FINITE VOLUME APPROXIMATION FOR AN IMMISCIBLE TWO-PHASE FLOW IN POROUS MEDIA WITH DISCONTINUOUS CAPILLARY PRESSURE[*]

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Abstract. We consider an immiscible incompressible two-phase flow in a porous medium composed of two different rocks. The flows of oil and water are governed by the Darcy-Muskat law and a capillary pressure law, where the capillary pressure field may be discontinuous at the interface between the rocks. Using the concept of multi-valued phase pressures, we introduce a notion of weak solution for the flow. We discretize the problem by means of a numerical scheme which reduces to a standard finite volume scheme in each rock and prove the convergence of an approximate solutions towards a weak solution. The numerical experiments show that the scheme can reproduce the oil trapping phenomenon.

Key words. Finite volume schemes, degenerate parabolic convection–reaction–diffusion equation, two-phase flow in porous media, discontinuous capillarity

AMS subject classifications. 35K65, 35R05, 65M12, 76M12

1. Introduction.

1.1. Multivalued phase pressures.

Models of incompressible immiscible two-phase flows are widely used in oil engineering to predict the motion of oil in the subsoil. They have been widely studied from a mathematical point of view (see e.g. [1], [2], [4], [5], [16]) and from a numerical point of view (see e.g. [15], [18], [19], [17], [26], [28]). In this model, sometimes called *dead-oil* approximation, it is assumed that there are only two phases oil and water, and that each phase is only composed of a single component.

The governing equations are derived by substituting the Darcy-Muskat (or diphasic Darcy) law into the conservation equations for both phases, that is that for each phase $\alpha \in \{o, w\}$ (o corresponds to the oil phase, while w corresponds to the water phase):

$$\phi\partial_t s_\alpha - \operatorname{div}\left(K\frac{k_\alpha(s_\alpha)}{\mu_\alpha}\left(\nabla p_\alpha - \rho_\alpha \mathbf{g}\right)\right) = 0, \tag{1.1}$$

where $\phi = \phi(\mathbf{x})$ is the porosity of the rock ($\phi \in (0,1)$ in the domain Ω), s_α is the saturation of the phase α, the permeability of the porous medium K is supposed to be a positive scalar function, the relative permeability k_α of the phase α is a increasing function of the saturation s_α, satisfying $k_\alpha(0) = 0$ and $k_\alpha(1) = 1$, μ_α, p_α and ρ_α denote respectively the viscosity, the pressure and the density of the phase α, and $\mathbf{g}$ is the gravity vector. Assuming that the porous medium is saturated, one has

$$s_o + s_w = 1. \tag{1.2}$$

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This relation allows to eliminate the water saturation. We note $s := s_o$, so that $s_w = 1 - s$.

Classically, it is assumed that the phase pressures are connected by the equality

$$p_o - p_w = \pi(s_o), \tag{1.3}$$

where π is the capillary pressure function, which is supposed to be strictly increasing on $(0, 1)$.

As it has been stressed in [1], the natural topology for the phase pressures in such a flow is prescribed by the quantity

$$\sum_{\alpha\in\{o,w\}} \int_0^T \int_\Omega K \frac{k_\alpha(s_\alpha)}{\mu_\alpha} (\nabla p_\alpha)^2 \,\mathrm{d}\mathbf{x}\mathrm{d}t. \tag{1.4}$$

Note that when the phase α vanishes, i.e. $s_\alpha = 0$, then (1.4) provides no control on the pressure p_α —it is indeed difficult to define the pressure of a missing phase—. As a consequence, if $s_\alpha = 0$ and p_β is known ($\beta \neq \alpha$), then p_α is not defined in a unique way, but it is multivalued, i.e. it can take any value lower than a threshold value, for which the phase α would appear. This point of view, developed in [14], leads to

$$p_o \in [-\infty, p_w + \pi(0)] \quad \text{if } s_o = 0 \tag{1.5}$$

and

$$p_w \in [-\infty, p_o - \pi(1)] \quad \text{if } s_o = 1. \tag{1.6}$$

We will take advantage of the multivalued formalism in order to deal with the case where the porous medium is composed of several rock types, and where the functions describing the porous medium depend in a discontinuous way of space. As it was stressed in [29], [6], [10], [11] and [3], the discontinuities of the capillary pressure function (1.3) have a large influence on the behavior of the flow, leading to *oil-trapping*. Several numerical methods ([23], [21], [9], [22]) have been proposed to approximate this problem, but, to our knowledge, no convergence proof has been provided for the full problem in several space dimensions. Indeed, the convergence result stated in [9] holds in the one-dimensional case, where the problem (1.1)-(1.3) becomes a single degenerate parabolic equation. In [21], simplifying assumptions are made so that the problem also reduces to a single degenerate parabolic equation. The scope of this paper is to deal with the two balance equations (1.1). In the case where the porous medium has smooth variations, it is possible to show the continuity of p_α, in the sense that on each Lipschitz continuous hypersurface of Ω, p_α admits strong traces on both sides $p_{\alpha,1}$, $p_{\alpha,2}$, and that these traces coincide:

$$p_{\alpha,1} = p_{\alpha,2}, \quad \text{for } \alpha \in \{o, w\}. \tag{1.7}$$

If at the level of such a hypersurface Γ, the characteristic of the medium change in a discontinuous way, then, unless we have a compatibility condition on the capillary pressure functions (see [14]), the continuity of the pressure can not be prescribed by (1.7), but, taking advantage of the multivalued definitions (1.5),(1.6) of the phase pressures, one should use the graph formalism:

$$p_{\alpha,1} \cap p_{\alpha,2} \neq \emptyset, \quad \text{for } \alpha \in \{o, w\}. \tag{1.8}$$

Note that if the phase pressures are single-valued, then the conditions (1.7) and (1.8) are equivalent. Moreover, because of the conservation of mass, one has

$$\sum_{i\in\{1,2\}} K_i \frac{k_{\alpha,i}(s_i)}{\mu_\alpha} (\nabla p_{\alpha,i} - \rho_\alpha \mathbf{g}) \cdot \mathbf{n}_i = 0, \tag{1.9}$$

where K_i and $k_{\alpha,i}(s_i)$ denote the traces of K and $k_\alpha(s)$ on each side of the interface, and $\mathbf{n}_i$ denotes the outward normal to Γ with respect to the side i of the interface.

The fact that the phase pressures p_α are multivalued when $s_\alpha = 0$ implies that the capillary pressure functions $s \mapsto \pi_i(s)$ must also be multivalued for $s = 0$ and $s = 1$. Thus we introduce the monotonous graphs $\tilde{\pi}_i$ defined by

$$\tilde{\pi}_i(s) = \begin{cases} [-\infty, \pi_i(0)] & \text{if } s = 0, \\ \pi_i(s) & \text{if } s \in (0,1), \\ [\pi_i(1), +\infty] & \text{if } s = 1, \end{cases}$$

for $i = 1, 2$. The capillary pressure graphs admit a continuous inverse functions, denoted by $\tilde{\pi}_i^{-1}$, which is defined on $\mathbb{R}$ by

$$\tilde{\pi}_i^{-1}(p) := \begin{cases} 0 & \text{if } p \le \pi_i(0), \\ \pi_i^{-1}(p) & \text{if } p \in (\pi_i(0), \pi_i(1)), \\ 1 & \text{if } p \ge \pi_i(1). \end{cases}$$

Requiring (1.8) implies that, at an interface where the porous medium is discontinuous, one has

$$\tilde{\pi}_1(s_1) \cap \tilde{\pi}_2(s_2) \neq \emptyset, \tag{1.10}$$

where $\pi_1(s_1)$ and $\pi_2(s_2)$ denote the traces of $\pi(s)$ on both sides of the interface. It has been shown in [13] and [8] that this interface condition is natural, at least in the one-dimensional case. But this single relation is not sufficient to deal with the case of two equations such as in (1.1), and further information has to be given along the discontinuity lines.

1.2. The model problem and assumptions on the data. We assume that the porous medium Ω is a connex open bounded polygonal subset of $\mathbb{R}^d$, and is made of two disjoint homogeneous rocks Ω_i, $i \in \{1, 2\}$, which are both open polygonal subsets of $\mathbb{R}^d$. We denote by Γ the interface between Ω_1 and Ω_2, i.e.

$$\overline{\Gamma} = \partial\Omega_1 \cap \partial\Omega_2.$$

For all functions a depending on the physical characteristics of the rock, we use the notation $a_i = a(\cdot, x)$ if $x \in \Omega_i$.

Also remark that as it has been stressed in [**?**], the gravity plays a crucial role in the so called *oil-trapping* phenomenon.

We assume that at the initial time $t = 0$, the composition of the fluid is known

$$s_{|t=0} = s_0 \in L^\infty(\Omega; [0, 1]). \tag{1.11}$$

We also assume that both phase fluxes are equal to 0 through the boundary $\partial\Omega \times [0, T]$ where $T > 0$ is a positive fixed (but arbitrary) final time:

$$K_i \frac{k_{\alpha,i}(s_i)}{\mu_\alpha} (\nabla p_{\alpha,i} - \rho_\alpha \mathbf{g}) \cdot \mathbf{n}_i = 0, \qquad \text{on } (\partial\Omega \cap \partial\Omega_i) \times (0, T). \tag{1.12}$$

Nevertheless, it should be possible to deal with other types of boundary conditions, such as Dirichlet conditions on a part of the boundary and Neumann conditions on the remaining part.

We denote by Q_T and $Q_{i,T}$ the cylinders

$$Q_T := \Omega \times (0,T), \qquad Q_{i,T} := \Omega_i \times (0,T).$$

In order to control the energy of the flow, we have to make the following natural assumption on the capillary pressure functions.

Assumption 1. *The functions π_i belong to $\mathcal{C}^1((0,1)) \cap L^1((0,1))$.*

Remark 1.1. *As a direct consequence of Assumption 1,*

$$\tilde{\pi}_i^{-1} \in L^1(\mathbb{R}_-) \qquad \text{and} \qquad (1-\tilde{\pi}_i^{-1}) \in L^1(\mathbb{R}_+), \qquad i \in \{1,2\}.$$

1.3. Global pressure formulation of the problem. The lack of control on the phase pressures, described in Section 1.1 and in [14], leads to important mathematical difficulties. A classical mathematical tool consists in introducing the so-called *global pressure* P to circumvent some of them. We define, for $\mathbf{x} \in \Omega_i$

$$P = p_w + \int_0^{\pi_i(s)} \frac{k_{o,i}(\tilde{\pi}_i^{-1}(a))}{k_{o,i}(\tilde{\pi}_i^{-1}(a)) + \frac{\mu_o}{\mu_w} k_{w,i}(\tilde{\pi}_i^{-1}(a))} \mathrm{d}a, \tag{1.13}$$

$$= p_o - \int_0^{\pi_i(s)} \frac{k_{w,i}(\tilde{\pi}_i^{-1}(a))}{k_{w,i}(\tilde{\pi}_i^{-1}(a)) + \frac{\mu_w}{\mu_o} k_o(\tilde{\pi}_i^{-1}(a))} \mathrm{d}a. \tag{1.14}$$

While the phase pressures p_α shall be defined as multivalued, it has been pointed out in [14] that the global pressure P is always single valued, and is therefore much easier to work with. It is well known that in the case where the domain Ω is homogeneous ([15]), or if Ω varies smoothly ([5], [16]), then the global pressure belongs to the space $L^\infty(0,T;H^1(\Omega))$. This regularity result does not remain true, as it will be shown in the sequel, in the case of a discontinuous capillary pressure.

We define the fractional function $f_i(s) = \dfrac{k_{o,i}(s)}{k_{o,i}(s) + \frac{\mu_o}{\mu_w} k_{w,i}(s)}$ and we introduce the Kirchhoff transform

$$\varphi_i(s) = \int_0^s K_i \frac{k_{o,i}(a) k_{w,i}(a)}{\mu_w k_{o,i}(a) + \mu_o k_{w,i}(a)} \pi_i'(a) \mathrm{d}a, \qquad \forall s \in (0,1), \tag{1.15}$$

that we extend in a continuous way by constants outside of $(0,1)$. We make moreover the following assumption on the functions φ_i.

Assumption 2. *For $i \in \{1,2\}$, the functions φ_i are Lipschitz continuous and increasing on $[0,1]$.*

It is well known that the system (1.1)–(1.3) can be rewritten in $Q_{i,T}$ under the form

$$\left\{ \begin{array}{l} \phi_i \partial_t s + \mathrm{div}\left(f_i(s)\mathbf{q}_i + \gamma_i(s)\mathbf{g} - \nabla\varphi_i(s)\right) = 0, \\ \mathrm{div}\mathbf{q}_i = 0, \\ \mathbf{q}_i = -M_i(s)\nabla P + \zeta_i(s)\mathbf{g}, \end{array} \right. \tag{1.16}$$

where

$$\gamma_i(s) = K_i(\rho_o - \rho_w) \frac{k_{o,i}(s) k_{w,i}(s)}{\mu_w k_{o,i}(s) + \mu_o k_{w,i}(s)} \tag{1.17}$$

and

$$\zeta_i(s) = K_i\left(\frac{k_{o,i}(s)}{\mu_o}\rho_o + \frac{k_{w,i}(s)}{\mu_w}\rho_w\right).$$

The total mobility is defined by $M_i(s) = K_i\left(\frac{k_{o,i}(s)}{\mu_o} + \frac{k_{w,i}(s)}{\mu_w}\right)$. Since the relative permeabilities $k_{\alpha,i}$ are supposed to be strictly monotone, one has $k_{\alpha,i}(s) > 0$ if $s \in (0,1)$. As a consequence,

$$\begin{aligned}&\text{there exists } \alpha_M > 0 \text{ such that, for } i \in \{1,2\}, \text{ and for all } s \in [0,1],\\ &\text{one has } M_i(s) \geq \alpha_M.\end{aligned} \tag{1.18}$$

The boundary conditions on the phase fluxes (1.12) are given by

$$\mathbf{q}_i\cdot\mathbf{n}_i = 0, \qquad (f_i(s)\mathbf{q}_i + \gamma_i(s)\mathbf{g} - \nabla\varphi_i(s))\cdot\mathbf{n}_i = 0, \quad \text{on } (\partial\Omega\cap\partial\Omega_i)\times(0,T). \tag{1.19}$$

Concerning the transmission conditions on the interface Γ, prescribing the relation (1.10) is not sufficient. It has to be replaced by: there exists π such that

$$\pi \in \tilde{\pi}_1(s_1)\cap\tilde{\pi}_2(s_2), \tag{1.20}$$

$$P_1 - W_1(\pi) = P_2 - W_2(\pi), \tag{1.21}$$

where $W_i(p) = \int_0^p f_i\circ\tilde{\pi}_i^{-1}(u)du$. Note that (1.20) ensures that (1.10) holds. Equation (1.21) consists in requiring the continuity of the water pressure in some weak sense implying (1.8) for $\alpha = w$. On the other hand, adding π on both side of (1.21) leads to the continuity in the same weak sense of the oil pressure (see [14]).

The conservation of the total mass and of the oil mass give

$$\sum_{i\in\{1,2\}} \mathbf{q}_i\cdot\mathbf{n}_i = 0 \quad \text{on } \Gamma, \tag{1.22}$$

$$\sum_{i\in\{1,2\}} (f_i(s)\mathbf{q}_i + \gamma_i(s)\mathbf{g} - \nabla\varphi_i(s))\cdot\mathbf{n}_i = 0 \quad \text{on } \Gamma, \tag{1.23}$$

where $\mathbf{n}_i$ denotes the outward normal to Γ with respect to Ω_i.

Since the global pressure P is defined up to a constant, we have to impose a condition to select a solution. More precisely we impose that

$$m_{\Omega_1}(P)(t) = 0, \text{ where } m_{\Omega_i}(P)(t) := \frac{1}{m(\Omega_i)}\int_{\Omega_i} P(\mathbf{x},t)\mathrm{d}\mathbf{x} \text{ for } i \in \{1,2\}. \tag{1.24}$$

We now define a weak solution of Problem (1.16)-(1.24).

Definition 1.1. *We say that a function pair* (s,P) *is a weak solution of Problem (1.16)-(1.24) if:*

1. $s \in L^\infty(Q_T;[0,1])$;
2. $\varphi_i(s), P \in L^2(0,T;H^1(\Omega_i))$, *with* $m_{\Omega_1}(P)(t) = 0$ *for almost every* $t \in (0,T)$;
3. *there exists a measurable function* π *on* $\Gamma\times(0,T)$ *such that, for a.e.* $(\mathbf{x},t) \in \Gamma\times(0,T)$, (1.20)–(1.21) *hold;*
4. *for all* $\psi \in \mathcal{C}_c^\infty\left(\overline{\Omega}\times[0,T)\right)$, *the following integral equalities hold:*

$$\int_0^T \sum_{i\in\{1,2\}} \int_{\Omega_i} \mathbf{q}_i\cdot\nabla\psi\mathrm{d}\mathbf{x}\mathrm{d}t = 0, \tag{1.25}$$

$$\int_0^T \int_\Omega \phi s \partial_t \psi \mathrm{d}\mathbf{x}\mathrm{d}t + \int_\Omega \phi s_0 \psi(\cdot,0)\mathrm{d}\mathbf{x}$$
$$= \int_0^T \sum_{i\in\{1,2\}} \int_{\Omega_i} \left(f_i(s)\mathbf{q}_i + \gamma_i(s)\mathbf{g} + \nabla\varphi_i(s)\right)\cdot\nabla\psi \mathrm{d}\mathbf{x}\mathrm{d}t, \quad (1.26)$$

where

$$\mathbf{q}_i = -M_i(s)\nabla P + \zeta_i(s)\mathbf{g}.$$

We will use several time the following lemma, which ensures that the global pressure jump $P_1 - P_2$ at the interface belongs to $L^\infty(\Gamma\times(0,T))$.

