SEMI-LAGRANGIAN DISCONTINUOUS GALERKIN SCHEMES FOR SOME FIRST AND SECOND ORDER PARTIAL DIFFERENTIAL EQUATIONS

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Abstract. Explicit, unconditionally stable, high order schemes for the approximation of some first and second order linear, time-dependent partial differential equations (PDEs) are proposed. The schemes are based on a weak formulation of a semi-Lagrangian scheme using discontinuous Galerkin elements. It follows the ideas of the recent works of Crouseilles, Mehrenberger and Vecil (2010) and of Qiu and Shu (2011), for first order equations, based on exact integration, quadrature rules, and splitting techniques. In particular we obtain high order schemes, unconditionally stable and convergent, in the case of linear second order PDEs with constant coefficients. In the case of non-constant coefficients, we construct "almost" unconditionally stable second order schemes and give precise convergence results. The schemes are tested on several academic examples, including the Black and Scholes PDE in finance.

1. Introduction

In this paper we consider equations of the form

$$u_t - \frac{1}{2}Tr(\sigma\sigma^T D^2 u) + b\cdot\nabla u + ru = 0, \quad x\in\Omega,\ t\in(0,T), \tag{1}$$

where $\Omega\subset\mathbb{R}^d$ is a box (with some boundary conditions on $\partial\Omega$), σ (matrix), b (vector) and r (scalar) may be x-dependent, together with an initial condition

$$u(0,x) = u_0(x), \quad x\in\Omega.$$

The matrix σ may be zero or degenerated.

We study and propose new Semi-Lagrangian Discontinuous Galerkin schemes ("Lagrange-Galerkin" in short, also abbreviated "SLDG" in this work) in order to treat Partial Differential Equations (PDE) of type (1).

The Semi-Lagrangian approach [13] is based on the approximation of the "method of characteristics".

By considering a weak formulation of this principle, an explicit SLDG scheme is obtained. Our approach is based on a similar method for the approximation of first order PDEs such as the Vlasov equation in plasma physics as in the recent works of Crouseilles et al [7] and Qiu et al [18] (see also [20] for the original approach). Here we introduce a new SLDG scheme for second order PDEs on which we prove stability and convergence results, and obtain higher-order accuracy when possible.

First, in Section 2, we revisit the one-dimensional first order advection equation with non-constant advection term $b(x)$ (case $\sigma = 0$ in (1)). We give a new unconditional stability result, and convergence proof, extending a result of [18] that

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was obtained for the case of a constant advection term. The unconditional stability property can be interesting when compared to a standard DG approach where small enough CFL condition (and a small enough time step) must in general be considered for stability [6]. We also show how to combine the scheme with higher-order splitting techniques in order to treat two-dimensional (or more) first order PDEs.

Based on the operator construction for first order advection and splitting techniques, we then introduce new schemes for linear second order PDEs of type (1), in the form of explicit high order SLDG schemes (see Section 3). These schemes are based on the idea of Menaldi [16] (see also Camilli et al. [5], and Debrabant et al [9] for general SL schemes). In particular, for second order PDEs with constant coefficients, we propose explicit and unconditionally stable schemes, which can be of any order in space and up to third order in time (higher order can be obtained [3]). In the case of non-constant coefficients, we construct a scheme that is second order in space (using time step and mesh step of the same order), stable and convergent under a weak CFL condition (of the form $\Delta x^4 \leq \lambda \Delta t$ for some constant λ, where Δt and Δx denote the time and mesh steps).

High-order schemes for stochastic differential equations also exists [15, 8], that could be adapted to our context, but our basic constructions seem new.

The advantage of the proposed schemes is that they combine the DG framework which allows, in principle, high-order spatial accuracy, the potential of degree adaptivity, together with unconditional stability properties in the L^2 norm from the weak semi-Lagrangian formulation of the scheme.

We show the relevance of our approach on several academic numerical examples in one and two dimensions (using Cartesian meshes), including also the Black and Scholes PDE in mathematical Finance (see Section 4).

Altough the present setting concerns linear PDEs, we think the estimates and ideas proposed here could be useful to analyse similar semi-Lagrangian schemes in non linear settings. Ongoing works concern the obtention of higher order schemes for second order PDEs [3], as well as extensions to nonlinear PDEs arising from deterministic control [4] or from stochastic control.

2. Advection equation

We first consider the Semi-Lagrangian Discontinuous Galerkin scheme (SLDG for short) for the following one-dimensional first-order PDEs, as in [7]

$$\begin{cases} v_t + b(x) v_x = 0, & (t,x) \in (0,T) \times \Omega \\ v(0,x) = v_0(x), & x \in \Omega \end{cases} \tag{2}$$

where $\Omega = (x_{min}, x_{max})$, together with periodic boundary conditions on Ω.

2.1. **Constant drift coefficient, and notations.** We consider here the case when b is a constant:

$$b(x) = b, \quad x \in \Omega.$$

The methods of characteristics gives that $v(t,x)$, the solution of (2), is constant along the characteristics ($t \to v(t, x + bt)$ is constant). Let $N \in \mathbb{N}$, $N \geq 1$, $\Delta t = \frac{T}{N}$ a time step and $t_n = n\Delta t$. Let $v^n(x) := v(t_n, x)$. We have in particular

$$\begin{cases} v^{n+1}(x) = v^n(x - b\Delta t), & n = 0, \ldots, N-1, \ x \in \Omega \\ v^0(x) = v_0(x) \end{cases} \tag{3}$$

Now, we define a space discretization that is considered uniform for simplification of presentation. Let $\Delta x = \frac{x_{max} - x_{min}}{M}$ for some integer $M \geq 1$, $x_{i-\frac{1}{2}} := x_{min} + i\Delta x$,

$\forall i = 0,..,M$, and $I_i := (x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}})$. Let $k \in \mathbb{N}$. We define V_k as the space of discontinuous-Galerkin elements on Ω with polynomials of degree k, that is:

$$V_k \quad = \quad \{v \in L^2(\Omega, \mathbb{R}) : v|_{I_i} \in P_k, \forall i = 0,..,M-1\} \tag{4}$$

whereas P_k denotes the set of polynomials of degree at most k.

In the classical Semi-Lagrangian approach, looking for $u^n(x)$, an approximation of $v(t_n, x)$, a first "direct" iterative scheme for (3) would be

$$u^{n+1}(x_i) = [u^n](x_i - b\Delta t) \tag{5}$$

where $[u^n](x)$ denotes some interpolation of the function u^n at point x. We could take for instance a set of $k+1$ values $(x^i_\alpha)_{\alpha=0,\ldots,k}$ in each interval I_i, and define the new polynomial u^{n+1} such that $u^{n+1}(x^i_\alpha) = [u^n](x^i_\alpha - b\Delta t)$ for all $\alpha = 0, \ldots, k$. However, given the discontinuities between the intervals I_i, this may lead to instabilities in the scheme ([18]). For instance, taking x^i_α to be the Gauss quadrature points on each interval I_i is in general unstable, see Appendix A (see also [17]).

Here we consider exact Lagrange-Galerkin approach by considering the weak form of (3): for $n = 0, \ldots, N-1$, find $u^{n+1} \in V_k$ such that

$$\int_\Omega u^{n+1}(x)\varphi(x)dx = \int_\Omega u^n(x - b\Delta t)\varphi(x)dx, \quad \forall \varphi \in V_k \tag{6}$$

and for $n = 0$, find $u^0 \in V_k$ such that:

$$\int_\Omega u^0(x)\varphi(x)dx = \int_\Omega v_0(x)\varphi(x)dx, \quad \forall \varphi \in V_k. \tag{7}$$

In the case of a constant coefficient b, the new function u^{n+1} (as well as u^0) can be computed by solving exactly (6) (resp. (7)).

Otherwise, if $b(x)$ is not a constant, a precise EDO integration for the characteristics and a quadrature rule can be used [17].

The weak form (6) gives the stability of the scheme in the L^2 norm. Indeed, taking $\varphi = u^{n+1}$ in (6) we get

$$\|u^{n+1}\|_2^2 = (u^n(\cdot - b\Delta t),\ u^{n+1}) \le \|u^n(\cdot - b\Delta t)\|_2\, \|u^{n+1}\|_2,$$

where $\|\cdot\|_2$ denotes the L^2 norm on Ω and $(\,.,\,.)$ is the associated scalar product. Then, by the periodic boundary condition, $\|u^n(\cdot - b\Delta t)\|_2 = \|u^n\|_2$ and therefore

$$\|u^{n+1}\|_2 \le \|u^n\|_2. \tag{8}$$

This proof works only for b constant, however.

For any $w \in L^2$, we denote its projection on V_k by Πw, corresponding to the unique element of V_k such that

$$\|w - \Pi w\|_2 = \inf_{f \in V_k} \|w - f\|_2. \tag{9}$$

Remark 2.1. *The function u^{n+1} defined by* (6) *corresponds to the projection of the function $x \to u^n(x - b\Delta t)$ on the space V_k:*

$$u^{n+1} = \Pi(u^n(\cdot - b\Delta t)),$$

and, in the same way, we have $u^0 = \Pi v_0$.

From now on, we rewrite the scheme (6), corresponding to a constant advection term b, in the following abstract form :

$$u^{n+1} = \mathcal{T}_{b\Delta t}(u^n).$$

Now, we recall how the scheme can be exactly implemented, as in [7]. Let $\{x_\alpha\}_{\alpha=0,..,k}$ be the set of Gauss points in the interval $(-1,1)$, with its corresponding weights $\{w_\alpha\}_{\alpha=0,..,k}$ ($w_\alpha > 0$), such that:

$$\forall p \in P_{2k+1}, \quad \int_{-1}^{1} p(x)dx = \sum_{\alpha=0}^{k} w_\alpha p(x_\alpha). \tag{10}$$

In particular, we get on the interval I_i,

$$\forall p \in P_{2k+1}, \quad \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} p(x)dx = \sum_{\alpha=0}^{k} w_\alpha^i p(x_\alpha^i), \tag{11}$$

where $x_\alpha^i := x_i + x_\alpha \Delta x \equiv x_{i-\frac{1}{2}} + \frac{1}{2}(1+x_\alpha)\Delta x$ and $w_\alpha^i := \frac{\Delta x}{2} w_\alpha$.

From each set of Gauss points $\{x_\alpha^i\}_{\alpha=0,..,k}$ in I_i, we can associate the corresponding Lagrange polynomials (dual basis) $\{\varphi_\alpha^i\}_{\alpha=0,..,k}$ defined by

$$\varphi_\alpha^i(x) := 1_{I_i}(x) \prod_{\substack{0\le\beta\le k \\ \beta\ne\alpha}} \frac{x - x_\beta}{x_\alpha - x_\beta}. \tag{12}$$

For any $u^n \in V_k$, there exists coefficients $(u_{\alpha,i}^n)_{\alpha=0,..,k}^{i=0,..,M-1} \in \mathbb{R}$ such that:

$$u^n(x) = \sum_{i=0}^{M-1} \sum_{\alpha=0}^{k} u_{\alpha,i}^n \varphi_\alpha^i(x). \tag{13}$$

In particular, the left-hand side of (6) for $\varphi = \varphi_\alpha^i$ becomes

$$\int_\Omega u^{n+1}(x)\varphi_\alpha^i(x)dx \quad = \quad \int_{I_i} u^{n+1}(x)\varphi_\alpha^i(x)dx = u_{\alpha,i}^{n+1} w_\alpha^i.$$

Then, due to the discontinuities of u^n, an exact integration of the right-hand side of (6) is obtained by separating the integral into several parts. Considering the case of constant b, we will have in general two regular parts. On each part, the Gaussian quadrature rule with $k+1$ points is applied and is exact for integrating $x \to u^n(x - b\Delta t)\varphi_\alpha^i(x)$ since it is a polynomial of degree at most $2k$ (see [7]).

2.2. **Error estimate for constant drift coefficient.** We first recall a simple estimate for the L^2 projection on V_k.

Lemma 2.1 (Projection error)**.** *Let $k \ge 0$ and $\ell \le k$. If $w \in \mathcal{C}^{\ell+1}$, then*

$$\|w - \Pi w\|_{L^2} \le |\Omega|^{1/2} C_\ell(w)\ \Delta x^{\ell+1}$$

where $C_\ell(w) := \frac{1}{2^{\ell+1}(\ell+1)!}\|w^{(\ell+1)}\|_\infty$.

