)| \le Cs^2 e^{-3s\alpha}\xi^5,$$

for every $s \ge C$, and Young's inequality, we get

$$\begin{aligned} & s^3 \iint\limits_{\omega_1\times(0,T)} e^{-3s\alpha}\xi^3|\nabla\Delta\phi_1|^2 dx\,dt \\ & \le \varepsilon s \iint\limits_{\widehat{\omega}\times(0,T)} e^{-3s\alpha}\xi|\Delta^2\phi_1|^2 dx\,dt + C(\varepsilon)s^5 \iint\limits_{\widehat{\omega}\times(0,T)} e^{-3s\alpha}\xi^5|\Delta\phi_1|^2 dx\,dt, \end{aligned}$$

for every $s \ge C$ and any $\varepsilon > 0$, which combined with (3.19), (3.18) and an interpolation argument between the spaces $L^2(Q)$ and $L^2(0,T;H^3(\Omega))$, gives (3.4). This ends the proof of Proposition 3.2. $\square$

3.2 New Carleman estimate for ψ

Now, we deal with the Stokes system

$$\begin{cases} \psi_t - \Delta\psi + \nabla\kappa = g^1,\ \nabla\cdot\psi = 0, & \text{in } Q, \\ \psi = 0, & \text{on } \Sigma \\ \psi(0) = \psi^0, & \text{in } \Omega. \end{cases} \tag{3.20}$$

Let us start by introducing (ψ^*, κ^*) and $(\widetilde{\psi}, \widetilde{\kappa})$ the solutions of the following systems:

$$\begin{cases} \psi_t^* - \Delta\psi^* + \nabla\kappa^* = \rho_1\, g^1,\ \nabla\cdot\psi^* = 0, & \text{in } Q,\\ \psi^* = 0, & \text{on } \Sigma\\ \psi^*(0) = 0, & \text{in } \Omega. \end{cases} \tag{3.21}$$

and

$$\begin{cases} \widetilde{\psi}_t - \Delta\widetilde{\psi} + \nabla\widetilde{\kappa} = \rho_1'\, \psi,\ \nabla\cdot\widetilde{\psi} = 0, & \text{in } Q,\\ \widetilde{\psi} = 0, & \text{on } \Sigma\\ \widetilde{\psi}(0) = 0, & \text{in } \Omega, \end{cases} \tag{3.22}$$

where $\rho_1(t) := e^{-1/2s\alpha^*}$. It is not hard to see that $(\psi^* + \widetilde{\psi}, \kappa^* + \widetilde{\kappa})$ solves the same system as $(\rho_1\,\psi, \rho_1\,\kappa)$. Thus, by uniqueness of the Cauchy problem we have

$$\rho_1\,\psi = \psi^* + \widetilde{\psi} \text{ and } \rho_1\,\kappa = \kappa^* + \widetilde{\kappa}. \tag{3.23}$$

Notice that, from Lemma 2.6 applied to system (3.21), we have $\psi^* \in L^2(0,T;H^2(\Omega)^N)\cap H^1(0,T;H)$ and

$$\|\psi^*\|^2_{L^2(0,T;H^2(\Omega)^N)} + \|\psi^*\|^2_{H^1(0,T;L^2(\Omega)^N)} \leq C \left\|\rho_1\, g^1\right\|^2_{L^2(Q)^N}. \tag{3.24}$$

Furthermore, from Lemma 2.7 (see (2.9)), since $g^1 \in L^2(0,T;V)$, we have $\psi^* \in L^2(0,T;H^3(\Omega)^N)\cap H^1(0,T;V)$ and

$$\|\psi^*\|^2_{L^2(0,T;H^3(\Omega)^N)} + \|\psi^*\|^2_{H^1(0,T;H^1(\Omega)^N)} \leq C \left\|\rho_1\, g^1\right\|^2_{L^2(0,T;V)}. \tag{3.25}$$

We prove the following estimate for the solutions of system (3.22):

Proposition 3.5. *Let $\widetilde{\omega} \subset \Omega$ be a nonempty open set such that $\omega_0 \Subset \widetilde{\omega}$. Then, there exists a constant $\widehat{\lambda}_3$, such that for any $\lambda \geq \widehat{\lambda}_3$ there exists a constant $C(\lambda) > 0$ such that for any $i \in \{1, \dots, N\}$, any $g^1 \in L^2(0,T;V)$ and any $\psi^0 \in H$, the solution of (3.22) satisfies*

$$\begin{aligned} &\sum_{j=1, j\neq i}^{N} \left[\iint\limits_Q e^{-3s\alpha}(s^5\xi^5|\Delta\widetilde{\psi}_j|^2 + s^3\xi^3|\nabla\Delta\widetilde{\psi}_j|^2 + s\xi|\nabla\nabla\Delta\widetilde{\psi}_j|^2 + s^{-1}\xi^{-1}|\nabla\nabla\nabla\Delta\widetilde{\psi}_j|^2)dx\,dt\right] \\ &\qquad + s^5 \iint\limits_Q e^{-3s\alpha^*}(\xi^*)^5|\widetilde{\psi}|^2 dx\,dt \leq C\left(\iint\limits_Q e^{-s\alpha^*}(|g^1|^2 + |\nabla g^1|^2)dx\,dt\right. \\ &\qquad\qquad \left. + s^5 \sum_{j=1, j\neq i}^{N} \iint\limits_{\widetilde{\omega}\times(0,T)} e^{-3s\alpha}\xi^5|\Delta\widetilde{\psi}_j|^2 dx\,dt\right), \end{aligned} \tag{3.26}$$

for every $s \geq C$.

Proof. As mentioned in Remark 3.4, we consider $N = 2$ and $i = 2$. We apply the Proposition 3.2 to system (3.22) with $f = \rho_1'\,\psi$, $g = 0$ and $\widehat{\omega} = \widetilde{\omega}$. This gives (see (3.10) for the definition of $I_1(s,\widetilde{\psi})$):

$$I_1(s,\widetilde{\psi}) \leq C \left(s^{5/2} \iint\limits_Q e^{-3s\alpha^*} (\xi^*)^{3-2/m} |\rho_1' \, \psi|^2 \, dx \, dt \right.$$

$$+ s^{-1/2} \int_0^T e^{-3s\alpha^*} (\xi^*)^{-5/m} \left\| \rho_1' \, \psi \right\|^2_{H^3(\Omega)^2} \, dt + s^{-1/2} \int_0^T \left\| \left(e^{-3/2s\alpha^*} (\xi^*)^{-5/(2m)} \rho_1' \, \psi \right)_t \right\|^2_{H^1(\Omega)^2} dt$$

$$\left. + \iint\limits_Q e^{-3s\alpha} |\nabla \Delta(\rho_1' \, \psi_1)|^2 dx \, dt + s^5 \iint\limits_{\widetilde{\omega}\times(0,T)} e^{-3s\alpha} \xi^5 |\Delta \widetilde{\psi}_1|^2 dx \, dt \right), \quad (3.27)$$

for every $s \geq C$.
Now, we estimate the global terms in the right-hand side of (3.27) by the $L^2(0,T;V)$-norm of g^1 and $\varepsilon \, I_1(s,\widetilde{\psi})$, $\varepsilon > 0$ to be chosen small enough.

Note that from (2.3), we have

$$|\rho_1'| \leq Cs(\xi^*)^{1+1/m} \rho_1 \quad (3.28)$$

for every $s \geq C$. Thus, from (3.23), the fact that $s^{9/4} e^{-3/2s\alpha^*} (\xi^*)^{5/2}$ is bounded and (3.24), we obtain

$$\begin{aligned} \left\| s^{5/4} e^{-3/2s\alpha^*} (\xi^*)^{3/2-1/m} \rho_1' \, \psi \right\|^2_{L^2(Q)^2} &\leq C \left(\left\| s^{9/4} e^{-3/2s\alpha^*} (\xi^*)^{5/2} \, \widetilde{\psi} \right\|^2_{L^2(Q)^2} + \left\| \rho_1 \, g^1 \right\|^2_{L^2(Q)^2} \right) \\ &\leq \varepsilon \, I_1(s,\widetilde{\psi}) + C \left\| \rho_1 \, g^1 \right\|^2_{L^2(Q)^2}, \end{aligned} \quad (3.29)$$

for every $s \geq C^2/\varepsilon^2$.