LEMMA 1.1. *The function $p\mapsto W_1(p) - W_2(p)$ belongs to $\mathcal{C}^1(\mathbb{R};\mathbb{R})$, is uniformly bounded on $\mathbb{R}$ and admits finite limits as $p\to\pm\infty$.*

Proof. Define

$$\Upsilon_i(p) = \begin{cases} \displaystyle\int_0^p \left(f_i\circ\tilde\pi_i^{-1}(p) - 1\right)\mathrm{d}p & \text{if } p\ge 0,\\ \displaystyle\int_0^p f_i\circ\tilde\pi_i^{-1}(p)\ \mathrm{d}p & \text{if } p<0,\end{cases}$$

then $W_1(p) - W_2(p) = \Upsilon_1(p) - \Upsilon_2(p)$. Hence, we deduce that if $\Upsilon_1(p),\Upsilon_2(p)$ have finite limits for $p\to\pm\infty$, then $W_1 - W_2$ also does, since $f_i(1) = 1$. Since Υ_1,Υ_2 are nonincreasing functions, it only remains to check that they are bounded. Let $p\ge 0$, then

$$\begin{aligned} 0\ge\Upsilon_i(p) &\ge -\int_0^p |f_i\circ\tilde\pi_i^{-1}(p) - f_i(1)|\mathrm{d}p\\ &\ge -L_{f_i}\int_0^p |\tilde\pi_i^{-1}(p) - 1|\mathrm{d}p \ge -L_{f_i}\|\tilde\pi_i^{-1} - 1\|_{L^1(\mathbb{R}_+)}.\end{aligned}$$

Similarly, for $p<0$, one has

$$0\le\Upsilon_i(p)\le L_{f_i}\|\tilde\pi_i^{-1}\|_{L^1(\mathbb{R}_-)}.$$

We conclude the proof of Lemma 1.1 by applying Remark 1.1. ◻

2. The Finite Volume approximation.

2.1. Discretization of $\mathbf{Q_T}$.

DEFINITION 2.1. *An admissible mesh of Ω is given by a set $\mathcal{T}$ of open bounded convex subsets of Ω called control volumes, a family $\mathcal{E}$ of subsets of $\overline{\Omega}$ contained in hyperplanes of $\mathbb{R}^d$ with strictly positive measure, and a family of points $(x_K)_{K\in\mathcal{T}}$ (the "centers" of control volumes) satisfying the following properties:*

1. *there exists $i\in\{1,2\}$ such that $K\subset\Omega_i$. We note $\mathcal{T}_i = \{K\in\mathcal{T}, K\subset\Omega_i\}$;*
2. *$\overline{\bigcup_{K\in\mathcal{T}_i} K} = \overline{\Omega_i}$. Thus, $\overline{\bigcup_{K\in\mathcal{T}} K} = \overline{\Omega}$;*
3. *for any $K\in\mathcal{T}$, there exists a subset $\mathcal{E}_K$ of $\mathcal{E}$ such that $\partial K = \bigcup_{\sigma\in\mathcal{E}_K}\overline{\sigma}$. Furthermore, $\mathcal{E} = \bigcup_{K\in\mathcal{T}}\mathcal{E}_K$;*
4. *for any $(K,L)\in\mathcal{T}^2$ with $K\ne L$, either the "length"(i.e. the $(d-1)$ Lebesgue measure) of $\overline{K}\cap\overline{L}$ is 0 or $\overline{K}\cap\overline{L} = \overline{\sigma}$ for some $\sigma\in\mathcal{E}$. In the latter case, we write $\sigma = K|L$, and*
   - $\mathcal{E}_i = \{\sigma\in\mathcal{E},\ \exists(K,L)\in\mathcal{T}_i^2,\ \sigma = K|L\}$, $\mathcal{E}_{\rm int} = \mathcal{E}_1\cup\mathcal{E}_2$, $\mathcal{E}_{K,{\rm int}} = \mathcal{E}_K\cap\mathcal{E}_{\rm int}$,
   - $\mathcal{E}_{\rm ext} = \{\sigma\in\mathcal{E},\ \sigma\subset\partial\Omega\}$, $\mathcal{E}_{K,{\rm ext}} = \mathcal{E}_K\cap\mathcal{E}_{\rm ext}$,

- $\mathcal{E}_\Gamma = \{\sigma \in \mathcal{E},\ \exists (K,L) \in \mathcal{T}_1 \times \mathcal{T}_2,\ \sigma = K|L\}$, $\mathcal{E}_{K,\Gamma} = \mathcal{E}_K \cap \mathcal{E}_\Gamma$;

5. *The family of points* $(x_K)_{K\in\mathcal{T}}$ *is such that* $x_K \in K$ *(for all* $K \in \mathcal{T}$*) and, if* $\sigma = K|L$*, it is assumed that the straight line* (x_K, x_L) *is orthogonal to* σ*.*

For a control volume $K \in \mathcal{T}_i$*, we denote by* $\mathcal{N}_K = \{L \in \mathcal{T}_i, \sigma = K|L \in \mathcal{E}_{K,i}\}$ *the set of the neighbors and by* $m(K)$ *its measure. For all* $\sigma \in \mathcal{E}$*, we denote by* $m(\sigma)$ *the* $(d-1)$*-Lebesgue measure of* σ*. If* $\sigma \in \mathcal{E}_K$*, we note* $d_{K,\sigma} = d(x_K, \sigma)$*, and we denote by* $\tau_{K,\sigma}$ *the transmissibility of* K *through* σ*, defined by* $\tau_{K,\sigma} = \frac{m(\sigma)}{d_{K,\sigma}}$*. If* $\sigma = K|L$*, we note* $d_{K,L} = d(x_K, x_L)$ *and* $\tau_{KL} = \frac{m(\sigma)}{d_{K,L}}$*. The size of the mesh is defined by:*

$$size(\mathcal{T}) = \max_{K\in\mathcal{T}} diam(K),$$

and a geometrical factor, connected with the regularity of the mesh, is defined by

$$reg(\mathcal{T}) = \max_{K\in\mathcal{T}} \left(\sum_{\sigma=K|L\in\mathcal{E}_{K,\mathrm{int}}} \frac{m(\sigma) d_{K,L}}{m(K)} \right).$$

Definition 2.2. *A uniform time discretization of* $(0,T)$ *is given by an integer value* N *and a sequence of real values* $(t^n)_{n\in\{0,\ldots,N+1\}}$*. We define* $\delta t = \frac{T}{N+1}$ *and,* $\forall n \in \{0,\ldots,N\}$*,* $t^n = n\delta t$*. Thus we have* $t^0 = 0$ *and* $t^{N+1} = T$*.*

Remark 2.1. *We can easily prove all the results of this paper for a general time discretization, but for the sake of simplicity, we choose to only consider uniform time discretizations.*

Definition 2.3. *A finite volume discretization* $\mathcal{D}$ *of* Q_T *is a family*

$$\mathcal{D} = (\mathcal{T}, \mathcal{E}, (\mathbf{x}_K)_{K\in\mathcal{T}}, N, (t^n)_{n\in\{0,\ldots,N\}}),$$

where $(\mathcal{T}, \mathcal{E}, (\mathbf{x}_K)_{K\in\mathcal{T}})$ *is an admissible mesh of* Ω *in the sense of definition 2.1 and* $(N, (t^n)_{n\in\{0,\ldots,N\}})$ *is a discretization of* $(0,T)$ *in the sense of definition 2.2. For a given mesh* $\mathcal{D}$*, one defines:*

$$size(\mathcal{D}) = max(size(\mathcal{T}), \delta t), \qquad \text{and } reg(\mathcal{D}) = reg(\mathcal{T}).$$

2.2. Definition of the scheme and main result. For $K \in \mathcal{T}_i$, we denote by $g_K(s) = g_i(s)$ for all function g whose definition depends on the subdomain Ω_i, as for example $\phi_i, \varphi_i, M_i, f_i, W_i, \ldots$. For a function $f : \mathbb{R} \subset \mathbb{R} \to \mathbb{R}$ and for $(a,b) \in \mathbb{R}^2$ we denote by $\mathcal{R}(f; a, b)$ the Riemann solver

$$\mathcal{R}(f; a, b) = \begin{cases} \min_{c\in[a,b]} f(c) & \text{if } a \le b, \\ \max_{c\in[b,a]} f(c) & \text{if } b \le a. \end{cases} \tag{2.1}$$

The total flux balance equation is discretized by

$$\sum_{\sigma\in\mathcal{E}_K} m(\sigma) Q^{n+1}_{K,\sigma} = 0, \qquad \forall n \in \{0,\ldots,N\}, \forall K \in \mathcal{T}, \tag{2.2}$$

with

$$Q^n_{K,\sigma} = \begin{cases} \frac{M_{K,L}(s^n_K, s^n_L)}{d_{K,L}} (P^n_K - P^n_L) + \mathcal{R}(Z_{K,\sigma}; s^n_K, s^n_L) & \text{if } \sigma = K|L \in \mathcal{E}_{K,i}, \\ \frac{M_K(s^n_K)}{d_{K,\sigma}} \left(P^n_K - P^n_{K,\sigma}\right) + \mathcal{R}\left(Z_{K,\sigma}; s^n_K, s^n_{K,\sigma}\right) & \text{if } \sigma \in \mathcal{E}_{K,\Gamma}, \\ 0 & \text{if } \sigma \in \mathcal{E}_{K,\mathrm{ext}}, \end{cases} \tag{2.3}$$

where $M_{K,L}(s_K^{n+1}, s_L^{n+1}) = M_{L,K}(s_L^{n+1}, s_K^{n+1})$ is a mean value between $M_K(s_K^{n+1})$ and $M_L(s_L^{n+1})$. For example, we can consider, as in [28], the harmonic mean

$$M_{K,L}(s_K^{n+1}, s_L^{n+1}) = \frac{M_K(s_K^{n+1}) M_K(s_L^{n+1}) d_{K,L}}{d_{L,\sigma} M_K(s_K^{n+1}) + d_{K,\sigma} M_K(s_L^{n+1})}.$$

The function $Z_{K,\sigma}$ is defined by $Z_{K,\sigma}(s) = \zeta_K(s)\mathbf{g} \cdot \mathbf{n}_{K,\sigma}$, where $\mathbf{n}_{K,\sigma}$ denotes the outward normal to σ with respect to K.

The oil-flux balance equation is discretized as follows:

$$\phi_K \frac{s_K^{n+1} - s_K^n}{\delta t} m(K) + \sum_{\sigma \in \mathcal{E}_K} m(\sigma) F_{K,\sigma}^{n+1} = 0, \tag{2.4}$$

with

$$F_{K,\sigma}^n = \begin{cases} Q_{K,\sigma}^n \ f_K(\overline{s}_{K,\sigma}^n) + \mathcal{R}(G_{K,\sigma}; s_K^n, s_L^n) + \dfrac{\varphi_K(s_K^n) - \varphi_K(s_L^n)}{d_{K,L}} \text{ if } \sigma = K|L \in \mathcal{E}_{K,i}, \\ Q_{K,\sigma}^n \ f_K(\overline{s}_{K,\sigma}^n) + \mathcal{R}(G_{K,\sigma}; s_K^n, s_{K,\sigma}^n) + \dfrac{\varphi_K(s_K^n) - \varphi_K(s_{K,\sigma}^n)}{d_{K,\sigma}} \text{ if } \sigma \in \mathcal{E}_{K,\Gamma}, \\ 0 \text{ if } \sigma \in \mathcal{E}_{K,\text{ext}}, \end{cases} \tag{2.5}$$

where $G_{K,\sigma}(s) = \gamma_K(s)\mathbf{g} \cdot \mathbf{n}_{K,\sigma}$ and $\overline{s}_{K,\sigma}^{n+1}$ is the upwind value defined by

$$\overline{s}_{K,\sigma}^{n+1} = \begin{cases} s_K^{n+1} & \text{if } Q_{K,\sigma}^{n+1} \geq 0, \\ s_L^{n+1} & \text{if } Q_{K,\sigma}^{n+1} < 0 \text{ and } \sigma = K|L \in \mathcal{E}_{K,i}, \\ s_{K,\sigma}^{n+1} & \text{if } Q_{K,\sigma}^{n+1} < 0 \text{ and } \sigma \in \mathcal{E}_{K,\Gamma}. \end{cases} \tag{2.6}$$

The interface values $\left(s_{K,\sigma}^{n+1}, s_{L,\sigma}^{n+1}, P_{K,\sigma}^{n+1}, P_{L,\sigma}^{n+1}\right)$ for $\sigma = K|L \in \mathcal{E}_\Gamma$ are defined by the following nonlinear system. For all $\sigma = K|L \in \mathcal{E}_\Gamma$, for all $n \in \{0, \dots, N\}$, there exists $\pi_\sigma^{n+1} \in \mathbb{R}$ such that

$$\pi_\sigma^{n+1} \in \tilde{\pi}_K(s_{K,\sigma}^{n+1}) \cap \tilde{\pi}_L(s_{L,\sigma}^{n+1}), \tag{2.7}$$

$$P_{K,\sigma}^{n+1} - W_K\left(\pi_\sigma^{n+1}\right) = P_{L,\sigma}^{n+1} - W_L\left(\pi_\sigma^{n+1}\right), \tag{2.8}$$

$$Q_{K,\sigma}^{n+1} + Q_{L,\sigma}^{n+1} = 0, \tag{2.9}$$

$$F_{K,\sigma}^{n+1} + F_{L,\sigma}^{n+1} = 0. \tag{2.10}$$

Moreover, we impose the discrete counterpart of the equation (1.24), that is, for all $n \in \{0, \dots, N\}$,

$$\sum_{K \in \mathcal{T}_1} m(K) P_K^{n+1} = 0. \tag{2.11}$$

We will show below in Section 2.3 that the system (2.7)-(2.10) possesses a solution. We denote by $\mathcal{X}(\mathcal{D}, i)$ the finite dimensional space of piecewise constant functions $u_\mathcal{D}$ defined almost everywhere in $Q_{i,T}$ having a trace on the interface Γ, i.e.

$$\mathcal{X}(\mathcal{D}, i) := \big\{u_{\mathcal{D},i} \in L^\infty(Q_{i,T}) \text{ and for all } (K, \sigma, n) \in \mathcal{T} \times \mathcal{E}_\Gamma \times \{0, \dots, N\}, \\ u_{\mathcal{D},i} \text{ is constant on } K \times (t^n, t^{n+1}], u_{\mathcal{D},i} \text{ is constant on } \sigma \times (t^n; t^{n+1})\big\},$$

and by $\mathcal{X}(\mathcal{D})$ the space of the functions $u_\mathcal{D}$ whose restriction $(u_\mathcal{D})_{|_{\overline{Q}_{i,T}}}$ belongs to $\mathcal{X}(\mathcal{D},i)$. We define the solution $(s_\mathcal{D},P_\mathcal{D})\in\mathcal{X}(\mathcal{D})^2$ of the scheme by

$$s_\mathcal{D}(\mathbf{x},t)=s_K^{n+1},\quad P_\mathcal{D}(\mathbf{x},t)=P_K^{n+1}\quad \text{if } (\mathbf{x},t)\in K\times(t^n,t^{n+1}],$$

and, for $\mathbf{x}\in\sigma=K|L\subset\Gamma$ for some $K\in\mathcal{T}_1$, $L\in\mathcal{T}_2$, for $t\in(t^n,t^{n+1})$, the traces

$$s_{\mathcal{D}_{|\Gamma,1}}(\mathbf{x},t)=s_{K,\sigma}^{n+1},\qquad s_{\mathcal{D}_{|\Gamma,2}}(\mathbf{x},t)=s_{L,\sigma}^{n+1}.$$

In this paper we prove the following convergence result.