Proof. Let us write $w = P + R$ where P is the element of V_k corresponding, on each interval I_i, to the Taylor expansion of w centered at x_i and of degree ℓ. We have $\|w - \Pi w\|_{L^2} \le \|w - P\|_{L^2} = \|R\|_{L^2} \le |\Omega|^{1/2}\|R\|_{L^\infty}$. By the definition of R and usual Taylor estimates, we have $\|R\|_{L^\infty} \le C_\ell \Delta x^{\ell+1}$. □

Let $v^n(x) := v(t_n, x)$ where v denotes the exact solution of (2). Using the L^2-stability of the projection, it is straigthforward to show that $\|u^{n+1} - \Pi v^{n+1}\|_{L^2} = \|\Pi(u^n(\cdot - b\Delta t) - v^n(\cdot - b\Delta t)\|_{L^2} \le \|u^n - v^n\|_{L^2}$, therefore we have

$$\|u^{n+1} - v^{n+1}\|_{L^2} \le \|u^n - v^n\|_{L^2} + \|v^{n+1} - \Pi v^{n+1}\|_{L^2}.$$

By using Lemma 2.1, this leads to the following known convergence result [18].

Theorem 2.1. *Let $k \geq 0$ and b be a constant. Assume the initial condition v_0 is 1-periodic and in $\mathcal{C}^{k+1}$. The following estimate holds:*

$$\|u^n - v^n\|_{L^2} \leq \|u^0 - v^0\|_{L^2} + CT\frac{\Delta x^{k+1}}{\Delta t}, \quad \forall n \leq N, \tag{14}$$

where the constant C depends only of $|\Omega|$ and k.

Remark 2.2. *By taking $\Delta t = \Delta x$ this leads to an error estimate in $O(\Delta x^k)$. However the examples (such as in Example 1) will show a numerical behavior in $O(\Delta x^{k+1})$ (as already remarked also in* [18]*). We still do not understand this gap.*

2.3. **Nonconstant drift coefficient** $b(x)$**.** In order to simplify the presentation and the proofs, without loss of generality, we can assume here that $\Omega = (0,1)$ and that b is a 1-periodic function.

We denote $y = y_x$ as the solution of the differential equation

$$\begin{cases} \dot{y}(t) = b(y(t)), & t \in \mathbb{R} \\ y(0) = x \end{cases} \tag{15}$$

where we assume that $b(\cdot)$ is at least Lipschitz continuous. The solution of (2) satisfies

$$v(t_{n+1}, x) = v(t_n, y_x(-\Delta t)).$$

Hence, a scheme definition should be: $\forall n = 0, \dots, N-1$, find $u^{n+1} \in V_k$ such that

$$\int_\Omega u^{n+1}(x)\varphi(x)dx \ = \ \int_\Omega u^n(y_x(-\Delta t))\varphi(x)dx, \quad \forall \varphi \in V_k, \tag{16}$$

or, equivalently, $u^{n+1} = \mathcal{T}_{b\Delta t} u^n$ where

$$\mathcal{T}_{b\Delta t} u := \Pi\big(u(y_\cdot(-\Delta t))\big).$$

However, the computing procedure for the R.H.S. can no more be exact, because $x \to u^n(y_x(-\Delta t))$ is no more a piecewise polynomial.

Following [18], we consider the scheme where the R.H.S. of (16) is approximated by the Gaussian quadrature rule on each sub-interval where $u^n(y_x(-\Delta t))$ is a regular function.

More precisely, for a given mesh cell I_i and for a polynomial $\varphi \in V_k$, we first consider the points $(x_{i,q})_{1 \leq q \leq p_i}$ (in finite number) of the interval $(x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}})$, such that for $1 \leq q \leq p_i$, $y_{x_{i,q}}(-\Delta t) = x_{\ell_{i,q}-\frac{1}{2}}$ for some $\ell_{i,q} \in \mathbb{Z}$, and $x_{i,0} := x_{i-\frac{1}{2}}$, $x_{i,p_i+1} := x_{i+\frac{1}{2}}$. Then we apply the Gaussian quadrature rule on each interval $J_{i,q} = (x_{i,q}, x_{i,q+1})$ and obtain the following quadrature rule:

$$\int_{I_i} u^n(y_x(-\Delta t))\varphi(x)dx \ = \ \sum_{q=0}^{p_i} \int_{x_{i,q}}^{x_{i,q+1}} u^n(y_x(-\Delta t))\varphi(x)dx \tag{17}$$

$$\simeq \ \sum_{q=0}^{p_i}\sum_{\alpha=0}^{k} \tilde{w}^i_{q,\alpha} u^n(y_{\tilde{x}^i_{q,\alpha}}(-\Delta t))\varphi(\tilde{x}^i_{q,\alpha}), \tag{18}$$

with $\tilde{w}^i_{q,\alpha} := \frac{w_\alpha}{2}(x_{i,q+1} - x_{i,q})$ and $\tilde{x}^i_{q,\alpha} := x_{i,q} + \frac{1}{2}(1 + x_\alpha)(x_{i,q+1} - x_{i,q}) \equiv \frac{x_{i,q}+x_{i,q+1}}{2} + x_\alpha(\frac{x_{i,q+1}-x_{i,q}}{2})$.

Scheme definition : u^{n+1} is the unique element of V_k satisfying for all $\varphi \in V_k$,

$$\int_\Omega u^{n+1}(x)\varphi(x)dx \ = \ \sum_{i=0}^{M-1}\sum_{q=0}^{p_i}\sum_{\alpha=0}^{k} \tilde{w}^i_{q,\alpha} u^n\big(y_{\tilde{x}^i_{q,\alpha}}(-\Delta t)\big)\varphi(\tilde{x}^i_{q,\alpha}). \tag{19}$$

The scheme is made explicit by using formula (19) on each $\varphi = \varphi^j_\beta$. We extend also the definition of the operator $\mathcal{T}_{b\Delta t}$, by $\widetilde{\mathcal{T}}_{b\Delta t}$ such that

$$u^{n+1} = \widetilde{\mathcal{T}}_{b\Delta t} u^n.$$

Definition 1. *For further analysis, let us introduce the following scalar product on V_k (where the index "G" stands for the use of the Gaussian quadrature rule):*

$$(a,b)_G := \sum_{i=0}^{M-1} \sum_{q=0}^{p_i} \sum_{\alpha=0}^{k} \tilde{w}^i_{q,\alpha}\ a(\tilde{x}^i_{q,\alpha})\ b(\tilde{x}^i_{q,\alpha}). \tag{20}$$

In particular the scheme is equivalently defined by

$$(u^{n+1}, \varphi) = (u^n(y.(-\Delta t)), \varphi)_G, \quad \forall \varphi \in V_k.$$

For $u \in V_k$, we shall need the following approximation result, which controls the error between the desired formula (16) and the implementable scheme (19).

Proposition 2.1. *Let $k \geq 0$ and let b be of class $\mathcal{C}^{2k+2}$ and 1-periodic.*
(i) For all $u \in V_k$,

$$\Big| (u(y.(-\Delta t)), \varphi)_G - (u(y.(-\Delta t)), \varphi) \Big| \leq C \Delta t \Delta x^2 \|u\|_{L^2} \|\varphi\|_{L^2}, \quad \forall \varphi \in V_k.$$

where $C \geq 0$ is a constant. In particular, we have, in the L^2-norm:

$$\widetilde{\mathcal{T}}_{b\Delta t} u^n \equiv u^{n+1} = \mathcal{T}_{b\Delta t} u^n + O(\Delta t \Delta x^2 \|u^n\|_{L^2}), \quad \forall n \geq 0. \tag{21}$$

(ii) For all $u \in V_k$, for any ψ in $\mathcal{C}^{k+1}$, 1-periodic,

$$\Big| (u(y.(-\Delta t)) - \psi(y.(-\Delta t)), \varphi)_G - (u(y.(-\Delta t)) - \psi(y.(-\Delta t)), \varphi) \Big|$$
$$\leq C \Delta t \Delta x^2 \|u - \psi\|_{L^2} \|\varphi\|_{L^2} + C M_{k+1}(\psi) \Delta x^{k+1} \|\varphi\|_{L^2}, \quad \forall \varphi \in V_k, \tag{22}$$

where $C \geq 0$ is a constant which depends only of k, and

$$M_p(\psi) := \max_{0 \leq r \leq p} \|\psi^{(r)}\|_{L^\infty}. \tag{23}$$

(iii) For any regular $\psi \in \mathcal{C}^{k+1}$, for any $\varphi \in V_k$,

$$(\psi, \varphi)_G = (\psi, \varphi) + O(M_{k+1}(\psi) \Delta x^{k+1} \|\varphi\|_{L^2}). \tag{24}$$

(iv) Furthermore, $\exists C \geq 0$, for any $\psi \in \mathcal{C}^{k+1}$, 1-periodic,

$$\|\widetilde{\mathcal{T}}_{b\Delta t} \psi - \mathcal{T}_{b\Delta t} \psi\|_{L^2} \leq C M_{k+1}(\psi) \Delta x^{k+1}. \tag{25}$$

Remark 2.3. *Some assumptions can be weakened, for instance (i) and (ii) are still valid using that $b^{(2k+1)}$ is in L^∞, then in the error bounds (21) and (22) the $\Delta t \Delta x^2$ term should be replaced by a $\Delta t \Delta x$. However these bound will be used in Section 3 and the form (21) and (22) is preferred. Also, it is possible to prove that the error term in $O(M_{k+1}(\psi)\Delta x^{k+1})$ in (ii), (iii) and (iv) can be improved to $O(M_{2k+1}(\psi)\Delta x^{k+2})$ provided that $\psi \in \mathcal{C}^{2k+1}$.*

2.4. **Proof of Proposition 2.1.** Notice that the estimates of (i) and (iii) are a consequence of (ii) (either by choosing $\psi \equiv 0$ to obtain (i), or by choosing $\Delta t \equiv 0$ and $u \equiv 0$ to obtain (iii)). Then (iv) is deduced from (iii) when applied to the regular function $\psi_1(x) := \psi(y_x(-\Delta t))$.

The plan is first to prove (i), and then to generalize to (ii). Precise estimates for the $2k+2$ derivative of $x \to u(y_x(-\Delta t))$ will be needed in order to estimate the error when using Gaussian quadrature formula. In the following, we first bound the derivatives of $x \to y_x(-t)$.

Lemma 2.2. *Assume that $b \in \mathcal{C}^k$, 1-periodic, for some $k \geq 1$, and let $t \in \mathbb{R}$. Then $x \to y \equiv y_x(-t)$ is of class $\mathcal{C}^k$, 1-periodic, and*

$$\begin{cases} \|y\|_{L^\infty} \leq C, \ \|\frac{\partial}{\partial x} y\|_{L^\infty} \leq C, \\ \text{and, if } k \geq 2, \quad \|\frac{\partial^q}{\partial x^q} y\|_{L^\infty} \leq C|t|, \quad \forall q \in [2, \ldots, k], \end{cases} \tag{26}$$

for some constant C.

Proof. We consider y as a function of the time t and of x. We can assume that $|t| \leq C_0 = 1$ to obtain the bounds (since y is 1-periodic). We denote $y^{(k)} \equiv \frac{\partial^k}{\partial x^k} y$ the k-th derivative of y with respect to x.

If $k = 1$ and $b \in \mathcal{C}^1$, then $\frac{\partial}{\partial t} \frac{\partial}{\partial x} y = b'(y) \frac{\partial}{\partial x} y$ and $\frac{\partial}{\partial x} y(0) = 1$, therefore $|\frac{\partial}{\partial x} y(t)| = \exp\big(\int_0^t b'(y(s))ds\big) \leq e^{C_0 \|b'\|_\infty}$ and is bounded.

For $k \geq 2$, we have

$$\begin{aligned} \frac{\partial}{\partial t} y^{(k)} &= (b'(y) y^{(1)})^{(k-1)} \\ &= b'(y) y^{(k)} + \sum_{\ell=1}^{k-1} C_{k-1}^\ell (b'(y))^{(\ell)} y^{(k-\ell)}. \end{aligned}$$

Then, for $b \in \mathcal{C}^k$, all the spatial derivatives $y^{(\ell)}$ are bounded for $\ell \leq k-1$, $|t| \leq C_0$ and $x \in [0,1]$. Therefore, for $\ell = 1, \ldots, k$, all the terms $(b'(y))^{(\ell)}$ and $y^{(k-\ell)}$ are bounded and the function $f := \sum_{\ell=1}^{k-1} C_{k-1}^\ell (b'(y))^{(\ell)} y^{(k-\ell)}$ is bounded independently of $|t| \leq C_0$ and of x, by some constant C. By using the formula

$$y^{(k)}(t) = e^{\int_0^t b'(y(s))ds} y^{(k)}(0) + \int_0^t e^{\int_s^t b'(y(\theta))d\theta} f(s) ds,$$

the fact that $y^{(k)}(0) = 0$ for $k \geq 2$ and $|e^{\int_s^t b'(y(\theta))d\theta} f(s)| \leq C e^{C_0 \|b'\|_\infty}$, we conclude to $|y^{(k)}(t)| \leq C e^{C_0 \|b'\|_\infty} |t|$. □

Lemma 2.3. *Assume $q \geq k+1$, and $u \in V_k$. On any interval J where u is regular,*

$$\|\frac{d^q}{dx^q}(u(y))\|_{L^\infty(J)} \leq C \Delta t \sum_{p=1}^k \|u^{(p)}\|_{L^\infty(y(J))}.$$

Proof. We first recall an expression for the q-th derivative of the composite function $u(y)$, also known as "Faà di Bruno's formula" [12, 1] :

$$\frac{1}{q!} \frac{d^q}{dx^q}(u(y(x))) = \sum_{p=1}^k u^{(p)}(y(x)) \Bigg(\sum_{(\alpha_j),\ \sum_j \alpha_j = p,\ \sum_j j\alpha_j = q} \frac{(y^{(1)}/1!)^{\alpha_1} \cdots (y^{(q)}/q!)^{\alpha_q}}{\alpha_1! \cdots \alpha_q!} \Bigg). \tag{27}$$

Here the sum is limited to $p \leq k$ (instead of $p \leq q$) since $u \in V_k$.