In order to estimate the second and third terms in the right-hand side of (3.27), we define

$$\zeta(t) := s^{3/4} e^{-3/2s\alpha^*} (\xi^*)^{1-3/(2m)} \rho_1$$

and take a look to the system satisfied by $(\zeta \, \psi)$. Since the right-hand side of this equation belongs to $L^2(0,T;V)$, we apply the estimate (2.9) in Lemma 2.7 and we get

$$\|\zeta \, \psi\|^2_{L^2(0,T;H^3(\Omega)^2)} + \|\zeta \, \psi\|^2_{H^1(0,T;H^1(\Omega)^2)} \leq C \left(\left\| \zeta \, g^1 \right\|^2_{L^2(0,T;V)} + \|\zeta' \, \psi\|^2_{L^2(0,T;V)} \right). \quad (3.30)$$

From the estimate $|\zeta'| \leq Cs^{7/4} e^{-3/2s\alpha^*} (\xi^*)^{2-1/(2m)} \rho_1$ and the interpolation inequality

$$\|v\|_{H^1(\Omega)} \leq C \, \|v\|^{2/3}_{L^2(\Omega)} \, \|v\|^{1/3}_{H^3(\Omega)}, \; \forall v \in H^3(\Omega),$$

we have that

$$\|\zeta' \, \psi\|^2_{L^2(0,T;V)} \leq \varepsilon \left\| s^{5/2} e^{-3/2s\alpha^*} (\xi^*)^{5/2} \rho_1 \, \psi \right\|^2_{L^2(Q)^2} + s^{-1} C_\varepsilon \, \|\zeta \, \psi\|^2_{L^2(0,T;H^3(\Omega)^2)}.$$

Combining this with (3.30) and, using (3.23) and (3.25), we have that

$$\|\zeta \, \psi\|^2_{L^2(0,T;H^3(\Omega)^2)} + \|\zeta \, \psi\|^2_{H^1(0,T;H^1(\Omega)^2)} \leq \varepsilon \, I_1(s,\widetilde{\psi}) + C \left\| \rho_1 \, g^1 \right\|^2_{L^2(0,T;V)}, \quad (3.31)$$

for every $s \geq C$.

For the second last term, we use again (3.23), (3.25) and the fact that $\xi \geq C > 0$ in Q, to obtain

$$\iint\limits_Q e^{-3s\alpha}|\nabla\Delta(\rho_1'\,\psi_1)|^2 dx\,dt \leq \varepsilon I_1(s,\widetilde{\psi}) + \left\|\rho_1\, g^1\right\|^2_{L^2(0,T;V)} \tag{3.32}$$

for every $s \geq C/\varepsilon$.

Putting together (3.29), (3.31) and (3.32) in (3.27), and choosing $\varepsilon > 0$ small enough, we obtain (3.26) and conclude the proof of Proposition 3.5.

$\square$

3.3 Carleman for φ and final estimate

To finish the proof of Proposition 3.1, we turn now to the equation satisfied by φ :

$$\begin{cases} -\varphi_t - \Delta\varphi + \nabla\pi = g^0 + \psi 1\!\!1_{\mathcal{O}},\ \nabla\cdot\varphi = 0, & \text{in } Q, \\ \varphi = 0, & \text{on } \Sigma \\ \varphi(T) = 0, & \text{in } \Omega. \end{cases} \tag{3.33}$$

Assuming that ψ is given, we apply estimate (2.4) in Lemma 2.1 to φ:

$$I_0(s,\varphi) := s^3 \iint\limits_Q e^{-8/3s\alpha-4s\alpha^*}\xi^3|\Delta\varphi_1|^2 dx\,dt + s^4 \iint\limits_Q e^{-20/3s\alpha^*}(\xi^*)^4|\varphi|^2 dx\,dt$$

$$\leq C\left(\iint\limits_{\mathcal{O}\times(0,T)} e^{-4s\alpha^*}|\psi|^2 dx\,dt + \iint\limits_Q e^{-4s\alpha^*}|g^0|^2 dx\,dt + s^7 \iint\limits_{\omega\times(0,T)} e^{-8/3s\widehat{\alpha}-4s\alpha^*}\widehat{\xi}^7|\varphi_1|^2 dx\,dt\right) \tag{3.34}$$

for every $s \geq C$. Recall that $N = 2$ and $i = 2$.

Notice that from (3.23) we have

$$\iint\limits_{\mathcal{O}\times(0,T)} e^{-4s\alpha^*}|\psi|^2 dx\,dt = \iint\limits_{\mathcal{O}\times(0,T)} e^{-4s\alpha^*}|\rho_1|^{-2}|\psi^* + \widetilde{\psi}|^2 dx\,dt.$$

Since $e^{-4s\alpha^*}|\rho_1|^{-2} = e^{-3s\alpha^*}$, using estimate (3.24) and $s^5(\xi^*)^5 \geq C$, we have

$$\iint\limits_{\mathcal{O}\times(0,T)} e^{-4s\alpha^*}|\psi|^2 dx\,dt \leq C\left(s^5\iint\limits_Q e^{-3s\alpha^*}(\xi^*)^5|\widetilde{\psi}|^2 dx\,dt + \iint\limits_Q |\rho_1|^2|g^1|^2 dx\,dt\right),$$

for every $s \geq C$. Combining this with (3.34) and (3.26) from Proposition 3.5, we obtain

$$I_1(s,\widetilde{\psi}) + I_0(s,\varphi) \leq C\left(\iint\limits_Q e^{-4s\alpha^*}|g^0|^2 dx\,dt + \iint\limits_Q |\rho_1|^2(|g^1|^2 + |\nabla g^1|^2)dx\,dt\right.$$
$$\left. + s^5 \iint\limits_{\widetilde{\omega}\times(0,T)} e^{-3s\alpha}\xi^5|\Delta\widetilde{\psi}_1|^2 dx\,dt + s^7 \iint\limits_{\omega\times(0,T)} e^{-8/3s\widehat{\alpha}-4s\alpha^*}\widehat{\xi}^7|\varphi_1|^2 dx\,dt\right), \tag{3.35}$$

for every $s \geq C$ and $\widetilde{\omega} \Subset \omega \cap \mathcal{O}$.

To conclude the proof of Proposition 3.1, we estimate the local term in $\Delta\widetilde{\psi}_1$ in terms of local integrals of φ_1, global terms in g^0 and g^1, and $\varepsilon(I_1(s,\widetilde{\psi})+I_0(s,\varphi))$ with ε a small positive constant.

We start by looking at the equation satisfied by φ_1 in $\mathcal{O}\times(0,T)$ where, by applying the laplacian and multiplying by ρ_1, we find

$$\rho_1\Delta\psi_1=-\rho_1(\Delta\varphi_1)_t-\rho_1\Delta(\Delta\varphi_1)+\rho_1\partial_1\nabla\cdot g^0-\rho_1\Delta g_1^0\quad\text{in}\quad\mathcal{O}\times(0,T),$$

where we have used that $\Delta\pi=\nabla\cdot g^0$ in $\mathcal{O}\times(0,T)$. By (3.23), we have

$$\rho_1\,\Delta\psi_1=\Delta\psi_1^*+\Delta\widetilde{\psi}_1,$$

and therefore

$$\Delta\widetilde{\psi}_1=-\Delta\psi_1^*-\rho_1(\Delta\varphi_1)_t-\rho_1\Delta(\Delta\varphi_1)-\rho_1\Delta g_1^0+\rho_1\partial_1\nabla\cdot g^0\text{ in }\mathcal{O}\times(0,T).\tag{3.36}$$