THEOREM 1. *Assume that Assumptions 1 and 2 hold. Let $(\mathcal{D}_m)_m$ be a sequence of admissible discretizations of Q_T in the sense of Definition 2.3, then for all $m\in\mathbb{N}$, there exists a discrete solution $(s_{\mathcal{D}_m},P_{\mathcal{D}_m})\in\mathcal{X}(\mathcal{D}_m)^2$ to the scheme. Moreover, if $\lim_{m\to\infty} size(\mathcal{D}_m)=0$, and if there exists $\zeta>0$ such that, for all m, $reg(\mathcal{D}_m)\le\zeta$, then up to a subsequence, $s_{\mathcal{D}_m}$ converges, towards $s\in L^\infty(Q_T;[0,1])$ in the $L^p(Q_T)$ topology for all $p\in[1,\infty)$, $P_{\mathcal{D}_m}$ converges to P weakly in $L^2(Q_T)$, where (s,P) is a weak solution of Problem* (1.16)-(1.24) *in the sense of Definition 1.1.*

2.3. The interface conditions system. Define, for all $\sigma=K|L\in\mathcal{E}_\Gamma$, for all $n\in\{0,\dots,N\}$,

$$P_\sigma^{n+1}:=P_{K,\sigma}^{n+1}-W_K\left(\pi_\sigma^{n+1}\right)=P_{L,\sigma}^{n+1}-W_L(\pi_\sigma^{n+1}), \tag{2.12}$$

and

$$Q_{K,\sigma}^{n+1}(\pi_\sigma^{n+1}):=\alpha_K^{n+1}\left(P_K^{n+1}-P_\sigma^{n+1}-W_K(\pi_\sigma^{n+1})\right)+\mathcal{R}(Z_{K,\sigma};s_K^{n+1},\tilde\pi_K^{-1}(\pi_\sigma^{n+1})), \tag{2.13}$$

where $\alpha_K^{n+1}=\frac{M_K(s_K^{n+1})}{d_{K,\sigma}}$. Then, the balance of the fluxes on the interface (2.9)–(2.10) can be rewritten as

$$Q_{K,\sigma}^{n+1}(\pi_\sigma^{n+1})+Q_{L,\sigma}^{n+1}(\pi_\sigma^{n+1})=0 \tag{2.14}$$

$$\begin{aligned}&Q_{K,\sigma}^{n+1}(\pi_\sigma^{n+1})f_K\left(\overline{s}_{K,\sigma}^{n+1}(\pi_\sigma^{n+1})\right)+Q_{L,\sigma}^{n+1}(\pi_\sigma^{n+1})f_L\left(\overline{s}_{L,\sigma}^{n+1}(\pi_\sigma^{n+1})\right)\\&\quad+\mathcal{R}(G_{K,\sigma};s_K^{n+1},\tilde\pi_K^{-1}(\pi_\sigma^{n+1}))+\mathcal{R}(G_{L,\sigma};s_L^{n+1},\tilde\pi_L^{-1}(\pi_\sigma^{n+1}))\\&\quad+\frac{\varphi_K(s_K^{n+1})-\varphi_K\circ\tilde\pi_K^{-1}(\pi_\sigma^{n+1})}{d_{K,\sigma}}+\frac{\varphi_L(s_L^{n+1})-\varphi_L\circ\tilde\pi_L^{-1}(\pi_\sigma^{n+1})}{d_{L,\sigma}}=0,\end{aligned} \tag{2.15}$$

where

$$\overline{s}_{K,\sigma}^{n+1}(p)=\begin{cases}s_K^{n+1} & \text{if}\quad Q_{K,\sigma}^{n+1}(p)\ge0,\\ \pi_K^{-1}(p) & \text{if}\quad Q_{K,\sigma}^{n+1}(p)<0.\end{cases} \tag{2.16}$$

We deduce from (2.14) that

$$\begin{aligned}P_\sigma^{n+1}&=\frac{\alpha_K^{n+1}(P_K^{n+1}-W_K(\pi_\sigma^{n+1}))+\alpha_L^{n+1}(P_L^{n+1}-W_L(\pi_\sigma^{n+1}))}{\alpha_K^{n+1}+\alpha_L^{n+1}}\\&\quad+\frac{\mathcal{R}(Z_{K,\sigma};s_K^{n+1},\tilde\pi_K^{-1}(\pi_\sigma^{n+1}))+\mathcal{R}(Z_{L,\sigma};s_L^{n},\tilde\pi_L^{-1}(\pi_\sigma^{n+1}))}{\alpha_K^{n+1}+\alpha_L^{n+1}}\end{aligned} \tag{2.17}$$

and thus that

$$Q_{K,\sigma}^{n+1}(\pi_\sigma^{n+1}) = \frac{\alpha_K^{n+1}\alpha_L^{n+1}}{\alpha_K^{n+1}+\alpha_L^{n+1}}\left(P_K^{n+1}-P_L^{n+1}-W_K(\pi_\sigma^{n+1})+W_L(\pi_\sigma^{n+1})\right) + \frac{\alpha_L^{n+1}\mathcal{R}(Z_{K,\sigma};s_K^{n+1},\tilde\pi_K^{-1}(\pi_\sigma^{n+1}))-\alpha_K^{n+1}\mathcal{R}(Z_{L,\sigma};s_L^{n},\tilde\pi_L^{-1}(\pi_\sigma^{n+1}))}{\alpha_K^{n+1}+\alpha_L^{n+1}}. \tag{2.18}$$

As a direct consequence of Lemma 1.1, $Q_{K,\sigma}^{n+1}$ belong to $\mathcal{C}^1(\mathbb{R};\mathbb{R})$ and admits finite limits as $p\to\pm\infty$.

Denote by

$$\Psi_\sigma^{n+1}(p) := Q_{K,\sigma}^{n+1}(p)\left(f_K(\overline{s}_{K,\sigma}(p))-f_L(\overline{s}_{L,\sigma}(p))\right) + \mathcal{R}(G_{K,\sigma};s_K^{n+1},\tilde\pi_K^{-1}(p))+\mathcal{R}(G_{L,\sigma};s_L^{n+1},\tilde\pi_L^{-1}(p)) + \frac{\varphi_K(s_K^{n+1})-\varphi_K\circ\pi_K^{-1}(p)}{d_{K,\sigma}}+\frac{\varphi_L(s_L^{n+1})-\varphi_L\circ\pi_L^{-1}(p)}{d_{L,\sigma}},$$

then Ψ_σ is continuous on $\mathbb{R}$.

Lemma 2.1. *Let $(s_K^{n+1},s_L^{n+1})\in[0,1]^2$, there exists $\pi_\sigma^{n+1}\in[\min_i\pi_i(0),\max_i\pi_i(1)]$ such that $\Psi_\sigma^{n+1}(\pi_\sigma^{n+1})=0$.*

Proof. From the definition (2.16) of $\overline{s}_{K,\sigma}^{n+1}(p)$, since $\lim_{p\to\min_i\pi_i(0)}\pi_K^{-1}(p)=0$, and since $Q_{K,\sigma}^{n+1}(p)$ admits a limit as $p\to\min_i\pi_i(0)$, one has

$$\lim_{p\to\min_i\pi_i(0)} Q_{K,\sigma}^{n+1}(p)\left(f_K(\overline{s}_{K,\sigma}^{n+1}(p))-f_L(\overline{s}_{L,\sigma}^{n+1}(p))\right)\geq 0$$

and also

$$\lim_{p\to\min_i\pi_i(0)}\mathcal{R}(G_{M,\sigma};s_M^{n+1},\tilde\pi_M^{-1}(p))=\max_{s\in[0,s_M]}G_{M,\sigma}(s)\geq 0,\qquad\text{with}\qquad M\in\{K,L\}.$$

This yields that

$$\lim_{p\to\min_i\pi_i(0)}\Psi_\sigma^{n+1}(p)\geq\frac{\varphi_K(s_K^{n+1})}{d_{K,\sigma}}+\frac{\varphi_L(s_L^{n+1})}{d_{L,\sigma}}\geq 0.$$

One obtains similarly that $\lim_{p\to\max_i\pi_i(1)}\Psi_\sigma^{n+1}(p)\leq 0$. One conclude thanks to the continuity of Ψ_σ^{n+1}. ◻

Proposition 2.2. *Let $\sigma=K|L\in\mathcal{E}_\Gamma$, and let $\left(s_K^{n+1},s_L^{n+1},P_K^{n+1},P_L^{n+1}\right)\in\mathbb{R}^4$, then there exists a solution $\left(\pi_\sigma^{n+1},s_{K,\sigma}^{n+1},s_{L,\sigma}^{n+1},P_{K,\sigma}^{n+1},P_{L,\sigma}^{n+1}\right)\in[\min_i\pi_i(0),\max_i\pi_i(1)]\times[0,1]^2\times\mathbb{R}^2$ to the nonlinear system* (2.7)–(2.10).

Proof. Let $\pi_\sigma^{n+1}\in\overline{\mathbb{R}}$ be a solution of the equation $\Psi_\sigma^{n+1}(\pi_\sigma^{n+1})=0$, whose existence has been claimed in Lemma 2.1. Firstly, defining $s_{K,\sigma}^{n+1}:=\pi_K^{-1}(\pi_\sigma^{n+1})$ and $s_{L,\sigma}^{n+1}:=\pi_L^{-1}(\pi_\sigma^{n+1})$, one has directly that

$$\pi_\sigma^{n+1}\in\tilde\pi_K(s_{K,\sigma}^{n+1})\cap\tilde\pi_L(s_{L,\sigma}^{n+1}).$$

As it was noticed in Lemma 1.1, the function $p \mapsto W_K(p) - W_L(p)$ is uniformly bounded. Hence, the values

$$P_{K,\sigma}^{n+1} := \frac{\alpha_K^{n+1} P_K^{n+1} + \alpha_L^{n+1} P_L^{n+1} + \alpha_L^{n+1}\left(W_K(\pi_\sigma^{n+1}) - W_L(\pi_\sigma^{n+1})\right)}{\alpha_K^{n+1} + \alpha_L^{n+1}} + \frac{\mathcal{R}\left(Z_{K,\sigma}; s_K^{n+1}, \tilde\pi_K^{-1}(\pi_\sigma^{n+1})\right) + \mathcal{R}\left(Z_{L,\sigma}; s_L^{n}, \tilde\pi_L^{-1}(\pi_\sigma^{n+1})\right)}{\alpha_K^{n+1} + \alpha_L^{n+1}}$$

and

$$P_{L,\sigma}^{n+1} := \frac{\alpha_K^{n+1} P_K^{n+1} + \alpha_L^{n+1} P_L^{n+1} + \alpha_K^{n+1}\left(W_L(\pi_\sigma^{n+1}) - W_K(\pi_\sigma^{n+1})\right)}{\alpha_K^{n+1} + \alpha_L^{n+1}} + \frac{\mathcal{R}\left(Z_{K,\sigma}; s_K^{n+1}, \tilde\pi_K^{-1}(\pi_\sigma^{n+1})\right) + \mathcal{R}\left(Z_{L,\sigma}; s_L^{n}, \tilde\pi_L^{-1}(\pi_\sigma^{n+1})\right)}{\alpha_K^{n+1} + \alpha_L^{n+1}}$$

are finite. It is now easy to check that $\left(\pi_\sigma^{n+1}, s_{K,\sigma}^{n+1}, s_{L,\sigma}^{n+1}, P_{K,\sigma}^{n+1}, P_{L,\sigma}^{n+1}\right)$ is a solution to the system (2.7)–(2.10) thanks to the analysis carried out above. ☐

3. *A priori* estimates and existence of a discrete solution.

3.1. $\mathbf{L^\infty(Q_T)}$ estimate on the saturation.

PROPOSITION 3.1. *Let $(s_\mathcal{D}, P_\mathcal{D})$ be a solution to the scheme* (2.2)–(2.11), *then*

$$0 \le s_\mathcal{D} \le 1 \qquad a.e.\ in\ Q_T. \tag{3.1}$$

Proof. We will prove that for all $K \in \mathcal{T}$, for all $n \in \{0, \dots, N\}$,

$$s_K^{n+1} \le 1.$$

The proof for obtaining $s_K^{n+1} \ge 0$ is similar.

Using the definition (2.5) of $F_{K,\sigma}^{n+1}$, one can rewrite (2.4) under the form

$$H_K\left(s_K^{n+1}, s_K^n, \left(s_L^{n+1}\right)_{L\in\mathcal{N}_K}, \left(s_{K,\sigma}^{n+1}\right)_{\sigma\in\mathcal{E}_{K,\Gamma}}, \left(Q_{K,\sigma}^{n+1}\right)_{\sigma\in\mathcal{E}_K}\right) = 0, \tag{3.2}$$

where H_K is non increasing with respect to $s_K^n, \left(s_L^{n+1}\right)_{L\in\mathcal{N}_K}, \left(s_{K,\sigma}^{n+1}\right)_{\sigma\in\mathcal{E}_{K,\Gamma}}$. Making use of the notations $a\top b = \max(a,b)$, we obtain that

$$H_K\left(s_K^{n+1}, s_K^n\top 1, \left(s_L^{n+1}\top 1\right)_{L\in\mathcal{N}_K}, \left(s_{K,\sigma}^{n+1}\top 1\right)_{\sigma\in\mathcal{E}_{K,\Gamma}}, \left(Q_{K,\sigma}^{n+1}\right)_{\sigma\in\mathcal{E}_K}\right) \le 0.$$

We remark that for all $K \in \mathcal{T}$ and for all $s \in [0,1]$ one has

$$\sum_{\sigma\in\mathcal{E}_K} m(\sigma) G_{K,\sigma}(s) = 0. \tag{3.3}$$

Combining (3.3) and (2.2) we have

$$H_K\left(1, 1, (1)_{L\in\mathcal{N}_K}, (1)_{\sigma\in\mathcal{E}_{K,i}}, \left(Q_{K,\sigma}^{n+1}\right)_{\sigma\in\mathcal{E}_K}\right) = 0.$$

Hence, using once again the monotonicity of H_K, one obtains

$$H_K\left(1, s_K^n \top 1, \left(s_L^{n+1}\top 1\right)_{L\in\mathcal{N}_K}, \left(s_{K,\sigma}^{n+1}\top 1\right)_{\sigma\in\mathcal{E}_{K,\Gamma}}, \left(Q_{K,\sigma}^{n+1}\right)_{\sigma\in\mathcal{E}_K}\right) \le 0.$$

Since $a\top b$ is either equal to a or to b, one has

$$H_K\left(s_K^{n+1}\top 1, s_K^n \top 1, \left(s_L^{n+1}\top 1\right)_{L\in\mathcal{N}_K}, \left(s_{K,\sigma}^{n+1}\top 1\right)_{\sigma\in\mathcal{E}_{K,\Gamma}}, \left(Q_{K,\sigma}^{n+1}\right)_{\sigma\in\mathcal{E}_K}\right) \le 0. \quad (3.4)$$

Next we remark that for any $\sigma = K|L \in \mathcal{E}_\Gamma$ the the equation (2.10) can be written as

$$H_\sigma\left(s_K^{n+1}, s_L^{n+1}, \left(s_{M,\sigma}^{n+1}\right)_{M\in\{K,L\}}, \left(Q_{M,\sigma}^{n+1}\right)_{M\in\{K,L\}}\right) = 0,$$

where H_σ is non increasing with respect to s_K^{n+1}, s_L^{n+1} and $\left(s_{M,\sigma}^{n+1}\right)_{M\in\{K,L\}}$. Thanks to (2.9) and using $\gamma_i(1) = 0$ for $i \in \{1,2\}$ we obtain

$$H_\sigma\left(1, 1, (1)_{M\in\{K,L\}}, \left(Q_{M,\sigma}^{n+1}\right)_{M\in\{K,L\}}\right) = 0.$$

Using the same arguments as for (3.4) one has that

$$H_\sigma\left(s_K^{n+1}\top 1, s_L^{n+1}\top 1, \left(s_{M,\sigma}^{n+1}\top 1\right)_{M\in\{K,L\}}, \left(Q_{M,\sigma}^{n+1}\right)_{M\in\{K,L\}}\right) \le 0. \quad (3.5)$$

Multiplying (3.4) by δt and summing over $K \in \mathcal{T}$ provides, using (3.5) and the conservativity of the scheme,

$$\sum_{K\in\mathcal{T}} \phi_K (s_K^{n+1} - 1)^+ m(K) \le \sum_{K\in\mathcal{T}} \phi_K (s_K^n - 1)^+ m(K).$$

Since $s_0 \in L^\infty(Q_T; [0,1])$, $s_K^0 \in [0,1]$ for all $K \in \mathcal{T}$. A straightforward induction allows us to conclude. ☐

3.2. Energy estimate.

DEFINITION 3.1. *We define the discrete $L^2(0,T;H^1(\Omega_i))$ semi-norm of an element $u_\mathcal{D} \in \mathcal{X}(\mathcal{D}, i)$ by*

$$|u_\mathcal{D}|_{\mathcal{D},i}^2 := \sum_n \delta t \sum_{\sigma=K|L\in\mathcal{E}_i} \tau_{KL}\left(u_K^{n+1} - u_L^{n+1}\right)^2 + \sum_n \delta t \sum_{K\in\mathcal{T}_i}\sum_{\sigma\in\mathcal{E}_{K,\Gamma}} \tau_{K\sigma}\left(u_K^{n+1} - u_{K,\sigma}^{n+1}\right)^2.$$

LEMMA 3.2. *The following inequalities hold:*

- *for all $\sigma = K|L \in \mathcal{E}_{\text{int}}$,*

$$Q_{K,\sigma}^{n+1} f_K\left(\overline{s}_{K,\sigma}^{n+1}\right)\left(\pi_K(s_K^{n+1}) - \pi_K(s_L^{n+1})\right) \ge Q_{K,\sigma}^{n+1}\left(W_K(\pi_K(s_K^{n+1})) - W_K(\pi_K(s_L^{n+1}))\right); \quad (3.6)$$

- *for all $\sigma \in \mathcal{E}_{K,\Gamma}$,*

$$Q_{K,\sigma}^{n+1} f_K\left(\overline{s}_{K,\sigma}^{n+1}\right)\left(\pi_K(s_K^{n+1}) - \pi_\sigma^{n+1}\right) \ge Q_{K,\sigma}^{n+1}\left(W_K(\pi_K(s_K^{n+1})) - W_K(\pi_\sigma^{n+1})\right). \quad (3.7)$$

Proof. Since $f_K \circ \pi_K^{-1}$ is a non decreasing function, then function $W_K : p \mapsto \int_0^p f_K \circ \pi_K^{-1}(a)\mathrm{d}a$ is convex, so that for all $(a,b) \in \mathbb{R}^2$,

$$f_K \circ \pi_K^{-1}(a)\,(b-a) \le W_K(b) - W_K(a) \le f_K \circ \pi_K^{-1}(b)\,(b-a)\,.$$