Therefore, together with Lemma 2.2, we obtain the bound

$$\|\frac{d^q}{dx^q}(u(y))\|_{L^\infty(J)} \leq C \sum_{p=1}^k \|u^{(p)}\|_{L^\infty(y(J))} \Bigg(\sum_{(\alpha_j),\ \sum_{j=1}^q \alpha_j = p,\ \sum_{j=1}^q j\alpha_j = q} \Delta t^{\alpha_2 + \cdots + \alpha_q} \Bigg)$$

The case when $\alpha_2 = \cdots = \alpha_q = 0$ happens only if $\alpha_1 = p = q$. Since $q \geq k+1$, and $p \leq k$, this case never occurs. Therefore the power of Δt is at least 1, which concludes the proof. □

Proof of Proposition 2.1(i): Let ε be the error term, defined by

$$\varepsilon := \int_0^1 u(y_x(-\Delta t)) \varphi(x) dx - \sum_{i=0}^{M-1} \sum_{q=0}^{p_i} \sum_{\alpha=0}^k \tilde{w}_{q,\alpha}^i u(y_{\tilde{x}_{q,\alpha}^i}(-\Delta t)) \varphi(\tilde{x}_{q,\alpha}^i).$$

We have $\varepsilon = \sum_i \sum_{q=0}^{p_i} \varepsilon_{i,q}$ where

$$\varepsilon_{i,q} := \int_{J_{i,q}} u\big(y_x(-\Delta t)\big)\varphi(x)\,dx - \sum_{\alpha=0}^{k} \tilde{w}^i_{q,\alpha} u\big(y_{\tilde{x}^i_{q,\alpha}}(-\Delta t)\big)\varphi(\tilde{x}^i_{q,\alpha}) \tag{28}$$

and with $J_{i,q} := (x_{i,q}, x_{i,q+1})$.

Let $u(y)$ be the function $x \to u(y_x(-\Delta t))$. Since $u(y)$ is $\mathcal{C}^{2k+2}$ regular on $J_{i,q}$ for each fixed i, $q \in [0,\dots,p_i]$, and that the R.H.S. of (28) corresponds to the Gaussian quadrature rule on $J_{i,q}$, we have in particular

$$|\varepsilon_{i,q}| \le C\Delta x_{i,q}^{2k+3} \, \|[u(y)\varphi]^{(2k+2)}\|_{L^\infty(J_{i,q})},$$

where $\Delta x_{i,q} := x_{i,q+1} - x_{i,q}$.

On the other hand, since $\varphi \in V_k$,

$$\|[u(y)\varphi]^{(2k+2)}\|_{L^\infty(J_{i,q})} \le C \sum_{r=0}^{k} \|\varphi^{(r)}\|_{L^\infty(J_{i,q})} \|[u(y)]^{(2k+2-r)}\|_{L^\infty(J_{i,q})}.$$

For all $r \in [0,\dots,k]$ we have $2k+2-r \ge k+2 \ge k+1$, hence we can use Lemma 2.3 and obtain the bound

$$\|[u(y)\varphi]^{(2k+2)}\|_{L^\infty(J_{i,q})} \le C\big(\sum_{r=0}^{k} \|\varphi^{(r)}\|_{L^\infty(J_{i,q})}\big)\Delta t\big(\sum_{p=1}^{k} \|u^{(p)}\|_{L^\infty(y(J_{i,q}))}\big).$$

In particular,

$$\sum_{i,q} |\varepsilon_{i,q}| \le C \sum_{r=0}^{k}\sum_{p=1}^{k}\sum_{i}\sum_{q=0}^{p_i} \Delta t \Delta x_{i,q}^{2k+3} \|\varphi^{(r)}\|_{L^\infty(J_{i,q})} \|u^{(p)}\|_{L^\infty(y(J_{i,q}))}$$

By a scaling argument and using that $\varphi \in V_k$ for fixed k, we have, $\forall 0 \le r \le k$

$$\|\varphi^{(r)}\|_{L^\infty(J_{i,q})} \le \frac{C}{\Delta x_{i,q}^{r+1/2}} \|\varphi\|_{L^2(J_{i,q})} \le \frac{C}{\Delta x_{i,q}^{k+1/2}} \|\varphi\|_{L^2(J_{i,q})}, \tag{29}$$

for some constant C (assuming for instance $\Delta x_{i,q} \le 1$). Denoting $|J|$ the length of any interval J, we have also

$$|J_{i,q}|e^{-L\Delta t} \le |y(J_{i,q})| \le |J_{i,q}|e^{L\Delta t}, \quad L := \|b'\|_{L^\infty},$$

where $|J_{i,q}| = \Delta x_{i,q}$. Hence, for $r \le k$ and $p \le k$,

$$\begin{aligned}
\Delta x_{i,q}^{2k+3} \sum_{i,q} \|\varphi^{(r)}\|_{L^\infty(J_{i,q})} \|u^{(p)}\|_{L^\infty(y(J_{i,q}))} &\le C\Delta x_{i,q}^{2k+3} \sum_{i,q} \frac{\|\varphi\|_{L^2(J_{i,q})}}{\Delta x_{i,q}^{r+1/2}} \frac{\|u\|_{L^2(y(J_{i,q}))}}{|y(J_{i,q})|^{p+1/2}} \\
&\le C\Delta x_{i,q}^{2} \sum_{i,q} \|\varphi\|_{L^2(J_{i,q})} \|u\|_{L^2(y(J_{i,q}))}.
\end{aligned}$$

Finally, by the Cauchy-Schwartz inequality,

$$\begin{aligned}
\sum_{i,q} \|\varphi\|_{L^2(J_{i,q})} \|u\|_{L^2(y(J_{i,q}))} &\le \Big(\sum_{i,q} \|\varphi\|^2_{L^2(J_{i,q})}\Big)^{1/2} \Big(\sum_{i,q} \|u\|^2_{L^2(y(J_{i,q}))}\Big)^{1/2} \\
&\le \|\varphi\|_{L^2} \|u\|_{L^2}.
\end{aligned}$$

since $\bigcup_{i,q} J_{i,q}$ is a covering of $[0,1]$. Hence we obtain

$$\sum_{i,q} |\varepsilon_{i,q}| \le C\Delta t \Delta x^2 \|\varphi\|_{L^2} \|u\|_{L^2},$$

which concludes the proof of (i).

Proof of Proposition 2.1(ii): Let us write $\psi = P + R$ where $P \in V_k$ is defined as the Taylor expansion of ψ on each $J_{i,q} = (x_{i,q}, x_{i,q+1})$, around $x_{i,q}$. We consider the decomposition

$$u(y_\cdot(-\Delta t)) - \psi(y_\cdot(-\Delta t)) \equiv (u - P)(y_\cdot(-\Delta t)) - R(y_\cdot(-\Delta t)) \tag{30}$$

Then by Proposition 2.1(i), for any $\varphi \in V_k$,

$$|((u-P)(y_\cdot(-\Delta t)), \varphi)_G - ((u-P)(y_\cdot(-\Delta t)), \varphi)| \le C\Delta t\Delta x^2 \|u - P\|_{L^2}\|\varphi\|_{L^2}.$$

Using the fact that $\|R\|_{L^2} \le C\|R\|_{L^\infty} \le CM_{k+1}(\psi)\Delta x^{k+1}$, we obtain the bound

$$\begin{aligned}|((u-P)(y_\cdot(-\Delta t)), \varphi)_G - ((u-P)(y_\cdot(-\Delta t)), \varphi)| \\ \le\ C\Delta t\Delta x\|u-\psi\|_{L^2}\|\varphi\|_{L^2} + CM_{k+1}(\psi)\Delta t\Delta x^{k+3}\|\varphi\|_{L^2}.\end{aligned} \tag{31}$$

There remains to bound the error

$$(R(y_\cdot(-\Delta t)), \varphi)_G - (R(y_\cdot(-\Delta t)), \varphi).$$

This is easily bounded by $C\|R\|_\infty\|\varphi\|_{L^2} = O(\Delta x^{k+1}\|\varphi\|_{L^2})$. Combined with (30) and (31), we obtain the desired bound. □

2.5. **Stability and error analysis.** We now turn on the stability and convergence analysis. The following result shows the *unconditional* stability of the scheme, for any $k \ge 1$.

Proposition 2.2 (Stability)**.** *Let $k \ge 0$ and let b be Lipschitz continuous and* 1-*periodic.*
(i) *For any $u \in L^2$,*

$$\|x \to u(y_x(-t))\|_{L^2} \le e^{\frac{1}{2}L|t|}\ \|u\|_{L^2}, \quad \textit{where } L := \|b'\|_{L^\infty}. \tag{32}$$

(ii) *If furthermore b is of class $\mathcal{C}^{2k+2}$, there exists a constant $C_1 \ge 0$ such that, $\forall u \in V_k$,*

$$\|\widetilde{\mathcal{T}}_{b\Delta t}u\|_{L^2} \le e^{C_1\Delta t}\|u\|_{L^2} \quad \forall u \in V_k.$$

(iii) *In particular for the scheme $u^{n+1} = \widetilde{\mathcal{T}}_{b\Delta t}u^n$,*

$$\|u^n\|_{L^2} \le e^{C_1 t_n}\|u^0\|_{L^2}, \quad \forall n \ge 0,$$

where $t_n = n\Delta t$.

Proof. (i) We make use of the change of variable $x \to z := y_x(-t)$, with periodic boundary conditions for the integrants. Therefore we have $x = y_z(-t)$ and

$$\frac{\partial x}{\partial z}(t) = \exp\left(\int_0^t b'(y_z(s))ds\right) \le e^{L|t|}.$$

We then obtain

$$\int_\Omega |u(y_x(-t))|^2 dx = \int_\Omega |u(z)|^2 \left|\frac{\partial x}{\partial z}(t)\right| dz \le e^{L|t|}\int_\Omega |u(z)|^2 dz.$$

(ii) By using (21), we have

$$\|\widetilde{\mathcal{T}}_{b\Delta t}u\|_{L^2} \quad \le \quad \|u(y_\cdot(-\Delta t))\|_{L^2} + C\Delta t\Delta x^2\|u\|_{L^2}. \tag{33}$$

Together with (32) we get a stability constant

$$e^{\frac{L}{2}\Delta t} + C\Delta t\Delta x^2 \le e^{\frac{L}{2}\Delta t}(1 + C\Delta t\Delta x^2) \le e^{\frac{L}{2}\Delta t}e^{C\Delta t\Delta x^2},$$

hence the desired result for any $C_1 \ge 0$ such that $C_1 \ge \frac{1}{2}L + C\Delta x^2$. □

We now state our convergence result. It generalizes the error estimate of Theorem 2.1 established in the case when b is constant, to the non-constant case.