Now, let $\theta\in C_0^4(\omega)$ be a nonnegative function with $\theta\equiv1$ in $\widetilde{\omega}$ (recall that $\widetilde{\omega}\Subset\omega$). Using (3.36) in the third term in the right-hand side of (3.35), since $\widetilde{\omega}\subset\mathcal{O}$, integration by parts leads to

$$\begin{aligned}
&s^5\iint\limits_{\widetilde{\omega}\times(0,T)}e^{-3s\alpha}\xi^5|\Delta\widetilde{\psi}_1|^2dx\,dt\\
&\le s^5\iint\limits_{\omega\times(0,T)}\theta e^{-3s\alpha}\xi^5\Delta\widetilde{\psi}_1(-\Delta\psi_1^*-\rho_1(\Delta\varphi_1)_t-\rho_1\Delta\Delta\varphi_1-\rho_1\Delta g_1^0+\rho_1\partial_1\nabla\cdot g^0)dx\,dt\\
&\quad=-s^5\iint\limits_{\omega\times(0,T)}\theta e^{-3s\alpha}\xi^5\Delta\widetilde{\psi}_1\,\Delta\psi_1^*dx\,dt+s^5\iint\limits_{\omega\times(0,T)}\theta(e^{-3s\alpha}\xi^5\rho_1)_t\Delta\widetilde{\psi}_1\Delta\varphi_1dx\,dt\\
&-s^5\iint\limits_{\omega\times(0,T)}\Delta(\theta e^{-3s\alpha}\xi^5)\rho_1\Delta\widetilde{\psi}_1\Delta\varphi_1dx\,dt-2s^5\iint\limits_{\omega\times(0,T)}\rho_1\nabla(\theta e^{-3s\alpha}\xi^5)\cdot\nabla\Delta\widetilde{\psi}_1\Delta\varphi_1dx\,dt\\
&\quad+s^5\iint\limits_{\omega\times(0,T)}\theta\rho_1e^{-3s\alpha}\xi^5\rho_1'\Delta\psi_1\Delta\varphi_1dx\,dt-s^5\iint\limits_{\omega\times(0,T)}\theta\rho_1e^{-3s\alpha}\xi^5\Delta^2\widetilde{\psi}_1g_1^0dx\,dt\\
&\quad-s^5\iint\limits_{\omega\times(0,T)}\Delta(\theta e^{-3s\alpha}\xi^5)\rho_1\Delta\widetilde{\psi}_1g_1^0dx\,dt-2s^5\iint\limits_{\omega\times(0,T)}\rho_1\nabla(\theta e^{-3s\alpha}\xi^5)\cdot\nabla\Delta\widetilde{\psi}_1g_1^0dx\,dt\\
&+s^5\iint\limits_{\omega\times(0,T)}\rho_1(\theta e^{-3s\alpha}\xi^5)\partial_1\nabla\Delta\widetilde{\psi}_1\cdot g^0dx\,dt+s^5\iint\limits_{\omega\times(0,T)}\rho_1\partial_1\nabla(\theta e^{-3s\alpha}\xi^5)\cdot\Delta\widetilde{\psi}_1g^0dx\,dt\\
&+s^5\iint\limits_{\omega\times(0,T)}\rho_1\nabla(\theta e^{-3s\alpha}\xi^5)\partial_1\Delta\widetilde{\psi}_1\cdot g^0dx\,dt+s^5\iint\limits_{\omega\times(0,T)}\rho_1\partial_1(\theta e^{-3s\alpha}\xi^5)\nabla\Delta\widetilde{\psi}_1\cdot g^0dx\,dt\\
&\quad=:\sum_{k=1}^{12}J_k,
\end{aligned}\tag{3.37}$$

for every $s\ge C$, where we have used the equation satisfied by $\Delta\widetilde{\psi}_1$ to obtain J_5.

To estimate J_1, we use Young's inequality and (3.24). We obtain

$$J_1\le\varepsilon s^5\iint\limits_Q e^{-3s\alpha}\xi^5|\Delta\widetilde{\psi}_1|^2dx\,dt+C\iint\limits_Q|\rho_1|^2|g^1|^2dx\,dt,\tag{3.38}$$

for every $s \geq C$ and any $\varepsilon > 0$.

For J_2, we perform another integration by parts

$$J_2 = s^5 \iint\limits_{\omega\times(0,T)} \theta(e^{-3s\alpha}\xi^5\rho_1)_t \Delta^2\widetilde{\psi}_1\varphi_1 dx\,dt + 2s^5 \iint\limits_{\omega\times(0,T)} \nabla(\theta(e^{-3s\alpha}\xi^5\rho_1)_t)\cdot\nabla\Delta\widetilde{\psi}_1\varphi_1 dx\,dt$$
$$+ s^5 \iint\limits_{\omega\times(0,T)} \Delta(\theta(e^{-3s\alpha}\xi^5\rho_1)_t)\Delta\widetilde{\psi}_1\varphi_1 dx\,dt.$$

Using Young's inequality and the estimates

$$|\theta(e^{-3s\alpha}\xi^5\rho_1)_t| \leq Cse^{-3s\alpha}\xi^{6+1/m}\rho_1,$$

$$|\nabla[\theta(e^{-3s\alpha}\xi^5\rho_1)_t]| \leq Cs^2e^{-3s\alpha}\xi^{7+1/m}\rho_1,$$

and

$$|\Delta[\theta(e^{-3s\alpha}\xi^5\rho_1)_t]| \leq Cs^3e^{-3s\alpha}\xi^{8+1/m}\rho_1,$$

for every $s \geq C$, we have

$$\begin{aligned} J_2 \leq& \varepsilon \iint\limits_{Q} e^{-3s\alpha}(s^5\xi^5|\Delta\widetilde{\psi}_1|^2 + s^3\xi^3|\nabla\Delta\widetilde{\psi}_1|^2 + s\xi|\Delta^2\widetilde{\psi}_1|^2)dx\,dt \\ &+ Cs^{11} \iint\limits_{\omega\times(0,T)} e^{-3s\alpha}\xi^{11+2/m}|\rho_1|^2|\varphi_1|^2 dx\,dt, \end{aligned} \tag{3.39}$$

for every $s \geq C$ and any $\varepsilon > 0$.

For J_3, we integrate by parts twice in space:

$$J_3 = -s^5 \iint\limits_{\omega\times(0,T)} \Delta(\theta e^{-3s\alpha}\xi^5)\rho_1\Delta^2\widetilde{\psi}_1\varphi_1 dx\,dt - 2s^5 \iint\limits_{\omega\times(0,T)} \rho_1\nabla\Delta(\theta e^{-3s\alpha}\xi^5)\cdot\nabla\Delta\widetilde{\psi}_1\varphi_1 dx\,dt$$
$$- s^5 \iint\limits_{\omega\times(0,T)} \Delta^2(\theta e^{-3s\alpha}\xi^5)\rho_1\Delta\widetilde{\psi}_1\varphi_1 dx\,dt.$$

Using young's inequality and the estimates

$$|\Delta(\theta e^{-3s\alpha}\xi^5)| \leq Cs^2e^{-3s\alpha}\xi^7, \tag{3.40}$$

$$|\nabla\Delta(\theta e^{-3s\alpha}\xi^5)| \leq Cs^3e^{-3s\alpha}\xi^8, \tag{3.41}$$

and

$$|\Delta^2(\theta e^{-3s\alpha}\xi^5)| \leq Cs^4e^{-3s\alpha}\xi^9, \tag{3.42}$$

for every $s \geq C$, we have

$$\begin{aligned} J_3 \leq& \varepsilon \iint\limits_{Q} e^{-3s\alpha}(s^5\xi^5|\Delta\widetilde{\psi}_1|^2 + s^3\xi^3|\nabla\Delta\widetilde{\psi}_1|^2 + s\xi|\Delta^2\widetilde{\psi}_1|^2)dx\,dt \\ &+ Cs^{13} \iint\limits_{\omega\times(0,T)} e^{-3s\alpha}\xi^{13}|\rho_1|^2|\varphi_1|^2 dx\,dt, \end{aligned} \tag{3.43}$$

for every $s \geq C$ and any $\varepsilon > 0$.