The inequalities (3.6) and (3.7) follow from the definition (2.6) of $\overline{s}_{K,\sigma}^{n+1}$. ∎

LEMMA 3.3. *Let us define*

$$\mathcal{G}_{K,\sigma}(p) := \int_0^p G_{K,\sigma}\left(\tilde{\pi}_K^{-1}(\tau)\right)\mathrm{d}\tau \tag{3.8}$$

for all $K \in \mathcal{T}$ and $\sigma \in \mathcal{E}_K$. Then, the following estimates hold:

- *for all $\sigma = K|L \in \mathcal{E}_{\text{int}}$,*

$$\mathcal{R}(G_{K,\sigma}; s_K^{n+1}, s_L^{n+1})\left(\pi_K(s_K^{n+1}) - \pi_K(s_L^{n+1})\right) \ge \mathcal{G}_{K,\sigma}(\pi_K(s_K^{n+1})) - \mathcal{G}_{K,\sigma}(\pi_K(s_L^{n+1})) \tag{3.9}$$

- *for all $\sigma \in \mathcal{E}_{K,\Gamma}$,*

$$\mathcal{R}(G_{K,\sigma}; s_K^{n+1}, s_{K,\sigma}^{n+1})\left(\pi_K(s_K^{n+1}) - \pi_\sigma^{n+1}\right) \ge \mathcal{G}_{K,\sigma}(\pi_K(s_K^{n+1})) - \mathcal{G}_{K,\sigma}(\pi_\sigma^{n+1}). \tag{3.10}$$

Proof. For any $a, b \in \mathbb{R}$ one has

$$\begin{aligned}\mathcal{R}(G_{K,\sigma}; \tilde{\pi}_K^{-1}(a), \tilde{\pi}_K^{-1}(b))\,(a-b)) \int_b^a G_{K,\sigma}\left(\tilde{\pi}_K^{-1}(p)\right)\mathrm{d}p \\ + \int_b^a \mathcal{R}\left(G_{K,\sigma}; \tilde{\pi}_K^{-1}(a), \tilde{\pi}_K^{-1}(b)\right) - G_{K,\sigma}\left(\tilde{\pi}_K^{-1}(p)\right)\mathrm{d}p.\end{aligned} \tag{3.11}$$

We only have to remark that in view of (2.1) the last term in the right hand side of (3.11) is positive. ∎

LEMMA 3.4. *For all $K \in \mathcal{T}$, for all $n \in \{0,\dots,N\}$ and for all $\sigma \in \mathcal{E}_{K,\Gamma}$, one has*

$$\begin{aligned}\left(\varphi_K(s_K^{n+1}) - \varphi_K(s_{K,\sigma}^{n+1})\right)\left(\pi_K(s_K^{n+1}) - \pi_\sigma^{n+1}\right) \\ \ge \left(\varphi_K(s_K^{n+1}) - \varphi_K(s_{K,\sigma}^{n+1})\right)\left(\pi_K(s_K^{n+1}) - \pi_K(s_{K,\sigma}^{n+1})\right).\end{aligned} \tag{3.12}$$

Proof. Assume that $s_{K,\sigma}^{n+1} \in (0,1)$, then $\tilde{\pi}_K(s_{K,\sigma}^{n+1}) = \{\pi_K(s_{K,\sigma}^{n+1})\}$, thus the inequality (3.12) is in fact an equality. Assume now that $s_{K,\sigma}^{n+1} = 0$, then $\pi_\sigma^{n+1} \le \pi_K(s_{K,\sigma}^{n+1}) \le \pi_K(s_K^{n+1})$, and $\varphi_K(s_{K,\sigma}^{n+1}) \le \varphi_K(s_K^{n+1})$. The inequality (3.12) follows. Similarly, if $s_{K,\sigma}^{n+1} = 1$, then $\pi_\sigma^{n+1} \ge \pi_K(s_{K,\sigma}^{n+1}) \ge \pi_K(s_K^{n+1})$, and $\varphi_K(s_{K,\sigma}^{n+1}) \ge \varphi_K(s_K^{n+1})$, leading also to (3.12). ∎

PROPOSITION 3.5. *There exists C_1, depending only on α_M, $\min_i K_i$, μ_o, μ_w , $\max_i \|\pi_i\|_{L^1((0,1))}$ and Ω, such that*

$$\sum_{i\in\{1,2\}} \left(|P_\mathcal{D}|_{\mathcal{D},i}^2 + |\varphi(s_\mathcal{D})|_{\mathcal{D},i}^2\right) \le C_1. \tag{3.13}$$

Proof. Multiplying the equation (2.4) by $\delta t \pi_K(s_K^{n+1})$ and summing over $K \in \mathcal{T}$ and $n \in \{0, \dots, N\}$ yield, after reorganizing the sum,

$$A + B = 0, \tag{3.14}$$

where

$$A = \sum_{n=0}^{N} \sum_{K \in \mathcal{T}} \phi_K \pi_K(s_K^{n+1}) \left(s_K^{n+1} - s_K^n\right) m(K),$$

$$B = \sum_{n=0}^{N} \delta t \sum_{\sigma = K|L \in \mathcal{E}_{\text{int}}} m(\sigma) F_{K,\sigma}^{n+1} \left(\pi_K(s_K^{n+1}) - \pi_K(s_L^{n+1})\right)$$
$$+ \sum_{n=0}^{N} \delta t \sum_{K \in \mathcal{T}} \sum_{\sigma \in \mathcal{E}_{K,\Gamma}} m(\sigma) F_{K,\sigma}^{n+1} \left(\pi_K(s_K^{n+1}) - \pi_\sigma^{n+1}\right),$$

where we have used (2.10). The definition (2.5) of $F_{K,\sigma}^{n+1}$ gives

$$B = B_1 + B_2 + B_3, \tag{3.15}$$

where

$$B_1 = \sum_{n=0}^{N} \delta t \sum_{\sigma = K|L \in \mathcal{E}_{\text{int}}} m(\sigma) Q_{K,\sigma}^{n+1} f_K(\overline{s}_{K,\sigma}^{n+1}) \left(\pi_K(s_K^{n+1}) - \pi_K(s_L^{n+1})\right)$$
$$+ \sum_{n=0}^{N} \delta t \sum_{K \in \mathcal{T}} \sum_{\sigma \in \mathcal{E}_{K,\Gamma}} m(\sigma) Q_{K,\sigma}^{n+1} f_K(\overline{s}_{K,\sigma}^{n+1}) \left(\pi_K(s_K^{n+1}) - \pi_\sigma^{n+1}\right),$$

$$B_2 = \sum_{n=0}^{N} \delta t \sum_{\sigma = K|L \in \mathcal{E}_{\text{int}}} m(\sigma) \mathcal{R}(G_{K,\sigma}; s_K^{n+1}, s_L^{n+1}) \left(\pi_K(s_K^{n+1}) - \pi_K(s_L^{n+1})\right)$$
$$+ \sum_{n=0}^{N} \delta t \sum_{K \in \mathcal{T}} \sum_{\sigma \in \mathcal{E}_{K,\Gamma}} m(\sigma) \mathcal{R}(G_{K,\sigma}; s_K^{n+1}, s_{K,\sigma}^{n+1}) \left(\pi_K(s_K^{n+1}) - \pi_\sigma^{n+1}\right),$$

$$B_3 = \sum_{n=0}^{N} \delta t \sum_{\sigma = K|L \in \mathcal{E}_{\text{int}}} \tau_{KL} \left(\varphi_K(s_K^{n+1}) - \varphi_K(s_L^{n+1})\right) \left(\pi_K(s_K^{n+1}) - \pi_K(s_L^{n+1})\right)$$
$$+ \sum_{n=0}^{N} \delta t \sum_{K \in \mathcal{T}} \sum_{\sigma \in \mathcal{E}_{K,\Gamma}} \tau_{K\sigma} \left(\varphi_K(s_K^{n+1}) - \varphi_K(s_{K,\sigma}^{n+1})\right) \left(\pi_K(s_K^{n+1}) - \pi_\sigma^{n+1}\right).$$

It follows from Lemma 3.2 that

$$B_1 \geq \sum_{n=0}^{N} \delta t \sum_{\sigma = K|L \in \mathcal{E}_{\text{int}}} m(\sigma) Q_{K,\sigma}^{n+1} \left(W_K(\pi_K(s_K^{n+1})) - W_K(\pi_K(s_L^{n+1}))\right)$$
$$+ \sum_{n=0}^{N} \delta t \sum_{K \in \mathcal{T}} \sum_{\sigma \in \mathcal{E}_{K,\Gamma}} m(\sigma) Q_{K,\sigma}^{n+1} \left(W_K(\pi_K(s_K^{n+1})) - W_K(\pi_\sigma^{n+1})\right).$$

Multiplying the equation (2.2) by $\delta t\left(P_K^{n+1} - W_K(\pi_K(s_K^{n+1}))\right)$ and summing over $K \in \mathcal{T}$ and $n \in \{0,\ldots,N\}$ yields, after reorganizing the sum and using (2.8) and (2.9),

$$\begin{aligned}
&\sum_{n=0}^{N}\delta t \sum_{\sigma=K|L\in\mathcal{E}_{\rm int}} m(\sigma)Q_{K,\sigma}^{n+1}(P_K^{n+1}-P_L^{n+1})\\
&+\sum_{n=0}^{N}\delta t \sum_{K\in\mathcal{T}}\sum_{\sigma\in\mathcal{E}_{K,\Gamma}} m(\sigma)Q_{K,\sigma}^{n+1}(P_K^{n+1}-P_{K,\sigma}^{n+1})\\
&\quad=\sum_{n=0}^{N}\delta t \sum_{\sigma=K|L\in\mathcal{E}_{\rm int}} m(\sigma)Q_{K,\sigma}^{n+1}\left(W_K(\pi_K((s_K^{n+1}))-W_K(\pi_K(s_L^{n+1}))\right)\\
&\quad+\sum_{n=0}^{N}\delta t \sum_{K\in\mathcal{T}}\sum_{\sigma\in\mathcal{E}_{K,\Gamma}} m(\sigma)Q_{K,\sigma}^{n+1}\left(W_K(\pi_K(s_{K,}^{n+1}))-W_K(\pi_\sigma^{n+1})\right).
\end{aligned}$$

Therefore, using the definition (2.3) of $Q_{K,\sigma}^n$, we deduce that

$$B_1 \geq B_4 + B_5, \tag{3.16}$$

where

$$\begin{aligned}
B_4 &= \sum_{n=0}^{N}\delta t \sum_{\sigma=K|L\in\mathcal{E}_{\rm int}} \frac{m(\sigma)M_{K,L}}{d_{K,L}}(P_K^{n+1}-P_L^{n+1})^2\\
&+\sum_{n=0}^{N}\delta t \sum_{K\in\mathcal{T}}\sum_{\sigma\in\mathcal{E}_{K,\Gamma}} \frac{m(\sigma)M_K}{d_{K,\sigma}}(P_K^{n+1}-P_{K,\sigma}^{n+1})^2
\end{aligned}$$

and

$$\begin{aligned}
B_5 \quad &= \sum_{n=0}^{N}\delta t \sum_{\sigma=K|L\in\mathcal{E}_{\rm int}} m(\sigma)\mathcal{R}(Z_{K,\sigma};s_K^{n+1},s_L^{n+1})(P_K^{n+1}-P_L^{n+1})\\
&+\sum_{n=0}^{N}\delta t \sum_{K\in\mathcal{T}}\sum_{\sigma\in\mathcal{E}_{K,\Gamma}} m(\sigma)\mathcal{R}(Z_{K,\sigma};s_K^{n+1},s_{K,\sigma}^{n+1})(P_K^{n+1}-P_{K,\sigma}^{n+1}).
\end{aligned} \tag{3.17}$$

Using (1.18), i.e the fact that for all $s\in\mathbb{R}$, $M_i(s)\geq \alpha_M > 0$ we obtain

$$B_4 \geq \alpha_M \sum_{i\in\{1,2\}} |P_{\mathcal{D}}|_{\mathcal{D},i}^2\,. \tag{3.18}$$

The Cauchy-Schwarz inequality applied to the right hand side of (3.17) implies

$$\begin{aligned}
|B_5| \leq E_{int}&\left(\sum_{n=0}^{N}\delta t \sum_{\sigma=K|L\in\mathcal{E}_{\rm int}} \frac{m(\sigma)}{d_{K,L}}(P_K^{n+1}-P_L^{n+1})^2\right)^{\frac12}\\
&+E_\Gamma\left(\sum_{n=0}^{N}\delta t \sum_{K\in\mathcal{T}}\sum_{\sigma\in\mathcal{E}_{K,\Gamma}} \frac{m(\sigma)}{d_{K,\sigma}}(P_K^{n+1}-P_{K,\sigma}^{n+1})^2\right)^{\frac12},
\end{aligned}$$

where

$$(E_{int})^2 = \sum_{n=0}^{N} \delta t \sum_{\sigma=K|L\in\mathcal{E}_{\rm int}} m(\sigma) d_{K,L} \mathcal{R}(Z_{K,\sigma}; s_K^{n+1}, s_L^{n+1})^2$$

and

$$(E_\Gamma)^2 = \sum_{n=0}^{N} \delta t \sum_{K\in\mathcal{T}} \sum_{\sigma\in\mathcal{E}_{K,\Gamma}} m(\sigma) d_{K,\sigma} \mathcal{R}(Z_{K,\sigma}; s_K^{n+1}, s_{K,\sigma}^{n+1})^2.$$

Therefore we deduce that,

$$B_5^2 \le \frac{3}{2} T |\mathbf{g}|^2 d \sum_{i\in\{1,2\}} m(\Omega_i) \|\zeta_i\|^2_{L^\infty((0,1))} \sum_{i\in\{1,2\}} |P_\mathcal{D}|^2_{\mathcal{D},i}\,, \tag{3.19}$$

where d stands for the dimension of Ω. Combining (3.16), (3.18) and (3.19) one has

$$B_1 \ge \alpha_M \sum_{i\in\{1,2\}} |P_\mathcal{D}|^2_{\mathcal{D},i} - \left(\frac{3}{2} T |\mathbf{g}|^2 d \sum_{i\in\{1,2\}} m(\Omega_i) \|\zeta_i\|^2_{L^\infty((0,1))} \right)^{\frac{1}{2}} \left(\sum_{i\in\{1,2\}} |P_\mathcal{D}|^2_{\mathcal{D},i} \right)^{\frac{1}{2}}. \tag{3.20}$$

We now will show the estimates on the term B_2. Using Lemma 3.3 we have

$$\begin{aligned} B_2 \quad &\ge \sum_{n=0}^{N} \delta t \sum_{\sigma=K|L\in\mathcal{E}_{\rm int}} m(\sigma) \left(\mathcal{G}_{K,\sigma}\left(\pi_K(s_K^{n+1})\right) - \mathcal{G}_{K,\sigma}\left(\pi_K(s_L^{n+1})\right) \right) \\ &+ \sum_{n=0}^{N} \delta t \sum_{K\in\mathcal{T}} \sum_{\sigma\in\mathcal{E}_{K,\Gamma}} m(\sigma) \left(\mathcal{G}_{K,\sigma}\left(\pi_K(s_K^{n+1})\right) - \mathcal{G}_{K,\sigma}\left(\pi_\sigma^{n+1}\right) \right). \end{aligned} \tag{3.21}$$

Recombining terms we obtain

$$\begin{aligned} B_2 \quad &\ge \sum_{n=0}^{N} \delta t \sum_{K\in\mathcal{T}} \sum_{\mathcal{E}_{K,\rm int}} m(\sigma) \mathcal{G}_{K,\sigma}\left(\pi_K(s_K^{n+1})\right) \\ &+ \sum_{n=0}^{N} \delta t \sum_{K\in\mathcal{T}} \sum_{\sigma\in\mathcal{E}_{K,\Gamma}} m(\sigma) \left(\mathcal{G}_{K,\sigma}\left(\pi_K(s_K^{n+1})\right) - \mathcal{G}_{K,\sigma}\left(\pi_\sigma^{n+1}\right) \right), \end{aligned}$$

which in view of (3.8) and (3.3) implies

$$B_2 \ge -\sum_{n=0}^{N} \delta t \sum_{K\in\mathcal{T}} \sum_{\sigma\in\mathcal{E}_{K,\Gamma}} m(\sigma) \mathcal{G}_{K,\sigma}\left(\pi_\sigma^{n+1}\right).$$

Remark that if $\sigma = K|L \in \mathcal{E}_\Gamma$ then the function $\mathcal{G}_{K,\sigma}(p) + \mathcal{G}_{L,\sigma}(p)$ in general is not equal to zero. However we can write an lower bound for the term B_2. Indeed, comparing the definition (1.15) of φ_i with the definition (1.17) of γ_i, and using the fact that $\gamma_i(0) = 0$ and $\gamma_i(1) = 0$ one has

$$\int_0^{\pi_\sigma^n} \gamma_K \circ \tilde{\pi}_K^{-1}(p) \mathrm{d}p = \int_0^{s_{K,\sigma}^n} \gamma_K(a) \pi_K'(a) \mathrm{d}a = (\rho_o - \rho_w) \varphi_K(s_{K,\sigma}^n)$$

and thus, in view of Proposition 3.1

$$B_2 \geq -|\rho_o - \rho_w||\mathbf{g}| \max_{i\in\{1,2\}} \varphi_i(1)\ m(\Gamma)T.$$