Theorem 2.2 (Convergence)**.** *Let $k \geq 0$. Assume the initial condition v_0 is 1-periodic and of class $\mathcal{C}^{k+1}$. Let b be 1-periodic and of class $\mathcal{C}^{2k+2}$. There exists constants $C_1 \geq 0$, $C \geq 0$ such that*

$$\|u^n - v^n\|_{L^2} \leq e^{C_1 T}\left(\|v^0 - u^0\|_{L^2} + CT\frac{\Delta x^{k+1}}{\Delta t}\right), \quad \forall n \leq N. \tag{34}$$

Proof of Theorem 2.2. By using the regularity of v^{n+1} and Proposition 2.1(iv) we have

$$\Pi v^{n+1} = \mathcal{T}_{b\Delta t} v^n = \widetilde{\mathcal{T}}_{b\Delta t} v^n + O(\Delta x^{k+2}). \tag{35}$$

Because of the projection error $\|v^{n+1} - \Pi v^{n+1}\| = O(\Delta x^{k+1})$ we obtain the following consistency estimate:

$$v^{n+1} = \widetilde{\mathcal{T}}_{b\Delta t} v^n + O(\Delta x^{k+1}). \tag{36}$$

Therefore

$$u^{n+1} - v^{n+1} \quad = \quad \widetilde{\mathcal{T}}_{b\Delta t}(u^n - v^n) + O(\Delta x^{k+1}). \tag{37}$$

By the stability bound of Proposition 2.2(ii),

$$\|u^{n+1} - v^{n+1}\|_{L^2} \leq e^{C_1 \Delta t}\|u^n - v^n\|_{L^2} + C\Delta x^{k+1}.$$

We conclude by induction. □

2.6. **Two-dimensional advection equation : splitting strategies.** We consider a square box $\Omega = [x_{1,min}, x_{1,max}] \times [x_{2,min}, x_{2,max}]$ and a spatial discretization into cells $I_{i,j} := I_i \times J_j$ where I_i and J_j are as in the one-dimensional case, using M_1 (resp. M_2) points in the x_1 direction (resp x_2 direction). We define the corresponding space of 2d discontinuous Galerkin element by using the Q_k basis ($v \in Q_k$ if $v(x) = \sum_{i,j \leq k} v_{ij} x_1^i x_2^j$ for some real coefficients v_{ij}):

$$V_k^{(2)} := \left\{ v \in L^2(\Omega, \mathbb{R}),\, v|_{I_{i,j}} \in Q_k,\ \forall (i,j) \right\} \tag{38}$$

In principle, one could consider $\mathcal{T}_{b\Delta t}$, the SLDG advection scheme for solving $u_t + b(x) \cdot \nabla u = 0$ during time step Δt, defined by

$$\int_\Omega (\mathcal{T}_{b\Delta t} u)(x)\varphi(x)\, dx = \int_\Omega u(y_x(-\Delta t))\varphi(x)\ dx, \quad \forall \varphi \in V_k^{(2)}$$

where $\dot{y} = b(y)$ and $y(0) = x$. Equivalently, $\mathcal{T}_{b\Delta t} u := \Pi\big(u(y_{.}(-\Delta t))\big)$. However, $\mathcal{T}$ has then no explicit analytical form in general, some approximation procedure must be considered (see for instance Restelli et al [19]), and the stability and convergence analysis can be more difficult.

In the case of constant coefficients, such as

$$u_t + b_1 u_{x_1} + b_2 u_{x_2} = 0,$$

with b_1, b_2 constant, we know that the Trotter splitting is exact. Let us denote $\mathcal{T}^k_{b_k \Delta t}$ the SLDG advection scheme in direction x_k, that is, for solving during time step Δt the equation

$$u_t + b_k u_{x_k} = 0,$$

then we have

$$\mathcal{T}_{b\Delta t} = \mathcal{T}^2_{b_2 \Delta t}\mathcal{T}^1_{b_1 \Delta t}.$$

We can thus use explicit one-dimensional advection SLDG schemes to compute iteratively $u^{n+1} = \mathcal{T}_{b\Delta t} u^n$.

In the case when $b(x)$ is non-constant, we consider splitting strategies. We recall the following approximations of the exponential $e^{(A+B)h}$ for A and B matrices and for small h:

$$e^{(A+B)h} = e^{Bh}e^{Ah} + O(h^2) \quad \text{(Trotter spitting)}, \tag{39}$$

$$e^{(A+B)h} = e^{B\frac{h}{2}}e^{Ah}e^{B\frac{h}{2}} + O(h^3) \quad \text{(Strang's spitting)}, \tag{40}$$

as well as

$$e^{(A+B)h} = \frac{2}{3}(e^{A\frac{h}{2}}e^{Bh}e^{A\frac{h}{2}} + e^{B\frac{h}{2}}e^{Ah}e^{B\frac{h}{2}}) - \frac{1}{6}(e^{Bh}e^{Ah} + e^{Ah}e^{Bh}) + O(h^4), \tag{41}$$

$$e^{(A+B)h} = \frac{4}{3}e^{A\frac{h}{4}}e^{B\frac{h}{2}}e^{A\frac{h}{2}}e^{B\frac{h}{2}}e^{A\frac{h}{4}} - \frac{1}{3}e^{A\frac{h}{2}}e^{Bh}e^{A\frac{h}{2}} + O(h^5) \tag{42}$$

(see for instance [2]). We can then propose the following splitting schemes for the approximation of $\mathcal{T}^b_{\Delta t}$ when $b = b(x)$ is non-constant, based on the one-dimensional scheme in each direction:

$$\mathcal{T}_{b\Delta t} \simeq \mathcal{T}^2_{b_2\Delta t}\mathcal{T}^1_{b_1\Delta t} \quad \text{(Trotter)} \tag{43}$$

$$\mathcal{T}_{b\Delta t} \simeq \mathcal{T}^2_{b_2\frac{\Delta t}{2}}\mathcal{T}^1_{b_1\Delta t}\mathcal{T}^2_{b_2\frac{\Delta t}{2}} \quad \text{(Strang)} \tag{44}$$

of expected consistency error $O(\Delta t)$ and $O(\Delta t^2)$ respectively. These last two splitting schemes are similar to the ones used in [18]. We can also go further:

3rd order splitting

$$\begin{aligned}\mathcal{T}_{b\Delta t} \quad \simeq \quad & \frac{2}{3}(\mathcal{T}^1_{b_1\frac{\Delta t}{2}}\mathcal{T}^2_{b_2\Delta t}\mathcal{T}^1_{b_1\frac{\Delta t}{2}} + \mathcal{T}^2_{b_2\frac{\Delta t}{2}}\mathcal{T}^1_{b_1\Delta t}\mathcal{T}^2_{b_2\frac{\Delta t}{2}}) \\ & -\frac{1}{6}(\mathcal{T}^2_{b_2\Delta t}\mathcal{T}^1_{b_1\Delta t} + \mathcal{T}^1_{b_1\Delta t}\mathcal{T}^2_{b_2\Delta t}),\end{aligned} \tag{45}$$

or

4th order splitting

$$\mathcal{T}_{b\Delta t} \quad \simeq \quad \frac{4}{3}\mathcal{T}^1_{b_1\frac{\Delta t}{4}}\mathcal{T}^2_{b_2\frac{\Delta t}{2}}\mathcal{T}^1_{b_1\frac{\Delta t}{2}}\mathcal{T}^2_{b_2\frac{\Delta t}{2}}\mathcal{T}^1_{b_1\frac{\Delta t}{4}} - \frac{1}{3}\mathcal{T}^1_{b_1\frac{\Delta t}{2}}\mathcal{T}^2_{b_2\Delta t}\mathcal{T}^1_{b_1\frac{\Delta t}{2}}, \tag{46}$$

of expected consistency error $O(\Delta t^3)$ and $O(\Delta t^4)$, respectively. We refer to the works of Descombes et al. [10, 11] for such splitting techniques and analysis in a different context.

Stability in the L^2-norm (and expected order of convergence) are easily obtained for Trotter and Strang's splittings, by using the L^2-stability of the one-directional advection operators $\mathcal{T}^k_{b_k\Delta t}$. We do not know about the stability for the two other splitting (45) and (46), which are no more a convex combination of stable schemes, but we will show numerical evidence of convergence (see Section 4, Example 4).

3. Second order PDEs

This section deals with new SLDG schemes for second order PDEs.

3.1. 1d-diffusion equations with constant diffusion.

We consider a diffusion equation with a constant coefficient $\sigma \in \mathbb{R}$:

$$v_t - \frac{\sigma^2}{2}v_{xx} = 0, \quad x \in \Omega,\ t \in (0, T), \tag{47}$$

$$v(0, x) = v_0(x), \quad x \in \Omega. \tag{48}$$

A first scheme, in semi-discrete form, is

$$u^{n+1}(x) = \frac{1}{2}\left(u^n(x - \sigma\sqrt{\Delta t}) + u^n(x + \sigma\sqrt{\Delta t})\right) \equiv S^0_{\Delta t}u^n\,(x). \tag{49}$$

(see Menaldi [16], Camilli et al. [5] for such schemes). It is easy to see that, taking $v^n(x) := v(t_n, x)$ where v is solution of (47) and is assumed sufficiently regular, the following consistency error estimate holds:

$$\|\frac{v^{n+1} - S^0_{\Delta t} v^n}{\Delta t}\|_{L^2} = O(\Delta t).$$

The basic SLDG scheme (also called hereafter SLDG-RK1) is to consider the weak formulation of the relation (49), leading to define recursively u^{n+1} in V_k such that

SLDG-RK1 scheme:

$$\int u^{n+1}(x)\varphi(x)dx = \int \frac{1}{2}\bigg(u^n(x - \sigma\sqrt{\Delta t}) + u^n(x + \sigma\sqrt{\Delta t})\bigg)\varphi(x)\,dx, \quad \forall \varphi \in V_k.$$

(The initialization of u^0 is done as before). The scheme will be also written in abstract form as follows:

$$u^{n+1} = \mathcal{S}_{\Delta t}(u^n),$$

where

$$\mathcal{S}_{\Delta t} := \Pi \mathcal{S}^0_{\Delta t} \equiv \frac{1}{2}\bigg(\mathcal{T}_{-\sigma\sqrt{\Delta t}} + \mathcal{T}_{\sigma\sqrt{\Delta t}}\bigg).$$

We will show, for this scheme, an L^2-error estimate of order

$$O(\Delta t) + O(\frac{\Delta x^{k+1}}{\Delta t})$$

(see Theorem 3.1), therefore being high order in space but only first order in time.

Our aim is now to improve the accuracy with respect to the time discretization.

Let us denote $h = \Delta t$. Using Taylor expansions, for u sufficiently regular, we have, for h small,

$$S^0_h u = u + h\frac{\sigma^2}{2}u_{xx} + h^2\frac{\sigma^4}{24}u^{(4)}_x + O(h^3), \tag{50}$$

$$S^0_h S^0_h u = u + h\sigma^2 u_{xx} + h^2\frac{\sigma^4}{3}u^{(4)}_x + O(h^3), \tag{51}$$

where $u^{(q)}_x$ denotes the q-th derivative of u w.r.t. x.

On the other hand, if $v^n = v(t_n, x)$ where v is the exact solution of $v_t = \frac{\sigma^2}{2}v_{xx}$, and $h \equiv \Delta t$, we have

$$v^{n+1} = v^n + hv_t + \frac{h^2}{2}v_{tt} + O(h^3) \tag{52}$$

$$= v^n + h\frac{\sigma^2}{2}v^n_{xx} + h^2\frac{\sigma^4}{8}v^{n,(4)} + O(h^3) \tag{53}$$

Now, looking for coefficients a, b, c such that $av^n + bS^0_h v^n + cS^0_h S^0_h v^n$ be equal to v^{n+1} up to $O(h^3)$, using (50) and (51), we obtain the system

$$\begin{cases} a + b + c = 1 \\ \frac{b}{2} + c = \frac{1}{2} \\ \frac{b}{24} + \frac{c}{3} = \frac{1}{8} \end{cases} \tag{54}$$

and we find that $a = b = c = \frac{1}{3}$. Therefore, a second order scheme is now given by

SLDG-RK2 scheme:

$$u^{n+1} = S^{RK2}_{\Delta t}u^n := \frac{1}{3}(u^n + S_{\Delta t}u^n + S_{\Delta t}S_{\Delta t}u^n). \tag{55}$$

We will refer to this scheme as "SLDG-RK2", for a second order PDE.

Remark 3.1. *A variant of this scheme can be*

$$u^{n+1} = \Pi\frac{1}{3}\Big(u^n + S^0_{\Delta t}u^n + S^0_{\Delta t}S^0_{\Delta t}u^n\Big). \tag{56}$$

This is in general slightly different from (55) *because* $S_{\Delta t}S_{\Delta t} = \Pi S^0_{\Delta t}\Pi S^0_{\Delta t}$ *may differ from* $\Pi S^0_{\Delta t}S^0_{\Delta t}$. *Nevertheless, the difference between the two will be of the order of the projection error* $O(\Delta x^{k+1})$ *when applied to a regular data.*

In order to obtain a third order scheme, we can proceed in a similar way. First, we obtain the following expansions:

$$S^0_h u = u + h\frac{\sigma^2}{2}u_{xx} + h^2\frac{\sigma^4}{24}u_x^{(4)} + h^3\frac{\sigma^6}{6!}u_x^{(6)} + O(h^4), \tag{57}$$

$$S^0_h S^0_h u = u + h\sigma^2 u_{xx} + h^2\frac{\sigma^4}{3}u_x^{(4)} + \frac{2}{45}h^3\sigma^6 u_x^{(6)} + O(h^4), \tag{58}$$

$$S^0_h S^0_h S^0_h u = u + \frac{3}{2}h\sigma^2 u_{xx} + \frac{7}{8}h^2\sigma^4 u_x^{(4)} + \frac{61}{240}h^3\sigma^6 u_x^{(6)} + O(h^4). \tag{59}$$

Looking for coefficients a, b, c, d such that $av^n + S^0_h v^n + S^0_h S^0_h v^n + S^0_h S^0_h S^0_h v^n$ be equal to v^{n+1} up to $O(h^4)$, we find the system

$$\left\{\begin{array}{ccccccc} a & + & b & + & c & + & d = 1 \\ & & \frac{b}{2} & + & c & + & \frac{3}{2}d = \frac{1}{2} \\ & & \frac{b}{24} & + & \frac{c}{3} & + & \frac{7}{8}d = \frac{1}{8} \\ & & \frac{b}{6!} & + & \frac{2}{45}c & + & \frac{61}{240}d = \frac{1}{48} \end{array}\right. \tag{60}$$

and its solution

$$(a, b, c, d) := \frac{1}{45}(13, 21, 9, 2).$$

Thus, the following scheme is of 3rd order in time:

<u>SLDG-RK3 scheme:</u>

$$u^{n+1} = S^{RK3}_{\Delta t}u^n := \frac{13}{45}u^n + \frac{7}{15}S_{\Delta t}u^n + \frac{1}{5}S_{\Delta t}S_{\Delta t}u^n + \frac{2}{45}S_{\Delta t}S_{\Delta t}S_{\Delta t}u^n.$$

We will refer to this scheme as "SLDG-RK3", for a second order PDE.