We integrate by parts again in J_4:

$$J_4 = -2s^5 \iint\limits_{\omega\times(0,T)} \rho_1 \nabla(\theta e^{-3s\alpha}\xi^5)\cdot\nabla\Delta^2\widetilde{\psi}_1\varphi_1 dx\,dt - 4s^5 \iint\limits_{\omega\times(0,T)} \rho_1\nabla\nabla(\theta e^{-3s\alpha}\xi^5) : \nabla\nabla\Delta\widetilde{\psi}_1\varphi_1 dx\,dt$$
$$- 2s^5 \iint\limits_{\omega\times(0,T)} \rho_1\nabla\Delta(\theta e^{-3s\alpha}\xi^5)\cdot\nabla\Delta\widetilde{\psi}_1\varphi_1 dx\,dt,$$

Young's inequality and estimates

$$|\nabla(\theta e^{-3s\alpha}\xi^5)| \le Cse^{-3s\alpha}\xi^6, \tag{3.44}$$

$$|\nabla\nabla(\theta e^{-3s\alpha}\xi^5)| \le Cs^2e^{-3s\alpha}\xi^7, \tag{3.45}$$

for every $s \ge C$, and (3.41) yield

$$\begin{aligned} J_4 \le& \varepsilon \iint\limits_{Q} e^{-3s\alpha}(s^{-1}\xi^{-1}|\nabla\Delta^2\widetilde{\psi}_1|^2 + s\xi|\nabla\nabla\Delta\widetilde{\psi}_1|^2 + s^3\xi^3|\nabla\Delta\widetilde{\psi}_1|^2)dx\,dt \\ &+ Cs^{13} \iint\limits_{\omega\times(0,T)} e^{-3s\alpha}\xi^{13}|\rho_1|^2|\varphi_1|^2 dx\,dt, \end{aligned} \tag{3.46}$$

for every $s \ge C$ and any $\varepsilon > 0$.

Using (3.23) and integration by parts in J_5 gives

$$\begin{aligned} J_5 =& s^5 \iint\limits_{\omega\times(0,T)} \theta e^{-3s\alpha}\xi^5\rho_1'\,(\Delta\psi_1^* + \Delta\widetilde{\psi}_1)\Delta\varphi_1 dx\,dt \\ =& s^5 \iint\limits_{\omega\times(0,T)} \theta e^{-3s\alpha}\xi^5\rho_1'\,\Delta\psi_1^*\Delta\varphi_1 dx\,dt + s^5 \iint\limits_{\omega\times(0,T)} \theta\rho_1' e^{-3s\alpha}\xi^5\Delta^2\widetilde{\psi}_1\varphi_1 dx\,dt \\ &+ 2s^5 \iint\limits_{\omega\times(0,T)} \rho_1'\nabla(\theta e^{-3s\alpha}\xi^5)\cdot\nabla\Delta\widetilde{\psi}_1\varphi_1 dx\,dt + s^5 \iint\limits_{\omega\times(0,T)} \rho_1'\Delta(\theta e^{-3s\alpha}\xi^5)\Delta\widetilde{\psi}_1\varphi_1 dx\,dt, \end{aligned}$$

Using Young's inequality and estimates (3.24), (3.44) and (3.45), we obtain

$$\begin{aligned} J_5 \le& \varepsilon \iint\limits_{Q} e^{-3s\alpha}(s^5\xi^5|\Delta\widetilde{\psi}_1|^2 + s^3\xi^3|\nabla\Delta\widetilde{\psi}_1|^2 + s\xi|\Delta^2\widetilde{\psi}_1|^2)dx\,dt \\ &+ \varepsilon\, s^3 \iint\limits_{\omega\times(0,T)} e^{-8/3s\alpha - 4s\alpha^*}\xi^3|\Delta\varphi_1|^2)dx\,dt \\ &+ C\left(s^9 \iint\limits_{\omega\times(0,T)} e^{-3s\alpha}\xi^9|\rho_1'|^2|\varphi_1|^2 dx\,dt + \iint\limits_{Q} |\rho_1|^2|g^1|^2 dx\,dt\right), \end{aligned} \tag{3.47}$$

for every $s \ge C$ and any $\varepsilon > 0$.

Finally, the rest of the terms in (3.37) are estimated using Young's inequality and the estimates (3.44) and (3.45). Namely, we obtain

$$J_7 + J_{10} \le \varepsilon\, s^5 \iint\limits_{Q} e^{-3s\alpha}\xi^5|\Delta\widetilde{\psi}_1|^2 dx\,dt + Cs^9 \iint\limits_{Q} e^{-3s\alpha}\xi^9|\rho_1|^2|g^0|^2 dx\,dt, \tag{3.48}$$

$$J_8 + J_{11} + J_{12} \le \varepsilon\, s^3 \iint\limits_Q e^{-3s\alpha}\xi^3|\nabla\Delta\widetilde{\psi}_1|^2 dx\,dt + Cs^9 \iint\limits_Q e^{-3s\alpha}\xi^9|\rho_1|^2|g^0|^2 dx\,dt, \quad (3.49)$$

and

$$J_6 + J_9 \le \varepsilon\, s \iint\limits_Q e^{-3s\alpha}\xi|\nabla\nabla\Delta\widetilde{\psi}_1|^2 dx\,dt + Cs^9 \iint\limits_Q e^{-3s\alpha}\xi^9|\rho_1|^2|g^0|^2 dx\,dt. \quad (3.50)$$

Combining (3.38), (3.39), (3.43) and (3.46)- (3.50) in (3.37), and then in (3.35), together with the fact that

$$s^{11}e^{-3s\alpha}\xi^{11+2/m}|\rho_1|^2 + s^9 e^{-3s\alpha}\xi^9|\rho_1'|^2 + s^7 e^{-8/3s\widehat{\alpha}-4s\alpha^*}\widehat{\xi}^7 \le Cs^{13}e^{-3s\alpha}\xi^{13}|\rho_1|^2,$$

for every $s \ge C$, and $e^{-7s\alpha^*} \le e^{-20/3s\alpha^*}$, we deduce (3.2). This concludes the proof of Proposition 3.1.

4 Null controllability of the linear system

In this section we deal with the null controllability of system

$$\begin{cases} Lw + \nabla p^0 = f^0 + v\mathbb{1}_\omega,\ \nabla\cdot w = 0 & \text{in } Q,\\ L^*z + \nabla q = f^1 + w\mathbb{1}_{\mathcal{O}},\ \nabla\cdot z = 0 & \text{in } Q,\\ w = z = 0 & \text{on } \Sigma,\\ w(0) = 0,\ z(T) = 0 & \text{in } \Omega. \end{cases} \quad (4.1)$$

Here, we will assume that f_0 and f_1 are in appropriate weighted functional spaces,

$$Lw := w_t - \Delta w,$$

and

$$L^*z := -z_t - \Delta z,$$

which is the adjoint operator of L. We look for a control v such that $v_i \equiv 0$, for some given $i \in \{1,\dots,N\}$, such that the associated solution of (4.1) satisfies $z(0) = 0$.

To do this, let us first state a Carleman inequality with weight functions not vanishing in $t = T$. We introduce the following weight functions:

$$\begin{aligned} &\beta(x,t) = \frac{e^{2\lambda\|\eta\|_\infty} - e^{\lambda\eta(x)}}{\widetilde{\ell}(t)^m},\ \gamma(x,t) = \frac{e^{\lambda\eta(x)}}{\widetilde{\ell}(t)^m},\\ &\beta^*(t) = \max_{x\in\overline{\Omega}}\alpha(x,t),\ \gamma^*(t) = \min_{x\in\overline{\Omega}}\gamma(x,t),\\ &\widehat{\beta}(t) = \min_{x\in\overline{\Omega}}\beta(x,t),\ \widehat{\gamma}(t) = \max_{x\in\overline{\Omega}}\gamma(x,t), \end{aligned} \quad (4.2)$$

where

$$\widetilde{\ell}(t) = \begin{cases} \ell(t) & 0 \le t \le T/2,\\ \|\ell\|_\infty & T/2 < t \le T. \end{cases} \quad (4.3)$$

Lemma 4.1. *Let $i \in \{1,\dots,N\}$ and let s and λ be like in Proposition 3.1. Then, there exists a constant $C > 0$ (depending on s and λ) such that every solution (φ,ψ) of (3.1)* *satisfies*

$$\iint\limits_Q e^{-7s\beta^*}(\gamma^*)^4|\varphi|^2dx\,dt + \iint\limits_Q e^{-4s\beta^*}(\gamma^*)^5|\psi|^2dx\,dt$$
$$\le C\left(\iint\limits_Q e^{-3s\widehat{\beta}-s\beta^*}(\widehat{\gamma})^9|g^0|^2 + \iint\limits_Q e^{-s\beta^*}(|g^1|^2+|\nabla g^1|^2)dx\,dt \right.$$
$$\left. + \sum_{j=1,j\neq i}^N \iint\limits_{\omega\times(0,T)} e^{-3s\widehat{\beta}-s\beta^*}(\widehat{\gamma})^{13}|\varphi_j|^2dx\,dt\right). \quad (4.4)$$

To prove estimate (4.4) it suffices to combine (3.2) and classical energy estimates for the Stokes system satisfied by φ and ψ. For simplicity, we omit the proof. For more details on how to obtain (4.4), please see [3], [4] or [12].