Because of the definition (1.15) of the function φ_i, then, for all $(a,b) \in [0,1]^2$,

$$(\varphi_i(a) - \varphi_i(b))(\pi_i(a) - \pi_i(b)) \geq \frac{\max(\mu_o,\mu_w)}{K_i} \left(\varphi_i(a) - \varphi_i(b)\right)^2. \tag{3.22}$$

Then it follows from Lemma 3.4 and for inequality (3.22) that

$$B_3 \geq \frac{\max(\mu_o,\mu_w)}{\min_{i\in\{1,2\}} K_i} \sum_{i\in\{1,2\}} |\varphi_i(s_{\mathcal{D}})|^2_{\mathcal{D},i}\,. \tag{3.23}$$

We define $\Pi_i(s) = \int_0^s \pi_i(a)\mathrm{d}a$, then Π_i is a continuous convex function. As a consequence, for all $(a,b) \in [0,1]^2$,

$$\pi_i(b)(b-a) \geq \Pi_i(b) - \Pi_i(a).$$

Therefore,

$$\begin{aligned} A &\geq \sum_{n=0}^{N} \sum_{K\in\mathcal{T}} \phi_K \left(\Pi_K(s_K^{n+1}) - \Pi_K(s_K^n)\right) m(K) \\ &= \sum_{K\in\mathcal{T}} \phi_K \left(\Pi_K(s_K^{N+1}) - \Pi_K(s_K^0)\right) m(K). \end{aligned}$$

Using the fact that, for all $(a,b) \in [0,1]^2$, one has

$$\Pi_i(b) - \Pi_i(a) = \int_a^b \pi_i(u)\mathrm{d}u \geq -\int_0^1 |\pi_i(u)|\mathrm{d}u,$$

it follows from Proposition 3.1 that

$$A \geq -\sum_{i\in\{1,2\}} \phi_i m(\Omega_i) \|\pi_i\|_{L^1((0,1))}. \tag{3.24}$$

Taking (3.20), (3.23), (3.23) and (3.24) into account in (3.14) we have.

$$\begin{aligned} \alpha_M \sum_{i\in\{1,2\}} |P_{\mathcal{D}}|^2_{\mathcal{D},i} \quad &- \left(\frac{3}{2}\frac{T|\mathbf{g}|^2}{d} \sum_{i\in\{1,2\}} m(\Omega_i)\|\zeta_i\|^2_{L^\infty((0,1))}\right)^{\frac{1}{2}} \left(\sum_{i\in\{1,2\}} |P_{\mathcal{D}}|^2_{\mathcal{D},i}\right)^{\frac{1}{2}} \\ &+ \frac{\max(\mu_o,\mu_w)}{\min_{i\in\{1,2\}} K_i} \sum_{i\in\{1,2\}} |\varphi_i(s_{\mathcal{D}})|^2_{\mathcal{D},i} \leq C. \end{aligned} \tag{3.25}$$

Applying Young's inequality to (3.21) we complete the proof of Proposition 3.5. Indeed,

$$\frac{\alpha_M}{2} \sum_{i\in\{1,2\}} |P_{\mathcal{D}}|^2_{\mathcal{D},i} + \frac{\max(\mu_o,\mu_w)}{\min_{i\in\{1,2\}} K_i} \sum_{i\in\{1,2\}} |\varphi_i(s_{\mathcal{D}})|^2_{\mathcal{D},i} \leq C.$$

□

PROPOSITION 3.6. *There exists C only depending on Ω_i, C_1 and $\|W_1 - W_2\|_\infty$ such that*

$$\|P_\mathcal{D}\|_{L^2(Q_T)} \le C.$$

Proof. In view of the discrete Poincar-Wirtinger inequality [27] and Proposition 3.5, there exists C depending only on Ω_i and C_1 such that

$$\iint_{Q_{i,T}} (P_\mathcal{D} - m_{\Omega_i}(P_\mathcal{D}))^2 \,\mathrm{d}\mathbf{x}\mathrm{d}t \le C.$$

In order to conclude the proof, it only remains to check that $m_{\Omega_2}(P_\mathcal{D})$ is uniformly bounded. □

3.3. Existence of a discrete solution. PROPOSITION 3.7. *There exists (at least) a solution to the scheme* (2.4)-(2.11).

Proof. The proof is based on a topological degree argument (see for example [20]). For $\nu \in [0,1]$, we introduce the functions

- $f_i^\nu(s) = \nu f_i(s) + (1-\nu)s$,
- $\zeta_i^\nu(s) = \nu\zeta_i(s)$, $\gamma_i^\nu(s) = \nu\gamma_i(s)$
- $M_i^\nu(s) = \nu M_i(s) + (1-\nu)\alpha_M$,
- $\lambda_i^\nu(s) = \nu\lambda_i(s) + (1-\nu)\alpha_M s(1-s)$,
- $\pi_i^\nu(s) = \nu\pi_i(s) + (1-\nu)\pi_1(s)$,
- $\varphi_i^\nu(s) = \displaystyle\int_0^s \lambda_i^\nu(a) \left(\pi_i^\nu\right)'(a)\mathrm{d}a$,
- $W_i^\nu(s) = \displaystyle\int_{s^\star}^s f_i^\nu(a) \left(\pi_i^\nu\right)'(a)\mathrm{d}a$.

We denote by $(s_\mathcal{D}^\nu, P_\mathcal{D}^\nu)$ the solution to the modified scheme. For $\nu = 0$, the problem becomes homogeneous, corresponding to the equations

$$\begin{cases} \partial_t s^0 - \mathrm{div}\left(s^0 \nabla P^0 - \nabla\varphi^0(s^0)\right) = 0, \\ -\alpha_M \Delta P^0 = 0. \end{cases} \tag{3.26}$$

The pressure equation provides a classical linear Finite Volume scheme which is completely uncoupled from the saturation equation. The transmission conditions (2.9),(2.8) turn to

$$P_{K,\sigma}^{n+1,0} = P_{L,\sigma}^{n+1,0} = \frac{\tau_{K\sigma} P_K^{n+1,0} + \tau_{L\sigma} P_L^{n+1,0}}{\tau_{K\sigma} + \tau_{L\sigma}},$$

and thus

$$Q_{K,\sigma}^{n+1,0} = \tau_{KL}\left(P_K^{n+1,0} - P_L^{n+1,0}\right).$$

Note that the *a priori* estimates (3.1) and (3.13) still hold for $(s_\mathcal{D}^\nu, P_\mathcal{D}^\nu)$ instead of $(s_\mathcal{D}, P_\mathcal{D})$. We introduce now a new parameter $\eta \in [0,1]$, and we approximate the problem

$$\begin{cases} \partial_t s^{0,\eta} - \eta\mathrm{div}\left(s^{0,\eta} \nabla P^0 - \nabla\varphi^0(s^{0,\eta})\right) = 0, \\ -\alpha_M \Delta P^0 = 0. \end{cases}$$

The corresponding discrete solution $s_\mathcal{D}^{0,\eta}$ satisfies

$$0 \le s_\mathcal{D}^{0,\eta} \le 1, \quad \forall \eta \in [0,1]. \tag{3.27}$$

We introduce the compact set

$$\mathcal{K} = \left\{ (u_\mathcal{D}, v_\mathcal{D}) \in (\mathcal{X}(\mathcal{D}))^2 \;\Big|\; \|u_\mathcal{D}\|_\infty \le 2 \text{ and } |v_\mathcal{D}|_\mathcal{D} \le 2C_1 \right\},$$

where C_1 is the quantity introduced in Proposition 3.5. Since, for $\nu = \eta = 0$, the problem turns to an invertible linear problem, we can claim that the corresponding topological degree is equal to $+1$ (since the determinant of the underlying matrix is positive). One can let first η go to 1, and thanks to (3.13),(3.27), $\left(s_\mathcal{D}^{0,\eta}, P_\mathcal{D}^0\right)$ never belongs to the boundary $\partial\mathcal{K}$ of $\mathcal{K}$. Hence, the topological degree is constant for $\eta \in [0,1]$, and, for $\eta = 1$, the discrete counterpart of (3.26) admits at least a solution. Letting then ν tend to 1 provides thanks to similar arguments the existence of a solution to the scheme (2.4)-(2.7). ◻

4. Compactness properties of the discrete solution. In order to prove the convergence of the scheme, we will use the method presented in [24] to derive the relative compactness of the sequencies $(s_{\mathcal{D}_m})_{m\in\mathbb{N}}$ and $(P_{\mathcal{D}_m})_{m\in\mathbb{N}}$, where $(\mathcal{D}_m)_{m\in\mathbb{N}}$ is a sequence of admissible discretizations of $\Omega \times (0,T)$ in the sense of Definition 2.3, for which the discretization parameter $h_m := \text{size}(\mathcal{D}_m)$ tends to 0 as $m \to \infty$, while the regularity parameter $\text{reg}(\mathcal{D}_m)$ remains bounded.

Firstly, since $0 \le s_{\mathcal{D}_m} \le 1$ almost everywhere in Q_T, we can claim that there exists $s \in L^\infty(Q_T;[0,1])$, such that, up to a subsequence,

$$s_{\mathcal{D}_m} \rightharpoonup s \text{ in the } L^\infty(Q_T) \text{ weak-}\star \text{ sense as } m \to \infty.$$

This is of course not sufficient to pass to the limit, so that we seek for additional compactness on the family of approximate solutions $(s_{\mathcal{D}_m}, P_{\mathcal{D}_m})_m$.

4.1. Estimates on differences of space and time translates. We recall here the lemmas 4.2 and 4.3 of [24].

LEMMA 4.1. *Let $u_\mathcal{D}$ be an element of $\mathcal{X}(\mathcal{D})$, then for all $\xi \in \mathbb{R}^d$,*

$$\int_0^T \int_{\Omega_{i,\xi}} (u_\mathcal{D}(\mathbf{x}+\xi,t) - u_\mathcal{D}(\mathbf{x},t))^2 \,\mathrm{d}\mathbf{x}\mathrm{d}t \le |u_\mathcal{D}|^2_{\mathcal{D},i} |\xi| \left(|\xi| + 2\text{size}(\mathcal{D})\right),$$

where $\Omega_{i,\xi} = \{x \in \Omega_i \mid [\mathbf{x}, \mathbf{x}+\xi] \subset \Omega_i\}$.

LEMMA 4.2. *Let $u_\mathcal{D}$ be an element of $\mathcal{X}(\mathcal{D})$, and let $T_i(u_\mathcal{D})$ the function of $L^2(\mathbb{R}^{d+1})$ defined by*

$$T_i(u_\mathcal{D})(\mathbf{x},t) = \begin{cases} u_\mathcal{D}(\mathbf{x},t) & \textit{if } (\mathbf{x},t) \in \Omega_i \times (0,T), \\ 0 & \textit{otherwise}, \end{cases}$$

then for all $\xi \in \mathbb{R}^d$,

$$\int_0^T \int_{\mathbb{R}^d} (T_i(u_\mathcal{D})(\mathbf{x}+\xi,t) - T_i(u_\mathcal{D})(\mathbf{x},t))^2 \,\mathrm{d}\mathbf{x}\mathrm{d}t \\ \le |u_\mathcal{D}|^2_{\mathcal{D},i} |\xi| \left(|\xi| + 2\text{size}(\mathcal{D}) + 2m(\partial\Omega_i)\, \|u_\mathcal{D}\|_\infty\right),$$

where $\Omega_{i,\xi} = \{x \in \Omega_i \mid [\mathbf{x}, \mathbf{x}+\xi] \subset \Omega_i\}$.

LEMMA 4.3. *There exists C_3, which does not depend on size$(\mathcal{T})$, δt nor on τ such that for all $\tau \in (0,T)$,*

$$\int_0^{T-\tau} \sum_{i\in\{1,2\}} \int_{\Omega_i} (\varphi_i(s_\mathcal{D})(\mathbf{x},t+\tau) - \varphi_i(s_\mathcal{D})(\mathbf{x},t))^2 \,\mathrm{d}\mathbf{x}\mathrm{d}t \le C_3\tau. \tag{4.1}$$

Lemma 4.3 is an extension of Lemma 4.6 of [24] (see also Proposition 5.1 in [26]).

PROPOSITION 4.4. *The sequence* $(\varphi_i(s_{\mathcal{D}_m}))_m$ *converges strongly in* $L^2(Q_{i,T})$, *up to a subsequence, towards the function* $\varphi_i(s) \in L^2(0,T;H^1(\Omega_i))$.

Proof. First recall that, by Proposition 3.1, $(\varphi_i(s_{\mathcal{D}_m}))_m$ is bounded in $L^\infty(Q_{i,T})$ for $i \in \{1,2\}$ and that by Proposition 3.5 the sequence $(|\varphi_i(s_{\mathcal{D}_m})|_{\mathcal{D}_m,i})_m$ is bounded. Thanks to the lemmas 4.2 and 4.3 and the Kolmogorov compactness criterion (see e.g. [7] or [24, Theorem 3.9]), it follows that $(T_i(\varphi_i(s_{\mathcal{D}_m})))_m$ is relatively compact in $L^2(\mathbb{R}^{d+1})$ for $i \in \{1,2\}$. Thus we can extract a subsequence, still denoted by $(T_i(\varphi_i(s_{\mathcal{D}_m})))_m$, such that both $T_1(\phi_1(s_{\mathcal{D}_m}))$ and $T_2(\phi_2(s_{\mathcal{D}_m}))$ converge to their limit strongly in $L^2(Q_{1,T})$ and $L^2(Q_{2,T})$ respectively. As a direct consequence, $(\varphi_i(s_{\mathcal{D}_m}))_m$ converges in $L^2(Q_{i,T})$ for $i \in \{1,2\}$ towards a function ϕ, which satisfies, thanks to Lemma 4.1,

$$\int_0^T \int_{\Omega_{i,\xi}} (\phi(\mathbf{x}+\xi,t) - \phi(\mathbf{x},t))^2 \mathrm{d}\mathbf{x}\mathrm{d}t \le C|\xi|^2, \qquad \forall \xi \in \mathbb{R}^d.$$

This implies (see [7]) that $\phi \in L^2(0,T;H^1(\Omega_i))$. It remains to identify ϕ as $\varphi_i(s)$, $i \in \{1,2\}$. This can be done using Minty's lemma (see e.g. [25, Theorem 4.1]). ☐

COROLLARY 4.5. *Up to a subsequence,* $(s_{\mathcal{D}_m})_m$ *converges towards* s *strongly in* $L^p(Q_T)$ *for all* $p \in [1,\infty)$.

Proof. Since $(\varphi_i(s_{\mathcal{D}_m}))_m$ converges in $L^2(Q_T)$ towards $\varphi_i(s)$, it converges (up to a new subsequence) almost everywhere in Q_T. Since φ_i^{-1} is continuous, $s_{\mathcal{D}_m}$ tends to s almost everywhere. The result then follows from the uniform bound on $(s_{\mathcal{D}_m})_m$ stated in Proposition 3.1. ☐

LEMMA 4.6. *There exists* $\mathfrak{P} \in L^2(0,T;H^1(\Omega_i))$ *such that, up to a subsequence,*

$$P_{\mathcal{D}_m} - m_{\Omega_i}(P_{\mathcal{D}_m}) \rightharpoonup \mathfrak{P} \text{ weakly in } L^2(Q_{i,T}) \text{ as } m \to \infty.$$

Proof. In view of the discrete Poincar-Wirtinger inequality [27], there exists C depending only on Ω_i and on the quantity C_1 introduced in Proposition 3.5 such that

$$\|P_{\mathcal{D}_m} - m_{\Omega_i}(P_{\mathcal{D}_m})\|_{L^2(Q_{i,T})} \le C, \text{ for } i \in \{1,2\}.$$

Hence the sequence $(P_{\mathcal{D}_m} - m_{\Omega_i}(P_{\mathcal{D}_m}))_m$ converges weakly in $L^2(Q_{i,T})$ towards a function $\mathfrak{P}$. Therefore, for all $\xi \in \mathbb{R}^d$,

$$P_{\mathcal{D}_m}(\cdot+\xi,\cdot) - P_{\mathcal{D}_m} \rightharpoonup \mathfrak{P}(\cdot+\xi,\cdot) - \mathfrak{P} \quad \text{weakly in } L^2(\Omega_{i,\xi}\times(0,T)) \text{ as } m \to \infty.$$

The lower semi-continuity of the norm for the weak L^2 topology implies that

$$\int_0^T \int_{\Omega_{i,\xi}} (\mathfrak{P}(\mathbf{x}+\xi,t) - \mathfrak{P}(\mathbf{x},t))^2 \mathrm{d}\mathbf{x}\mathrm{d}t \le \liminf_{m\to\infty} \int_0^T \int_{\Omega_{i,\xi}} (P_{\mathcal{D}_m}(\mathbf{x}+\xi,t) - P_{\mathcal{D}_m}(\mathbf{x},t))^2 \mathrm{d}\mathbf{x}\mathrm{d}t.$$

We deduce from Proposition 3.5 and Lemma 4.1 that

$$\int_0^T \int_{\Omega_{i,\xi}} (\mathfrak{P}(\mathbf{x}+\xi,t) - \mathfrak{P}(\mathbf{x},t))^2 \mathrm{d}\mathbf{x}\mathrm{d}t \le C_1|\xi|^2,$$

ensuring that $\mathfrak{P}$ belongs to $L^2(0,T;H^1(\Omega_i))$. ☐

4.2. Convergence of the traces. We denote by $s_{\mathcal{D}_{|\Gamma,i}}$ (resp. $P_{\mathcal{D}_{|\Gamma,i}}$) the trace of $s_{\mathcal{D}}$ (resp. $P_{\mathcal{D}}$) on Γ from the side of Ω_i, defined by

$$s_{\mathcal{D}_{|\Gamma,i}}(\mathbf{x},t) = s_{K,\sigma}^{n+1}, \quad P_{\mathcal{D}_{|\Gamma,i}}(\mathbf{x},t) = P_{K,\sigma}^{n+1}, \qquad \forall(\mathbf{x},t) \in \sigma \times (t^n, t^{n+1}],$$

where $\sigma \in \mathcal{E}_{K,\Gamma}$, $K \subset \Omega_i$.