As in Remark 3.1, a variant of the scheme can be

$$u^{n+1} = \Pi\Big(\frac{13}{45}u^n + \frac{7}{15}S^0_{\Delta t}u^n + \frac{1}{5}S^0_{\Delta t}S^0_{\Delta t}u^n + \frac{2}{45}S^0_{\Delta t}S^0_{\Delta t}S^0_{\Delta t}u^n\Big). \tag{61}$$

Since we are using a convex combination of a stable schemes ($S_{\Delta t}$, $S_{\Delta t}S_{\Delta t}$ or $S_{\Delta t}S_{\Delta t}S_{\Delta t}$), the schemes SLDG-RK2 and SLDG-RK3 are all stable in the L^2 norm.

Remark 3.2. *Up to 5th order schemes - in time - can also be obtained (see* [3]*), using convex combinations of the form* $u^{n+1} = \sum_{i=0}^p a_i(S^0_{\Delta t})^i u^n$.

We now state our convergence result.

Theorem 3.1. *Let* $k \geq 0$ *and let* σ *be a constant, and assume that the exact solution* v *has bounded derivative* $\frac{\partial^q v}{\partial x^q}$ *for* $q = \max(k+2, 2p+2)$. *We consider the SLDG-RKp schemes for* $p = 1, 2$ *or* 3 *(or the variants* (56) *or* (61) *for* $p = 2, 3$*). Then*

$$\|v^n - u^n\|_{L^2} \leq \|v^0 - u^0\|_{L^2} + CT(\frac{\Delta x^{k+1}}{\Delta t} + \Delta t^p), \quad \forall n \leq N. \tag{62}$$

Proof. We will consider the proof in the case of the SLDG-RK2 scheme, with $p = 2$ (the other cases when $p = 1$ or $p = 3$ being similar). By using the regularity of the exact solution ($\frac{\partial^3 v}{\partial t^3}$ and $v_x^{n,(6)}$ bounded), we have the following consistency estimate:

$$v^{n+1} = a_0 v^n + a_1 S^0_{\Delta t}v^n + a_2 S^0_{\Delta t}S^0_{\Delta t}v^n + O(\Delta t^3), \tag{63}$$

where $a_0 = a_1 = a_2 = \frac{1}{3}$, and the bound $O()$ is in the norm $\|.\|_{L^2}$. Since $\Pi S^0_{\Delta t}\psi = \Pi S^0_{\Delta t}\Pi\psi + O(\Delta x^{k+1})$ for regular data ψ, we have also $S^2_{\Delta t}v^n = \Pi(S^0_{\Delta t})^2 v^n + O(\Delta x^{k+1})$, and thus

$$v^{n+1} = a_0 v^n + a_1 S_{\Delta t} v^n + a_2 S_{\Delta t} S_{\Delta t} v^n + O(\Delta t^3) + O(\Delta x^{k+1}). \quad (64)$$

By the scheme definition we have

$$u^{n+1} = \sum_{i=0}^{2} a_i (S_{\Delta t})^i u^n. \quad (65)$$

We deduce, using the consistency estimate (63),

$$\begin{aligned} \|u^{n+1} - v^{n+1}\|_{L^2} &\leq \|\sum_{i\leq 2} a_i (S_{\Delta t})^i (u^n - v^n)\|_{L^2} + C\Delta t^3 + C\Delta x^{k+1} \\ &\leq \sum_{i\leq 2} a_i \|(S_{\Delta t})^i (u^n - v^n)\|_{L^2} + C\Delta t^3 + C\Delta x^{k+1} \\ &\leq \|u^n - v^n\|_{L^2} + C\Delta t^3 + C\Delta x^{k+1}, \end{aligned}$$

(since $a_i \geq 0$ and $\sum_i a_i = 1$). The result follows by induction. □

Remark 3.3. *The chosen denomination "RK1", "RK2" and "RK3" is because of the relationship with Runge-Kutta schemes for first order equations. It is indeed related to Strong Stability Preserving (SSP) schemes* [14, 22, 21]. *For instance, if we consider the Euler scheme for the ODE $\dot{u} = L(u)$ during time step Δt written as $u^{n+1} = S(u^n)$, then it is known that the Heun scheme can also be written $u^{n+1} = \frac{1}{2}(u^n + SSu^n)$ and is of order $O(\Delta t^2)$, it corresponds to a so-called "SSP-RK2" approximation. This can be compared to the present SLDG-RK2 scheme. A known "SSP-RK3" approximation for $\dot{u} = L(u)$ is given by $u^{n+1} = \frac{1}{6}(2u^n + 3Su^n + SSSu^n)$, and can be compared to the present SLDG-RK3 scheme.*

3.2. **$1d$-diffusion equations with non-constant diffusion term $\sigma(x)$.** We now consider the case of

$$v_t + \frac{1}{2}\sigma^2(x) v_{xx} = 0, \quad t > 0, \quad x \in \Omega. \quad (66)$$

On the first hand, if $v^n = v(t_n, x)$ where v is the exact solution of (66), and for $h \equiv \Delta t$, we have

$$\begin{aligned} v^{n+1} &= v^n + h v_t + \frac{h^2}{2} v_{tt} + O(h^3) \\ &= v^n + h\frac{\sigma^2}{2} v_{xx} + h^2 \frac{\sigma^2}{8}\left(\sigma^2 v^n_{xx}\right)_{xx} + O(h^3) \\ &= v^n + h\frac{\sigma^2}{2} v_{xx} + h^2\frac{\sigma^4}{8}(v^n)^{(4)}_x + h^2\left\{\frac{\sigma^2(\sigma^2)'}{4}(v^n)^{(3)}_x + \frac{\sigma^2(\sigma^2)''}{8} v^n_{xx}\right\} \\ &\qquad + O(h^3). \end{aligned} \quad (67)$$

Now, we observe that

$$\begin{aligned} \frac{1}{3}(u + S^0_h u + S^0_h S^0_h u) &= \quad (68) \\ u + h\frac{\sigma^2}{2} u_{xx} &+ h^2 \frac{\sigma^4}{8} u^{(4)}_x + h^2\left\{\frac{\sigma^2(\sigma^2)'}{6} u^{(3)}_x + \frac{\sigma^2(\sigma^2)''}{12} u_{xx}\right\} + O(h^3). \end{aligned}$$

We have new terms depending of $h^2 u^{(3)}_x$ and $h^2 u^{(4)}_x$ and that will differ between (67) and (68).

In order to take these new terms into account, we introduce the following approximations. For a given $h > 0$, let D_h^0 and D_h be defined by

$$D_h^0 u(x) := \frac{1}{2}(u(x+h) - u(x-h)) \quad \text{and} \quad D_h := \Pi D_h^0.$$

We have for a regular u:

$$D_h^0 u = h u_x + \frac{1}{6} h^3 u_x^{(3)} + O(h^5 \|u_x^{(5)}\|_\infty),$$

from which it can be obtained that

$$D_h^0 D_h^0 D_h^0 u = h^3 u_x^{(3)} + \frac{1}{2} h^5 u_x^{(5)} + O(h^7 \|u_x^{(7)}\|_\infty),$$

and

$$D_h^0 D_h^0 D_h^0 D_h^0 D_h^0 u = h^5 u_x^{(5)} + O(h^7 \|u_x^{(7)}\|_\infty).$$

Therefore,

$$\begin{aligned} h(D^0_{h^{1/3}} D^0_{h^{1/3}} D^0_{h^{1/3}} u - \frac{1}{2} D^0_{h^{1/3}} D^0_{h^{1/3}} D^0_{h^{1/3}} D^0_{h^{1/3}} D^0_{h^{1/3}} u) \\ = h^2 u_x^{(3)} + O(h^{10/3} \|u_x^{(7)}\|), \end{aligned} \tag{69}$$

and, in the same way,

$$h D^0_{h^{1/2}} D^0_{h^{1/2}} u = h^2 u_{xx} + O(h^3 \|u_x^{(4)}\|_\infty) \tag{70}$$

((70) corresponds also to the usual second order finite difference approximation). Also, if u is regular and that we replace the operator D_h^0 by D_h, they the above approximations (69) or (70) are still valid up to a complementary error term in $O(\Delta x^{k+1})$ coming from the projection on V_k. Hence we propose the following.

SLDG-RK2 "modified" scheme:

$$\begin{aligned} u^{n+1} &= \frac{1}{3}(u^n + S_{\Delta t} u^n + S_{\Delta t} S_{\Delta t} u^n) \\ &+ \Delta t \frac{\sigma^2 (\sigma^2)'}{12} \big(D_{\Delta t^{1/3}} D_{\Delta t^{1/3}} D_{\Delta t^{1/3}} u^n - \frac{1}{2} D_{\Delta t^{1/3}} D_{\Delta t^{1/3}} D_{\Delta t^{1/3}} D_{\Delta t^{1/3}} D_{\Delta t^{1/3}} u^n \big) \\ &+ \Delta t \frac{\sigma^2 (\sigma^2)''}{24} D_{\Delta t^{1/2}} D_{\Delta t^{1/2}} u^n. \end{aligned} \tag{71}$$

Indeed, the missing part for the term $\frac{1}{6}\sigma^2(\sigma^2)' u_x^{(3)}$, with factor $\frac{1}{6}$, from the desired factor $\frac{1}{4}$ in (67), is $\frac{1}{4} - \frac{1}{6} = \frac{1}{12}$. Also, the missing part for the term $\frac{1}{12}\sigma^2(\sigma^2)'' u_{xx}$, with factor $\frac{1}{12}$, from the desired factor $\frac{1}{8}$ in (67) is $\frac{1}{24}$. The expected error is of order $O(\Delta t^3) + O(\Delta x^{k+1})$ at each time step, hence the expected global error is of order $O(\Delta t^2) + O(\frac{\Delta x^{k+1}}{\Delta t})$.

3.3. **Implementation details.** Now the following SLDG-RK1 scheme

$$u^{n+1} \equiv S_{\Delta t} u^n := \frac{1}{2} \Pi \bigg(u^n(\cdot - \sigma(\cdot)\sqrt{\Delta t}) + u^n(\cdot + \sigma(\cdot)\sqrt{\Delta t}) \bigg) \tag{72}$$

is no more exactly implementable because $\sigma(x)$ is not constant. We consider for the scheme definition the use of the Gaussian quadrature rule on each interval of regularity of the data.