Now we are ready to prove the null controllability of system (4.1). The idea is to look for a solution in an appropriate weighted functional space. To this end, we introduce the space

$$\begin{aligned} E_i = \{(w,p^0,z,q,v) :\, & e^{3/2s\widehat{\beta}+1/2s\beta^*}\widehat{\gamma}^{-9/2}w \in L^2(Q)^N,\ e^{1/2s\beta^*}z \in L^2(0,T;H^{-1}(\Omega)^N),\\ & e^{3/2s\widehat{\beta}+1/2s\beta^*}\widehat{\gamma}^{-13/2}v\mathbb{1}_\omega \in L^2(Q)^N,\ v_i\equiv 0, z(T)\equiv 0,\\ & e^{7/4s\beta^*}w \in L^2(0,T;H^2(\Omega)^N)\cap L^\infty(0,T;V),\\ & e^{1/2s\beta^*}(\gamma^*)^{-2-2/m}z \in L^2(0,T;H^2(\Omega)^N)\cap L^\infty(0,T;V),\\ & e^{7/2s\beta^*}(\gamma^*)^{-2}(Lw+\nabla p^0 - v\mathbb{1}_\omega)\in L^2(Q)^N,\\ & e^{2s\beta^*}(\gamma^*)^{-5/2}(L^*z+\nabla q - w\mathbb{1}_{\mathcal{O}})\in L^2(Q)^N\}. \end{aligned}$$

It is clear that E_i is a Banach space with the norm:

$$\begin{aligned} \|(w,p^0,z,q,v)\|_{E_i} := \Bigg(&\left\|e^{3/2s\widehat{\beta}+1/2s\beta^*}\widehat{\gamma}^{-9/2}w\right\|^2_{L^2(Q)^N} + \left\|e^{1/2s\beta^*}z\right\|^2_{L^2(0,T;H^{-1}(\Omega)^N)}\\ & + \left\|e^{3/2s\widehat{\beta}+1/2s\beta^*}\widehat{\gamma}^{-13/2}v\mathbb{1}_\omega\right\|^2_{L^2(Q)^N}\\ & + \left\|e^{7/4s\beta^*}w\right\|^2_{L^2(0,T;H^2(\Omega)^N)} + \left\|e^{7/4s\beta^*}w\right\|^2_{L^\infty(0,T;V)}\\ & + \left\|e^{1/2s\beta^*}(\gamma^*)^{-2-2/m}z\right\|^2_{L^2(0,T;H^2(\Omega)^N)} + \left\|e^{1/2s\beta^*}(\gamma^*)^{-2-2/m}z\right\|^2_{L^\infty(0,T;V)}\\ & + \left\|e^{7/2s\beta^*}(\gamma^*)^{-2}(Lw+\nabla p^0 - v\mathbb{1}_\omega)\right\|^2_{L^2(Q)^N}\\ & + \left\|e^{2s\beta^*}(\gamma^*)^{-5/2}(L^*z+\nabla q - w\mathbb{1}_{\mathcal{O}})\right\|^2_{L^2(Q)^N}\Bigg)^{1/2}. \end{aligned}$$

Remark 4.2. *In particular, an element (w,p^0,z,q,v) of E_i satisfies $w(0)=0$, $z(0)=0$ and $v_i\equiv 0$. Moreover, we have that*

$$\begin{aligned} & e^{7/2s\beta^*}(\gamma^*)^{-2}(w\cdot\nabla)w \in L^2(Q)^N,\\ & e^{2s\beta^*}(\gamma^*)^{-5/2}(z\cdot\nabla^t)w \in L^2(Q)^N,\\ & e^{2s\beta^*}(\gamma^*)^{-5/2}(w\cdot\nabla)z \in L^2(Q)^N. \end{aligned} \quad (4.5)$$

Proposition 4.3. *Assume the hypothesis of Lemma 4.1 and*

$$e^{7/2s\beta^*}(\gamma^*)^{-2}f^0 \in L^2(Q)^N \text{ and } e^{2s\beta^*}(\gamma^*)^{-5/2}f^1 \in L^2(Q)^N. \tag{4.6}$$

Then, we can find a control v such that the associated solution (w, p^0, z, q) to (4.1) satisfies $(w, p^0, z, q, v) \in E_i$. In particular, $v_i \equiv 0$ and $z(0) = 0$.

Proof. Following the arguments in [9] and [13], we introduce the space

$$\begin{aligned} P_0 = \{(\chi, \sigma, \mu, \nu) \in C^3(\overline{Q})^{2N+2} : \quad & \nabla\cdot\chi = \nabla\cdot\mu = 0 \text{ in } Q, \Delta\nu = 0 \text{ in } Q, \chi|_\Sigma = \mu|_\Sigma = 0,\\ & \chi(T) = \mu(0) = 0,\ L\mu + \nabla\nu|_\Sigma = 0\} \end{aligned}$$

and consider the operators

$$a((\widehat{\chi}, \widehat{\sigma}, \widehat{\mu}, \widehat{\nu}), (\chi, \sigma, \mu, \nu)) := \iint\limits_Q e^{-3s\widehat{\beta}-s\beta^*}\widehat{\gamma}^9(L^*\widehat{\chi} + \nabla\widehat{\sigma} - \widehat{\mu}1\!\!1_{\mathcal{O}})\cdot(L^*\chi + \nabla\sigma - \mu 1\!\!1_{\mathcal{O}})\,dx\,dt$$

$$+ \iint\limits_Q e^{-s\beta^*}\left[(L\widehat{\mu} + \nabla\widehat{\nu})\cdot(L\mu + \nabla\nu) + \nabla(L\widehat{\mu} + \nabla\widehat{\nu}) : \nabla(L\mu + \nabla\nu)\right]\,dx\,dt$$

$$+ \sum_{j=1, j\neq i}^{N} \iint\limits_{\omega\times(0,T)} e^{-3s\widehat{\beta}-s\beta^*}\widehat{\gamma}^{13}\widehat{\chi}_j\cdot\chi_j\,dx\,dt,$$

and

$$\langle G, (\chi, \sigma, \mu, \nu)\rangle := \iint\limits_Q f^0\cdot\chi\,dx\,dt + \iint\limits_Q f^1\cdot\mu\,dx\,dt.$$

Thanks to (4.4), we have that $a(\cdot,\cdot) : P_0 \times P_0 \mapsto \mathbb{R}$ is a symmetric, definite positive bilinear form on P_0. We denote by P the completion of P_0 for the norm induced by $a(\cdot,\cdot)$. Then, $a(\cdot,\cdot)$ is well-defined, continuous and definite positive on P. Furthermore, in view of the Carleman estimate (4.4) and the assumptions (4.6), the linear form $(\chi, \sigma, \mu, \nu) \mapsto \langle G, (\chi, \sigma, \mu, \nu)\rangle$ is well-defined and continuous on P. Hence, from Lax-Milgram's lemma, we deduce that the variational problem:

$$\begin{cases} \text{Find } (\widehat{\chi}, \widehat{\sigma}, \widehat{\mu}, \widehat{\nu}) \in P \text{ such that} \\ a((\widehat{\chi}, \widehat{\sigma}, \widehat{\mu}, \widehat{\nu}), (\chi, \sigma, \mu, \nu)) = \langle G, (\chi, \sigma, \mu, \nu)\rangle \quad \forall\,(\chi, \sigma, \mu, \nu) \in P, \end{cases} \tag{4.7}$$

possesses exactly one solution $(\widehat{\chi}, \widehat{\sigma}, \widehat{\mu}, \widehat{\nu})$.