It has been proven in Proposition 4.4 that $\varphi_i(s_{\mathcal{D}_m})$ converges strongly in $L^2(Q_{i,T})$ towards $\varphi_i(s) \in L^2(0,T;H^1(\Omega_i))$. Hence, $\varphi_1(s)$ and $\varphi_2(s)$ admits a traces in the sense of $L^2(\Gamma \times (0,T))$. Since φ_i^{-1} is continuous, s also admits a traces on the interface, denoted by s_1 and s_2. We claim in Corollary 4.10 below that $s_{\mathcal{D}_{m|\Gamma,i}}$ converges *strongly* in $L^p(\Gamma \times (0,T))$ towards s_i for all $p \in [1,\infty)$.

We now introduce another definition of the trace, denoted by $\tilde{u}_{|\Gamma,i}$. For a function u of $\mathcal{X}(\mathcal{D})$ we define

$$\tilde{u}_{|\Gamma,i}(\mathbf{x},t) := u_K^{n+1} \text{ if } (\mathbf{x},t) \in \sigma \times (t^n,t^{n+1}], \sigma \subset \Gamma \cap \partial K, K \subset \Omega_i.$$

LEMMA 4.7. *Let $u \in \mathcal{X}(\mathcal{D})$, then*

$$\int_0^T \int_\Gamma |u_{|\Gamma,i} - \tilde{u}_{|\Gamma,i}| \mathrm{d}\mathbf{x}\mathrm{d}t \le |u|_{\mathcal{D}} \left(Tm(\Gamma)\mathrm{size}(\mathcal{D})\right)^{1/2}.$$

Proof. From the definitions of the traces of u,

$$\int_0^T \int_\Gamma |u_{|\Gamma,i} - \tilde{u}_{|\Gamma,i}| \mathrm{d}\mathbf{x}\mathrm{d}t = \sum_{n=0}^N \delta t \sum_{K\in\mathcal{T}_i} \sum_{\sigma\in\mathcal{E}_{K,\Gamma}} m(\sigma)|u_{K,\sigma}^{n+1} - u_K^{n+1}|.$$

Cauchy-Schwarz inequality yields that

$$\int_0^T \int_\Gamma |u_{|\Gamma,i} - \tilde{u}_{|\Gamma,i}| \mathrm{d}\mathbf{x}\mathrm{d}t \le \left(\sum_{n=0}^N \delta t \sum_{K\in\mathcal{T}_i} \sum_{\sigma\in\mathcal{E}_{K,\Gamma}} \tau_{K,\sigma}(u_{K,\sigma}^{n+1} - u_K^{n+1})^2\right)^{1/2} \times \left(\sum_{n=0}^N \delta t \sum_{K\in\mathcal{T}_i} \sum_{\sigma\in\mathcal{E}_{K,\Gamma}} m(\sigma) d_{K,\sigma}\right)^{1/2}.$$

The result follows. ◻

Since Ω_i is supposed to be polygonal, Γ is made of a finite number of faces $(\Gamma_j)_{1\le j\le J}$ contained in affine hyperplanes of $\mathbb{R}^d$. We denote by $\mathbf{n}_{i,j}$ the outward normal to Γ_j with respect to Ω_i. For $\varepsilon > 0$ and $j \in \{1,\dots,J\}$, we define the open subset $\omega_{i,j,\varepsilon}$ of Ω_i as the largest cylinder of width ε generate by Γ_j and $n_{i,j}$ included in Ω_i, that is

$$\omega_{i,j,\varepsilon} := \left\{x - h\mathbf{n}_{i,j} \in Q_{i,T} \;|x \in \Gamma_j, 0 < h < \varepsilon \text{ and } [\mathbf{x}, x - \varepsilon\mathbf{n}_{i,j}] \subset \overline{\Omega}_i\right\}. \tag{4.2}$$

We also define the subset $\Gamma_{i,j,\varepsilon} = \partial\omega_{i,j,\varepsilon} \cap \Gamma_j$ of Γ_j, that satisfies

$$m(\Gamma_j \setminus \Gamma_{i,j,\varepsilon}) \le C\varepsilon, \tag{4.3}$$

where C only depends on Ω.

LEMMA 4.8. *Let $u \in \mathcal{X}(\mathcal{D})$, then for all $j \in \{1, \ldots, J\}$,*

$$\int_0^T \frac{1}{\varepsilon} \int_{G_{i,j,\varepsilon}} \int_0^\varepsilon \left(\tilde{u}_{|\Gamma,i}(\mathbf{x}, t) - u(\mathbf{x} - h\mathbf{n}_{i,j}, t)\right)^2 \mathrm{d}h \mathrm{d}\mathbf{x} \mathrm{d}t \le |u|_{\mathcal{D}}^2 \left(\varepsilon + \mathrm{size}(\mathcal{D})\right).$$

Proof. For all $\sigma \in \mathcal{E}_{\mathrm{int}}$, we denote by

$$\chi_\sigma(\mathbf{x}, y) := \begin{cases} 1 & \text{if } (\mathbf{x}, y) \cap \sigma \text{ is reduced to a single point,} \\ 0 & \text{otherwise.} \end{cases}$$

We also introduce, for almost all $\mathbf{x} \in \Gamma_{i,j,\varepsilon}$, for almost $h \in (0, \varepsilon)$ and for all $t \in (0, T)$, the quantity

$$\begin{aligned} T_{\mathcal{D}}(\mathbf{x}, h, t) &:= \left|\tilde{u}_{|\Gamma,i}(\mathbf{x}, t) - u_{\mathcal{D}}(\mathbf{x} - h\mathbf{n}_{i,j}, t)\right| \\ &\le \sum_{\sigma = K|L \in \mathcal{E}_i} \chi_\sigma(\mathbf{x}, \mathbf{x} - h\mathbf{n}_{i,j}) \left|u_K^{n+1} - u_L^{n+1}\right| \end{aligned}$$

if $t \in (t^n, t^{n+1}]$. It follows from the Cauchy-Schwarz inequality that, for $t \in (t^n, t^{n+1}]$,

$$\begin{aligned} (T_{\mathcal{D}}(\mathbf{x}, h, t))^2 \le & \left(\sum_{\sigma = K|L \in \mathcal{E}_i} \chi_\sigma(\mathbf{x}, \mathbf{x} - h\mathbf{n}_{i,j}) \frac{\left(u_K^{n+1} - u_L^{n+1}\right)^2}{d_{KL}|\mathbf{n}_{i,j} \cdot \mathbf{n}_{KL}|} \right) \\ & \times \left(\sum_{\sigma = K|L \in \mathcal{E}_i} \chi_\sigma(\mathbf{x}, \mathbf{x} - h\mathbf{n}_{i,j}) d_{KL} |\mathbf{n}_{i,j} \cdot \mathbf{n}_{KL}| \right). \end{aligned}$$

For almost all $\mathbf{x} \in G_{i,j,\varepsilon}$, there exists a unique $K_1 \in \mathcal{T}_i$ such that $\mathbf{x} \in \partial K_1$. Moreover, for almost all $h \in (0, \varepsilon)$, there exists a unique $K_2 \in \mathcal{T}_i$ such that $\mathbf{x} - hn_i$ belongs to K_2 (possibly K_2 coincides with K_1). Hence,

$$\begin{aligned} \sum_{\sigma = K|L \in \mathcal{E}_{\mathrm{int}}} \chi_\sigma(x, x - h\mathbf{n}_{i,j}) d_{KL} |\mathbf{n}_{i,j} \cdot \mathbf{n}_{KL}| &= (x_{K_1} - x_{K_2}) \cdot \mathbf{n}_{i,j} \\ &\le (x_{K_1} - x) \cdot \mathbf{n}_{i,j} + h + |(x_{K_2} - (x - h\mathbf{n}_{i,j})) \cdot \mathbf{n}_{i,j}|. \end{aligned} \tag{4.4}$$

Since $\mathbf{x} - h\mathbf{n}_{i,j}$ belongs to K_2, we have

$$|(\mathbf{x}_{K_2} - (\mathbf{x} - h\mathbf{n}_{i,j})) \cdot \mathbf{n}_{i,j}| \le \mathrm{size}(\mathcal{D}),$$

and since $\mathbf{x}$ belongs to Γ_i, $(\mathbf{x}_{K_1} - x) \cdot \mathbf{n}_{i,j} \le 0$. Then we obtain

$$\sum_{\sigma = K|L \in \mathcal{E}_{\mathrm{int}}} \chi_\sigma(\mathbf{x}, \mathbf{x} - h\mathbf{n}_{i,j}) d_{KL} |\mathbf{n}_{i,j} \cdot \mathbf{n}_{KL}| \le \varepsilon + \mathrm{size}(\mathcal{D}). \tag{4.5}$$

For all $\sigma \in \mathcal{E}_{\mathrm{int}}$ with $\sigma \cap \omega_{i,j,\varepsilon} = \emptyset$ and all $h \in (0, \varepsilon)$, one has

$$\int_{\Gamma_i^\varepsilon} \chi_\sigma(\mathbf{x}, \mathbf{x} - hn_i) \mathrm{d}\mathbf{x} = 0.$$

For all $\sigma \in \mathcal{E}_{i,j,\varepsilon} := \{\sigma \in \mathcal{E}_i \mid \sigma \cap \omega_{i,j,\varepsilon} \neq \emptyset\}$, one has

$$\forall h \in (0, \varepsilon), \qquad \int_{G_{i,j,\varepsilon}} \chi_\sigma(\mathbf{x}, \mathbf{x} - h\mathbf{n}_{i,j}) \mathrm{d}\mathbf{x} \le m(\sigma) |\mathbf{n}_{i,j} \cdot \mathbf{n}_{KL}|. \tag{4.6}$$

We obtain from (4.5) and (4.6) that for all $t \in (t^n, t^{n+1}]$, for all $h \in (0, \varepsilon)$,

$$\int_{G_{i,j,\varepsilon}} (T_{\mathcal{D}}(\mathbf{x}, h, t))^2 \, \mathrm{d}\mathbf{x} \le (\varepsilon + \mathrm{size}(\mathcal{D})) \sum_{\sigma = K|L \in \mathcal{E}_{i,j,\varepsilon}} \tau_{KL} \left(u_K^{n+1} - u_L^{n+1}\right)^2,$$

which complete the proof. □

PROPOSITION 4.9. *The sequence* $\left(\varphi_i(s_{\mathcal{D}_{m|\Gamma,i}})\right)_m$ *converges towards* $\varphi_i(s_i)$ *strongly in* $L^1(\Gamma \times (0, T))$ *as* $m \to \infty$.

Proof. For notation convenience, we remove the subscripts m in the proof. Denote by

$$A_{i,j,\mathcal{D}} := \int_0^T \int_{\Gamma_j} \left|\varphi_i(s_{\mathcal{D}_{|\Gamma,i}}) - \varphi_i(s_i)\right| \mathrm{d}\mathbf{x}\mathrm{d}t, \tag{4.7}$$

then in view of Lemma 4.7 and Proposition 3.5, there exists C not depending on $\mathcal{D}$ such that

$$A_{i,j,\mathcal{D}} = \int_0^T \int_{\Gamma_j} \left|\varphi_i(\tilde{s}_{\mathcal{D}_{|\Gamma,i}}) - \varphi_i(s_i)\right| \mathrm{d}\mathbf{x}\mathrm{d}t + C\mathrm{size}(\mathcal{D})^{1/2}. \tag{4.8}$$

By (4.3), one has

$$\int_0^T \int_{\Gamma_j} \left|\varphi_i(\tilde{s}_{\mathcal{D}_{|\Gamma,i}}) - \varphi_i(s_i)\right| \mathrm{d}\mathbf{x}\mathrm{d}t \le \int_0^T \int_{\Gamma_{i,j,\varepsilon}} \left|\varphi_i(\tilde{s}_{\mathcal{D}_{|\Gamma,i}}) - \varphi_i(s_i)\right| \mathrm{d}\mathbf{x}\mathrm{d}t + \varphi_i(1)C\varepsilon. \tag{4.9}$$

Next we apply the triangle inequality to deduce that

$$\int_0^T \int_{\Gamma_{i,j,\varepsilon}} \left|\varphi_i(\tilde{s}_{\mathcal{D}_{|\Gamma,i}}) - \varphi_i(s_i)\right| \mathrm{d}\mathbf{x}\mathrm{d}t \le B_{1,\mathcal{D},\varepsilon} + B_{2,\mathcal{D},\varepsilon} + B_{3,\varepsilon}, \tag{4.10}$$

where

$$B_{1,\mathcal{D},\varepsilon} = \frac{1}{\varepsilon}\int_0^T \int_{G_{i,j,\varepsilon}} \int_0^\varepsilon \left|\varphi_i(\tilde{s}_{\mathcal{D}_{|\Gamma,i}})(\mathbf{x}, t) - \varphi_i(s_{\mathcal{D}})(\mathbf{x} - h\mathbf{n}_{i,j}, t)\right| \mathrm{d}h\mathrm{d}\mathbf{x}\mathrm{d}t,$$

$$B_{2,\mathcal{D},\varepsilon} = \frac{1}{\varepsilon}\int_0^T \int_{\omega_{i,j,\varepsilon}} \left|\varphi_i(s_{\mathcal{D}}) - \varphi_i(s)\right| \mathrm{d}\mathbf{x}\mathrm{d}t,$$

$$B_{3,\varepsilon} = \frac{1}{\varepsilon}\int_0^T \int_{G_{i,j,\varepsilon}} \int_0^\varepsilon \left|\varphi_i(s_i)(\mathbf{x}, t) - \varphi_i(s)(\mathbf{x} - h\mathbf{n}_{i,j}, t)\right| \mathrm{d}h\mathrm{d}\mathbf{x}\mathrm{d}t,$$

where we have used (4.2). From Cauchy-Schwarz inequality, one has

$$(B_{1,\mathcal{D},\varepsilon})^2 \le m(G_{i,j,\varepsilon})T \int_0^T \int_{G_{i,j,\varepsilon}} \frac{1}{\varepsilon}\int_0^\varepsilon (\varphi_i(\tilde{s}_{\mathcal{D}_{|\Gamma,i}})(\mathbf{x}, t) - \varphi_i(s_{\mathcal{D}})(\mathbf{x} - h\mathbf{n}_{i,j}, t))^2 \mathrm{d}h\mathrm{d}\mathbf{x}\mathrm{d}t,$$

and then, from Proposition 3.5 and Lemma 4.8, one has

$$|B_{1,\mathcal{D},\varepsilon}| \le \left(C_1(\mathrm{size}(\mathcal{D}) + \varepsilon)m(\Gamma_i)T\right)^{1/2}. \tag{4.11}$$

We can now let size($\mathcal{D}$) tend to 0 in (4.10). Thanks to Proposition 4.4, we can claim that

$$\lim_{\mathrm{size}(\mathcal{D}) \to 0} B_{2,\mathcal{D},\varepsilon} = 0.$$

Then it follows from (4.9) and (4.11) that

$$\limsup_{\mathrm{size}(\mathcal{D})\to 0} \int_0^T \int_{\Gamma_j} \left|\varphi_i(\tilde{s}_{\mathcal{D}_{|\Gamma,i}}) - \varphi_i(s_i)\right| \mathrm{d}\mathbf{x}\mathrm{d}t \le C(\varepsilon + \sqrt{\varepsilon}) + B_{3,\varepsilon}. \tag{4.12}$$

Since $\varphi_i(s_i)$ is the trace of $\varphi_i(s)$ on Γ, $\lim_{\varepsilon\to 0} B_{3,\varepsilon} = 0$. Therefore, letting ε tend to 0 in (4.12) implies that

$$\lim_{\mathrm{size}(\mathcal{D})\to 0} \int_0^T \int_{\Gamma_j} \left|\varphi_i(\tilde{s}_{\mathcal{D}_{|\Gamma,i}}) - \varphi_i(s_i)\right| \mathrm{d}\mathbf{x}\mathrm{d}t = 0.$$

Then the result follows from (4.7) and (4.2). □

Corollary 4.10. *The sequence* $\left(s_{\mathcal{D}_{m|\Gamma,i}}\right)_m$ *converges towards* s_i *strongly in* $L^p(\Gamma \times (0,T))$ *for all* $p \in [1,\infty)$.

Proof. This corollary is just a consequence from the fact that $\varphi_i(s_{\mathcal{D}_{m|\Gamma,i}})$ converges, up to a subsequence, almost everywhere on $\Gamma \times (0,T)$, from the fact that φ_i^{-1} is continuous and from the fact that $s_{\mathcal{D}_{m|\Gamma,i}}$ is essentially uniformly bounded between 0 and 1. □

Lemma 4.11. *The sequence* $\left((P_{\mathcal{D}_m})_{|\Gamma,i} - m_{\Omega_i}(P_{\mathcal{D}})\right)_m$ *converges towards* $\mathfrak{P}_i$ *weakly in* $L^2(\Gamma \times (0,T))$.