Let us introduce the following notation

$$y^{\pm}(x) := x \pm \sigma(x)\sqrt{\Delta t}. \tag{73}$$

Remark 3.4. *Notice that if $\sqrt{\Delta t}\|\sigma'\|_{L^\infty} < 1$ and with σ 1-periodic, then the function $x \to y^{\pm}(x)$ is a one-to-one and onto function (it also satisfies $y^{\pm}(x+q) = y^{\pm}(x) + q$ $\forall q \in \mathbb{Z}$, $\forall x$). Furthermore, its inverse can be easily and rapidly computed by using a fixed point / Newton's algorithm for instance (details are left to the reader).*

Now for each given $\eta \in \{\pm\}$, we consider a partition of I_i into intervals $J^\eta_{i,q}$ such that all $y^\eta(J^\eta_{i,q})$ be subintervals of some I_j. We then define Gauss points $\tilde{x}^{i,\eta}_{q,\alpha}$ and the bilinear product $(a,b)_{G^\eta}$ in a similar way as in (20), that is, using the Gaussian quadrature rule on each $J^\eta_{i,q}$. Hence we define $\widetilde{S}_{\Delta t} u^n$ in V_k such that

SLDG-RK1 scheme (implementable version):

$$(\widetilde{S}_{\Delta t} u^n, \varphi) = \frac{1}{2} \sum_{\eta=\pm} (u^n(y^\eta), \varphi)_{G^\eta}, \quad \forall \varphi \in V_k. \tag{74}$$

Formula (74) involves two different quadrature rules, because the discontinuity points of $u^n(y^+(x))$ and $u^n(y^-(x))$ are not the same. It differs from the definition of $S_{\Delta t} u$, which satisfies

$$(S_{\Delta t} u, \varphi) = \frac{1}{2} \sum_{\eta=\pm} (u(y^\eta), \varphi), \quad \forall \varphi \in V_k. \tag{75}$$

In order to implement the SLDG-RK2 modified scheme, the operator $\widetilde{S}_{\Delta t}$ is used instead of $S_{\Delta t}$ in (71). New terms appear also in (71) and that are computed naturally in the following way. We first compute the DG polynomials for $D_{\Delta t^{1/3}} D_{\Delta t^{1/3}} D_{\Delta t^{1/3}} u^n$ (resp. $D_{\Delta t^{1/2}} D_{\Delta t^{1/2}} u^n$, etc.). The operators $D_{\Delta t^{1/3}}$ or $D_{\Delta t^{1/2}}$ involve only exact quadrature rules since they use constant advection parameters. Then we pointwise multiply the polynomials at each Gauss quadrature points x^i_α of the intervals I_i, by the corresponding value of $\Delta t\, \sigma^2 (\sigma^2)'$ (resp. $\Delta t\, \sigma^2 (\sigma^2)''$).

These implementation steps create errors with respect to the theoretical schemes, (71) or (72), and that need to be controlled.

3.4. **Stability and convergence, case $\sigma(x)$ non-constant.** We first state some useful estimates for the operator $\widetilde{S}_{\Delta t}$. The proof is similar to the one of Proposition 2.1.

Proposition 3.1. *Let $k \ge 0$ and let σ be of class $\mathcal{C}^{2k+2}$ and 1-periodic.*
(i) There exists a constant $C \ge 0$, for all $u \in V_k$,

$$\left| \frac{1}{2} \sum_{\eta=\pm} (u(y^\eta), \varphi)_{G^\eta} - \frac{1}{2} \sum_{\eta=\pm} (u(y^\eta), \varphi) \right| \tag{76}$$
$$\le C\sqrt{\Delta t}\Delta x^2 \|u\|_{L^2} \|\varphi\|_{L^2}. \quad \forall \varphi \in V_k$$

In particular, for any $u \in V_k$,

$$\widetilde{S}_{\Delta t} u = S_{\Delta t} u + O\big(\sqrt{\Delta t}\Delta x^2 \|u\|_{L^2}\big). \tag{77}$$

(ii) For all $u \in V_k$, for any ψ in $\mathcal{C}^{k+1}$, 1-periodic,

$$\widetilde{S}_{\Delta t}(u-\psi) = S_{\Delta t}(u-\psi) + O(\sqrt{\Delta t}\Delta x^2 \|u-\psi\|_{L^2}) + O(M_{k+1}(\psi)\Delta x^{k+1}), \tag{78}$$

where $C \ge 0$ is a constant.
(iii) For any regular $\psi \in \mathcal{C}^{k+1}$, 1-periodic, we have in the L^2 norm

$$\widetilde{S}_{\Delta t} \psi = S_{\Delta t} \psi + O(M_{k+1}(\psi)\Delta x^{k+1}) \tag{79}$$

We now study the stability of the operator $\tilde{S}_{\Delta t}$.

Proposition 3.2. *Assume $\sqrt{\Delta t}\|\sigma'\|_{L^\infty} \le \eta$ for some $\eta < 1$, and let $k \ge 0$.*
(i) If we use exact integration, as in (72), then for all $u \in L^2$,

$$\|S_{\Delta t} u\|_{L^2} \le (1 + C\Delta t)\|u\|_{L^2},$$

where $C \ge 0$ is a constant.
(ii) If we use Gaussian quadrature rule, as in (74), then, for any $u \in V_k$,

$$\|\widetilde{S}_{\Delta t} u\|_{L^2} \le (1 + C\Delta t + C\sqrt{\Delta t}\Delta x^2)\|u\|_{L^2}.$$

In particular the scheme SLDG-RK1 is L^2 stable if $\Delta x^4 \le \lambda \Delta t$ for some $\lambda > 0$.

Proof. (i) By making use of the change of variable formula for $x \to y^{\pm}(x)$, we have

$$\begin{aligned}
\|S_{\Delta t} u\|_{L^2}^2 &= \|\frac{1}{2}\sum_{\varepsilon=\pm 1} \mathcal{T}_{\varepsilon\sigma\sqrt{\Delta t}} u\|_{L^2}^2 \\
&\le \frac{1}{2}\sum_{\varepsilon=\pm 1} \int |u(x+\varepsilon\sigma(x)\sqrt{\Delta t})|^2 dx \\
&\le \int \Big(\frac{1}{2}\sum_{\varepsilon=\pm 1} \frac{1}{1+\varepsilon\sigma'(x^{\varepsilon}(z))\sqrt{\Delta t}}\Big)|u(z)|^2 dz \\
&= \int \frac{1}{1-|\sigma'(x^{\varepsilon}(z))|^2 \Delta t}|u(z)|^2 dz \\
&\le \int (1+C_\eta \Delta t)|u(z)|^2 dz
\end{aligned}$$

for some constant $C_\eta \ge 0$, where $z \to x^{\pm}(z)$ denotes the inverse function of $y^{\pm}$. Thus, the desired result.

(ii) This is a consequence of (i) and of the bound (77) of Proposition 3.1. □

Now we can state our convergence result for the approximation of (66).

Theorem 3.2. *Let $k \ge 0$ and let σ be a 1-periodic function, of class $\mathcal{C}^{2k+2}$. We consider the schemes SLDG-RKp for $p = 1, 2$ (implementable version). We assume that the exact solution v has a bounded derivative $\frac{\partial^q v}{\partial x^q}$, with $q \ge \max(2p+2, k+1)$.*
(i) The schemes are L^2 stable under the "weak" CFL condition

$$\Delta x^4 \le \lambda \Delta t, \quad \textit{for some } \lambda > 0, \tag{80}$$

(ii) If the weak CFL condition is satisfied, then

$$\|u^n - v^n\|_{L^2} \le e^{L_1 T}\Big(\|u^0 - v^0\|_2 + CT(\frac{\Delta x^{k+1}}{\Delta t} + \Delta t^p)\Big), \quad \forall n \le N, \tag{81}$$

for some constant $L_1 \ge 0$.

Although if it is not always the optimal choice, we will typically use $\Delta t = \Delta x$, and $p = k$ in the numerical examples, so the order $O(\Delta x^p)$ is expected for the schemes SLDG-RKp, $p = 1, 2$.

Proof of Theorem 3.2. (i) The stability of SLDG-RK1 has already been obtained by Proposition 3.2(ii). For SLDG-RK2, using the stability bound $\|\widetilde{S}_{\Delta t}\|_{L^2} \le e^{C\Delta t}\|u\|_{L^2}$, we first obtain

$$\|\sum_i a_i (\widetilde{S}_{\Delta t})^i u^n\|_{L^2} \le \sum_{i\le 2} a_i e^{iC\Delta t}\|u^n\|_{L^2} \le e^{2C\Delta t}\|u^n\|_{L^2}.$$

On the other hand, we have supplementary terms involving operators D_h, times a regular function, times a factor Δt. We first remark that $\|D_h u\|_{L^2} \le \|u\|_{L^2}$. Then, let us define, for a given (continuous) function a, the operator $a \bullet u$ in V_k by

$$(a \bullet u)(x_\alpha^i) := (au)(x_\alpha^i), \quad \forall i, \alpha. \tag{82}$$

One can establish using similar estimates as in the proof of Proposition 2.1, if a is in C^{2k+2} and is a 1-periodic function:

$$a \bullet u = au + O(M_{2k+2}(a)\, \Delta x^2 \|u\|_{L^2}), \quad \forall u \in V_k, \tag{83}$$

and, if furthermore $\psi \in C^{k+1}$, 1-periodic,

$$a \bullet \psi = a\psi + O(\Delta x^{k+1} M_{k+1}(a\psi)). \tag{84}$$

For the SLDG-RK2 scheme, denoted $u^{n+1} = S^{RK2}u^n$, by using the estimate (83), all supplementary corrections are controlled by (at most) $C\Delta t\|u^n\|_{L^2}$ for some constant $C \geq 0$. In the end we obtain a bound of the form $\|u^{n+1}\|_{L^2} \leq (e^{L\Delta t} + C\Delta t)\|u^n\|_{L^2}$, which is enough to conclude to the stability.

(ii) We first consider the SLDG-RK1 scheme $u^{n+1} = \widetilde{S}_{\Delta t}u^n$. By making use of the consistency error estimate, it holds

$$v^{n+1} = \Pi S^0_{\Delta t}v^n + O(\Delta t^2) + O(\Delta x^{k+1}) = S_{\Delta t}v^n + O(\Delta t^2) + O(\Delta x^{k+1}). \tag{85}$$

Furthermore, by proposition 3.1(iii),

$$\|\widetilde{S}_{\Delta t}v^n - S_{\Delta t}v^n\|_{L^2} \leq CM_{k+1}(v^n)\Delta x^{k+1}. \tag{86}$$

Hence

$$v^{n+1} = \widetilde{S}_{\Delta t}v^n + O(\Delta t^2) + O(\Delta x^{k+1}), \tag{87}$$

and taking the difference with the scheme $u^{n+1} = \widetilde{S}_{\Delta t}u^n$, and using (i),

$$\begin{aligned} \|u^{n+1} - v^{n+1}\| &= \|\widetilde{S}_{\Delta t}u^n - \widetilde{S}_{\Delta t}v^n\|_{L^2} + C(\Delta t^2 + \Delta x^{k+1}) && (88)\\ &\leq e^{C\Delta t}\|u^n - v^n\|_{L^2} + C(\Delta t^2 + \Delta x^{k+1}), && (89)\end{aligned}$$

for some constant $C \geq 0$, where we have made use of the stability estimate for $\widetilde{S}_{\Delta t}$. Therefore we obtain the desired error bound.

For the SLDG-RK2 scheme, the estimates are similar, now using furthermore (84) in order to get the consistency estimate $v^{n+1} = S^{RK2}v^n + O(\Delta t^3) + O(\Delta x^{k+1})$, and then using the stability estimate. □

3.5. ***d*-dimensional diffusion equations.** In order to treat a PDE in several space variables, a scheme can be constructed by using a splitting strategy of one-dimensional type schemes. We consider the case of

$$u_t - \frac{1}{2}Tr(\sigma\sigma^T D^2 u) = 0, \quad x \in \Omega,\ t \in (0,T), \tag{90a}$$

$$u(0,x) = u_0(x), \quad x \in \Omega \tag{90b}$$

where $\sigma(x) \in \mathbb{R}^{d\times d}$, and $Tr(A)$ denotes the trace of the matrix A. Indeed, in what follows we could also treat equivalently any term of the form $Tr(AD^2u)$ where A is a given constant positive semi-definite matrix of $\mathbb{R}^{d\times d}$, as long as we know how to decompose A into the form $A = \sigma\sigma^T$ with some $\sigma \in \mathbb{R}^{d\times p}$ for some $p \geq 1$.

We introduce the following decomposition, as in Menaldi [16] (see also Debrabant and Jakobsen [9]) :

$$\sigma\sigma^T = \sum_{k=1}^d \Sigma_k\Sigma_k^T$$

where $\Sigma_k = (\sigma_{1,k}, \ldots, \sigma_{d,k})^T$ is the kth column vector of the matrix $\sigma = (\sigma_{ij})$. Thus (90a) is equivalent to

$$u_t - \frac{1}{2}\sum_{k=1,\ldots,d} Tr(\Sigma_k\Sigma_k^T D^2u) = 0. \tag{91}$$

Each term $Tr(\Sigma^k\Sigma_k^T D^2u)$ corresponds to a diffusion in the direction Σ_k. Therefore, a natural scheme is to combine a splitting approach with a semi-Lagrangian scheme in each direction Σ_k as in the one-dimensional case.

For the one-directional problem

$$u_t - \frac{1}{2}Tr(\Sigma_k\Sigma_k^T D^2u) = 0 \tag{92}$$

we consider the scheme

$$u^{n+1}(x) = \Pi\frac{1}{2}\Big(u^n(x - \Sigma_k(x)\sqrt{\Delta t}) + u^n(x + \Sigma_k(x)\sqrt{\Delta t})\Big) =: (S^{\Sigma_k}_{\Delta t} u^n)(x),$$

and for the general problem (90), we can consider Trotter's splitting:

$$u^{n+1}(x) = \Big(S^{\Sigma_d}_{\Delta t}\ \cdots\ S^{\Sigma_2}_{\Delta t}\ S^{\Sigma_1}_{\Delta t} u^n\Big)(x).$$

Other approximations are possible, as laid down in [9].