Let $\widehat{w}$, $\widehat{z}$ and $\widehat{v}$ be given by

$$\begin{cases} \widehat{w} = e^{-3s\widehat{\beta}-s\beta^*}(\widehat{\gamma})^9(L^*\widehat{\chi} + \nabla\widehat{\sigma} - \widehat{\mu}1\!\!1_{\mathcal{O}}) & \text{in } Q,\\ \widehat{z} = e^{-s\beta^*}(L\widehat{\mu} + \nabla\widehat{\nu} - \Delta(L\widehat{\mu} + \nabla\widehat{\nu})) & \text{in } Q,\\ \widehat{v}_j = -e^{-3s\widehat{\beta}-s\beta^*}(\widehat{\gamma})^{13}\widehat{\chi}_j 1\!\!1_\omega,\ j \neq i,\ \widehat{v}_i \equiv 0 & \text{in } Q. \end{cases} \tag{4.8}$$

Note that

$$\int_0^T e^{s\beta^*}\,\|\widehat{z}\|^2_{H^{-1}(\Omega)^N}\;dt=\int_0^T e^{s\beta^*}\sup_{\|\zeta\|_{H^1_0(\Omega)}=1}\langle\widehat{z},\zeta\rangle^2_{H^{-1}(\Omega)^N\times H^1_0(\Omega)}\,dt$$
$$=\int_0^T e^{-s\beta^*}\sup_{\|\zeta\|_{H^1_0(\Omega)}=1}\langle L\widehat{\mu}+\nabla\widehat{\nu}-\Delta(L\widehat{\mu}+\nabla\widehat{\nu}),\zeta\rangle^2_{H^{-1}(\Omega)^N\times H^1_0(\Omega)}\,dt$$
$$=\int_0^T e^{-s\beta^*}\sup_{\|\zeta\|_{H^1_0(\Omega)}=1}((L\widehat{\mu}+\nabla\widehat{\nu},\zeta)_{L^2(\Omega)^N}+(\nabla(L\widehat{\mu}+\nabla\widehat{\nu}),\nabla\zeta)_{L^2(\Omega)^N})^2\,dt$$
$$\leq\iint_Q e^{-s\beta^*}(|L\widehat{\mu}+\nabla\widehat{\nu}|^2+|\nabla(L\widehat{\mu}+\nabla\widehat{\nu})|^2)dx\,dt.$$

Furthermore, the equality can be achieved, and thus, it is readily seen that we have

$$\iint_Q e^{3s\widehat{\beta}+s\beta^*}(\widehat{\gamma})^{-9}|\widehat{w}|^2\,dx\,dt+\int_0^T e^{s\beta^*}\,\|\widehat{z}\|^2_{H^{-1}(\Omega)^N}\;dt+\sum_{j=1,j\neq i}^N\;\iint_{\omega\times(0,T)} e^{3s\widehat{\beta}+s\beta^*}(\widehat{\gamma})^{-13}|\widehat{v}_j|^2\,dx\,dt$$
$$=a((\widehat{\chi},\widehat{\sigma},\widehat{\mu},\widehat{\nu}),(\widehat{\chi},\widehat{\sigma},\widehat{\mu},\widehat{\nu}))<+\infty,$$

Now, let us introduce the weak solution $(\widetilde{w},\widetilde{z},\widetilde{p},\widetilde{q})$ of the Stokes system (4.1) with $v\equiv\widehat{v}$. It is readily seen that this is also the (unique) solution defined by transposition, i.e., it satisfies

$$\iint_Q \widetilde{w}\cdot g^0\,dx\,dt+\iint_Q \widetilde{z}\cdot g^1\,dx\,dt=\iint_Q (f^0+\widehat{v})\cdot\varphi\,dx\,dt+\iint_Q f^1\cdot\psi\,dx\,dt,\ \forall\,(g^0,g^1)\in L^2(Q)^{2N}, \tag{4.9}$$

where (φ,ψ) is, together with some (π,κ), the solution of

$$\begin{cases} L^*\varphi+\nabla\pi=g^0+\psi 1\!\!1_{\mathcal{O}},\ \nabla\cdot\varphi=0 & \text{in } Q,\\ L\psi+\nabla\kappa=g^1,\ \ \nabla\cdot\psi=0 & \text{in } Q,\\ \varphi=\psi=0 & \text{on } \Sigma,\\ \varphi(T)=0,\ \ \psi(0)=0 & \text{in } \Omega.\end{cases} \tag{4.10}$$

Notice that, since $L^2(\Omega)=H\oplus H^\perp$ ($H^\perp=\{\nabla p: p\in H^1(\Omega)\}$, see for instance [20]) and $\nabla\cdot\widetilde{z}=0$, we have an equivalent formulation for all $(g^0,g^1)\in L^2(Q)^N\times L^2(0,T;H)$ in (4.9).

The next task is to check that $(\widehat{w},\widehat{z})$ coincides with the weak solution of the Stokes system (4.1). For this, we are going to prove that $(\widehat{w},\widehat{z})$ satisfies (4.9). It is not difficult to prove that, from (4.7), (4.8) and performing an integration by parts in space, $(\widehat{w},\widehat{z})$ satisfies

$$\iint_Q \widehat{w}\cdot(L^*\chi+\nabla\sigma-\mu 1\!\!1_{\mathcal{O}})\,dx\,dt+\iint_Q \widehat{z}\cdot(L\mu+\nabla\nu)\,dx\,dt$$
$$=\iint_Q (f^0+\widehat{v})\cdot\chi\,dx\,dt+\iint_Q f^1\cdot\mu\,dx\,dt,\quad\forall\,(\chi,\sigma,\mu,\nu)\in P_0. \tag{4.11}$$

By a density argument, we will show that this is equivalent to (4.9) for all $(g^0,g^1)\in L^2(Q)^N\times L^2(0,T;H)$. Indeed, for such a pair (g^0,g^1), there exist a sequence

$$(g^0_k,g^1_k)\in C^\infty_0(Q)^N\times C^\infty_0((0,T);\mathcal{V})$$

converging to (g^0, g^1) in $L^2(Q)^{2N}$, where $\mathcal{V} = \{y \in C_0^\infty(\Omega)^N : \nabla \cdot y = 0 \text{ in } \Omega\}$.

Now, let $(\chi_k, \sigma_k, \mu_k, \nu_k)$ be the solution to

$$\begin{cases} L^*\chi_k + \nabla\sigma_k = g_k^0 + \mu_k\,\theta_k,\ \nabla\cdot\chi_k = 0 & \text{in } Q,\\ L\mu_k + \nabla\nu_k = g_k^1,\ \nabla\cdot\mu_k = 0 & \text{in } Q,\\ \chi_k = \mu_k = 0 & \text{on } \Sigma,\\ \chi_k(T) = 0,\ \mu_k(0) = 0 & \text{in } \Omega, \end{cases}$$

where $\theta_k \in C^\infty(\overline{\Omega})$ satisfies $\theta_k \to 1\!\!1_{\mathcal{O}}$ in $L^2(\Omega)$ as $k \to \infty$. Then, it is not difficult to see that $(\chi_k, \sigma_k, \mu_k, \nu_k) \in P_0$. Thanks to regularity estimates for the Stokes system (Lemma 2.6), we obtain that (χ_k, μ_k) converges to (φ, ψ) (solution of (4.10)) in $L^2(0,T;H^2(\Omega)^{2N}) \cap H^1(0,T;L^2(\Omega)^{2N})$. Then, we observe that

$$\iint_Q \mu_k \cdot (1\!\!1_{\mathcal{O}} - \theta_k)\,dx\,dt \to 0 \text{ as } k \to \infty,$$

and we can pass to the limit in (4.11) for $(\chi_k, \sigma_k, \mu_k, \nu_k)$ and establish that $(\widehat{w}, \widehat{z})$ is also a solution of (4.9) for any $(g^0, g^1) \in L^2(Q)^N \times L^2(0,T;H)$. Then $(\widehat{w}, \widehat{z}) = (\widetilde{w}, \widetilde{z})$ is, together with some $(\widehat{p}^0, \widehat{q})$, the weak solution of system (4.1) for $v = \widehat{v}$.