Proof. Let $\psi \in \mathcal{D}(\Gamma_i \times (0,T))$, then, there exists $\varepsilon_\star$ such that, for all $\varepsilon \in (0,\varepsilon_\star)$, $\mathrm{supp}(\psi) \subset \Gamma_{i,j,\varepsilon} \times (0,T)$. We aim to prove that

$$\lim_{\mathrm{size}(\mathcal{D})\to 0} \int_0^T \int_{\Gamma_j} \left(P_{\mathcal{D}_{|\Gamma,i}} - m_{\Omega_i}(P_{\mathcal{D}}) - \mathfrak{P}_i\right) \psi \mathrm{d}\mathbf{x}\mathrm{d}t = 0. \tag{4.13}$$

Thanks to Lemma 4.7 and to Proposition 3.5, it is sufficient to show that

$$\lim_{\mathrm{size}(\mathcal{D})\to 0} \int_0^T \int_{\Gamma_j} \left(\tilde{P}_{\mathcal{D}_{|\Gamma,i}} - m_{\Omega_i}(P_{\mathcal{D}}) - \mathfrak{P}_i\right) \psi \mathrm{d}\mathbf{x}\mathrm{d}t = 0.$$

Let $\varepsilon \in (0,\varepsilon_\star)$, then one has

$$\int_0^T \int_{\Gamma_j} \left(\tilde{P}_{\mathcal{D}_{|\Gamma,i}} - m_{\Omega_i}(P_{\mathcal{D}_m}) - \mathfrak{P}_i\right) \psi \mathrm{d}\mathbf{x}\mathrm{d}t = E_{1,\mathcal{D},\varepsilon} + E_{2,\mathcal{D},\varepsilon} + E_{3,\varepsilon},$$

where

$$E_{1,\mathcal{D},\varepsilon} = \int_0^T \frac{1}{\varepsilon} \int_{G_{i,j,\varepsilon}} \int_0^\varepsilon \left(\tilde{P}_{\mathcal{D}_{|\Gamma,i}}(\mathbf{x},t) - P_{\mathcal{D}}(\mathbf{x} - h\mathbf{n}_{i,j},t)\right) \psi(\mathbf{x},t) \mathrm{d}h\mathrm{d}\mathbf{x}\mathrm{d}t,$$

$$E_{2,\mathcal{D},\varepsilon} = \int_0^T \frac{1}{\varepsilon} \int_{G_{i,j,\varepsilon}} \int_0^\varepsilon (P_{\mathcal{D}}(\mathbf{x} - h\mathbf{n}_{i,j},t) - m_{\Omega_i}(P_{\mathcal{D}}) - \mathfrak{P}(\mathbf{x} - h\mathbf{n}_{i,j},t))\, \psi(\mathbf{x},t) \mathrm{d}h\mathrm{d}\mathbf{x}\mathrm{d}t,$$

$$E_{3,\varepsilon} = \int_0^T \frac{1}{\varepsilon} \int_{G_{i,j,\varepsilon}} \int_0^\varepsilon (\mathfrak{P}(\mathbf{x} - h\mathbf{n}_{i,j},t) - \mathfrak{P}_i)\, \psi(\mathbf{x},t) \mathrm{d}h\mathrm{d}\mathbf{x}\mathrm{d}t.$$

The Cauchy-Schwarz inequality gives that

$$\begin{aligned}(E_{1,\mathcal{D},\varepsilon})^2 \le & \int_0^T \frac{1}{\varepsilon} \int_{G_{i,j,\varepsilon}} \int_0^\varepsilon \left(\tilde{P}_{\mathcal{D}_{|\Gamma,i}}(\mathbf{x},t) - P_{\mathcal{D}}(\mathbf{x} - h\mathbf{n}_{i,j},t)\right)^2 \mathrm{d}h\mathrm{d}\mathbf{x}\mathrm{d}t \\ & \times \int_0^T \int_{\Gamma_j} (\psi(\mathbf{x},t))^2 \mathrm{d}\mathbf{x}\mathrm{d}t.\end{aligned}$$

Using Proposition 3.5 and Lemma 4.8 yields

$$|E_{1,\mathcal{D},\varepsilon}| \le \|\psi\|_{L^2(\Gamma_j\times(0,T))} \left(C_1(\varepsilon + \mathrm{size}(\mathcal{D}))\right)^{1/2}.$$

It has been stated in Lemma 4.6 that $P_{\mathcal{D}} - m_{\Omega_i}(P_{\mathcal{D}})$ tends to $\mathfrak{P}$ weakly in $L^2(Q_{i,T})$ as size($\mathcal{D}$) tends to 0, then

$$\lim_{\mathrm{size}(\mathcal{D})\to 0} E_{2,\mathcal{D},\varepsilon} = 0.$$

Therefore,

$$\limsup_{\mathrm{size}(\mathcal{D})\to 0} \left| \int_0^T \int_{\Gamma_j} \left(\tilde{P}_{\mathcal{D}_{|\Gamma,i}} - P_i\right)\psi \mathrm{d}\mathbf{x}\mathrm{d}t \right| \le C_\psi\sqrt{\varepsilon} + |E_{3,\varepsilon}|.$$

Since P_i is the trace on Γ of P from the side of Ω_i, one has

$$\lim_{\varepsilon\to 0} E_{3,\varepsilon} = 0.$$

Thus, letting $\varepsilon \to 0$, one obtains that for all $\psi \in \mathcal{D}(\Gamma_j \times (0,T))$,

$$\lim_{\mathrm{size}(\mathcal{D})\to 0} \int_0^T \int_{\Gamma_j} \left(\tilde{P}_{\mathcal{D}_{|\Gamma,i}} - m_{\Omega_i}(P_{\mathcal{D}}) - \mathfrak{P}_i\right)\psi \mathrm{d}\mathbf{x}\mathrm{d}t = 0. \tag{4.14}$$

A straightforward generalization of [24, Lemma 3.10] allows us to claim, using Proposition 3.5 and the discrete Poincar-Wirtinger inequality [27], that $\left(\tilde{P}_{\mathcal{D}_{|\Gamma,i}} - m_{\Omega_i}(P_{\mathcal{D}})\right)_{\mathcal{D}}$ is uniformly bounded in $L^2(\Gamma \times (0,T))$. Then, we conclude, using a classical density argument, that (4.14) holds for all $\psi \in L^2(\Gamma_j \times (0,T))$. ◻

PROPOSITION 4.12. *There exists $P \in L^2(0,T;H^1(\Omega_i))$ such that $P_{\mathcal{D}_m}$ tends to P weakly in $L^2(Q_T)$ as $m \to \infty$, and such that $\left(P_{\mathcal{D}_{m|\Gamma,i}}\right)_m$ converges weakly in $L^2(\Gamma \times (0,T))$ towards P_i.*

Proof. Firstly, since we have enforced $m_{\Omega_1}(P_{\mathcal{D}_m}) = 0$, we can set $P := \mathfrak{P}$ in $Q_{1,T}$. Next we search for a uniform bound on $\|P_{\mathcal{D}_m}\|_{L^2(Q_{2,T})}$. In view of the discrete Poincar-Wirtinger inequality

$$\|P_{\mathcal{D}_m}\|^2_{L^2(Q_{2,T})} \le \left(m_{\Omega_2}(P_{\mathcal{D}_m})\right)^2 + C, \tag{4.15}$$

it only remains to check that $m_{\Omega_2}(P_{\mathcal{D}_m})$ is uniformly bounded w.r.t. m. This is a consequence of the fact that, almost everywhere on $\Gamma \times (0,T)$, one has

$$m_{\Omega_2}(P_{\mathcal{D}_m}) = P_{\mathcal{D}_{m|\Gamma,1}} - \left(P_{\mathcal{D}_{m|\Gamma,2}} - m_{\Omega_2}(P_{\mathcal{D}_m})\right) - \left(W_1(\pi_{\mathcal{D}_m}) - W_2(\pi_{\mathcal{D}_m})\right).$$

Then, integrating on $\Gamma \times (0,T)$ and using Lemma 1.1 provides

$$|m_{\Omega_2}(P_{\mathcal{D}_m})| \le \frac{1}{m(\Gamma)T} \sum_{i\in\{1,2\}} \left\| P_{\mathcal{D}_{m|\Gamma,i}} - m_{\Omega_i}(P_{\mathcal{D}_m}) \right\|_{L^1(\Gamma\times(0,T))} + \|W_1 - W_2\|_\infty.$$

For all $i \in \{1,2\}$ the quantities $\left\| P_{\mathcal{D}_{m|\Gamma,i}} - m_{\Omega_i}(P_{\mathcal{D}_m}) \right\|_{L^1(\Gamma\times(0,T))}$ are bounded by the proof of Lemma 4.11. Hence, in view of (4.15), $(P_{\mathcal{D}_m})_m$ converges towards some function P weakly in $L^2(Q_{i,T})$. From the analysis performed in the proof of Lemma 4.6, we deduce that $P \in L^2(0,T;H^1(\Omega_i))$, and from the analysis of Lemma 4.11, we deduce the weak convergence of the traces. □

LEMMA 4.13. *Let $s_1, s_2 \in L^\infty(\Gamma\times(0,T))$ be the respective limits of $\left(s_{\mathcal{D}_m}{}_{|_{\Gamma,1}}\right)_m$ and $\left(s_{\mathcal{D}_m}{}_{|_{\Gamma,2}}\right)_m$, then,*

$$\tilde\pi_1(s_1)\cap\tilde\pi_2(s_2)\neq\emptyset \qquad a.e.\ on\ \Gamma\times(0,T). \tag{4.16}$$

Proof. For all $m\in\mathcal{N}$, one has

$$\tilde\pi_1(s_{\mathcal{D}_m}{}_{|_{\Gamma,1}})\cap\tilde\pi_2(s_{\mathcal{D}_m}{}_{|_{\Gamma,2}})\neq\emptyset.$$

Since the set $F=\{(a,b)\in[0,1]^2 \mid \tilde\pi_1(a)\cap\tilde\pi_2(b)\neq\emptyset\}$ is closed in $[0,1]^2$, we conclude that (4.16) holds. □

We now focus on the last technical difficulty for proving Theorem 1, that is the convergence of the sequence $(\pi_{\mathcal{D}_m})_m$. This is done by following the same path as in [14].

In the sequel, we denote by

$$T_{[A,B]}(s)=\begin{cases} s & \text{if } s\in[A,B],\\ A & \text{if } s\le A,\\ B & \text{if } s\ge B,\end{cases}$$

and by

$$\mathcal{U}=\{(\mathbf{x},t)\in\Gamma\times(0,T)\mid\{s_1,s_2\}=\{0,1\}\},\qquad \mathcal{V}=\mathcal{U}^c.$$

Note that, thanks to Lemma 4.13, the set $\mathcal{U}$ is empty if $\min_i\pi_i(1)>\max_i\pi_i(0)$.

LEMMA 4.14. *There exists a measurable function π defined on $\mathcal{V}$ with values in $\overline{\mathbb{R}}$, such that, up to a subsequence,*

$$\pi_{\mathcal{D}_m}\to\pi \qquad a.e.\ in\ \mathcal{V}.$$

Proof. We define the functions

$$\tilde\varphi_i : p\mapsto\int_{\pi_i(0)}^{p} K_i\frac{k_{o,i}(\tilde\pi_i^{-1}(a))k_{w,i}(\tilde\pi_i^{-1}(a))}{\mu_w k_{o,i}(\tilde\pi_i^{-1}(a))+\mu_o k_{w,i}(\tilde\pi_i^{-1}(a))}\mathrm{d}a,$$

that satisfy the properties

$$\pi\in\tilde\pi_i(s)\implies\tilde\varphi_i(\pi)=\tilde\varphi_i(\pi_i(s))=\varphi_i(s), \tag{4.17}$$

and

$$\text{its restriction } (\tilde\varphi_i)_{|_{[\pi_i(0),\pi_i(1)]}} \text{ admits a continuous inverse function.} \tag{4.18}$$

Thanks to Proposition 4.9 and to (4.17), we can claim that, up to a subsequence, $\tilde\varphi_i(\pi_{\mathcal{D}_m})$ converges almost everywhere on $\Gamma\times(0,T)$ towards $\tilde\varphi_i(\pi_i(s_i))$. For a.e. $(\mathbf{x},t)\in\mathcal{V}$, the set $\tilde\pi_1(s_1)\cap\tilde\pi_2(s_2)$ is reduced to the singleton $\{\pi_{i_0}(s_{i_0})\}$ for some $i_0\in\{1,2\}$. Thanks to (4.18), we can identify the limit π of $\pi_{\mathcal{D}_m}$ as $\pi_{i_0}(s_{i_0})$. □

LEMMA 4.15. *Assume that* $[\min_i \pi_i(1), \max_i \pi_i(0)] \neq \emptyset$*, then there exists* $\pi \in L^\infty(\mathcal{U}; [\min_i \pi_i(1), \max \pi_i(0)])$ *such that, for all bounded interval* $\mathcal{J} \subset \mathbb{R}$ *such that* $[\min_i \pi_i(1), \max_i \pi_i(0)] \subset \overset{\circ}{\mathcal{J}}$,

$$T_{\mathcal{J}}(\pi_{\mathcal{D}_m}) \to \pi \qquad \textit{in the } L^\infty(\mathcal{U}) \textit{ weak-}\star \textit{ sense.}$$

Proof. For the sake of simplicity, we assume, without loss of generality, that $\pi_1(1) \leq \pi_2(0)$, then thanks to Lemma 4.13, almost everywhere in $\mathcal{U}$, $s_1 = 1$ and $s_2 = 0$.

The sequence $(T_{\mathcal{J}}(\pi_{\mathcal{D}_m}))_m$ is bounded in $L^\infty(\mathcal{U})$, thus, up to a subsequence, it converges towards a function $\pi_{\mathcal{J}}$ in the $L^\infty(\mathcal{U})$ weak-$\star$ sense. Let us now show that $\pi_{\mathcal{J}}$ does not depend on the choice of the bounded interval $\mathcal{J}$. Because of Lemma 4.13, one has, for a.e. $(\mathbf{x}, t) \in \mathcal{U}$,

$$\liminf_m \pi_{\mathcal{D}_m} \geq \pi_1(1), \qquad \limsup_m \pi_{\mathcal{D}_m} \leq \pi_2(0). \tag{4.19}$$

Let $\mathcal{J}_1$ and $\mathcal{J}_2$ be two bounded intervals such that $[\pi_1(1), \pi_2(0)] \subset \overset{\circ}{\mathcal{J}}_k$ ($k \in \{1,2\}$). Then, it follows from (4.19) that, for a.e. $(\mathbf{x}, t) \in \mathcal{U}$, for m large enough (depending on $(\mathbf{x}, t)$),

$$T_{\mathcal{J}_1}(\pi_{\mathcal{D}_m}(\mathbf{x}, t)) - T_{\mathcal{J}_2}(\pi_{\mathcal{D}_m}(\mathbf{x}, t)) = 0.$$

As a consequence, the sequence $(T_{\mathcal{J}_1}(\pi_{\mathcal{D}_m}) - T_{\mathcal{J}_2}(\pi_{\mathcal{D}_m}))_m$ converges almost everywhere to 0 on $\mathcal{U}$, and is uniformly bounded in $L^\infty(\mathcal{U})$. The dominated convergence theorem yields that for all $\psi \in L^1(\mathcal{U})$,

$$\iint_{\mathcal{U}} (T_{\mathcal{J}_1}(\pi_{\mathcal{D}_m}) - T_{\mathcal{J}_2}(\pi_{\mathcal{D}_m}))\, \psi \mathrm{d}\mathbf{x}\mathrm{d}t \to 0 = \iint_{\mathcal{U}} (\pi_{\mathcal{J}_1} - \pi_{\mathcal{J}_2})\, \psi \mathrm{d}\mathbf{x}\mathrm{d}t.$$

Choosing $\psi = (\pi_{\mathcal{J}_1} - \pi_{\mathcal{J}_2})$ provides that $\pi_{\mathcal{J}_1} = \pi_{\mathcal{J}_2} = \pi$ almost everywhere in $\mathcal{U}$. ☐

LEMMA 4.16. *Assume that* $[\min_i \pi_i(1), \max_i \pi_i(0)] \neq \emptyset$*, then there exists* $\pi \in L^\infty(\mathcal{U})$ *such that, for all bounded interval* $\mathcal{J} \subset \mathbb{R}$ *such that* $[\min_i \pi_i(1), \max_i \pi_i(0)] \subset \overset{\circ}{\mathcal{J}}$*, the sequence* $(W_i(T_{\mathcal{J}}(\pi_{\mathcal{D}_m})))_m$ *converges towards* $W_i(\pi)$ *in the* $L^\infty(\mathcal{U})$ *weak-*$\star$ *sense.*

Proof. We suppose, without loss of generality, that $\pi_1(1) \leq \pi_2(0)$. Then on $\mathcal{U}$, $s_2 = 0$ and $s_1 = 1$. One has

$$W_2(T_{\mathcal{J}}(\pi_{\mathcal{D}_m})) = \int_0^{\pi_2(0)} f_2 \circ \pi_2^{-1}(p)\mathrm{d}p + \int_{\pi_2(0)}^{\pi_{\mathcal{D}_m}} f_2 \circ \pi_2^{-1}(p)\mathrm{d}p.$$