Case of a constant diffusion matrix σ: In that case, $\Sigma_k(x) \equiv \Sigma_k$ and equation (91) can be splitted exactly into equations (92) for $k = 1, \dots, d$. The expected global error is of order $O(\Delta t^p)$, depending on which SLDG-RKp scheme we use, plus the spatial error of order $O(\frac{\Delta x^{k+1}}{\Delta t})$.

On the other hand, for a given constant vector $\Sigma \in \mathbb{R}^d$, the evaluation of $\Pi(u(\cdot + \Sigma\sqrt{\Delta t})) \equiv \mathcal{T}^{\Sigma}_{\sqrt{\Delta t}} u$ can then be done exactly by using by an exact splitting as in Section 2.6, and using exact quadrature rule for each dimension. An illustration is given in the Numerical section (Example 7).

Case of a non-constant diffusion matrix $\sigma(x)$: In the more general case, there is an error $O(\Delta t)$ coming from the splitting. Assuming that $d = 2$, instead of $u^{n+1} = S^{\Sigma_2}_{\Delta t}\ S^{\Sigma_1}_{\Delta t} u^n$ we can consider Strang's splitting

$$u^{n+1} = S^{\Sigma_2}_{\frac{1}{2}\Delta t} S^{\Sigma_1}_{\Delta t} S^{\Sigma_2}_{\frac{1}{2}\Delta t} u^n$$

(with consistency error of order $O(\Delta t^2)$), where each $S^{\Sigma_k}_{\Delta t}$ is computed by using the *modified* SLDG-RK2 scheme (consistency error of order $O(\Delta t^2)$). In that case, the global expected error is of order $O(\Delta t^2) + O(\frac{\Delta x^{k+1}}{\Delta t})$. Adaptation of Strang's splitting when $d \geq 2$ can also be considered.

General convection-diffusion equation. Let us consider

$$u_t - \frac{1}{2}Tr(AD^2u) + b \cdot \nabla u + ru = 0,$$

with $A = \sigma\sigma^T = \sum_k \Sigma_k \Sigma_k^T$ and $B = (b_1, \dots, b_d)^T$. In the constant coefficient case, the spatial differential operators commute and therefore we can use Trotter's splitting:

$$u^{n+1} = e^{-r\Delta t} S^{\Sigma_d}_{\Delta t}\ \cdots\ S^{\Sigma_2}_{\Delta t}\ S^{\Sigma_1}_{\Delta t} \mathcal{T}^1_{b_1\Delta t} \cdots \mathcal{T}^d_{b_d\Delta t} u^n$$

where S is one of the SLDG-RKp schemes ($p = 1, 2, \dots$).

For non-constant coefficients, we can use a generalization of the previous splitting techniques. In particular, Strang's splitting for matrices become for instance

$$e^{(A+B+C)h} = e^{A\frac{h}{4}} e^{B\frac{h}{2}} e^{A\frac{h}{4}} e^{Ch} e^{A\frac{h}{4}} e^{B\frac{h}{2}} e^{A\frac{h}{4}} + O(h^3).$$

4. Numerical examples

The first four examples are devoted to advection problems, while the other examples concern second order equations.

For advection equations (2), the CFL number will be defined by

$$\text{CFL} := \|b\|_\infty \frac{\Delta t}{\Delta x}.$$

Example 1. In this test we consider an advection equation in one dimension with smooth initial data:

$$v_t + v_x = 0 \qquad x \in (0,1),\ t \in (0,T), \tag{93}$$

$$v(0,x) = v_0(x) \qquad x \in (0,1), \tag{94}$$

using periodic boundary conditions on $\Omega = (0,1)$, terminal time $T = 1$, and the following smooth initial data:

$$v_0(x) := \sin(2\pi(x + \sin(2\pi x))). \tag{95}$$

Results for the L^2 error and corresponding orders are given in Table 1.

In this test we have fixed the CFL number to be 2.3, thus $\Delta t \equiv C\Delta x$ for some constant C, and the expected error is $O(\frac{\Delta x^{k+1}}{\Delta t}) = O(\Delta x^k)$.

We see that the expected order of convergence is well recovered. We have observed that the L^∞ and L^1 errors give similar result here. In this example the scheme and hence the results are similar to [18, Example 1] expected for the initial data.

	k=1		k=2		k=3		k=4	
Mesh M	error	order	error	order	error	order	error	order
23	8.01E-2	-	3.52E-3	-	4.09E-04	-	3.57E-05	-
46	1.44E-2	2.47	4.05E-4	3.11	2.16E-05	4.24	1.28E-06	4.80
92	2.42E-3	2.57	5.31E-5	2.93	1.25E-06	4.10	4.15E-08	4.94
184	4.79E-4	2.33	6.75E-6	2.97	7.81E-08	4.00	1.31E-09	4.98
368	1.09E-4	2.12	8.47E-7	2.99	4.85E-09	4.00	4.10E-11	4.99

TABLE 1. *(Example 1) advection test,* L^2 errors with initial data $v_0(x) = \sin(2\pi(x + \sin(2\pi x)))$ and CFL= 2.3.

Example 2. We consider now the case of an advection equation with non-constant advection term

$$v_t + b(x)v_x = 0, \qquad x \in (0,1),\ t \in (0,T), \tag{96}$$

$$v(0,x) = sin(2\pi x), \qquad x \in (0,1), \tag{97}$$

and

$$b(x) := C_0 + C_1 \sin(2\pi x), \qquad \text{with } C_0 = 1 \text{ and } C_1 := 0.8, \tag{98}$$

together with periodic boundary conditions on $(0,1)$. [1]

Results are given in Table 2 for $\Delta t \equiv \Delta x$ with fixed $CFL = 1.8$ and terminal time $T = 1.3$. The numerical error behaves approximatively as the expected one when $\Delta t = \lambda\Delta x$: it is of the order of $O(\frac{\Delta x^{k+1}}{\Delta t}) \equiv O(\Delta x^k)$.

Example 3 (2D advection) We consider a two-dimensional advection PDE

$$v_t + b_1 v_x + b_2 v_y = 0 \qquad (x,y) \in \Omega,\ t \in (0,T), \tag{99}$$

$$v(0,x,y) = \sin(2\pi(x+y)) \qquad (x,y) \in \Omega \tag{100}$$

with $\Omega = (0,1)^2$ and periodic boundary conditions, and with the constant dynamics $b_1 = b_2 = 1$.

Results are shown in Table 3, for $T = 1$. Here the scheme is based on a Trotter splitting.

[1]The exact solution is given by $v(t,x) = \sin(2\pi y_x(-t))$, where

$$y_x(-t) = \frac{1}{r}\text{atan}\Big(-r + \text{atan}\Big(\text{atan}(\frac{\tan(\pi x) + r}{a}) - C_0\pi a t\Big)\Big)$$

with $r := \frac{C_1}{C_0}$ and $a := \sqrt{1 - r^2}$.

L^2 error		k=1		k=2		k=3		k=4	
M	N	error	order	error	order	error	order	error	order
10	10	1.95E-01	-	3.45E-02	-	1.45E-02	-	7.83E-03	-
20	20	2.67E-02	1.93	6.06E-03	2.50	1.38E-03	3.39	2.33E-04	5.07
40	40	7.80E-03	1.77	6.39E-04	3.24	3.22E-05	5.42	4.31E-06	5.75
80	80	1.47E-03	2.40	3.62E-05	4.13	1.52E-06	4.40	7.74E-08	5.80
160	160	2.27E-04	2.69	3.31E-06	3.45	7.13E-08	4.41	2.48E-09	4.96
320	320	3.92E-05	2.53	4.03E-07	3.04	3.92E-09	4.18	8.03E-11	4.95

TABLE 2. *(Example 2) non-constant advection,* $\Delta t \equiv \Delta x$ and CFL= 1.8, $T = 1.3$.

L^2 error		k=1		k=2		k=3		k=4	
M	N	error	order	error	order	error	order	error	order
12	10	5.69E-03	-	1.86E-04	-	2.65E-06	-	1.09E-07	-
18	15	2.45E-03	2.072	5.68E-05	2.931	5.39E-07	3.92	1.44E-08	4.99
26	22	1.21E-03	1.925	1.99E-05	2.843	1.58E-07	3.32	2.44E-09	4.84
39	33	5.29E-04	2.043	5.88E-06	3.018	3.42E-08	3.78	3.22E-10	4.99

TABLE 3. *(Example 3) 2D advection,* $u_0(x, y) = \sin(2\pi(x + y))$, terminal time $T = 1$, and CFL=1.2.

Example 4 (2D advection with non-constant coefficients). We consider the following rotation example:

$$u_t + 2\pi(-x_2, x_1) \cdot \nabla u = 0, \qquad x = (x_1, x_2) \in \Omega,\ t \in (0, T),$$
$$u(0, x) = 1 - e^{-20((x_1-1)^2+x_2^2-r_0^2)},$$

with $\Omega := (-2, 2)^2$, $r_0 = 0.25$ and terminal time $T = 0.9$. Since $b(x_1, x_2) = 2\pi(-x_2, x_1)$ is non-constant, Trotter's splitting is no longer exact. In Table 4, we test and compare the splitting algorithms as described in subsection 2.3, from order 1 to 4 (Trotter's and Strang's splitting, 3rd order and 4th order splittings, tested with $k = 1, 2, 3$, and 4 respectively). We do not have a proof of stability for the 3rd and 4th order splittings, yet it is interesting here to observe the numerical stability and convergence. We have avoided taking the particular case of $T = 1$ (full turn) because it gives better numerical results but prevents to well analyze the order of the method.

In this example, the initial data is sufficiently close to 1 outside a ball of radius 1.5, so that the error coming from the boundary treatment is negligible.

L^2 error		Trotter		Strang		3rd order split.		4th order split.	
M	N	error	order	error	order	error	order	error	order
10	10	6.89E-01	-	2.91E-01	-	1.94E+00	-	2.26E-02	-
20	20	3.90E-01	0.82	6.62E-02	2.13	1.81E-01	3.42	8.10E-04	4.80
40	40	1.92E-01	1.02	1.60E-02	2.05	1.99E-02	3.18	3.46E-05	4.55
80	80	9.49E-02	1.02	3.99E-03	2.01	2.45E-03	3.02	1.80E-06	4.27
160	160	4.71E-02	1.01	9.96E-04	2.00	3.06E-04	3.00	1.07E-07	4.07

TABLE 4. *(Example 4), 2D rotation,* with $T = 0.9$.

Example 5 (1D convection diffusion). Now, we consider a diffusion equation

$$v_t - \frac{1}{2}\sigma^2 v_{xx} + b v_x = 0, \quad \forall x \in (0,1), \forall t \in (0,T) \tag{101}$$

$$v(0,x) = \cos(2\pi x) + \frac{1}{2}\cos(4\pi x) \tag{102}$$

together with periodic boundary conditions on $(0,1)$, with constants $\sigma = 0.1$, $b = 0.3$, and $T = 0.2$. The exact solution is given by

$$v(t,x) = \sum_{k=1,2} c_k \exp(-2\sigma^2 k^2 \pi^2 t) \cos(2k\pi(x - bt)),$$

with $c_1 = 1$ and $c_2 = \frac{1}{2}$.

Since the operators $\frac{1}{2}\sigma^2 \partial_x^2$ and $b\partial_x$ commute, we use the simple scheme

$$u^{n+1} = S^{\sigma}_{\Delta t} \mathcal{T}_{b\Delta t} u^n.$$

In Table 5 we study the orders of the SLDG-RKp schemes when $\Delta t \equiv \Delta x$ and $p \in \{1,2,3\}$. The orders are as expected. We also give in Table 6 the errors when taking larger time steps ($\Delta t \gg \Delta x$), still showing good behavior.

We have numerically also tested the case when $b = 0$ (pure diffusion), the numerical results are very close to the present case.

L^2 error		SLDG-RK1 (P_1)		SLDG-RK2 (P_2)		SLDG-RK3 (P_3)	
M	N	error	order	error	order	error	order
10	10	9.94E-03	-	1.37E-03	-	8.66E-05	-
20	20	1.39E-03	2.84	1.08E-04	3.67	3.70E-06	4.55
40	40	2.93E-04	2.25	3.63E-06	4.90	1.03E-07	5.17
80	80	8.02E-05	1.87	6.28E-07	2.53	9.81E-09	3.39
160	160	2.35E-05	1.77	9.72E-08	2.69	7.00E-10	3.81
320	320	8.22E-06	1.52	2.60E-08	1.90	5.79E-11	3.60
640	640	4.06E-06	1.02	6.17E-09	2.08	5.81E-12	3.32

TABLE 5. *Example 5 (1D diffusion),* SLDG-RKp schemes with $\Delta t \equiv \Delta x$, $p \in \{1,2,3\}$.