It only remains to check that

$$e^{7/4s\beta^*}\widehat{w} \in L^2(0,T;H^2(\Omega)^N) \cap L^\infty(0,T;V)$$

and

$$e^{1/2s\beta^*}(\gamma^*)^{-2-2/m}\widehat{z} \in L^2(0,T;H^2(\Omega)^N) \cap L^\infty(0,T;V).$$

To this purpose, let us define the functions

$$w_* := e^{7/4s\beta^*}\widehat{w},\ p_*^0 := e^{7/4s\beta^*}\widehat{p}^0,$$
$$z_* := e^{1/2s\beta^*}(\gamma^*)^{-1-1/m}\widehat{z},\ q_* := e^{1/2s\beta^*}(\gamma^*)^{-1-1/m}\widehat{q},$$
$$f_*^0 := e^{7/4s\beta^*}(f^0 + \widehat{v}1\!\!1_\omega),\ f_*^1 := e^{1/2s\beta^*}(\gamma^*)^{-1-1/m}(f^1 + \widehat{w}1\!\!1_{\mathcal{O}}).$$

Then, (w_*, p_*^0, z_*, q_*) satisfies

$$\begin{cases} Lw_* + \nabla p_*^0 = f_*^0 + (e^{7/4s\beta^*})_t\widehat{w},\ \nabla\cdot w_* = 0 & \text{in } Q,\\ L^*z_* + \nabla q_* = f_*^1 + (e^{1/2s\beta^*}(\gamma^*)^{-1-1/m})_t\widehat{z},\ \nabla\cdot z_* = 0 & \text{in } Q,\\ w_* = z_* = 0 & \text{on } \Sigma,\\ w_*(0) = 0,\ z_*(T) = 0 & \text{in } \Omega. \end{cases}$$

From the fact that $f_*^0 + (e^{7/4s\beta^*})_t\widehat{w} \in L^2(Q)^N$ and $f_*^1 + (e^{1/2s\beta^*}(\gamma^*)^{-1-1/m})_t\widehat{z} \in L^2(0,T;H^{-1}(\Omega)^N)$, we have indeed

$$w_* \in L^2(0,T;H^2(\Omega)^N) \cap L^\infty(0,T;V)$$

and

$$z_* \in L^2(0,T;H^1(\Omega)^N) \cap L^\infty(0,T;H)$$

(see (2.8)). Finally let $(z_{**}, q_{**}) := e^{1/2s\beta^*}(\gamma^*)^{-2-2/m}(\widehat{z}, \widehat{q})$. Then, (z_{**}, q_{**}) satisfies

$$\begin{cases} L^*z_{**} + \nabla q_{**} = f_{**}^1 + (e^{1/2s\beta^*}(\gamma^*)^{-2-2/m})_t\widehat{z},\ \nabla\cdot z_{**} = 0 & \text{in } Q,\\ z_{**} = 0 & \text{on } \Sigma,\\ z_{**}(T) = 0 & \text{in } \Omega, \end{cases}$$

where $f_{**}^1 := e^{1/2s\beta^*}(\gamma^*)^{-2-2/m}(f^1 + \widehat{w}1\!\!1_{\mathcal{O}}) \in L^2(Q)^N$ and $(e^{1/2s\beta^*}(\gamma^*)^{-2-2/m})_t\widehat{z} \in L^2(Q)^N$. Using again (2.8), we deduce that

$$z_{**} \in L^2(0,T;H^2(\Omega)^N) \cap L^\infty(0,T;V).$$

This concludes the proof of Proposition 4.3. $\square$

5 Proof of Theorem 1.1

In this section we give the proof of Theorem 1.1 using similar arguments to those in [13] (see also [4], [8], [11] and [12]). The result of null controllability for the linear system (4.1) given by Proposition 4.3 will allow us to apply an inverse mapping theorem. Namely, we will use the following result (see [1]).

Theorem 5.1. *Let B_1 and B_2 be two Banach spaces and let $\mathcal{A} : B_1 \to B_2$ satisfy $\mathcal{A} \in C^1(B_1; B_2)$. Assume that $b_1 \in B_1$, $\mathcal{A}(b_1) = b_2$ and that $\mathcal{A}'(b_1) : B_1 \to B_2$ is surjective. Then, there exists $\delta > 0$ such that, for every $b' \in B_2$ satisfying $\|b' - b_2\|_{B_2} < \delta$, there exists a solution of the equation*

$$\mathcal{A}(b) = b', \quad b \in B_1.$$

Recall that we deal with the control system

$$\begin{cases} Lw + (w\cdot\nabla)w + \nabla p^0 = f + v1\!\!1_\omega, \ \nabla\cdot w = 0 & \text{in } Q, \\ L^*z + (z\cdot\nabla^t)w - (w\cdot\nabla)z + \nabla q = w1\!\!1_{\mathcal{O}}, \ \nabla\cdot z = 0 & \text{in } Q, \\ w = z = 0 & \text{on } \Sigma, \\ w(0) = 0, \ z(T) = 0 & \text{in } \Omega, \end{cases} \tag{5.1}$$

that is, we look for a control v, with $v_i \equiv 0$, such that $z(0) = 0$. We apply Theorem 5.1 setting

$$B_1 = E_i,$$
$$B_2 = L^2(e^{7/2s\beta^*}(\gamma^*)^{-2}(0,T); L^2(\Omega)^N) \times L^2(e^{2s\beta^*}(\gamma^*)^{-5/2}(0,T); L^2(\Omega)^N)$$

and the operator

$$\mathcal{A}(w,p^0,z,q,v) = (Lw + (w\cdot\nabla)w + \nabla p^0 - v1\!\!1_\omega, \, L^*z + (z\cdot\nabla^t)w - (w\cdot\nabla)z + \nabla q - w1\!\!1_{\mathcal{O}})$$

for $(w,p^0,z,q,v) \in E_i$.

In order to apply Theorem 5.1, it remains to check that the operator $\mathcal{A}$ is of class $C^1(B_1; B_2)$. Indeed, notice that all terms in $\mathcal{A}$ are linear, except for $(w\cdot\nabla)w$, $(z\cdot\nabla^t)w$ and $(w\cdot\nabla)z$. We will prove that the bilinear operator

$$((w^1,p^{0,1},z^1,q^1,v^1),(w^2,p^{0,2},z^2,q^2,v^2)) \to (w^1\cdot\nabla)w^2$$

is continuous from $B_1 \times B_1$ to $L^2(e^{7/2s\beta^*}(\gamma^*)^{-2}(0,T); L^2(\Omega)^N)$. To do this, notice that

$$e^{7/4s\beta^*}w \in L^2(0,T; H^2(\Omega)^N) \cap L^\infty(0,T;V),$$

for any $(w,p^0,z,q,v) \in B_1$, so we have

$$e^{7/4s\beta^*}w^1 \in L^2(0,T;L^\infty(\Omega)^N) \text{ and } \nabla(e^{7/4s\beta^*}w^2) \in L^\infty(0,T;L^2(\Omega)^N).$$

Consequently, we obtain

$$\begin{aligned} \left\| e^{7/2s\beta^*}(w^1\cdot\nabla)w^2 \right\|_{L^2(Q)^N} &= \left\| (e^{7/4s\beta^*}\, w^1\cdot\nabla)e^{7/4s\beta^*}\, w^2 \right\|_{L^2(Q)^N} \\ &\le \left\| e^{7/4s\beta^*}w^1 \right\|_{L^2(0,T;L^\infty(\Omega)^N)} \left\| e^{7/4s\beta^*}w^2 \right\|_{L^\infty(0,T;V)}, \end{aligned}$$

and the continuity in $L^2(e^{7/2s\beta^*}(\gamma^*)^{-2}(0,T); L^2(\Omega)^N)$ follows since $(\gamma^*)^{-2}$ is bounded.