Since for almost every $(\mathbf{x}, t) \in \mathcal{U}$,

$$\limsup_m \pi_{\mathcal{D}_m}(\mathbf{x}, t) \leq \pi_2(0),$$

and since $f_2 \circ \pi_2^{-1}(p) = 0$ for all $p \leq \pi_1(0)$, then for almost every $(\mathbf{x}, t) \in \mathcal{U}$,

$$\int_{\pi_2(0)}^{\pi_{\mathcal{D}_m}(\mathbf{x},t)} f_2 \circ \pi_2^{-1}(p)\mathrm{d}p \to 0 \text{ as } m \to \infty.$$

Since the function $W_2 \circ T_{\mathcal{J}}$ is uniformly bounded on $\mathbb{R}$, the dominated convergence theorem yields that, for all $\psi \in L^1(\mathcal{U})$,

$$\lim_{m\to\infty} \int_{\mathcal{U}} W_2(T_{\mathcal{J}}(\pi_{\mathcal{D}_m}))\psi \mathrm{d}\mathbf{x}\mathrm{d}t \to \iint_{\mathcal{U}} W_2(\pi_2(0))\psi \mathrm{d}\mathbf{x}\mathrm{d}t = \iint_{\mathcal{U}} W_2(\pi)\psi \mathrm{d}\mathbf{x}\mathrm{d}t.$$

Similarly, we obtain that

$$\iint_{\mathcal{U}} \left(W_1(T_{\mathcal{J}}(\pi_{\mathcal{D}_m})) - T_{\mathcal{J}}(\pi_{\mathcal{D}_m})\right) \psi \mathrm{d}\mathbf{x}\mathrm{d}t \to \iint_{\mathcal{U}} \left(W_1(\pi_1(1)) - \pi_1(1)\right) \psi \mathrm{d}\mathbf{x}\mathrm{d}t.$$

Since, thanks to Lemma 4.15, $T_{\mathcal{J}}(\pi_{\mathcal{D}_m})$ tends to π in the $L^\infty(\mathcal{U})$ weak-$\star$ sense, one has

$$\begin{aligned}\lim_{m\to\infty} \iint_{\mathcal{U}} W_1(T_{\mathcal{J}}(\pi_{\mathcal{D}_m}))\psi \mathrm{d}\mathbf{x}\mathrm{d}t &= \iint_{\mathcal{U}} \left(W_1(\pi_1(1)) + \pi - \pi_1(1)\right) \psi \mathrm{d}\mathbf{x}\mathrm{d}t \\ &= \iint_{\mathcal{U}} W_1(\pi)\psi \mathrm{d}\mathbf{x}\mathrm{d}t.\end{aligned}$$

□

Proposition 4.17. *There exists a measurable function π on $\Gamma \times (0,T)$, with $\pi \in \tilde{\pi}_1(s_1) \cap \tilde{\pi}_2(s_2)$ a.e. on $\Gamma \times (0,T)$, with value in $[\min_i(\pi_i(0)), \max_i(\pi_i(1))]$ such that,*

$$W_1(\pi_{\mathcal{D}_m}) - W_2(\pi_{\mathcal{D}_m}) \to W_1(\pi) - W_2(\pi) \text{ in the } L^\infty(\Gamma\times(0,T)) \text{ weak-}\star\text{sense as } n \to \infty.$$

Proof. We know, from Lemma 1.1, that $W_1(p) - W_2(p)$ is uniformly bounded on $[\min_i \pi_i(0), \max_i \pi_i(1)]$. Hence, the sequence $(W_1(\pi_{\mathcal{D}_m}) - W_2(\pi_{\mathcal{D}_m}))_m$ converges in the $L^\infty(\Gamma \times (0,T))$ weak-$\star$ sense towards a function $\mathfrak{Z}$. Let $\psi \in L^1(\Gamma \times (0,T))$, then

$$\begin{aligned}\int_0^T \int_\Gamma \left(W_1(\pi_{\mathcal{D}_m}) - W_2(\pi_{\mathcal{D}_m})\right) \psi \mathrm{d}\mathbf{x}\mathrm{d}t = &\iint_{\mathcal{U}} \left(W_1(\pi_{\mathcal{D}_m}) - W_2(\pi_{\mathcal{D}_m})\right) \psi \mathrm{d}\mathbf{x}\mathrm{d}t \\ &+ \iint_{\mathcal{V}} \left(W_1(\pi_{\mathcal{D}_m}) - W_2(\pi_{\mathcal{D}_m})\right) \psi \mathrm{d}\mathbf{x}\mathrm{d}t.\end{aligned}$$

Thanks to Lemma 4.14, $\pi_{\mathcal{D}_m}$ tends almost everywhere to π on $\mathcal{V}$, then for almost every $(\mathbf{x}, t) \in \mathcal{V}$, we can identify $\mathfrak{Z}(\mathbf{x}, t)$ as $W_1(\pi(\mathbf{x}, t)) - W_2(\pi(\mathbf{x}, t))$. Thus

$$\lim_{m\to\infty} \iint_{\mathcal{V}} \left(W_1(\pi_{\mathcal{D}_m}) - W_2(\pi_{\mathcal{D}_m})\right) \psi \mathrm{d}\mathbf{x}\mathrm{d}t = \iint_{\mathcal{V}} \left(W_1(\pi) - W_2(\pi)\right) \psi \mathrm{d}\mathbf{x}\mathrm{d}t.$$

We denote by

$$\begin{aligned}A_m &= \iint_{\mathcal{U}} \left(W_1(\pi_{\mathcal{D}_m}) - W_2(\pi_{\mathcal{D}_m}) - W_1(\pi) + W_2(\pi)\right) \psi \mathrm{d}\mathbf{x}\mathrm{d}t, \\ &= \iint_{\mathcal{U}} \left(\Upsilon_1(\pi_{\mathcal{D}_m}) - \Upsilon_1(\pi)\right) \psi \mathrm{d}\mathbf{x}\mathrm{d}t + \iint_{\mathcal{U}} \left(\Upsilon_2(\pi_{\mathcal{D}_m}) - \Upsilon_2(\pi)\right) \psi \mathrm{d}\mathbf{x}\mathrm{d}t.\end{aligned}$$

Let $R \in \mathbb{R}$ such that $[\min_i \pi_i(0), \max_i \pi_i(1)] \subset [-R, R]$, then

$$A_m = B_{1,m}(R) - B_{2,m}(R) + C_m(R),$$

where

$$B_{i,m}(R) = \iint_{\mathcal{U}} \left(\Upsilon_i(\pi_{\mathcal{D}_m}) - \Upsilon_i(T_{[-R,R]}(\pi_{\mathcal{D}_m}))\right) \psi \mathrm{d}\mathbf{x}\mathrm{d}t$$

and

$$C_m(R) = \iint_{\mathcal{U}} \left(W_1(T_{[-R,R]}(\pi_{\mathcal{D}_m})) - W_2(T_{[-R,R]}(\pi_{\mathcal{D}_m})) - W_1(\pi) + W_2(\pi)\right) \psi \mathrm{d}\mathbf{x}\mathrm{d}t.$$

Let $\varepsilon > 0$, then since Υ_i admits finite limits as $p \to \min_i \pi_i(0)$ and $p \to \max_i \pi_i(1)$, there exists $R_0(\varepsilon) > 0$ such that

$$R > R_0(\varepsilon) \implies \|\Upsilon_i - \Upsilon_i \circ T_{[-R,R]}\|_\infty \leq \varepsilon.$$

Thus, for $R > R_0(\varepsilon)$ fixed,

$$|B_{i,m}(R)| \leq Tm(\Gamma)\varepsilon.$$

Thanks to Lemma 4.16,

$$\lim_{m\to\infty} C_m(R) = 0,$$

then, for all $\varepsilon > 0$,

$$\limsup_{m\to\infty} |A_m| \leq 2Tm(\Gamma)\varepsilon.$$

As a consequence, since the above estimate holds for all $\varepsilon > 0$, A_m tends to 0, concluding the proof of Proposition 4.17. ☐

5. End of the proof of Theorem 1. We have proven in the section 4 that, up to a subsequence, the sequence of approximate solutions $(s_{\mathcal{D}_m}, P_{\mathcal{D}_m})_m$ converge towards (s, P) as $m \to \infty$. Moreover, it as been stated in Lemmata 4.14 and 4.15 that $(\pi_{\mathcal{D}_m})_m$ converges in some sense on $\Gamma \times (0, T)$ towards a measurable function π. In order to conclude the proof of Theorem 1, it remains to check that (s, P) satisfy the weak formulations (1.25) and (1.26), and that the transmission conditions (1.20) and (1.21) are fulfilled. Let us begin by this latter point.

It follows from the construction of the function π carried out in Lemmata 4.14 and 4.15 that, for almost every $(\mathbf{x}, t) \in \Gamma \times (0, T)$,

$$\pi(\mathbf{x}, t) \in \tilde{\pi}_1(s_1(\mathbf{x}, t)) \cap \tilde{\pi}_2(s_2(\mathbf{x}, t)). \tag{5.1}$$

Let $\psi \in L^2(\Gamma \times (0, T))$, then thanks to (2.8), one has, for all $\psi \in L^2(\Gamma \times (0, T))$,

$$\int_0^T \int_\Gamma \left(P_{\mathcal{D}_{m|\Gamma,1}} - P_{\mathcal{D}_{m|\Gamma,2}}\right) \psi \mathrm{d}\mathbf{x}\mathrm{d}t = \int_0^T \int_\Gamma (W_1(\pi_{\mathcal{D}_m}) - W_2(\pi_{\mathcal{D}_m})) \psi \mathrm{d}\mathbf{x}\mathrm{d}t.$$

Letting m tend to ∞ provides, thanks to Propositions 4.12 and 4.17, that

$$\int_0^T \int_\Gamma (P_1 - P_2) \psi \mathrm{d}\mathbf{x}\mathrm{d}t = \int_0^T \int_\Gamma (W_1(\pi) - W_2(\pi)) \psi \mathrm{d}\mathbf{x}\mathrm{d}t.$$

Hence,

$$P_1 - W_1(\pi) = P_2 - W_2(\pi) \quad \text{a.e. on } \Gamma \times (0, T). \tag{5.2}$$

In order to recover the weak formulations (1.25) and (1.26), we can apply to our case the analysis carried out in the proof of Theorem 5.1 in [28].

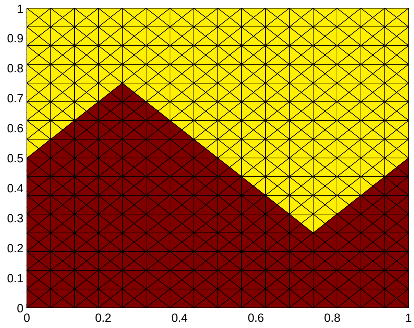

FIG. 6.1. *The model porous medium* $\Omega_1 \cup \Omega_2$

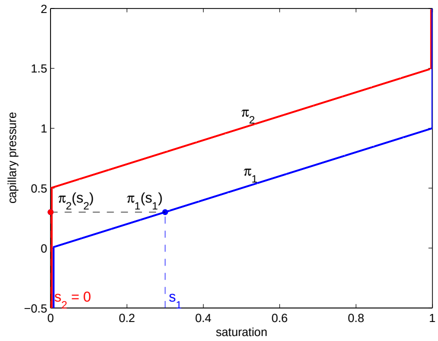

FIG. 6.2. *Capillary pressure connection at t = 0*

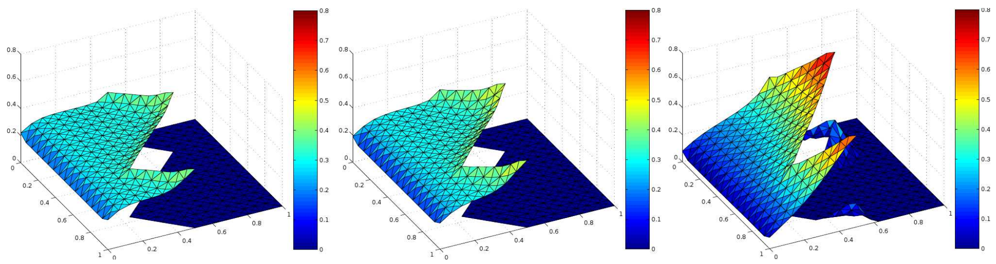

FIG. 6.3. *Saturation for* $t = 0.06$*,* $t = 0.11$ *and* $t = 0.6$

6. Numerical results. In this Section we consider a model porous medium $\Omega = (0,1)^2$ composed of two layers Ω_1 and Ω_2, which are separated by an "S-shaped" interface Γ (see Fig. 6.1), and which have different capillary pressure laws. The oil and water densities are given by $\rho_o = 0.81$, $\rho_w = 1$ respectively, and $\mathbf{g} = -9.81\mathbf{e}_y$. We suppose that the porosity is such that $\phi_i = 1, i \in \{1,2\}$, and we define the oil and water mobilities by

$$\eta_{o,i}(s) = 0.5s^2 \quad \text{and} \quad \eta_{w,i} = (1-s)^2, \quad i \in \{1,2\}.$$

Moreover we suppose that the capillary pressure curves have the form

$$\pi_1(s) = s \quad \text{and} \quad \pi_2(s) = 0.5 + s.$$

In the first test case we suppose that the layer Ω_1 contains some quantity of oil and it is situated below Ω_2, which is saturated with water, that is to say $\Omega_1 = \{(\mathbf{x},\mathbf{y}) \in \Omega \mid \mathbf{y} < \Gamma(\mathbf{x})\}$ and $\Omega_2 = \{(\mathbf{x},\mathbf{y}) \in \Omega \mid \mathbf{y} > \Gamma(\mathbf{x})\}$. The initial saturation is given by

$$s_0(\mathbf{x}) = \begin{cases} 0.3 & \text{if } x \in \Omega_1, \\ 0 & \text{otherwise.} \end{cases}$$

The flow is driven by buoyancy, making the oil move along $\mathbf{e}_y$ until it reaches the interface Γ. As one can see on the figures 6.3 and 6.4, for $t \le 0.11$, oil can not access the domain Ω_2, since the capillary pressure $\pi_1(s_1)$ is lower than the threshold value $\pi_2(0) = 0.5$, which is called *the entry pressure* (see Fig. 6.2). Hence the saturation below the interface s_1 increases, as well as the capillary pressure $\pi_1(s_1)$. As soon as the capillary pressure $\pi_1(s_1)$ reaches the entry pressure $\pi_2(0)$, oil starts to penetrate

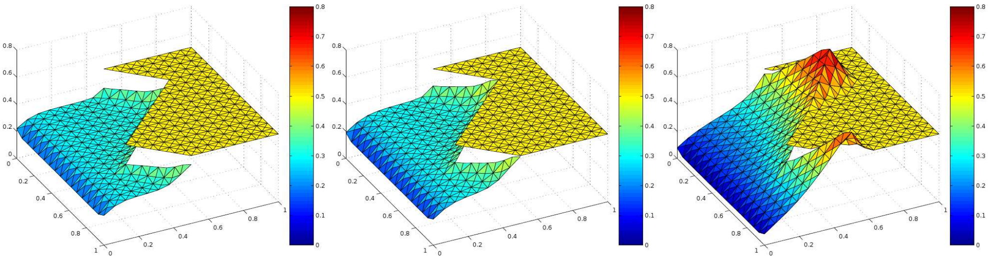

FIG. 6.4. *Capillary pressure for* $t = 0.06$, $t = 0.11$ *and* $t = 0.6$

in the domain Ω_2. Nevertheless, as pointed out in [6, 9], a finite quantity of oil remains trapped under the rock discontinuity. This phenomenon is called *oil trapping*. It is worth noting that the solution at $t = 0$ satisfies (1.20), thus in the absence of gravity the initial distribution of oil-phase would be a steady state solution $s(\mathbf{x}, t) = s_0(\mathbf{x})$.

In the second test case we assume that the oil is initially situated in the rock with a higher *entry pressure* pressure i.e.

$$s_0(\mathbf{x}) = \begin{cases} 0.3 & \text{if } x \in \Omega_2, \\ 0 & \text{otherwise.} \end{cases}$$

where this time $\Omega_1 = \{(\mathbf{x}, \mathbf{y}) \in \Omega \mid \mathbf{y} > \Gamma(\mathbf{x})\}$ and $\Omega_2 = \{(\mathbf{x}, \mathbf{y}) \in \Omega \mid \mathbf{y} < \Gamma(\mathbf{x})\}$. This time the flow is driven not only by a buoyancy, but also by a difference in the capillary pressure potential (the solution at $t = 0$ does not fulfill (1.20)). As a result the oil-phase can immediately penetrate the domain Ω_1. The figure 6.5 shows that the oil propagates in the domain Ω_1 with a finite speed.

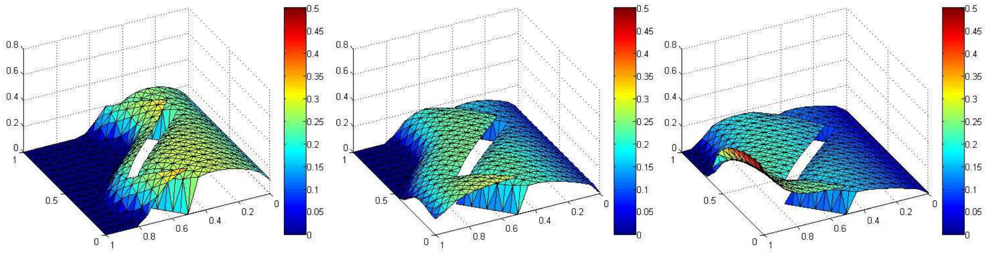

FIG. 6.5. *Saturation for* $t = 0.3$, $t = 1$ *and* $t = 1.7$

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