M	N	$\frac{\Delta t}{\Delta x}$	SLDG-RK1 (P_1)	SLDG-RK2 (P_2)	SLDG-RK3 (P_3)
20	10	0.40	1.37E-03	4.34E-05	1.79E-06
40	15	0.53	5.13E-04	6.87E-06	1.41E-07
80	20	0.80	1.39E-04	1.40E-06	1.11E-08
160	25	1.28	1.05E-04	1.83E-07	5.20E-10
320	30	2.13	8.49E-05	6.14E-08	3.09E-11
640	35	3.66	7.26E-05	4.35E-08	1.15E-11
1280	40	6.40	6.35E-05	3.31E-08	7.02E-12

TABLE 6. *Example 5 (1D diffusion),* SLDG-RKp with large time steps $\Delta t \gg \Delta x$, and $p \in \{1,2,3\}$.

Example 6 (1D diffusion with non-constant $\sigma(x)$). Now, we consider the following diffusion equation

$$v_t - \frac{1}{2}\sigma^2(x)v_{xx} = f(t,x), \qquad x \in (0,1),\ t \in (0,T) \tag{103}$$

$$v(0,x) = 0 \qquad x \in (0,1), \tag{104}$$

with periodic boundary conditions,

$$\sigma(x) := \sin(2\pi x),$$

and, for testing purposes, $f(t,x) := \bar{v}_t(t,x) - \frac{1}{2}\sigma^2(x)\bar{v}_{xx}(t,x)$ where $\bar{v}(t,x) := \sin(2\pi t)\cos(2\pi(x-t))$, which is therefore the exact solution ($v \equiv \bar{v}$).

In this case, in order to get higher than first-order accuracy in time, we use the "modified" SLDG-RK2 scheme. The source term $f(t,x)$ is integrated in the scheme in a standard way, by adding to the standard scheme value $v^{n+1}(x)$ the term

$$\Delta t f_t(t_n,x) + \frac{1}{2}\Delta t^2 f_{tt}(t_n,x)$$

at Gauss quadrature points.

In Table 7 we first check the accuracy with respect to time discretization, with fixed spatial mesh size.

Then, in Table 8 the errors are given for varying mesh sizes such that $\Delta t \equiv \Delta x$ and with P_1 or P_2 elements ($k = 1$ or $k = 2$). We find the expected orders, for SLDG-RKp, $p \in \{1,2\}$. However, the error is not as nice as in the Example 5.

L^2 error	SLDG-RK1		SLDG-RK2	
N	error	order	error	order
100	1.19E-03	-	1.89E-04	2.05
200	5.95E-04	1.01	4.57E-05	1.97
400	2.96E-04	1.01	1.16E-05	1.93
800	1.48E-04	1.00	3.07E-06	1.91
1600	7.40E-05	1.00	8.17E-07	1.92

TABLE 7. *Example 6 (1D diffusion with variable coefficient)*, varying time steps with fixed spatial mesh $M = 100$ and P_4 ($k = 4$); SLDG-RK2 is the modified scheme here.

L^2 error		SLDG-RK1 (P_1)		SLDG-RK2 (P_2)	
M	N	error	order	error	order
10	10	8.60E-02	-	4.13E-02	-
20	20	3.52E-02	1.29	7.30E-03	2.50
40	40	1.59E-02	1.15	1.39E-03	2.39
80	80	7.54E-03	1.08	3.03E-04	2.20
160	160	3.67E-03	1.04	7.17E-05	2.08
320	320	1.81E-03	1.02	1.80E-05	1.99

TABLE 8. *Example 6 (1D diffusion with variable coefficient)*, with $\Delta t \equiv \Delta x$; SLDG-RK2 is the modified scheme here.

Example 7 (2D diffusion) We consider the following two-dimensional diffusion equation:

$$u_t - \frac{1}{2}(5u_{xx} - 4u_{xy} + u_{yy}) = 0, \quad x \in \Omega,\ t \in (0,T), \tag{105}$$

$$u(0,x) = u_0(x), \quad x \in \Omega \tag{106}$$

set on $\Omega = (0,1)^2$ with periodic boundary conditions, and $T = 0.2$. The initial data is given by $u_0(x) = u_{01}(x+2y) + u_{02}(-y)$ and $u_{0i}(x) := \sum_{q=1,2} c_q^i \cos(2\pi q x)$ with the constant $c_q^i = \frac{1}{i+q}$. The exact solution is known.[2]

In order to define the numerical scheme, we use the fact that

$$A := \begin{bmatrix} 5 & -2 \\ -2 & 1 \end{bmatrix} = \sum_{k=1,2} \Sigma_k \Sigma_k^T, \qquad \text{with } \Sigma_1 := \begin{pmatrix} 1 \\ 0 \end{pmatrix},\ \Sigma_2 := \begin{pmatrix} 2 \\ -1 \end{pmatrix}.$$

Results are given in Table 9, where we consider variable time steps and mesh steps $\Delta t \sim \Delta x$, $p = k$, and expect a global error of order $O(\Delta t^p) + O(\frac{\Delta x^{k+1}}{\Delta t}) \equiv O(\Delta x^k)$.

L^2 error		SLDG-RK1 (P_1)		SLDG-RK2 (P_2)		SLDG-RK3 (P_3)	
M	N	error	order	error	order	error	order
10	10	6.66E-03	-	1.86E-04	-	2.20E-06	-
20	20	3.26E-03	1.02	4.52E-05	2.04	3.10E-07	2.83
40	40	1.61E-03	1.01	1.08E-05	2.06	3.20E-08	3.27
80	80	8.04E-04	1.00	2.69E-06	2.01	4.34E-09	2.88
160	160	4.01E-04	1.00	6.66E-07	2.01	4.90E-10	3.14

TABLE 9. *Example 7 (2D diffusion equation),* error table with $\Delta t \sim \Delta x$.

Example 8 (1D Black and Scholes and boundary conditions) This example deals with the one-dimensional Black-Scholes (B&S) PDE for the pricing of a European put option with one asset [23]. After a change of variable in logarithmic coordinates,[3] the equation for the European put option becomes on $\Omega := (x_{min}, x_{max})$:

$$\begin{cases} u_t - \frac{1}{2}\sigma^2 u_{xx} + b u_x + r u = 0, & x \in \Omega,\ t \in (0,T), \\ u(0,x) = u_0(x) = K\max(1-e^x, 0) & x \in \Omega, \\ u(t,x) = u_\ell(t) \equiv K e^{-rt} - K e^x & t \in (0,T),\ x \le x_{min}, \\ u(t,x) = u_r(t) \equiv 0 & t \in (0,T),\ x \ge x_{max}, \end{cases} \tag{107}$$

with $b := -(r - \frac{1}{2}\sigma^2)$ and where $x_{min} < 0$ and $x_{max} > 0$, and we have imposed boundary conditions outside of Ω. Numerically, the initial data presents a singular behavior at $x = 0$ (as it is only Lipschitz regular).

For this PDE the scheme reads

$$u^{n+1} = e^{-r\Delta t} S^{\sigma}_{\Delta t} \mathcal{T}^{b}_{\Delta t} u^n.$$

[2]Making the change of variable $\xi = (\xi_1, \xi_2)$ such that $\xi_1 = x + 2y$ and $\xi_2 = -y$ we find that $v(t,\xi) = u(t,x)$ satisfies $v_t - \frac{1}{2}(v_{\xi_1\xi_1} + v_{\xi_2\xi_2}) = 0$ and $v(0,\xi) = u_{01}(\xi_1) + u_{02}(\xi_2)$ and therefore the exact solution is given by $u(t,x) = v(t,\xi) = u_1(t,\xi_1) + u_2(t,\xi_2)$ where $u_i(t,\xi) = \sum_{q=1,2} c_q^i e^{-(2\pi q)^2 t/2} \cos(2\pi q \xi)$.

[3]The classical B&S PDE for the put option reads

$$v_t - \frac{1}{2}\sigma^2 s^2 v_{ss} - b s v_s + r v = 0, \qquad s \in (0,\infty),\ t \in (0,T),$$

(where $b = r - \frac{1}{2}\sigma^2$), with initial condition $v(0,s) = \varphi(s) \equiv \max(K - s, 0)$. Then using the change of variable $x = \log(s/K)$ and $u(t,x) := v(t,s)$, we obtain the PDE (107) on $x \in \mathbb{R}$.

The following financial parameters are used: $K = 100$ (strike price), $r = 0.10$ (interest rate), $\sigma = 0.2$ (volatility), and $T = 0.25$ (maturity). Since the interesting part of the solution lies in a neighborhood of $x = 0$ (notice that φ has a singularity at $x = 0$), for the computational domain we consider

$$\Omega = (x_{min}, x_{max}) :\equiv (-2, 2).$$

Results are reported in Table 10 for the L^1, L^2 and L^∞ errors, where Δt is chosen of the same order as Δx, and the SLDG-RK3 scheme is used ($k = 3$). We numerically observe an order between 4 and 5 (at least for the L^1 or L^2 errors), whereas the expected order is 3 (we also observe an order $\geq p + 1$ for SLDG-RKp schemes for $p = 1$ or $p = 2$). Presently, we do not understand why we have this unexpected "good" behavior.

Remark 4.1 (Boundary treatment)**.** *For semi-Lagrangian schemes, the knowledge of $u(t, x)$ for $x \leq x_{min}$ or $x \geq x_{max}$ can be used rather than only the boundary values $u(t, x_{min})$ or $u(t, x_{max})$. Here, out-of-bound values are needed for computing S^0v^n, $S^0S^0v^n$ and $S^0S^0S^0v^n$ for $v^n = \mathcal{T}^b_{\Delta t}u^n$. In particular, the values $u^n(x + k\sigma\sqrt{\Delta t} - b\Delta t)$ for $|k| \leq 3$ are used when $y := x + k\sigma\sqrt{\Delta t} - b\Delta t$ lies outside of (x_{min}, x_{max}). In that case, we simply directly use the "boundary" value $u_\ell(t_n, y)$ when $y \leq x_{min}$ or $u_r(t_n, y)$ when $y \geq x_{max}$.*

M	N	L^1-Error	Order	L^2-Error	Order	L^∞-Error	Order
10	10	2.99E-02	-	3.49E-02	-	6.22E-02	-
20	20	9.50E-04	4.98	1.50E-03	4.55	3.25E-03	4.26
40	40	5.04E-05	4.24	8.20E-05	4.19	2.82E-04	3.53
80	80	1.66E-06	4.92	2.75E-06	4.90	1.12E-05	4.65
160	160	4.79E-08	5.12	8.84E-08	4.96	4.49E-07	4.64

TABLE 10. *Example 8, 1D Black and Scholes PDE.* Error table with $\Delta t \sim \Delta x$, using SLDG-RK3 and P_3 polynomials.

APPENDIX A. INSTABILITY OF THE DIRECT SCHEME

Here we consider the "direct scheme", which defines naively at each time iteration a new piecewise polynomial $u^{n+1} \in V_k$ such that,

$$u^{n+1,i}_\alpha := u^n(x^i_\alpha - b\Delta t), \quad \text{for all Gauss points } x^i_\alpha.$$

In Figure 1, we consider again $v_t + v_x = 0$ with periodic boundary conditions on $(0, 1)$, and with the initial data $v_0(x) = \sin(2\pi x)$. We have represented two graphs with different choices of parameter N. In each graph is plotted the result of the direct scheme (green line) and of the SLDG scheme (red line) at time $T = 1$, with piecewise P_1 elements ($k = 1$) and fixed spatial mesh using $M = 46$ mesh steps. In the left graph, $N = 80$ time steps and both curves are confounded; in the right graph, $N = 320$, and the direct scheme becomes unstable. (We have found that the error behaves as $c^N\Delta x^{k+1}$ where $c > 1$, when using V_k elements.)

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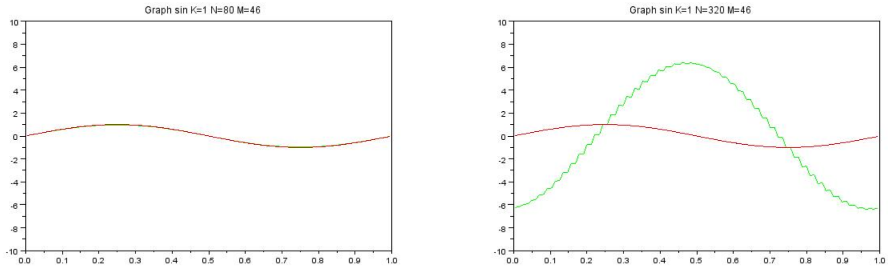

(A) $N = 80$ (B) $N = 320$

FIGURE 1. Results for $N = 80$ (left) and $N = 320$ (right), using P_1 elements with $M = 46$ in both cases. Instability appears on the right figure.

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