In a similar way, we prove that

$$((w^1,p^{0,1},z^1,q^1,v^1),(w^2,p^{0,2},z^2,q^2,v^2)) \to (w^1\cdot\nabla)z^2$$

is continuous from $B_1 \times B_1$ to $L^2(e^{2s\beta^*}(\gamma^*)^{-5/2}(0,T); L^2(\Omega)^N)$. Notice now that

$$e^{1/2s\beta^*}(\gamma^*)^{-2-2/m} z \in L^2(0,T; H^2(\Omega)^N) \cap L^\infty(0,T;V),$$

for any $(w, p^0, z, q, v) \in B_1$, thus $e^{1/2s\beta^*}(\gamma^*)^{-2-2/m} z^2 \in L^\infty(0,T;V)$. We have

$$\begin{aligned}
&\left\| e^{9/4s\beta^*}(\gamma^*)^{-2-2/m}(w^1 \cdot \nabla) z^2 \right\|_{L^2(Q)^N} \\
&= \left\| (e^{7/4s\beta^*} w^1 \cdot \nabla) e^{1/2s\beta^*}(\gamma^*)^{-2-2/m} z^2 \right\|_{L^2(Q)^N} \\
&\leq \left\| e^{7/4s\beta^*} w^1 \right\|_{L^2(0,T;L^\infty(\Omega)^N)} \left\| e^{1/2s\beta^*}(\gamma^*)^{-2-2/m} z^2 \right\|_{L^\infty(0,T;V)},
\end{aligned}$$

and the continuity follows since $9/4 > 2$.

By the same computations as before, we can prove that the bilinear operator

$$((w^1, p^{0,1}, z^1, q^1, v^1), (w^2, p^{0,2}, z^2, q^2, v^2)) \to (z^1 \cdot \nabla^t) w^2$$

is continuous from $B_1 \times B_1$ to $L^2(e^{2s\beta^*}(\gamma^*)^{-5/2}(0,T); L^2(\Omega)^N)$ just by taking into account that

$$e^{1/2s\beta^*}(\gamma^*)^{-2-2/m} z^1 \in L^2(0,T; L^\infty(\Omega)^N).$$

Notice that $\mathcal{A}'(0,0,0,0,0) : B_1 \to B_2$ is given by

$$\mathcal{A}'(0,0,0,0,0)(w, p^0, z, q, v) = (Lw + \nabla p^0 - v \mathbb{1}_\omega, L^* z + \nabla q - w \mathbb{1}_{\mathcal{O}})$$

for all $(w, p^0, z, q, v) \in B_1$, so this functional is surjective in view of the null controllability result for the linear system (4.1) given by Proposition 4.3.

We are now able to apply Theorem 5.1 for $b_1 = (0,0,0,0,0)$ and $b_2 = (0,0)$. In particular, this gives the existence of a positive number $\delta > 0$ such that, if $\|e^{C/t^m} f\|_{L^2(Q)^N} \leq \delta$, for some $C > 0$, then we can find a control v, with $v_i \equiv 0$, such that the associated solution (w, z) to (5.1) satisfies $z(0) = 0$.

This concludes the proof of Theorem 1.1.

Acknowledgments

The authors would like to thank the “Agence Nationale de la Recherche” (ANR), Project CISIFS, grant ANR-09-BLAN-0213-02, for partially supporting this work.

References

[1] V. M. Alekseev, V. M. Tikhomirov and S. V. Fomin, *Optimal Control*, Translated from the Russian by V. M. Volosov, Contemporary Soviet Mathematics. Consultants Bureau, New York, 1987.

[2] O. Bodart and C. Fabre, Controls insensitizing the norm of the solution of a semilinear heat equation, *J. Math. Anal. Appl.,* **195(3)** (1995), 658–683.

[3] N. Carreño, Local controllability of the N-dimensional Boussinesq system with $N-1$ scalar controls in an arbitrary control domain, to appear in *Mathematical Control and Related Fields.*

[4] N. Carreño and S. Guerrero, Local null controllability of the N-dimensional Navier-Stokes system with $N-1$ scalar controls in an arbitrary control domain, to appear in *Journal of Mathematical Fluid Mechanics.*

[5] J.-M. Coron and S. Guerrero, Null controllability of the N-dimensional Stokes system with $N-1$ scalar controls, *J. Differential Equations,* **246** (2009), 2908–2921.

[6] J.-M. Coron and S. Guerrero, Local null controllability of the two-dimensional Navier-Stokes system in the torus with a control force having a vanishing component, *J. Math. Pures Appl.,* **(9) 92** (2009), no. 5, 528–545.

[7] E. Fernández-Cara, S. Guerrero, O. Yu Imanuvilov and J.-P. Puel, Local exact controllability of the Navier-Stokes system,*J. Math. Pures Appl.,* **(9) 83** (2004) ,no. 12, 1501–1542.

[8] E. Fernández-Cara, S. Guerrero, O. Yu. Imanuvilov and J.-P. Puel, Some controllability results for the N-dimensional Navier-Stokes and Boussinesq systems with N-1 scalar controls, *SIAM J. Control Optim.,* **45** (2006), 146–173.

[9] A. V. Fursikov O. Yu. Imanuvilov, *Controllability of Evolution Equations*, Lecture Notes #34, Seoul National University, Korea, 1996.

[10] S. Guerrero, Null controllability of some systems of two parabolic equation with one control force, *SIAM Journal on Control and Optimization,* **46** (2007), no. 2, 379–394.

[11] S. Guerrero, Controllability of systems of Stokes equations with one control force: existence of insensitizing controls, *Annales de l'Institut Henri Poincaré Analyse Non Linéaire,* **24** (2007), no. 6, 1029–1054.

[12] M. Gueye, Insensitizing controls for the Navier-Stokes equations, to appear in *Annales de l'Institut Henri Poincaré (C) Non Linear Analysis.*

[13] O. Yu. Imanuvilov, Remarks on exact controllability for the Navier-Stokes equation, *ESAIM Control Optim. Calc. Var.,* **6** (2001), 39–37.

[14] O. Yu. Imanuvilov, J.-P. Puel and M. Yamamoto, Carleman estimates for parabolic equations with nonhomogeneous boundary conditions, *Chin. Ann. Math,* **30B(4)** (2009), 333–378.

[15] O. A. Ladyzenskaya, *The mathematical theory of viscous incompressible flow,* revised English edition, translated from the Russian by Richard A. Silverlman, Gordon and Breach Science Publishers, New York, London, 1963.

[16] J.-L. Lions, *Quelques notions dans l'analyse et le contrôle de systèmes à données incomplètes (Somes notions in the analysis and control of systems with incomplete data),* Proceedings of the XIth congress on Differential Equations and Applications/Fisrt Congress on Applied Mathematics, Málaga, 1989, Univ. Málaga, 1990, 43–54.

[17] J.-L. Lions, *Sentinelles pour les systèmes distribués à données incomplètes (Sentinelles for Distributed Systems with Incomplete Data), Recherches en Mathématiques Appliquées (Research in Applied Mathematics),* vol. 21, Masson, Paris, 1992.

[18] J.-L. Lions and E. Magenes, *Problèmes aux limites non homogènes et applications,* Volume 2, Travaux et Recherches Mathématiques, No. 18, Dunod, Paris, 1968.

[19] S. Micu, J. H. Ortega and L. de Teresa, An example of ε-insensitizing controls for the heat equation with no intersecting observation and control regions, *Appl. Math. Lett.,* **17 (8)** (2004), 927–932.

[20] R. Temam, *Navier-Stokes Equations,* Theory ans Numerical Analysis, Stud. Math. Appl., Vol. 2, North-Holland, Amsterdam-New York-Oxford, 1977.

[21] R. Temam, Behaviour at time $t = 0$ of the solutions of semilinear evolution equations, *J. Differential Equations,* **43(1)** (1982), 73–92

[22] L. de Teresa, Insensitizing controls for a semilinear heat equation, *Comm. Partial Differential Equations,* **25(1-2)** (2000), 39–72.

[23] L. de Teresa and O. Kavian, Unique continuation principle for systems of parabolic equations, *ESAIM: Control, Optimisation and Calculus of Variations,* **16** (2010), 247–274.